A Note on Logical Connectives with Weak Duality

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Dedicated to Enric Trillas

Abstract We study the weak duality $S(x, N(x)) = N(T(x, N(x)))$ where S is a continuous t-conorm, T is a continuous t-norm and N is a strong negation. We characterize completely the strict cases and show that in all other cases there is no direct relation between generators of T and S or between their possible Archimedean components in ordinal sums.

1 Introduction

The aim of this short paper is to study triplets (T, S, N), where T is a continuous t-norm, S is a continuous t-conorm and N is a strong negation, and one assumes the weak duality:

$$S(x, N(x)) = N(T(x, N(x)).$$

This condition is satisfied for $T = \text{Min}$, $S = \text{Max}$ and any strong negation N and, in particular yields

$$T(x, N(x)) = 0 \text{ if and only if } S(x, N(x)) = 1,$$

an interesting logical property when dealing with De Morgan triplets.

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2 Main Results

Let S be a continuous t-conorm, let T be a continuous t-norm and let N be a strong negation.

Definition 1 The t-conorm S and the t-norm T are said to have the *N-weak duality* if

$$S(x, N(x)) = N(T(x, N(x)), \tag{1}$$

for all x in [0, 1].

The usual N-*duality* between a t-norm T and a t-conorm S requires the strong condition (see [1]):

$$S(x, y) = N(T(N(x), N(y))), \tag{2}$$

for all x, y in [0, 1]. So clearly if the N-duality (2) holds, with the substitution $y = N(x)$ yields (1), i.e., the N-weak duality. But the converse does not hold as shown in the following example.

Example 1 Let N_0 the standard negation $N_0(x) = 1 - x$, $T = W$ the Lukasiewicz t-norm $W(x, y) = \text{Max}\ (x + y - 1, 0)$, and let S be a t-conorm isomorphic to the Lukasiewicz t-conorm but different from the standard one W^*. Precisely, let S be a non-strict Archimedean t-conorm $S(x, y) = s^{[-1]}(s(x) + s(y))$ generated by a continuous strictly increasing function $s : [0, 1] \to [0, 1]$ such that $s(0) = 0$, $s(1) = 1$ and $s(x) + s(1 - x) = 1$ (the graph of s is symmetric respect to the point $(1/2, 1/2)$) but $s(x) \neq x$ for all x in (0,1), $x \neq 1/2$. Then $S(x, 1 - x) = 1$ so (1) holds but clearly $S \neq W^*$ which is the N_0-dual of W.

Next, note that in general the functional equality (1) which involves only one variable may exhibit very bizarre unrelated solutions T and S.

Example 2 Fixed a strong negation N, let T be a non-strict Archimedean t-norm whose zero set is given by

$$Z(T) := \{(x, y) \in [0, 1]^2 \mid T(x, y) = 0\} = \{(x, y) \in [0, 1]^2 \mid y \leq N(x)\},$$

and let S be a non-strict Archimedean t-conorm whose one set is

$$\theta(S) := \{(x, y) \in [0, 1]^2 \mid S(x, y) = 1\} = \{(x, y) \in [0, 1]^2 \mid y \geq N(x)\}$$

Then T and S are N-weak dual.

Example 3 Let T be an ordinal sum of Archimedean t-norms $\{T_j\}_{j \in J}$ associated to a countable collection of intervals $\{[a_j, b_j]\}_{j \in J}$ and let S be an ordinal sum of Archimedean t-conorms $\{S_k\}_{k \in K}$ associated to another collection $\{[c_k, d_k]\}_{k \in K}$ with the only condition that $T(x, N(x)) = \text{Min}(x, N(x))$ and $S(x, N(x)) = \text{Max}(x, N(x))$, for all x in [0, 1]. Then (1) holds but (2) does not in most cases.

Thus all above examples show that in many instances, for non-strict Archimedean operations or ordinal sums, condition (1) does not imply a direct relation neither between additive generators or among the Archimedean blocks of the ordinal sums. In contrast with this situation we will find that in the strict case (1) allows us to determine an explicit relation between additive generators of T and S.

Lemma 1 *Let $f : [0, +\infty] \to [0, +\infty]$ be a continuous strictly increasing function such that $f(0) = 0$ and $f(+\infty) = +\infty$. Let $n : [0, +\infty] \to [0, +\infty]$ be a continuous strictly decreasing function such that $n(0) = +\infty$, $n(+\infty) = 0$ and $n = n^{-1}$. Then f and n satisfy the functional equation*

$$f(x + n(x)) = f(x) + f(n(x)), \tag{3}$$

for all x in $[0, +\infty]$ if and only if there exists a continuous strictly increasing function $g : [x_n, +\infty] \to [f(2x_n)/2, +\infty]$, where x_n denotes the fixed point of n, such that $g(x_n) = f(2x_n)/2$, $g(+\infty) = +\infty$ and

$$f(x) = \begin{cases} g(x), & \text{if } x \geq x_n, \\ g(x + n(x)) - g(n(x)), & \text{if } x \leq x_n. \end{cases} \tag{4}$$

Proof Note that substituting in (3) $x = x_n$ one obtains $f(2x_n) = 2f(x_n)$ and moreover by condition (3) when $x \leq x_n$ then $n(x) \geq x_n$ and $x + n(x) \geq x_n$ so the values of f on $[0, x_n]$ are determined by its values on $[x_n, +\infty]$. Thus the representation (4) holds. It is immediate to show that (4) satisfies (3).

Theorem 1 *Let S be a strict t-conorm generated by s, let T be a strict t-norm generated by t and let N be a strong negation with x_N as fixed point. Then (1) holds if and only if there exist a continuous strictly increasing function $g : [0, x_N] \to \mathbb{R}^+$ such that*

$$t(x) = \begin{cases} g(s(N(x))), & \text{if } x \leq x_N, \\ g(s(N(x)) + s(x)) - g(s(x))), & \text{if } x \geq x_N. \end{cases}$$

Proof If (1) holds, introduce the t-norm $S^*(x, y) = N(S(N(x), N(y))$, with generator $t_S = s \circ N$ where s generates S, so we have

$$S^*(x, N(x)) = N(S(N(x), x)) = T(x, N(x)),$$

and if t generates T we obtain

$$t_s^{-1}(t_s(x) + t_s(N(x))) = t^{-1}(t(x) + t(N(x))).$$

Introducing the new variable $t_s(x) = u$ in $\mathbb{R}^+$ and the functions $n = t_s \circ N \circ t_s^{-1}$, and $f = t \circ t_s^{-1}$ we obtain

$$\begin{aligned} f(u+n(u)) &= t\left[t_s^{-1}(t_s(x)+t_sN(x))\right] \\ &= t(x)+t(N(x)) = f(u)+f(n(u)). \end{aligned}$$

Therefore f and n satisfy the conditions of Lemma 1 so we have that there exists a function g as described above such that

$$f(x) = \begin{cases} g(x), & \text{if } x \geq x_n, \\ g(x+n(x)) - g(n(x)), & \text{if } x \leq x_n. \end{cases}$$

From this the claim of the theorem follows at once. The converse is a striaghtforward computation.

This very general result allow us to construct given a strict t-conorm S and the strong negation N all possible strict t-norms which are N-weak duals of S.

3 A Final Remark

Many equations involving t-norms, t-conorms and strong negations have been solved motivated by fuzzy logic. Enric Trillas has played a central role in this field since the 70's (see [2]). In general the equations considered had two variables and only in some cases Pexider equations involving many different functions. Stability has received also attention. Our case in this paper opens the possibility of studying equations of this type but in a *single variable*. It's a challenging question.

References

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