Lecture XX

Components with higher and lower risk in a reliability system[1]

J. Freixas and M. Pons

[1] Universitat Politècnica de Catalunya (Campus Manresa), Departament de Matemàtica Aplicada 3, Spain, josep.freixas@upc.edu; montserrat.pons@upc.edu.

Abstract

A new reliability importance measure for components in a system, that we call Representativeness measure, is introduced. It evaluates to which extent the performance of a component is representative of the performance of the whole system. Its relationship with Birnbaum's measure is analyzed, and the ranking of components given by both measures are compared. These rankings happen to be equal when all components have the same reliability but different in general. In contrast with Birnbaum's, the Representativeness reliability importance measure of a component does depend on its reliability.

1. Introduction

Many reliability importance measures (RIM) have been defined up to now, see Kuo and Zhu (2012) for a complete and detailed survey on importance measures. The most well–known, proposed by Birnbaum (1969), evaluates the contribution of a component to the system reliability by the rate at which the system reliability improves as the reliability of the component improves.
In this work we propose a new index for measuring the reliability importance of a component. This index evaluates the degree of "coupling" between one component and the whole system, i.e., the likelihood that the system and the component are in the same state. We call it Representativeness Measure because by selecting and analyzing the performance of components with a high value of this measure we can have an acceptable estimation of the performance of the system. In this sense we think that this measure can be useful for maintenance purposes. This measure gives rise to an associated structural importance measure when all components have the same reliability equal to one half.
In section 2 the context of the work is established and some basic definitions are given. The Representativeness RIM and SIM are defined in section 3 and their main properties are proved. In section 4 we compare these new measures with Birnbaum measures. All proofs of the different statements are omitted but available on request.

Notations:

N	set of components, which are assumed to be numbered consecutively from 1 to n.
x_i	binary random variable associated to component i: $x_i = 1$ if i is functioning, $x_i = 0$ otherwise.
$\phi(\mathbf{x})$	binary structure function of the system: $\phi(\mathbf{x}) = 1$ if system is functioning, $\phi(\mathbf{x}) = 0$ otherwise.
p_i	reliability of the ith component, $Pr\{x_i = 1\}$.
$\mathbf{p}$	reliability vector, $\mathbf{p} = (p_1, p_2, \dots, p_n)$.
$(a_i, \mathbf{p})$	vector $\mathbf{p}$ with the ith component of $\mathbf{p}$ changed to a.
π	collection of path sets of the system.
(N, π)	system defined by the collection π of path sets.
$h(\mathbf{p})$	system reliability, $Pr\{\phi(\mathbf{x}) = 1\}$.
$I_i^B(\mathbf{p})$	Birnbaum RIM of component i.
$I_i^{Bs}(\mathbf{p})$	Birnbaum SIM of component i.
$I_i^R(\mathbf{p})$	Representativeness RIM of component i.
$I_i^{Rs}(\mathbf{p})$	Representativeness SIM of component i.

Assumptions:

1) the random variables x_i are mutually statistically independent.
2) the systems we consider are coherent:
   - the structure function ϕ is nondecreasing,
   - all components are relevant.
3) $0 < p_i < 1$ for any component i.

2. Fundamentals

We consider binary systems, i.e., systems in which there is a random variable associated to each component which takes the value 1 if it is functioning and 0 otherwise. These random variables are assumed to be mutually statistically independent. The structure function of the system is a binary function of these random variables that takes the value 1 if the system is functioning and 0 otherwise. We assume that this structure function is nondecreasing.

The structure function of a system can be expressed in terms of its path sets. This is why we usually denote a system by (N, π), where $N = \{1, 2, \dots, n\}$ is the set of components and π is the collection of path sets. Recall that a path set is a set of components which by functioning ensure that the system is functioning. A path set is minimal if it does not have proper path subsets. A component i is relevant if it belongs to at least one minimal path set.

The systems we consider have a nondecreasing structure function and all the components are relevant. They are referred to as coherent systems.

The reliability $h(\mathbf{p})$ of the system, i.e., the probability of it being functioning, depends on its structure function and on the reliability of each component. If p_i is the reliability of component i (the probability of it being functioning), and $\mathbf{p}$ is the reliability vector $(p_1, p_2, \dots, p_n)$, then the system reliability can be expressed by means of path sets:

$$h(\mathbf{p}) = \sum_{S \in \pi} \prod_{i \in S} p_i \prod_{i \in N \setminus S} (1 - p_i).$$

Recall also that, by a conditional probability argument, for any component i we get:

$$h(\mathbf{p}) = p_i h(1_i, \mathbf{p}) + (1 - p_i) h(0_i, \mathbf{p})$$

and from this expression we obtain:

$$\frac{\partial h}{\partial p_i}(\mathbf{p}) = h(1_i, \mathbf{p}) - h(0_i, \mathbf{p}).$$

From now on we assume that (N, π) is a fixed coherent system, and all definitions and properties are to be thought in this system even though we do not specify their dependency on it. Also, $\mathbf{p}$ denote an arbitrary reliability vector with $\mathbf{p} \in (0,1)^n$, and i an arbitrary component in N.

Definition 2.1. Birnbaum Importance Measures

a) Birnbaum RIM: $I_i^B(\mathbf{p}) = \dfrac{\partial h}{\partial p_i}(\mathbf{p})$

b) Birnbaum SIM: $I_i^{Bs} = I_i^B(\mathbf{1/2})$ where $\mathbf{1/2} = (1/2, \ldots, 1/2)$

3. The Representativeness importance measures

In this section we start by defining the Representativeness reliability importance measure (RIM). The aim of introducing it is to have a measure of the degree of "agreement" between the performance of a given component and the performance of the whole system.

Definition 3.1. Representativeness Importance Measures

a) Representativeness RIM: $I_i^R(\mathbf{p}) = p_i h(1_i, \mathbf{p}) + (1 - p_i)(1 - h(0_i, \mathbf{p}))$

b) Representativeness SIM: $I_i^{Rs} = I_i^R(\mathbf{1/2})$ where $\mathbf{1/2} = (1/2, \ldots, 1/2)$

The first basic properties of the Representativeness RIM are established in the following proposition:

Proposition 3.1.

i) $0 \le min\{p_i, 1 - p_i\} \le I_i^R(\mathbf{p}) \le 1$.

ii) $I_i^R(\mathbf{p}) \ge I_i^B(\mathbf{p})$.

iii) $I_i^R(\mathbf{p}) = (1 - p_i)[1 - 2h(0_i, \mathbf{p})] + h(\mathbf{p})$.
$I_i^R(\mathbf{p}) = p_i[2h(1_i, \mathbf{p}) - 1] + 1 - h(\mathbf{p})$.

iv) $I_i^R(\mathbf{p})$ is a linear function of p_i.

v) $I_i^R(\mathbf{p})$ is non–decreasing in p_i if and only if $h\left((1/2)_i, \mathbf{p}\right) \ge 1/2$.

Let now focus our attention in the Representativeness SIM.

Proposition 3.2. For any component $i \in N$ it is

$$I_i^{Rs} = 1/2 + 1/2\ I_i^{Bs}$$

Observe that we can have $I_i^R(\mathbf{p}) > 1/2 + 1/2\ I_i^B(\mathbf{p})$ and $I_i^R(\mathbf{p}) < 1/2 + 1/2\ I_i^B(\mathbf{p})$ in a system, for the same components i, j, depending only on the reliability vector $\mathbf{p}$, as we show in the following example.

Example 3.1. Consider a 3-out-of-4 system.
If we consider all components having the same reliability p then the Birnbaum RIM and the Representativeness RIM of all components are the same (thus we omit here the arbitrary subscript i) and their respective values are:

$$I^B(p) = 3p^2(1-p) \quad \text{and} \quad I^R(p) = p^3(2-p) + 1 - p.$$

Now, if $p = 0.6$ it is $I^B(0.6) = 0.432$ and $I^R(0.6) = 0.7024$ so that $I^R(0.6) < 1/2 + 1/2\ I^B(0.6)$,
but for $p = 0.4$ it is $I^B(0.4) = 0.288$ and $I^R(0.4) = 0.7024$ so that $I^R(0.4) > 1/2 + 1/2\ I^B(0.4)$.

In the next theorem we prove that $\mathbf{p} = \mathbf{1/2}$ is the only reliability vector for which the linear relationship stated in Proposition 3.2 is valid between Birnbaum RIM and Representativeness RIM:

Theorem 3.1.

$$\begin{gathered} I_i^R(\mathbf{p}) = 1/2 + 1/2\ I_i^B(\mathbf{p}) \\ \text{for any system } \ \pi, \text{ and for any component } \ i \\ \Updownarrow \\ \mathbf{p} = \mathbf{1/2} \end{gathered}$$

Theorem 3.1 proves that the Representativeness RIM and the Birnbaum RIM are not only conceptually different but also analytically independent, as there is no way to derive one concept from the other, and only for the particular case $\mathbf{p} = (1/2, \ldots, 1/2)$ there is an affine relationship between the two measures.

4. Comparing the importance of two components

The Representativeness RIM and the Birnbaum RIM rank components of a series (parallel) system in the same way. But this is not true in general, as is shown in the following example.

Example 4.1. Consider the 3–component system given by the reliability function: $h(\mathbf{p}) = p_3 + p_1 p_2 - p_1 p_2 p_3$.

If we take, for instance, $\mathbf{p} = (0.6,\ 0.9,\ 0.7)$, then components 1 and 2 verify:

$$\begin{aligned} I_1^R(\mathbf{p}) = 0.702 \quad &< \quad 0.822 = I_2^R(\mathbf{p}) \\ I_1^B(\mathbf{p}) = 0.27 \quad &> \quad 0.18 = I_2^B(\mathbf{p}) \end{aligned}$$

and, thus, the two measures rank them in an opposite order.

It is well–known the behavior of the Birnbaum RIM on a series system (the most reliable component has the smallest Birnbaum RIM) and on a parallel system (the most reliable component has the largest Birnbaum RIM), which happens to be the same of the Representativeness RIM in both kinds of systems. Similar properties can be stated for systems which have a component in series (parallel) with the rest of the system, and we prove that they are verified also for the Representativeness RIM. These properties are proved in the next proposition:

Proposition 4.1.

a) If a component i is in series with the rest of the system then, for any $j \neq i$:

$$p_i \leq p_j \quad \Rightarrow \quad I_i^B(\mathbf{p}) \geq I_j^B(\mathbf{p}) \quad \text{and} \quad I_i^R(\mathbf{p}) \geq I_j^R(\mathbf{p})$$

b) If a component i is in parallel with the rest of the system then, for any $j \neq i$:

$$p_i \geq p_j \quad \Rightarrow \quad I_i^B(\mathbf{p}) \geq I_j^B(\mathbf{p}) \quad \text{and} \quad I_i^R(\mathbf{p}) \geq I_j^R(\mathbf{p})$$

Definition 4.1. Node criticality relation

- $i \succeq j \;\Leftrightarrow\; h(1_i, 0_j, \mathbf{p}) - h(0_i, 1_j, \mathbf{p}) \geq 0$ for any $\mathbf{p}$.
- $i \succ j \;\Leftrightarrow\; i \succeq j$ and $h(1_i, 0_j, \mathbf{p}) - h(0_i, 1_j, \mathbf{p}) > 0$ for some $\mathbf{p}$.

Proposition 4.2.

- If $i \succeq j$ and $p_i = p_j$ then $I_i^R(\mathbf{p}) \geq I_j^R(\mathbf{p})$.
- If $i \succ j$ and $p_i = p_j$ then $I_i^R(\mathbf{p}) > I_j^R(\mathbf{p})$.

Next result has a consequence which clarifies the similarities and differences between these two measures.

Theorem 4.1. Let i, j be different components in N.

i) If $p_i = p_j$ then
$\left[\, I_i^R(\mathbf{p}) > I_j^R(\mathbf{p}) \;\Leftrightarrow\; I_i^B(\mathbf{p}) > I_j^B(\mathbf{p}) \,\right]$ for any system π.

ii) If $N = \{i, j\}$ then
$\left[\, I_i^R(\mathbf{p}) > I_j^R(\mathbf{p}) \;\Leftrightarrow\; I_i^B(\mathbf{p}) > I_j^B(\mathbf{p}) \,\right]$ for any system π.

iii) If $p_i \neq p_j$ and there is a component $k \neq i, j$ with $p_k \neq 1/2$, then we can find a system π such that
$\left[I_i^R(\mathbf{p}) - I_j^R(\mathbf{p})\right] \left[I_i^B(\mathbf{p}) - I_j^B(\mathbf{p})\right] < 0$.

iv) If $p_i \neq p_j$ and for any component $k \neq i, j$ it is $p_k = 1/2$, then we can find a system π such that
$I_i^B(\mathbf{p}) \neq I_j^B(\mathbf{p})$ and $I_i^R(\mathbf{p}) = I_j^R(\mathbf{p})$.

Corollary 4.1. Let $n \geq 3$. Then,

$$\begin{array}{c}\left[I_i^B(\mathbf{p}) > I_j^B(\mathbf{p}) \;\; \Leftrightarrow \;\; I_i^R(\mathbf{p}) > I_j^R(\mathbf{p})\right] \\ \text{for any system } \pi \;\; \text{ and for any } \;\; i, j \in N \\ \Updownarrow \\ \text{all components of } \mathbf{p} \text{ coincide.}\end{array}$$

Corollary 4.1 states that, if all components of N have the same reliability then, the Representativeness RIM and Birnbaum RIM rank them in the same way, independently of the system taken, but this is the only case in which this property occurs. Thus, under equal reliability of components both measures are qualitatively similar but in any other case the rankings they produce are different. As the two measures order components in different ways for arbitrary component reliabilities we should take into account both measures when analyzing a system. The Representativeness RIM reflects the degree of concordance between the component and the system, or its likelihood of replicating the system performance. In some way it measures how sensitive is a component in the system, and therefore how small changes in this component can cause the loss of this sensitivity. We do think that designers should pay attention to this measure and that it should be taken into account in the near future.
Representativeness measure is the probability that the failure and functioning of the component coincide with system failure and functioning taking into account all possible situations.
and to analyze the possible rankings that can be obtained.

Acknowledgments

Research partially supported by Grant MTM2012-34426/FEDER "del Ministerio de Economía y Competitividad".

References

Kuo, W. and Zhu, X. (2012). Importance measures in reliability, risk, and optimization. UK: Wiley.

Birnbaum, Z. (1969). On the importance of different components in a multicomponent system in Multivariate Analysis, Vol 2. New York, USA: Academic Press.