

Generation and Characterization of Fuzzy T -preorders

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Abstract. This paper studies T -preorders than can be generated in a natural way by a single fuzzy subset. These T -preorders are called one-dimensional and are of great importance, since every T -preorder can be generated by combining one-dimensional ones.

In the paper, the relation between fuzzy subsets generating the same T -preorder is given and one-dimensional T -preorders are characterized in two different ways: They generate linear crisp orderings on X and they satisfy a Sincov like functional equation. This last characterization is used to approximate a given T -preorder by a one-dimensional one by relating the issue to Saaty matrices used in Analytical Hierarchical Process. Finally, strong complete T -preorders, important in Decision Making problems, are also characterized.

Keywords. T -preorder, Representation Theorem, generator, dimension, strong complete T -preorder

1. Introduction

T -preorders were introduced by Zadeh in [7] and are very important fuzzy relations, since they fuzzify the concept of preorder on a set. Although there are many works studying their properties and applications to different fields, starting with [7], [6] [1], authors have not paid much attention to their relationship with the very important Representation Theorem. Roughly speaking the Representation Theorem states that every fuzzy subset μ of a set X generates a T -preorder P_μ on X in a natural way and that every T -preorder can be generated by a family of such special T -preorders.

The Representation Theorem provides us with a method to generate a T -preorder from a family of fuzzy subsets. These fuzzy subsets can measure the degrees in which different features are fulfilled by the elements of a universe X or can be the degrees of compatibility with different prototypes. Reciprocally, from a T -preorder a family (in fact many families) of fuzzy subsets can be obtained providing thus semantics to the relation.

This paper provides some results of T -preorders related to the Representation Theorem. Special attention is paid to one-dimensional T -preorders (i.e.: T -preorders generated by a single fuzzy subset) because they are the bricks out of which T -preorders are built. The fuzzy subsets that generate the same T -preorder are determined (Propositions 3.1 and 3.3 and one-dimensional T -preorders are characterized in two different ways: In Section 4 as relations generating linear preorders on X and in Section 5 by the use of Sincov-like functional equations (Propositions 5.6 and 5.7). Also, the relation with T -preorders and reciprocal matrices [5] will allow us to find a one-dimensional T -preorder close to a given one as explained in Section 6.

A strong complete T -preorder P on a set X is a T -preorder satisfying that for $x, y \in X$, either $P(x, y) = 1$ or $P(y, x) = 1$. These are interesting fuzzy relations used in fuzzy preference structures [4]. It is a direct consequence of Lemma 2.6 that one-dimensional T -preorders are strong complete, but there are strong complete T -preorders that are not one-dimensional. In Section 7 strong complete T -preorders are characterized using the Representation Theorem (Propositions 7.4 and 7.5).

The last section of the paper contains some concluding remarks and an interesting open problem: Which conditions must a couple of fuzzy subsets μ and ν fulfill in order to exist a t -norm T with $P_\mu = P_\nu$. Also the possibility of defining two dimensions (right and left) is discussed.

A section of preliminaries with the results and definitions needed in the rest of the paper follows.

2. Preliminaries

This section contains the main definitions and properties related mainly to T -preorders that will be needed in the rest of the paper.

Definition 2.1. [7] *Let T be a t -norm. A fuzzy T -preorder P on a set X is a fuzzy relation $P : X \times X \rightarrow [0, 1]$ satisfying for all $x, y, z \in X$*

$$2.1.1. P(x, x) = 1 \text{ (Reflexivity)}$$

$$2.1.2. T(P(x, y), P(y, z)) \leq P(x, z) \text{ (} T\text{-transitivity).}$$

Definition 2.2. *The inverse or dual R^{-1} of a fuzzy relation R on a set X is the fuzzy relation on X defined for all $x, y \in X$ by*

$$R^{-1}(x, y) = R(y, x).$$

Proposition 2.3. *A fuzzy relation R on a set X is a T -preorder on X if and only if R^{-1} is a T -preorder on X .*

Definition 2.4. *The residuation \overrightarrow{T} of a t -norm T is defined for all $x, y \in [0, 1]$ by*

$$\overrightarrow{T}(x|y) = \sup\{\alpha \in [0, 1] \text{ such that } T(\alpha, x) \leq y\}.$$

Example 2.5.

1. If T is a continuous Archimedean t -norm with additive generator t , then $\overrightarrow{T}(x|y) = t^{[-1]}(t(y) - t(x))$ for all $x, y \in [0, 1]$.

As special cases,

- If T is the Łukasiewicz t -norm, then $\overrightarrow{T}(x|y) = \min(1 - x + y, 1)$ for all $x, y \in [0, 1]$.
- If T is the Product t -norm, then $\overrightarrow{T}(x|y) = \min(\frac{y}{x}, 1)$ for all $x, y \in [0, 1]$.

2. If T is the minimum t -norm, then $\overrightarrow{T}(x|y) = \begin{cases} y & \text{if } x > y \\ 1 & \text{otherwise.} \end{cases}$

Lemma 2.6. Let μ be a fuzzy subset of X and T a t -norm. The fuzzy relation P_μ on X defined for all $x, y \in X$ by

$$P_\mu(x, y) = \overrightarrow{T}(\mu(x)|\mu(y))$$

is a T -preorder on X .

Theorem 2.7. Representation Theorem [6]. A fuzzy relation R on a set X is a T -preorder on X if and only if there exists a family $(\mu_i)_{i \in I}$ of fuzzy subsets of X such that for all $x, y \in X$

$$R(x, y) = \inf_{i \in I} R_{\mu_i}(x, y).$$

Definition 2.8. A family $(\mu_i)_{i \in I}$ in Theorem 2.7 is called a generating family of R and an element of a generating family is called a generator of R . The minimum of the cardinalities of such families is called the dimension of R ($\dim R$) and a family with this cardinality a basis of R .

A generating family can be viewed as the degrees of accuracy of the elements of X to a family of prototypes. A family of prototypes with low cardinality, especially a basis, simplifies the computations and gives clarity to the structure of X .

The next proposition states a trivial but important result.

Proposition 2.9. μ is a generator of R if and only if $R_\mu \geq R$.

Definition 2.10. Two continuous t -norms T, T' are isomorphic if and only if there exists a bijective map $\varphi : [0, 1] \rightarrow [0, 1]$ such that $\varphi \circ T = T' \circ (\varphi \times \varphi)$.

It is well known that all strict continuous Archimedean t -norms T are isomorphic. In particular, they are isomorphic to the Product t -norm and $T(x, y) = \varphi^{-1}(\varphi(x) \cdot \varphi(y))$

Also, all non-strict continuous Archimedean t -norms T are isomorphic. In particular, they are isomorphic to the Łukasiewicz t -norm and $T(x, y) = \varphi^{-1}(\max(\varphi(x) + \varphi(y) - 1, 0))$.

Proposition 2.11. If T, T' are two isomorphic t -norms, then their residuations $\overrightarrow{T}, \overrightarrow{T}'$ also are isomorphic (i.e. there exists a bijective map $\varphi : [0, 1] \rightarrow [0, 1]$ such that $\varphi \circ \overrightarrow{T} = \overrightarrow{T}' \circ (\varphi \times \varphi)$).

3. Generators of One-dimensional T -preorders

Let us recall that according to Definition 2.8 a T -preorder P on X is one dimensional if and only if there exists a fuzzy subset μ of X such that for all $x, y \in X$, $P(x, y) = \overrightarrow{T}(\mu(x)|\mu(y))$.

For a one-dimensional T -preorder P it is interesting to find all fuzzy subsets μ that are a basis of P (i.e.: $P = P_\mu$). The two next propositions answer this question for continuous Archimedean t -norms and for the minimum t -norm.

Proposition 3.1. *Let T be a continuous Archimedean t -norm, t an additive generator of T and μ, ν fuzzy subsets of X . $P_\mu = P_\nu$ if and only if $\forall x \in X$ the following condition holds:*

$$t(\mu(x)) = t(\nu(x)) + k \text{ with } k \geq \sup\{-t(\nu(x)) | x \in X\}$$

Moreover, if T is non-strict, then $k \leq \inf\{t(0) - t(\nu(x)) | x \in X\}$.

Proof.

\Rightarrow

If $\mu(x) \geq \mu(y)$, then

$$P_\mu(x, y) = \overrightarrow{T}(\mu(x)|\mu(y)) = t^{-1}(t(\mu(y)) - t(\mu(x)))$$

$$P_\nu(x, y) = t^{-1}(t(\nu(y)) - t(\nu(x)))$$

where $t^{[-1]}$ is replaced by t^{-1} because all the values in brackets are between 0 and $t(0)$.

If $P_\mu = P_\nu$, then

$$t(\mu(y)) - t(\mu(x)) = t(\nu(y)) - t(\nu(x)).$$

Let us fix $y_0 \in X$. Then

$$t(\mu(y_0)) - t(\mu(x)) = t(\nu(y_0)) - t(\nu(x)).$$

and

$$t(\mu(x)) = t(\nu(x)) + t(\mu(y_0)) - t(\nu(y_0)) = t(\nu(x)) + k$$

\Leftrightarrow Trivial thanks to Example 2.5.1. □

Example 3.2. *With the previous notations,*

- *If T is the Łukasiewicz t -norm, then*

$$\mu(x) = \nu(x) + k \text{ with } \inf_{x \in X} \{1 - \nu(x)\} \geq k \geq \sup_{x \in X} \{-\nu(x)\}.$$

- *If T is the product t -norm, then*

$$\mu(x) = \frac{\nu(x)}{k} \text{ with } k \geq \sup_{x \in X} \{\nu(x)\}.$$

Proposition 3.3. Let T be the minimum t -norm, μ a fuzzy subset of X and x_M an element of X with $\mu(x_M) \geq \mu(x) \forall x \in X$. Let $Y \subset X$ be the set of elements x of X with $\mu(x) = \mu(x_M)$ and $s = \sup\{\mu(x) \text{ such that } x \in X - Y\}$. A fuzzy subset ν of X generates the same T -preorder than μ if and only if

$$\forall x \in X - Y \mu(x) = \nu(x) \text{ and } \nu(y) = t \text{ with } s \leq t \leq 1 \forall y \in Y.$$

Proof. It follows easily from the fact that

$$P_\mu(x, y) = \begin{cases} \mu(y) & \text{if } \mu(x) > \mu(y) \\ 1 & \text{otherwise.} \end{cases}$$

□

At this point, it seems that the dimension of P and of P^{-1} should coincide, but this is not true in general as we will show in the next example. Nevertheless, for continuous Archimedean t -norms they do coincide in most of the cases as will be proved in Proposition 3.5.

Example 3.4. Consider the one-dimensional min-preorder P of $X = \{x_1, x_2, x_3\}$ generated by the fuzzy subset $\mu = (0.8, 0.7, 0.4)$. Its matrix is

$$\begin{array}{ccc} & x_1 & x_2 & x_3 \\ \begin{array}{l} x_1 \\ x_2 \\ x_3 \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ 0.7 & 1 & 1 \\ 0.4 & 0.4 & 1 \end{pmatrix} \end{array}$$

while the matrix of P^{-1} is

$$\begin{array}{ccc} & x_1 & x_2 & x_3 \\ \begin{array}{l} x_1 \\ x_2 \\ x_3 \end{array} & \begin{pmatrix} 1 & 0.7 & 0.4 \\ 1 & 1 & 0.4 \\ 1 & 1 & 1 \end{pmatrix} \end{array}$$

which clearly is not one-dimensional.

Proposition 3.5. Let T be a continuous Archimedean t -norm, t an additive generator of T , μ a fuzzy subset of X and P_μ the T -preorder generated by μ . Then P_μ^{-1} is generated by the fuzzy subset ν of X such that $t(\nu(x)) = -t(\mu(x)) + k$.

Proof.

$$\begin{aligned} P_\mu^{-1}(x, y) &= P_\mu(y, x) = t^{[-1]}(t(\mu(x)) - t(\mu(y))) \\ &= t^{[-1]}(-t(\mu(y)) + k + t(\mu(x)) - k) \\ &= t^{[-1]}(t(\nu(y)) - t(\nu(x))) = P_\nu(x, y). \end{aligned}$$

□

Example 3.6.

- If T is the t -norm of Łukasiewicz, μ a fuzzy subset of X and P the T -preorder on X generated by μ (i.e.: $P = P_\mu$), then P^{-1} is generated by $k - \mu$, with $\sup_{x \in X} \{\mu(x)\} \leq k \leq 1 + \inf_{x \in X} \{\mu(x)\}$.
- If T is the product t -norm, μ a fuzzy subset of X such that $\inf_{x \in X} \{\mu(x)\} > 0$ and P the T -preorder on X generated by μ (i.e.: $P = P_\mu$), then P^{-1} is generated by $\frac{k}{\mu}$, with $0 < k \leq \inf_{x \in X} \{\mu(x)\}$.

Hence, the dimensions of a T -preorder P and its inverse P^{-1} coincide when T is the t -norm of Łukasiewicz (and any other continuous non-strict Archimedean t -norm) while for the product t -norm (and any other continuous strict t -norm) coincide when $\inf_{x,y \in X} \{P(x,y)\} \neq 0$.

4. A Characterization of One-dimensional T -preorders

For a continuous Archimedean t -norm, a one-dimensional T -preorder P on a set X is characterized by the fact that it generates a linear ordering \leq on X such that if $x \leq y \leq z$, then equality holds in 2.1.2. (i.e.: $T(P(x,y), P(y,z)) = P(x,z)$).

Proposition 4.1. *Let T be a continuous Archimedean t -norm and P a T -preorder on X such that $P(x,y) > 0$ for all $x,y \in X$. P is one-dimensional if and only if there exists a total order \leq_P on X with first element a and last element b such that if $x \leq_P y \leq_P z$, then $T(P(x,y), P(y,z)) = P(x,z)$.*

Proof.

\Rightarrow

Let μ be a fuzzy set of X generating P (i.e. $P = P_\mu$). Consider a total order \leq_P on X such that if $x \leq_P y$, then $\mu(x) \geq \mu(y)$.

If $x \leq_P y \leq_P z$, then

$$\begin{aligned} & T(P(x,y), P(y,z)) \\ &= t^{[-1]}(t(t^{-1}(t(\mu(y)) - t(\mu(x))) + t(t^{-1}(t(\mu(z)) - t(\mu(y)))))) \\ &= t^{[-1]}(t(\mu(z)) - t(\mu(x))) = P(x,z). \end{aligned}$$

\Leftarrow) Consider the fuzzy subset μ_a of X defined for all $x \in X$ by $\mu_a(x) = P(a,x)$.

If $a \leq_P x \leq_P y$, then

$$T(P(a,x), P(x,y)) = P(a,y).$$

Hence,

$$P(x,y) = \overrightarrow{T}(P(a,x)|P(a,y)) = \overrightarrow{T}(\mu_a(x)|\mu_a(y)).$$

and μ_a is a generator of P . □

5. Sincov Functional Equation

For continuous Archimedean t-norms, one-dimensionality is related to the fulfillment of the equation $T(P(x, y), P(y, z)) = P(x, z)$ constrained to the condition $x \leq_P y \leq_P z$. The resemblance of this equation to the functional equation of Sincov 5.1 will allow us to provide a new characterization of one-dimensional T -preorders relating them to Sincov-like functional equations (Propositions 5.6 and 5.7). In the next section, this characterization will be used to approximate a given T -preorder by a one-dimensional one.

Definition 5.1. [2] *A mapping $F : X \times X \rightarrow \mathbb{R}$ satisfies the Sincov functional equation if and only if for all $x, y, z \in X$ we have*

$$F(x, y) + F(y, z) = F(x, z).$$

The following result characterizes the mappings satisfying Sincov equation.

Proposition 5.2. [2] *A mapping $F : X \times X \rightarrow \mathbb{R}$ satisfies the Sincov functional equation if and only if there exists a mapping $g : X \rightarrow \mathbb{R}$ such that*

$$F(x, y) = g(y) - g(x)$$

for all $x, y \in X$.

Proposition 5.3. *The real line \mathbb{R} with the operation $*$ defined by $x * y = x + y - 1$ for all $x, y \in \mathbb{R}$ is an Abelian group with 1 as the identity element. The opposite of x is $-x + 2$.*

Replacing the addition by this operation $*$ we obtain a Sincov-like functional equation:

Proposition 5.4. *Let $F : X \times X \rightarrow \mathbb{R}$ be a mapping. F satisfies the functional equation*

$$F(x, y) * F(y, z) = F(x, z) \tag{1}$$

if and only if there exists a mapping $g : X \rightarrow \mathbb{R}$ such that

$$F(x, y) = g(y) - g(x) + 1$$

for all $x, y \in X$.

Proof. The mapping $G(x, y) = F(x, y) - 1$ satisfies the Sincov functional equation and so $G(x, y) = g(y) - g(x)$. \square

Replacing the addition by multiplication we obtain another Sincov-like functional equation:

Proposition 5.5. *Let $F : X \times X \rightarrow \mathbb{R}^+$ be a mapping. F satisfies the functional equation*

$$F(x, y) \cdot F(y, z) = F(x, z) \tag{2}$$

if and only if there exists a mapping $g : X \rightarrow \mathbb{R}^+$ such that

$$F(x, y) = \frac{g(y)}{g(x)}$$

for all $x, y \in X$.

Proof. Simply calculate the logarithm of both hand sides of the functional equation to transform it to Sincov functional equation. \square

If μ is a fuzzy subset of X , we can consider μ as a mapping from X to \mathbb{R} or to \mathbb{R}^+ . This will allow us to characterize one-dimensional T -preorders on X when T is the Łukasiewicz or the Product t-norm.

Proposition 5.6. *Let T be the Łukasiewicz t-norm and P a T -preorder on X . Consider $F : X \times X \rightarrow \mathbb{R}$ defined by*

$$F(x, y) = \begin{cases} 2 - P(x, y) & \text{if } P(x, y) \leq P(y, x) \\ P(x, y) & \text{otherwise.} \end{cases}$$

F satisfies equation (1) if and only if P is a one-dimensional T -preorder on X .

Proposition 5.7. *Let T be the Product t-norm and P a T -preorder on X . Consider $F : X \times X \rightarrow \mathbb{R}^+$ defined by*

$$F(x, y) = \begin{cases} \frac{1}{P(x, y)} & \text{if } P(x, y) \leq P(y, x) \\ P(x, y) & \text{otherwise.} \end{cases}$$

F satisfies equation (2) if and only if P is a one-dimensional T -preorder on X .

Due to the isomorphisms of continuous Archimedean t-norms to the Łukasiewicz and Product t-norms, the previous results allow us to characterize one-dimensional T -preorders for any continuous Archimedean t-norm.

6. Approximating T -preorders by One-dimensional Ones

Consider the cases where different sensor give values to a phenomenon (e.g., the temperature of a room during a day). Due to inaccuracy of the sensors, rounding and noise, the preorders generated by the data of the different sensors will not coincide and if we aggregate them, a not one-dimensional T -preorder will be obtained. In this situation, the obtained T -preorder should be replaced by a one-dimensional one close to it.

6.1. The Product t-norm Case

In this subsection we will tackle this problem using ideas from Saaty for the Product t-norm. The definition of reciprocal and consistent matrix will be recalled and related to the previous section. Then the eigenvalues method developed by Saaty [5] and the Logarithmic Least Squares Method [3] will be used to approximate a given T -preorder (in fact any reflexive fuzzy relation) by a one-dimensional T -preorder.

Definition 6.1. [5] An $n \times n$ real matrix A with entries $a_{ij} > 0$, $1 \leq i, j \leq n$, is reciprocal if and only if $a_{ij} = \frac{1}{a_{ji}} \forall i, j = 1, 2, \dots, n$. A reciprocal matrix is consistent if and only if $a_{ik} = a_{ij} \cdot a_{jk} \forall i, j = 1, 2, \dots, n$.

If the cardinality of X is finite (i.e.: $X = \{x_1, x_2, \dots, x_n\}$), then we can associate the matrix $A = (a_{ij})$ with entries $a_{ij} = F(x_i, x_j)$ with every map $F : X \times X \rightarrow \mathbb{R}^+$. Then F satisfies equation (2) if and only if A is a reciprocal consistent matrix as defined in [5].

Proposition 5.5 can be rewritten in this context by

Proposition 6.2. [5] An $n \times n$ real matrix A is reciprocal and consistent if and only if there exists a mapping g of X such that

$$a_{ij} = \frac{g(x_i)}{g(x_j)} \quad \forall i, j = 1, 2, \dots, n.$$

For a given reciprocal matrix A , Saaty obtains a consistent matrix A' close to A [5]. A' is generated by an eigenvector associated with the greatest eigenvalue of A (i.e.: a mapping g satisfying the previous Proposition 6.2) and fulfills the following properties.

1. If A is already consistent, then $A = A'$.
2. If A is a reciprocal positive matrix, then the sum of its eigenvalues is n .
3. If A is consistent, then there exist a unique eigenvalue $\lambda_{\max} = n$ different from zero.
4. Slight modifications of the entries of A produce slight changes to the entries of A' .

We can associate a reciprocal matrix with a T -preorder as in Proposition 5.7.

Definition 6.3. [5] The reciprocal matrix $A = (a_{ij})$ associated with a T -preorder $P = (p_{ij})$, $i, j = 1, 2, \dots, n$ is defined by

$$a_{ij} = p_{ij} \text{ if } p_{ij} \leq p_{ji}$$

$$a_{ij} = \frac{1}{p_{ji}} \text{ if } p_{ij} > p_{ji}.$$

Then in order to obtain a one dimensional T -preorder P' close to a given one P (T the Product t -norm), the following procedure can be used:

- Calculate the consistent reciprocal matrix A associated with P .
- Find an eigenvector v of the greatest eigenvalue of A .
- Divide the coordinates of v by the greatest one to obtain a fuzzy subset μ .
- $P' = P_\mu$.

Example 6.4. Let T be the Product t -norm and P the T -preorder on a set X of cardinality 5 given by the following matrix.

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0.74 & 1 & 1 & 1 & 1 \\ 0.67 & 0.87 & 1 & 1 & 1 \\ 0.50 & 0.65 & 0.74 & 1 & 1 \\ 0.41 & 0.53 & 0.60 & 0.80 & 1 \end{pmatrix}.$$

Its associated reciprocal matrix A is

$$A = \begin{pmatrix} 1 & 1.3514 & 1.4925 & 2.0000 & 2.4390 \\ 0.7400 & 1 & 1.1494 & 1.5385 & 1.8868 \\ 0.6700 & 0.8700 & 1 & 1.3514 & 1.6667 \\ 0.5000 & 0.6500 & 0.7400 & 1 & 1.2500 \\ 0.4100 & 0.5300 & 0.6000 & 0.8000 & 1 \end{pmatrix}.$$

Its greatest eigenvalue is 5.0003 and an eigenvector for 5.0003 is

$$\mu = (1, 0.76, 0.67, 0.50, 0.40).$$

This fuzzy set generates P_μ which is a one dimensional T -preorder close to P .

$$P_\mu = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0.76 & 1 & 1 & 1 & 1 \\ 0.67 & 0.88 & 1 & 1 & 1 \\ 0.50 & 0.66 & 0.74 & 1 & 1 \\ 0.40 & 0.53 & 0.60 & 0.81 & 1 \end{pmatrix}.$$

The second most popular method to approximate a reciprocal matrix by a reciprocal and consistent matrix is the Logarithmic Least Squares Method (LLSM) that will be recalled below and used as an alternative method to approximate a T -preorder by a one-dimensional one.

Definition 6.5. For a given reciprocal matrix $A = (a_{ij})$ $i, j = 1, 2, \dots, n$, the Logarithmic Least Squares Method (LLSM) obtains a vector of positive coordinates $w = (w_1, w_2, \dots, w_n)$ generating a reciprocal and consistent matrix close to A by minimizing the function

$$\sum_{i,j=1}^n \left(\ln a_{ij} - \ln \frac{w_i}{w_j} \right)^2$$

subject to the condition

$$\prod_{i=1}^n w_i = 1.$$

Proposition 6.6. [3] The solution of the LLSM is given by

$$w_i = \left(\prod_{j=1}^n a_{ij} \right)^{\frac{1}{n}} \quad i = 1, 2, \dots, n.$$

So, the different w_i are the geometric mean of the entries of the i^{th} file of A .

From this result, the following alternative method to approximate a reflexive fuzzy relation P by a one-dimensional T -preorder can be obtained.

- Calculate the consistent reciprocal matrix A associated with P .
- Find the vector w from A using the LLSM.
- Divide the coordinates of w by its greatest coordinate to obtain a vector μ with values between 0 and 1 (i.e. a fuzzy subset).
- $P' = P_\mu$.

Example 6.7. Consider the same T -preorder P as in Example 6.4.

The vector w is $w = (1.58, 1.20, 1.06, 0.79, 0.64)$ and dividing the coordinates of w by 1.58 we obtain $\mu = (1, 0.76, 0.67, 0.50, 0.40)$

The obtained one-dimensional T -preorder $P' = P_\mu$ coincides with the one obtained in Example 6.4.

The results of this subsection can be easily generalized to continuous strict Archimedean t-norms.

If T' is a continuous strict Archimedean t-norm, then it is isomorphic to the Product t-norm T . Let φ be this isomorphism. If P is a T' -preorder, then $\varphi \circ P$ is a T -preorder. We can find P' one-dimensional close to $\varphi \circ P$ as before. Since isomorphisms between continuous t-norms are continuous and preserve dimensions, $\varphi^{-1} \circ P'$ is a one-dimensional T' -preorder close to P .

6.2. The Łukasiewicz t-norm Case

In this subsection we will modify the last method in order to obtain a one-dimensional T -preorder close to a given reflexive fuzzy relation P when T is the Łukasiewicz t-norm. For that, instead of generating a reciprocal matrix associated with P we will generate the same matrix F of Proposition 5.6 and minimize a function of F . This procedure will be called the Łukasiewicz Least Squares Method (ŁLSM).

Definition 6.8. Given a matrix $A = (a_{ij})$ $i, j = 1, 2, \dots, n$, the Łukasiewicz Least Squares Method (ŁLSM) obtains a vector $w = (w_1, w_2, \dots, w_n)$ generating a matrix $B = (b_{ij})$, $i, j = 1, 2, \dots, n$ with $b_{ij} = 2 - b_{ji}$ close to A by minimizing the function

$$\sum_{i,j=1}^n (a_{ij} - (1 - w_i + w_j))^2$$

subject to the condition

$$\sum_{i=1}^n w_i = n.$$

Proposition 6.9. The solution of the Łukasiewicz Least Squares Method is given by

$$w_i = \frac{\sum_{j=1}^n a_{ij}}{n} \quad i = 1, 2, \dots, n.$$

Proof. Imposing the gradient of $\sum_{i,j=1}^n (a_{ij} - (1 - w_i + w_j))^2$ to be 0, we obtain

$$-nw_j + \sum_{i=1}^n a_{ij} + \sum_{i=1}^n w_i - n = 0.$$

Considering the condition $\sum_{i=1}^n w_i = n$ we obtain the desired result. \square

From this last result we can give a method to obtain a one-dimensional T -preorder P' close to a given reflexive fuzzy relation.

- Calculate the associated matrix A associated with P as in Proposition 5.6.
- Find the vector w from A using the $\mathbb{L}LSM$.
- Subtract the smallest coordinate w to all the coordinates of w to obtain a vector μ with values between 0 and 1 (i.e. a fuzzy subset).
- $P' = P_\mu$.

Example 6.10. Let T be the Łukasiewicz t -norm and P the T -preorder on a set X of cardinality 5 given by the following matrix.

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0.7 & 1 & 1 & 1 & 1 \\ 0.5 & 0.8 & 1 & 1 & 1 \\ 0.3 & 0.5 & 0.7 & 1 & 1 \\ 0.1 & 0.3 & 0.5 & 0.8 & 1 \end{pmatrix}.$$

Its associated matrix A is

$$A = \begin{pmatrix} 1 & 1.3 & 1.5 & 1.7 & 1.9 \\ 0.7 & 1 & 1.2 & 1.5 & 1.7 \\ 0.5 & 0.8 & 1 & 1.3 & 1.5 \\ 0.3 & 0.5 & 0.7 & 1 & 1.2 \\ 0.1 & 0.3 & 0.5 & 0.8 & 1 \end{pmatrix}.$$

The vector w is $w = (1.48, 1.22, 1.02, 0.74, 0.54)$. Subtracting 0.54 to the coordinates of w we get $\mu = (0.94, 0.68, 0.48, 0.20, 0.00)$ and P_μ is

$$P_\mu = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0.74 & 1 & 1 & 1 & 1 \\ 0.54 & 0.80 & 1 & 1 & 1 \\ 0.26 & 0.52 & 0.72 & 1 & 1 \\ 0.06 & 0.32 & 0.52 & 0.88 & 1 \end{pmatrix}.$$

Again, due to the isomorphisms between continuous Archimedean non-strict t -norms and the Łukasiewicz t -norm, the results of this subsection allow us to approximate reflexive fuzzy relations by one-dimensional T -preorders when T is continuous Archimedean non-strict.

7. Strong Complete T -preorders

Definition 7.1. [4] A T -preorder P on a set X is a strong complete T -preorder if and only if for all $x, y \in X$,

$$\max(P(x, y), P(y, x)) = 1.$$

Of course every one-dimensional fuzzy T -preorder is a strong complete T -preorder, but there are strong complete T -preorders that are not one-dimensional. In Propositions 7.3 and 7.5 these fuzzy relations will be characterized exploiting the fact that they generate crisp linear orderings.

Lemma 7.2. Let μ be a generator of a strong complete T -preorder P on X . If $P(x, y) = 1$, then $\mu(x) \leq \mu(y)$.

Proof. Trivial, since $\overrightarrow{T}(\mu(x)|\mu(y)) = P_\mu(x, y) \geq P(x, y) = 1$. □

Proposition 7.3. Let μ, ν be two generators of a strong complete T -preorder P on X . Then for all $x, y \in X$, $\mu(x) \leq \mu(y)$ if and only if $\nu(x) \leq \nu(y)$.

Proof. Given $x, y \in X$, $x \neq y$, let us suppose that $P(x, y) = 1$ (and $P(y, x) < 1$). Then $\mu(x) \leq \mu(y)$ and $\nu(x) \leq \nu(y)$. □

Proposition 7.4. Let P be a strong complete T -preorder on a set X . The elements of X can be totally ordered in such a way that if $x \leq y$, then $P(x, y) = 1$.

Proof. Consider the relation \leq on X defined by $x \leq y$ if and only if $\mu(x) \leq \mu(y)$ for any generator μ of P . (If for $x \neq y$, $\mu(x) = \mu(y)$ for any generator, then chose either $x < y$ or $y < x$). □

Reciprocally,

Proposition 7.5. If for any couple of generators μ and ν of a T -preorder P on a set X $\mu(x) \leq \mu(y)$ if and only if $\nu(x) \leq \nu(y)$, then P is strong complete.

Proof. Trivial. □

8. Concluding Remarks

T -preorders have been studied with the help of its Representation Theorem. The different fuzzy subsets generating the same T -preorder have been characterized and the relation between one-dimensional T -preorders, Sincov-like functional equations and Saaty's reciprocal matrices has been studied. Also strong complete T -preorders have been characterized.

We end pointing at two directions toward a future work.

- We can look at the results of Section 3 from a different point of view: Let us suppose that we obtain two different fuzzy subsets μ and ν of a universe X by two different measurements or by two different experts. It would be interesting to know in which conditions we could assure the existence of a (continuous Archimedean) t -norm for which $P_\mu = P_\nu$.

- A fuzzy subset μ of a set X generates a T -preorder P_μ by $P_\mu(x, y) = \overrightarrow{T}(\mu(x)|\mu(y))$, but also another T -preorder ${}_\mu P(x, y) = \overrightarrow{T}(\mu(y)|\mu(x))$ (in fact, the inverse P_μ^{-1} of P_μ). In this way we could define two dimensions of a T -preorder according whether we consider it generated by families $(P_{\mu_i})_{i \in I}$ or by families $({}_{\mu_i} P)_{i \in I}$. For instance the min-preorder of Example 3.4 would have right dimension 1 and left dimension 2.

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