The proportional partitional Shapley value

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Abstract

A new coalitional value is proposed under the hypothesis of isolated unions. The main difference between this value and the Aumann–Drèze value is that the allocations within each union are not given by the Shapley value of the restricted game but proportionally to the Shapley value of the original game. Axiomatic characterizations of the new value, examples illustrating its application and a comparative discussion are provided.

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1 Introduction

The cooperative game theory deals with situations where a group of agents (players) want to share the benefits derived from their cooperation. It offers mathematical tools to propose, according to different criteria, allocation vectors that could be acceptable for the agents. This theory has given rise to relevant applications in many fields (see e.g. Fiestras–Janeiro et al., 2011).

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Among those mathematical tools there are the so–called *values*. A value proposes for every cooperative game an allocation vector that represents a fair compromise for the players. Probably, the most important value is the Shapley value (Shapley, 1953), denoted here by Φ. Moretti and Patrone (2008) is a survey that shows the impact of the Shapley value in several scientific disciplines.

The notion of cooperative game with a *coalition structure* (a partition of the set N of players into *unions*) was considered in Aumann and Drèze (1974), and a modification of the Shapley value was proposed. Later on, other *coalitional values* (i.e. values for cooperative games with a coalition structure) have been introduced and analyzed in the game theoretical literature. The two most cited coalitional values are the Aumann–Drèze value, denoted here by α, and the Owen value (Owen, 1977), denoted here by Ω. They are based on two different interpretations of the coalition structure that give rise to two different approaches when defining coalitional values:

1. Aumann and Drèze consider that, once a partition $\{P_1,\ldots,P_m\}$ of N has been formed, m independent cooperative situations arise (*isolated unions*), so their value allocates the benefits generated by each P_k to its members by applying the Shapley value to the restricted game.

2. Instead, Owen considers the partition rather as a way to influence the negotiation among the agents (*bargaining unions*), so his value allocates the benefits generated by N by applying the Shapley value twice: first, to sharing the total utility among the unions and, then, to sharing among the members of each union the payoff obtained in the first step.[1]

Example 1 *(A glove game)*

To illustrate both approaches, let us consider an elementary glove game with three players where player 1 has two right gloves and players 2 and 3 have one left glove each. Only each left–and–right pair of gloves has a worth of 1; otherwise, the worth is 0. The cooperative game v associated to this situation is given by

$$v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0,$$

[1]The first sharing takes place in the *quotient game*, played by unions; the second sharing applies to games defined in each P_k that we will not describe. We refer the reader to Owen (1977).

$$v(\{1,2\}) = v(\{1,3\}) = 1, \quad v(N) = 2.$$

Consider now that partition $P = \{\{1,2\},\{3\}\}$ forms. The Aumann–Drèze value yields the allocation $\alpha(v,P) = (1/2,1/2,0)$. Indeed, once P is formed, this value merely takes into account that players 1 and 2 are symmetric (in P_1) and must share 1 unit, whereas player 3 is a null player (in P_3). Instead, the Owen value yields the allocation $\Omega(v,P) = (1,1/2,1/2)$. It first allocates to the unions $3/2$ and $1/2$, respectively, and assigns then 1 to player 1, $1/2$ to player 2, and $1/2$ to player 3. Note that the shared worth is different.

In this paper we adopt approach 1, thus leaving aside the Owen value definitely, and introduce a new coalitional value, called the proportional partitional Shapley value and denoted as π, as an alternative to the Aumann–Drèze value. Hence we assume that, once a partition forms, a new cooperative situation arises in each union independently of the remaining ones. However, we wish to take into account in some manner the outside options of the players, reflected by the Shapley value of the original game. More precisely, given a cooperative game v in N with a coalition structure $P = \{P_1,\ldots,P_m\}$, our value divides each worth $v(P_k)$ among the players in P_k proportionally to the Shapley value of these players in game v.

Thus, in Example 1 we obtain the allocation $\pi(v,P) = (2/3,1/3,0)$ since the Shapley value is $\Phi(v) = (1,1/2,1/2)$. It reflects that player 1 is in a better position than player 2 because he might join player 3 if $\{1,2\}$ collapses. We will restrict the domain of our value to the class of monotonic games in order to avoid some problems that often arise when using proportionality.

Example 2 *(A second glove game)*[2]

Let $N = \{r,r,\ell,\ell,\ell,\ell\}$ be, informally, the set of players, each one with a glove: r means righty, ℓ means lefty. Only each left–and–right pair of gloves has a worth of one. The glove game v describing this is a linear combination of 45 unanimity games that we omit. The Shapley value is

$$\Phi(v) = \frac{1}{15}(11,11,2,2,2,2)$$

[2]We are grateful to a reviewer for suggesting this numerical example.

and, for any partition $P = \{A, \dots\}$, where $A = \{r, r, \ell\}$, the Aumann–Drèze and proportional partitional Shapley values respectively yield

$$\alpha(v,P) = \frac{1}{6}(1,1,4,0,0,0) \qquad \text{and} \qquad \pi(v,P) = \frac{1}{24}(11,11,2,0,0,0).$$

These allocations do not depend on the way the remaining three players ℓ are arranged (a general property of α and π). Instead, for the Owen value, this greatly matters. There are three possibilities:

$$P^1 = \{A, \{\ell\}, \{\ell\}, \{\ell\}\}, \qquad P^2 = \{A, B, \{\ell\}\} \qquad \text{and} \qquad P^3 = \{A, C\},$$

where $B = \{\ell, \ell\}$ and $C = \{\ell, \ell, \ell\}$. Thus for the Owen value we obtain

$$\Omega(v,P^1) = \tfrac{1}{12}(9,9,3,1,1,1), \quad \Omega(v,P^2) = \tfrac{1}{36}(25,25,10,3,3,6), \quad \text{and}$$

$$\Omega(v,P^3) = \tfrac{1}{12}(7,7,4,2,2,2).$$

This example is interesting. First, because it shows a difference between the Aumann–Drèze value and the proportional partitional Shapley value: the former is concerned with the possibilities existing in $A = \{r, r, \ell\}$ only, and hence it gives the bulk of the payoff to player ℓ; instead, the latter recalls the strategic strength in the original game, thus avoiding a striking change in the payout ratios that would not satisfy the righties. Second, it shows the main difference between the Aumann–Drèze and proportional partitional Shapley values and the Owen value. The former two satisfy local efficiency, whereas the latter satisfies efficiency, as the Shapley value does.

The background for our new coalitional value shares ideas with the proportional coalitional Shapley value (Alonso–Meijide and Carreras, 2011), which follows approach 2. In Wiese (2007) and Casajus (2009) other variations of the Aumann–Drèze value can be found that also take into account, in a way different from ours, the players' outside options.

The organization of the paper is as follows. We assume that the reader is generally familiar with the basic ideas of the cooperative game theory (including simple games) and omit, therefore, a preliminary section. In Section 2 we formally define the Aumann–Drèze value and the proportional partitional Shapley value and study the properties of the latter. Section 3 includes several examples to illustrate the use of this new value. Section 4 is devoted to some comparative discussion and final remarks.

2 The proportional partitional Shapley value

Let $N = \{1, 2, \ldots, n\}$ represent a finite but otherwise arbitrary set of players. We will consider TU games only (just "games", in the sequel). The vector space of games in N will be denoted as $G(N)$, and as $MG(N)$ the subclass (cone) of monotonic games, which will be the domain of our new value. The set of partitions (coalition structures) in N will be denoted as $P(N)$.

For every nonempty coalition $T \subseteq N$, the unanimity game u_T is defined by $u_T(S) = 1$ if $T \subseteq S$ or else $u_T(S) = 0$. Every game $v \in MG(N)$ can be uniquely written as a linear combination of unanimity games using the Harsanyi dividends (Harsanyi, 1959):

$$v = \sum_{T \subseteq N\,:\,T \neq \emptyset} c_T u_T \quad \text{where} \quad c_T = \sum_{S \subseteq T} (-1)^{t-s} v(S), \;\; t = |T|, \;\; s = |S|.$$

The following relationship among monotonic games will be useful later:

$$v + v^- = v^+ \quad \text{where} \quad v^+ = \sum_{T\,:\,c_T > 0} c_T u_T \quad \text{and} \quad v^- = \sum_{T\,:\,c_T < 0} -c_T u_T.$$

The Shapley value is the map $\Phi : G(N) \to \mathbb{R}^N$ defined by

$$\Phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} [v(S \cup \{i\}) - v(S)],$$

for all $v \in G(N)$ and $i \in N$, where $s = |S|$ for every $S \subseteq N$.

The Shapley value is the only value on $MG(N)$ that satisfies the following properties[3]:

- *Efficiency*: $\sum_{i \in N} \Phi_i(v) = v(N)$ for all $v \in MG(N)$.
- *Null player property*[4]: if i is null in v then $\Phi_i(v) = 0$.
- *Symmetry*: if i, j are symmetric in v then $\Phi_i(v) = \Phi_j(v)$.
- *Additivity*: $\Phi(v + w) = \Phi(v) + \Phi(w)$ for all $v, w \in MG(N)$.

[3] An analogous characterization holds in $G(N)$.

[4] It is noteworthy that, if $v \in MG(N)$, then $\Phi_i(v) = 0$ if, *and only if*, i is a null player in v, so that the null player property could be so stated for Φ in this subclass of games.

Now, we introduce two key concepts for this paper: those of *partitional value*, using local efficiency, and *partitional Shapley value*, as a generalization of the Shapley value. We state them for $G(N)$ but will use both notions also in $MG(N)$.

Definition 3 *A* partitional value *on $G(N)$ is a map $\phi : G(N) \times P(N) \to \mathbb{R}^N$ such that*

$$\sum_{i \in P_k} \phi_i(v,P) = v(P_k)$$

for all $v \in G(N)$, $P \in P(N)$ and $P_k \in P$ (we call this local efficiency*).*

Definition 4 *A* partitional Shapley value *on $G(N)$ is a partitional value ϕ on $G(N)$ such that $\phi(v,P^N) = \Phi(v)$ for all $v \in G(N)$.*[5]

We next recall the Aumann–Drèze value (Aumann and Drèze, 1974) and introduce the proportional partitional Shapley value.

Definition 5 *The* Aumann–Drèze value *is the partitional Shapley value α defined on $G(N)$ by*

$$\alpha_i(v,P) = \Phi_i(v_{P_{(i)}})$$

for all $v \in G(N)$, $P \in P(N)$ and $i \in N$, where $P_{(i)}$ denotes the union of P to which i belongs, and $v_{P_{(i)}}$ denotes the restriction of game v to $P_{(i)}$.

Definition 6 *The* proportional partitional Shapley value *is the partitional Shapley value π defined on $MG(N)$ by*

$$\pi_i(v,P) = \begin{cases} \dfrac{\Phi_i(v)}{\sum_{j \in P_{(i)}} \Phi_j(v)} v(P_{(i)}) & \text{if } i \text{ is not a null player in } v, \\ 0 & \text{otherwise,} \end{cases}$$

for all $v \in MG(N)$, $P \in P(N)$ and $i \in N$, where $P_{(i)}$ denotes again the union of P to which i belongs.

The definition makes sense because, if i is not null in a monotonic game v, then $\Phi_i(v) > 0$ and hence the denominator does not vanish.

Our next goal will consist in establishing the basic properties of the new value and obtaining two axiomatic characterizations, which will be discussed in Section 4. We first state these properties for a generic partition value ϕ on $MG(N)$.

[5] P^N denotes the trivial partition $\{N\}$.

- *Nonnegativity (NN)*: $\phi_i(v,P) \geq 0$ for all $v \in MG(N)$, $P \in P(N)$ and $i \in N$.

- *Null player property (NPP)*: if i is null in v then $\phi_i(v,P) = 0$ for all $P \in P(N)$.

- *Symmetry within unions (SWU)*: if $i, j \in N$ are symmetric in v and $P_{(i)} = P_{(j)}$ then $\phi_i(v,P) = \phi_j(v,P)$.

- *Proportionality within unions (PWU)*: if $i, j \in N$ and $P_{(i)} = P_{(j)}$ then, for all $v \in MG(N)$,
$$\phi_i(v,P)\phi_j(v,P^N) = \phi_j(v,P)\phi_i(v,P^N).$$

- *Weighted additivity (WA)*[6]: for all $v, w \in MG(N)$ and $P \in P(N)$,
$$h^{\phi}(v+w,P) = h^{\phi}(v,P) + h^{\phi}(w,P),$$
where, for all $i \in N$ and $v \in MG(N)$,
$$h_i^{\phi}(v,P) = \begin{cases} \phi_i(v,P)\frac{\sum_{j\in P_{(i)}} \phi_j(v,P^N)}{v(P_{(i)})} & \text{if } v(P_{(i)}) > 0, \\ \phi_i(v,P^N) & \text{if } v(P_{(i)}) = 0. \end{cases}$$

The next results provide alternative characterizations of the new value.

Theorem 7 *(First axiomatic characterization of the proportional partitional Shapley value) The proportional partitional Shapley value π is the unique partitional Shapley value on $MG(N)$ that satisfies NPP and PWU.*

Proof. (Existence) It is straightforward to check that π is a partitional Shapley value on $MG(N)$ that satisfies NPP[7] and PWU.

(Uniqueness) Let ϕ be a partitional Shapley value on $MG(N)$ satisfying NPP and PWU. We show that $\phi = \pi$. Let $v \in MG(N)$, $P \in P(N)$ and $i \in N$.

- If i is a null player in v, then $\phi_i(v,P) = 0 = \pi_i(v,P)$ since ϕ and π satisfy NPP.

- If $P_{(i)} = \{i\}$, then $\phi_i(v,P) = v(\{i\}) = \pi_i(v,P)$ since ϕ and π are partitional values.

[6]This property recalls the classical additivity for ϕ: the difference lies in the attached weights.
[7]π also satisfies a null player *strong* property (NPP*): $\pi_i(v,P) = 0$ **iff** i is null in v or $v(P_{(i)}) = 0$.

- If every $j \in P_{(i)}$ different from i is a null player, then $\phi_i(v,P) = v(\{i\}) = \pi_i(v,P)$ since ϕ and π are partitional values and satisfy NPP.

- In any other case, take $j \in P_{(i)}$ such that j is not a null player. Then, since ϕ and π are partitional Shapley values and satisfy PWU,

$$\frac{\phi_i(v,P)}{\phi_j(v,P)} = \frac{\Phi_i(v)}{\Phi_j(v)} = \frac{\pi_i(v,P)}{\pi_j(v,P)} \,.$$

  This means that $\phi_j(v,P) = \lambda\pi_j(v,P)$ for every non–null player $j \in P_{(i)}$, where λ is a constant which does not depend on j. Since ϕ and π are partitional values, it is clear that $\lambda = 1$, and hence $\phi_i(v,P) = \pi_i(v,P)$.

We conclude that $\phi = \pi$. $\square$

Theorem 8 *(Second axiomatic characterization of the proportional partitional Shapley value) The proportional partitional Shapley value π is the unique partitional value on $MG(N)$ that satisfies NN, NPP, SWU and WA.*

Proof. (Existence) Again, it is straightforward to check that π satisfies NN, NPP and SWU. We proceed to prove WA. Let $v \in MG(N)$, $P \in P(N)$ and $i \in N$.

- If i is a null player and $v(P_{(i)}) = 0$, then $h_i^\pi(v,P) = \pi_i(v,P^N) = 0 = \Phi_i(v)$ since π is a partitional Shapley value.

- If i is a null player and $v(P_{(i)}) > 0$, then by NPP $h_i^\pi(v,P) = 0 = \Phi_i(v)$.

- If i is not a null player and $v(P_{(i)}) = 0$, then $h_i^\pi(v,P) = \pi_i(v,P^N) = \Phi_i(v)$ because π is a partitional Shapley value.

- Finally, if i is not a null player and $v(P_{(i)}) > 0$, by the definition of π, which is a partitional Shapley value, we get

$$h_i^\pi(v,P) = \frac{\Phi_i(v)}{\sum_{j\in P_{(i)}} \Phi_j(v)} v(P_{(i)}) \frac{\sum_{j\in P_{(i)}} \pi_j(v,P^N)}{v(P_{(i)})} = \Phi_i(v) \frac{\sum_{j\in P_{(i)}} \Phi_j(v)}{\sum_{j\in P_{(i)}} \Phi_j(v)}$$

  which reduces to $\Phi_i(v)$.

Summing up, we find $h^\pi(v,P) = \Phi(v)$ for all $v \in MG(N)$. Therefore, the relationship $h^\pi(v+w,P) = h^\pi(v,P) + h^\pi(w,P)$ for any $v,w \in MG(N)$ follows from the additivity of Φ.

(Uniqueness) Let ϕ be a partitional value on $MG(N)$ satisfying NN, NPP, SWU and WA. We show that ϕ is determined. Let $v \in MG(N)$ and $P \in P(N)$.

- If $v = 0$ then, by NPP, $\phi_i(v,P) = 0$ for all $i \in N$.

- Let u_T be the unanimity game for a given nonempty coalition $T \subseteq N$. Let $v = cu_T$ with $c > 0$ and $T_k = T \cap P_k$ for each $P_k \in P$. If $i \notin T$ then i is a null player in v and $\phi_i(v,P) = 0$ by NPP. Let $T_k \neq \emptyset$. Since ϕ is a partitional value,
$$\sum_{i \in P_k} \phi_i(v,P) = \sum_{i \in T_k} \phi_i(v,P) = v(P_k) = cu_T(P_k),$$
and hence
$$\sum_{i \in P_k} \phi_i(v,P) = \begin{cases} c & \text{if } T \subseteq P_k, \\ 0 & \text{otherwise}. \end{cases}$$
Since all players in T_k are symmetric in v, by SWU we have
$$\phi_i(v,P) = \begin{cases} c/t_k & \text{if } i \in T \text{ and } T \subseteq P_k, \\ 0 & \text{otherwise}, \end{cases}$$
for all $i \in P_k$, where $t_k = |T_k|$. This determines ϕ in this case.

- Let $P = P^N$ and $i \in N$. It is clear that $P_{(i)} = N$. Let v be any game. If $v(N) = 0$ then $h_i^\phi(v,P^N) = \phi_i(v,P^N)$. If, instead, $v(N) > 0$, using that ϕ is a partitional value we have
$$h_i^\phi(v,P^N) = \phi_i(v,P^N)\frac{\sum_{j \in N} \phi_j(v,P^N)}{v(N)} = \phi_i(v,P^N).$$
From WA it follows that $\phi(v+w,P^N) = \phi(v,P^N) + \phi(w,P^N)$ for all $v,w \in MG(N)$. Then, if we consider ϕ as a function only of v, once $P = P^N$ has been fixed, it is easily seen that ϕ satisfies efficiency, the null player property, symmetry and additivity, and the uniqueness of the Shapley value gives $\phi(v,P^N) = \Phi(v)$ for all $v \in MG(N)$.

- Let $v = \sum_{\ell=1}^r v_\ell$ in $MG(N)$ and $i \in N$. (a) If $v(P_{(i)}) = 0$ then $\phi_i(v,P) = 0$ since ϕ is a partitional value satisfying NN. (b) If $v(P_{(i)}) > 0$, let $R = \{1,2,\ldots,r\}$,

$R^+ = \{\ell \in R : v_\ell(P_{(i)}) > 0\}$ and $R^0 = R \setminus R^+$, so $R^0 = \{\ell \in R : v_\ell(P_{(i)}) = 0\}$ because v_ℓ is monotonic. By WA we have

$$\phi_i(v,P)\frac{\sum_{j\in P_{(i)}}\phi_j(v,P^N)}{v(P_{(i)})} =$$
$$\sum_{\ell\in R^+}\phi_i(v_\ell,P)\frac{\sum_{j\in P_{(i)}}\phi_j(v_\ell,P^N)}{v_\ell(P_{(i)})} + \sum_{\ell\in R^0}\phi_i(v_\ell,P^N).$$

By the previous item, $\sum_{j\in P_{(i)}}\phi_j(v,P^N) = \sum_{j\in P_{(i)}}\Phi_j(v)$, which is positive since $v(P_{(i)}) > 0$ implies that some $j \in P_{(i)}$ is not null in v (recall footnote 4). Thus, if $\phi_i(v_\ell,P)$ is uniquely determined for all $\ell \in R^+$ then so is $\phi_i(v,P)$ by solving the above equation for it.

- Finally, let $v \in MG(N)$ and $P \in P(N)$ be arbitrary. Using $v + v^- = v^+$, the decomposition of v^+ and v^- as linear combinations of unanimity games, and the preceding item, it follows that ϕ is completely determined on (v^+,P) and (v^-,P) and hence on (v,P). $\square$

3 Several examples

We sketch here some applications of the proportional partitional Shapley value.

Example 9 *(Allocating primary assistance centres)*

Due to budget constraints, the National Health Ministry (NHM) of a country restricts the creation of primary assistance centres (PACs) in regions with low population ($<$ 50000 inhabitants). Villages in such a region are allowed to freely form disjoint *unions*, and only unions with at least 5000 inhabitants will obtain a PAC. The final decision will be the location of the PAC for each such union. It will be placed on the centre of gravity of the concerned villages, which minimizes the weighted sum of squares of distances and is easy to compute; however, the “mass” attached to each village will not be its population but a different parameter related to it.

To fix ideas, let A(0,0), B(3,0), C(4,1), D(3,3) and E(1,3) be the locations of the five villages 1, 2, 3, 4 and 5 of a region (see Fig. 1), and 4230, 3160, 2120, 2005 and 1355 be, respectively, their populations (in all, 12870 inhabitants). The NHM takes into account the weighted majority game

$$v \equiv [5000; 4230, 3160, 2120, 2005, 1355]$$

(a simple but improper, i.e. not superadditive, game) because the unions that would get a PAC constitute, precisely, the family of winning coalitions in v:

$$W(v) = \{\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{3,4,5\}, \text{ and supersets}\}.$$

Since small differences between populations are considered not meaningful, in order to locate PACS the NHM prefers using parameters proportional to the Shapley value of this game, which disregards irrelevant weight differences and is given by $\Phi(v) = \frac{1}{12}(4,3,2,2,1)$.

However, at the end all will depend on the arrangement of villages into unions. Then, the NHM needs some *coalitional value* to attach weights and to compute centres of gravity. Once the unions are formed, the process stops since they are not interested in a further bargaining "at a higher level", so the Owen value is not suitable here. The alternative consists in using either the Aumann–Drèze value or the proportional partitional Shapley value that we have introduced in the previous section.

Requirements such as the reduction to the Shapley value whenever the partition is $P = P^N$, the null player property, symmetry within unions, and local efficiency make sense and are easily interpretable in this context, but they are satisfied by both values. Instead, the crucial property of interest for the NHM is proportionality within unions, and this leads to choosing the proportional partitional Shapley value. The reason is that, by social efficiency when computing centres of gravity of unions, the NHM must respect the priority of bigger towns and hence the relevant differences in population between the concerned towns, which are given by the Shapley value of v. (If the Aumann–Drèze value were applied e.g. to partition $P = \{\{1,5\}, \{2,3,4\}\}$, it would yield $1/2$ for both villages 1 and 5, and hence a PAC at the midpoint of the segment joining them, in spite of the great difference in population and in Shapley value.)

Thus, between 52 possible partitions of the villages, 6 of them imply no PAC, 36 give rise to one PAC, and the remaining 10 give rise to two PACs. Here are some examples:

- If $P = \{\{1\},\{2,5\},\{3,4\}\}$ then no PAC is assigned.
- If $P = \{\{1,4\},\{2\},\{3,5\}\}$ then one PAC is assigned to union $\{1,4\}$ at point $G_{14} = (1,1)$.
- If $P = \{\{1,5\},\{2,3,4\}\}$ then two PACs are assigned: to union $\{1,5\}$ at point G_{15} =(1/5, 3/5) and to union $\{2,3,4\}$ at point G_{234} =(23/7, 8/7) (see Fig. 1).

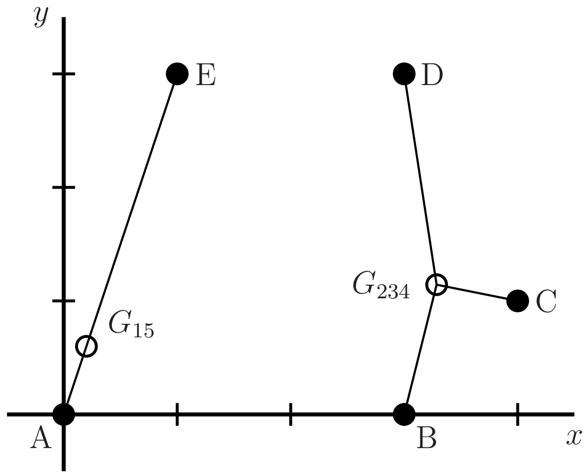

Fig. 1: Location of primary assistance centres for $P = \{\{1,5\},\{2,3,4\}\}$

Example 10 *(Sharing public funds)*

A specific industrial sector in a given region consists of a set N of enterprises. The regional government wishes to give financial support to collaboration projects endeavored by the enterprises, each one of which may remain isolated or intervene in one project at most. Thus, the set of projects is given by (i.e. equivalent to) a partition P of N.

The individual capabilities and the synergies derived from collaborations between the enterprises have been evaluated by the government in terms of expected benefits by means of a cooperative game v in N. Thus, the Shapley value $\Phi(v)$ describes the relative importance of each enterprise in the sector taking into account all possible collaborations.

Each project P_k will be rewarded by sharing $v(P_k)$ among the participants in the project. Since projects cannot be combined to give rise to "superprojects", the Owen value is not suitable here. The alternative consists in using either the Aumann–Drèze value or the proportional partitional Shapley value.

The governmental regulation establishes that the allocation to the members of a project must take into account the relevance of each member in the sector. The simplest way to do this consists in sharing each budget proportionally to the Shapley value $\Phi(v)$ (for the involved enterprises), so the proportional partitional Shapley value seems to be the suitable option.

For example, let $n = 4$ and assume that the individual capability (amounts expressed in thousands of US Dollars) is given by

$$v(\{1\}) = 36000, \quad v(\{2\}) = v(\{3\}) = 24000, \quad v(\{4\}) = 18000.$$

Game v is completed this way: if $|S| \geq 2$ then

$$v(S) = (1 + \sum_{i \in S} \sigma_i) \sum_{i \in S} v(\{i\}),$$

where $\sigma_1 = 0.15$, $\sigma_2 = \sigma_3 = 0.20$ and $\sigma_4 = 0.05$ are the *synergy coefficients*. For example, $v(N) = 163200$. The Shapley value of this game is

$$\Phi(v) = (53975, 41125, 41125, 26975).$$

Assume that partition $P = \{\{1,2\},\{3,4\}\}$ forms. Then the budgets to be shared are $v(P_1) = 81000$ and $v(P_2) = 52500$. The proportional partitional Shapley value yields

$$\pi_1(v,P) \approx 45972, \quad \pi_2(v,P) \approx 35028, \quad \pi_3(v,P) \approx 31704, \quad \pi_4(v,P) \approx 20796,$$

which keeps within each union the proportionality given by $\Phi(v)$. As a matter of comparison, the Aumann–Drèze value would give

$$\alpha_1(v,P) = 46500, \quad \alpha_2(v,P) = 34500, \quad \alpha_3(v,P) = 29250, \quad \alpha_4(v,P) = 23250.$$

We remark that the proportional partitional Shapley value reflects the effects of the synergy coefficients in the whole sector, whereas the Aumann–Drèze value takes into account only their effects within each union.

Example 11 *(Simple games and power indices)*

Values and coalitional values are often used as power indices by applying them to simple games. These games form a subclass $SG(N)$ of monotonic games and are useful for describing and analyzing binary voting procedures. We will discuss a bit the possibilities to act as power indices of the Aumann–Drèze value and the proportional partitional Shapley value.

We first remark that the axiomatic characterizations of the proportional partitional Shapley value established in Section 2 can be easily translated to subclass $SG(N)$, thus supporting the meaning of this value as a power index. Indeed, Theorem 5 (statement and proof) applies to $SG(N)$ without any change if the axioms are restricted to this class. In the case of Theorem 6, only WA does not make sense in $SG(N)$. In the Shapley value case, additivity was successfully replaced in Dubey (1975) with the *transfer property*:

$$\Phi(v \vee w) + \Phi(v \wedge w) = \Phi(v) + \Phi(w) \qquad \text{for all } v, w \in SG(N).$$

Here it is only necessary to replace WA with a "weighted transfer" property. Of course, also the proof requires some small modifications in this domain. We omit the details.

Let v be a proper (i.e. superadditive) simple game. This means that there are no disjoint winning coalitions. Some general rules hold for the Aumann–Drèze value α: given a coalition structure P, (a) if P_k is a minimal winning coalition in v then $\alpha(v,P)$ allocates $1/|P_k|$ to each member of P_k and 0 otherwise; (b) if P_k is winning but not minimal winning then α allocates in all 1 unit to the members of P_k, but the sharing depends on the minimal winning coalitions included in P_k; (c) if P_k is not winning then all its members get 0. Only property (c) holds for the proportional partitional Shapley value π. The following numerical example illustrates these assertions.

Let us consider the weighted majority game $v \equiv [3;2,1,1,1]$. The family of minimal winning coalitions is $W^m(v) = \{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\}$ and $\Phi(v) = (3/6, 1/6, 1/6, 1/6)$. Some partitions are studied in Table 1.

Now let v be an improper simple game. Rules (a), (b) and (c) above still hold for the Aumann–Drèze value α if only one (minimal or not) winning coalition, or none of them, appears in the partition. However, a new rule (d) says that if

partition P	$\alpha(v,P)$	$\pi(v,P)$
$\{\{1,2\},\{3\},\{4\}\}$	$(1/2,1/2,0,0)$	$(3/4,1/4,0,0)$
$\{\{1\},\{2,3,4\}\}$	$(0,1/3,1/3,1/3)$	$(0,1/3,1/3,1/3)$
$\{\{1,2,3\},\{4\}\}$	$(4/6,1/6,1/6,0)$	$(3/5,1/5,1/5,0)$
$\{\{1\},\{2\},\{3,4\}\}$	$(0,0,0,0)$	$(0,0,0,0)$

Table 1: α and π on $v \equiv [3;2,1,1,1]$

two or more winning coalitions are unions of P then a worth of one unit is shared in each of them. This is the main difference with the proper case. Here, the proportional partitional Shapley value π satisfies (c) and (d). A new numerical example illustrates these assertions.

Let us consider now $v \equiv [5;4,3,2,2,1]$, the game of Example 9. Here

$$W^m(v) = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{3,4,5\}\}$$

and $\Phi(v) = \frac{1}{12}(4,3,2,2,1)$. See some partitions in Table 2.

partition P	$\alpha(v,P)$	$\pi(v,P)$
$\{\{1,2\},\{3,4,5\}\}$	(1/6)$(3,3,2,2,2)$	(1/35)$(20,15,14,14,7)$
$\{\{1,2,3,4\},\{5\}\}$	(1/6)$(2,2,1,1,0)$	(1/11)$(4,3,2,2,0)$
$\{\{1,2,3\},\{4,5\}\}$	(1/3)$(1,1,1,0,0)$	(1/9)$(4,3,2,0,0)$
$\{\{1,5\},\{2,3,4\}\}$	(1/6)$(3,4,1,1,3)$	(1/35)$(28,15,10,10,7)$
$\{\{1,3\},\{2,4\},\{5\}\}$	(1/2)$(1,1,1,1,0)$	(1/15)$(10,9,5,6,0)$

Table 2: α and π on $v \equiv [5;4,3,2,2,1]$

Our conclusion is that the Aumann–Drèze value is not a suitable power index: it disregards a lot of information given by the original game and ends up being too “drastic”. On the contrary, we contend that, precisely because of the PWU property, the proportional partitional Shapley value looks more interesting as a measure of coalitional power. Indeed, due to PWU, all power relationships in the original game among players of the same union are kept after the coalition formation process. We feel that this should please politicians, who do not like too radical and troubling variations.

For example, in Table 1, it does not seem very reasonable that under partition $P = \{\{1,2\},\{3\},\{4\}\}$ player 1 obtains by means of α the same coalitional power as player 2. The same equal sharing of power would result if $v \equiv$

$[136;130,50,50,30,10]$ represents a parliamentary body and $P = \{\{1,2\},\dots\}$, in spite of being $W^m(v) = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3,4,5\}\}$ and $\Phi(v) = \frac{1}{10}(6,1,1,1,1)$. The Aumann–Drèze value yields here $\alpha(v,P) = \frac{1}{2}(1,1,0,0,0)$ whereas the proportional partitional Shapley value yields $\pi(v,P) = \frac{1}{7}(6,1,0,0,0)$.

Formally, let i, j be any two players with weights w_i, w_j. It is well known that if $w_i \geq w_j$ then $\Phi_i(v) \geq \Phi_j(v)$. In our coalitional structure framework, and provided that $P_{(i)} = P_{(j)}$, we have: if $\Phi_i(v) \geq \Phi_j(v)$ then $\alpha_i(v,P) \geq \alpha_j(v,P)$ and $\pi_i(v,P) \geq \pi_j(v,P)$, but the most interesting feature is that if $\Phi_i(v) > \Phi_j(v)$ and, moreover, $P_{(i)}$ is winning in v, then $\pi_i(v,P) > \pi_j(v,P)$ whereas it may well happen that $\alpha_i(v,P) = \alpha_j(v,P)$. And this holds even in simple games that are not weighted majority games.

Example 12 *(Extension of the new value to level coalition structures)*

A *level coalition structure* in N is a sequence of coalition structures

$$\mathcal{P} = \{P^{(0)}, P^{(1)}, \dots, P^{(r)}\},$$

where $P^{(0)} = P^N$ and $r \geq 1$ (thus including the basic case dealt with above, which arises for $r = 1$). We require that, for every h with $0 \leq h < r$, $P^{(h+1)}$ is a refinement of $P^{(h)}$, that is, each member of $P^{(h)}$ belongs to $P^{(h+1)}$ or splits into smaller pieces belonging to $P^{(h+1)}$. The extension of the Owen value to this new setup was treated in Owen (1977) and Winter (1989). We wish to discuss here the possibility of extending the notion of proportional partitional Shapley value to this more general concept of coalition structure. To this end, the coherent inductive definition of a new value, the *proportional partitional level value* $\pi^{\mathcal{P}}$, which will act on each level $h = 1, 2, \dots, r$, is as follows. If $i \in N$, $1 \leq h \leq r$, and $P^{(h)}_{(i)}$ is the union to which i belongs in $P^{(h)}$,

$$\pi^{\mathcal{P}}_i(v, P^{(h)}) = \begin{cases} \dfrac{\pi^{\mathcal{P}}_i(v, P^{(h-1)})}{\sum\limits_{j \in P^{(h)}_{(i)}} \pi^{\mathcal{P}}_j(v, P^{(h-1)})} v(P^{(h)}_{(i)}) & \text{if } i \text{ is not a null player in } v, \\ 0 & \text{otherwise.} \end{cases}$$

Coherence means that $\pi^{\mathcal{P}}(v, P^{(1)}) = \pi(v, P^{(1)})$. A numerical example will illustrate the procedure. Let us take $v \equiv [4;5,4,3,2,1,1]$, where

$$W^m(v) = \{\{1\},\{2\},\{3,4\},\{3,5\},\{3,6\},\{4,5,6\}\},$$

and let $\mathcal{P} = \{P^{(0)}, P^{(1)}, P^{(2)}\}$ with

$$P^{(0)} = P^N, \quad P^{(1)} = \{\{1,2\},\{3,4,5,6\}\}, \quad P^{(2)} = \{\{1\},\{2\},\{3,4,5\},\{6\}\}.$$

Then we have $\{1,2\},\{3,4,5,6\},\{1\},\{2\},\{3,4,5\} \in W(v)$ but $\{6\} \notin W(v)$. The results of applying the level value are displayed in Fig. 2.

A great difference between the Owen value and the Aumann–Drèze and the proportional partitional Shapley value is that, in the latter two, the allocation within any union is not affected by any change in other unions.

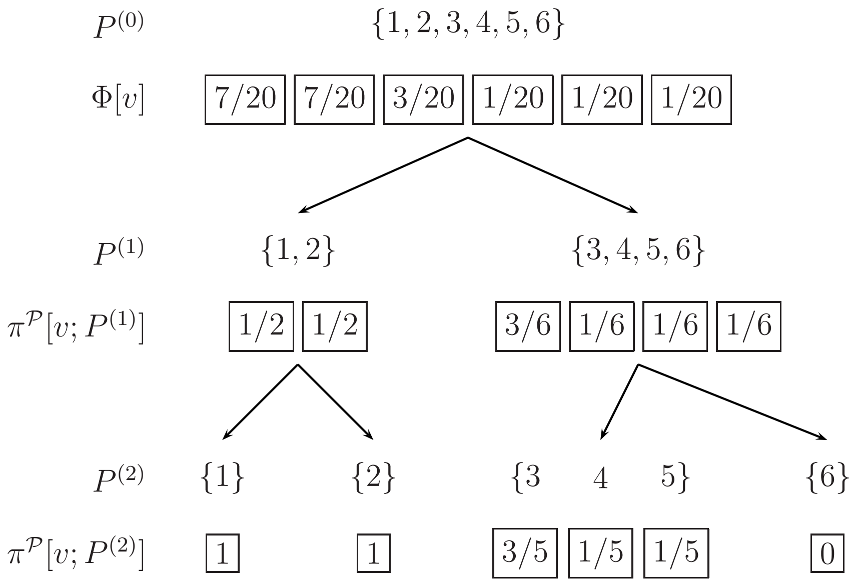

Fig. 2: Level coalition structure and level value for $v \equiv [4;5,4,3,2,1,1]$

4 Some discussion

We include here some suplementary information. First, two axiomatic characterizations of the Aumann–Drèze value on monotonic games using two classical properties.[8] Second, a remark supplying counterexamples to show that π does not

[8]Both characterizations are valid also on $G(N)$. We omit their proofs because they are quite similar to the classical ones. In particular, Theorem 14 is analogous to the original characterization of this value on $G(N)$ (Aumann and Drèze, 1974). The difference lies in the final part of the proof. Regarding Theorem 13, we recall that the balanced contributions property for the Shapley value asserts that $\Phi_i(v) - \Phi_i(v_{-j}) = \Phi_j(v) - \Phi_j(v_{-i})$ for all $v \in G(N)$ and $i, j \in N$. Here v_{-k} denotes the restriction of v to $N \setminus \{k\}$ for any $k \in N$. This property was introduced and proved in Myerson (1980).

satisfy these properties and α does not satisfy any of NPP*, PWU and WA. Third, two remarks on the logical independence of the axiomatic systems used in Theorems 7 and 8, respectively. Finally, a summary of all properties considered in the paper, displayed in Table 3.

Let us recall two classical coalitional properties not yet mentioned here that we state for a partitional value ϕ on $MG(N)$.

- *Additivity (ADD)*: $\phi(v+w,P) = \phi(v,P) + \phi(w,P)$ for all $v,w \in MG(N)$ and $P \in P(N)$.
- *Balanced contributions within unions (BCWU)*: if $P_{(i)} = P_{(j)}$ for some $i, j \in N$ then, for all $v \in MG(N)$,

$$\phi_i(v,P) - \phi_i(v,P^{-j}) = \phi_j(v,P) - \phi_j(v,P^{-i}),$$

where, for any $k \in N$, we define

$$P^{-k} = \{P_{(k)} \setminus \{k\}, \{k\}\} \cup \{P_i : P_i \in P, P_i \neq P_{(k)}\}.$$

Theorem 13 *(First axiomatic characterization of the Aumann–Drèze value) The Aumann–Drèze value α is the unique partitional value on $MG(N)$ that satisfies BCWU.* $\square$

Theorem 14 *(Second axiomatic characterization of the Aumann–Drèze value) The Aumann–Drèze value α is the unique partitional value on $MG(N)$ that satisfies NPP, SWU and ADD.* $\square$

Remark 15 *(Properties that distinguish between π and α)*

Let $n = 3$, $P = \{\{1,2\},\{3\}\}$, and $v,w \in MG(N)$ be the glove games defined by

$$\begin{aligned}&v(\{1,2\}) = v(\{1,3\}) = v(N) = 1, \text{ and } v(S) = 0 \text{ otherwise, and}\\ &w(\{1\}) = w(\{1,2\}) = w(\{1,3\}) = w(\{2,3\}) = 1, w(N) = 2,\\ &\text{and } w(S) = 0 \text{ otherwise.}\end{aligned}$$

(i) π does not satisfy ADD. Indeed,

$$\pi_1(v,P) + \pi_1(w,P) = 4/5 + 2/3 \neq 10/7 = \pi_1(v+w,P).$$

(ii) π does not satisfy BCWU. In effect,

$$\pi_1(v,P)-\pi_1(v,P^{-2}) = 4/5-0 \neq 1/5-0 = \pi_2(v,P)-\pi_2(v,P^{-1}).$$

(iii) α does not satisfy NPP* (see footnote 7). Player 2 is not null in w and $w(P_{(2)}) \neq 0$ but $\alpha_2(w,P)=0$.

(iv) α does not satisfy PWU. We have $\alpha(v,P^N)=\Phi(v)=(2/3,1/6,1/6)$ and $\alpha(v,P)=(1/2,1/2,0)$, so

$$\alpha_1(v,P)\alpha_2(v,P^N) = 1/12 \neq 1/3 = \alpha_2(v,P)\alpha_1(v,P^N).$$

(v) α does not satisfy WA. It is easily checked that

$$h_1^{\alpha}(v,P)+h_1^{\alpha}(w,P) = 5/12+3/2 \neq 7/4 = h_1^{\alpha}(v+w,P).$$

Remark 16 *(Independence of the axiomatic system in Theorem 7)*

(i) The value ϕ^1 defined for all $v \in MG(N)$, $P \in P(N)$ and $i \in N$ by

$$\phi_i^1(v,P) = \begin{cases} \dfrac{\beta_i(v)}{\sum_{j\in P_{(i)}}\beta_j(v)}v(P_{(i)}) & \text{if } i \text{ is not a null player in } v, \\ 0 & \text{otherwise,} \end{cases}$$

where β denotes the Banzhaf value (Owen, 1975), satisfies NPP and PWU and is a partitional value but not a partitional *Shapley* value.

(ii) The Aumann–Drèze value α is a partitional Shapley value on $MG(N)$ that satisfies NPP but not PWU.

(iii) As $N=\{1,2,\ldots,n\}$, for any nonempty subset $S \subseteq N$ we can consider the minimum and maximum members of S according to the ordering of natural numbers. Let us consider the partitional value ϕ^2 defined on $MG(N)$ as follows. For any (v,P), if $P \neq P^N$ and there exists $P_k \in P$ with $|P_k|>1$ and all $i \in P_k$ are null in v then, for each $i \in P_k$,

$$\phi_i^2(v,P) = \begin{cases} -1 & \text{if } i=\min P_k, \\ 1 & \text{if } i=\max P_k, \\ 0 & \text{otherwise,} \end{cases}$$

while, in any other case, $\phi_i^2(v,P)=\pi_i(v,P)$ for all $i \in P_k$. This value is a partitional Shapley value that satisfies PWU but not NPP.

Remark 17 *(Independence of the axiomatic system in Theorem 8)*

(i) Let us consider for $n=3$ the game w defined by $w(\emptyset)=w(\{1\})=w(\{2\})=w(\{3\})=w(\{1,2\})=0$, $w(\{1,3\})=1$, $w(\{2,3\})=w(N)=2$, partition $Q=\{\{1,2\},\{3\}\}$, and numbers $\tau_1=-1$, $\tau_2=1$, $\tau_3=0$. We define ϕ^3 on $MG(N)$ by

$$\phi_i^3(v,P)=\begin{cases}\tau_i & \text{if } (v,P)=(w,Q),\\ \pi_i(v,P) & \text{otherwise.}\end{cases}$$

ϕ^3 is a partitional value that satisfies NPP, SWU and WA but not NN.

(ii) The value ϕ^4 defined for all $v\in MG(N)$, $P\in P(N)$ and $i\in N$ by

$$\phi_i^4(v,P)=\frac{v\left(P_{(i)}\right)}{\left|P_{(i)}\right|}$$

is a partitional value that satisfies NN, SWU and WA but not NPP.

(iii) Let $\omega=(\omega_1,\omega_2)$ be a weighting vector such that $\omega_1\neq\omega_2$ and Φ^ω be the corresponding weighted Shapley value (Kalai and Samet, 1987). The partitional value ϕ^5 on $MG(N)$ defined by

$$\phi_i^5(v,P)=\begin{cases}\Phi^\omega(v) & \text{if } n=2 \text{ and } P=P^N,\\ \pi_i(v,P) & \text{otherwise}\end{cases}$$

satisfies NN, NPP and WA but not SWU.

(iv) The Aumann–Drèze value α is a partitional value on $MG(N)$ that satifies NN, NPP and SWU but not WA.

All properties considered in this paper are shown in Table 3.

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properties	A–D value α	PPS value π
efficiency	no	no
local efficiency (partitional value)	OK	OK
partitional Shapley value (P^N)	OK	OK
nonnegativity (NN)	OK	OK
null player property (NPP)	OK	OK
null player strong property (NPP*)	no	OK
symmetry within unions (SWU)	OK	OK
balanced contributions within unions (BCWU)	OK	no
proportionality within unions (PWU)	no	OK
additivity (ADD)	OK	no
weighted additivity (WA)	no	OK

Table 3: Comparison of properties for α and π

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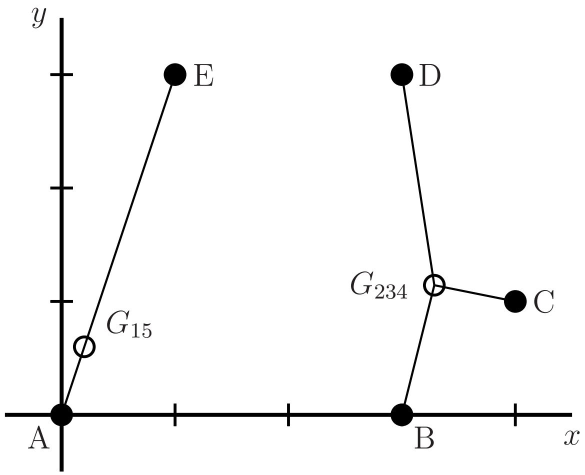

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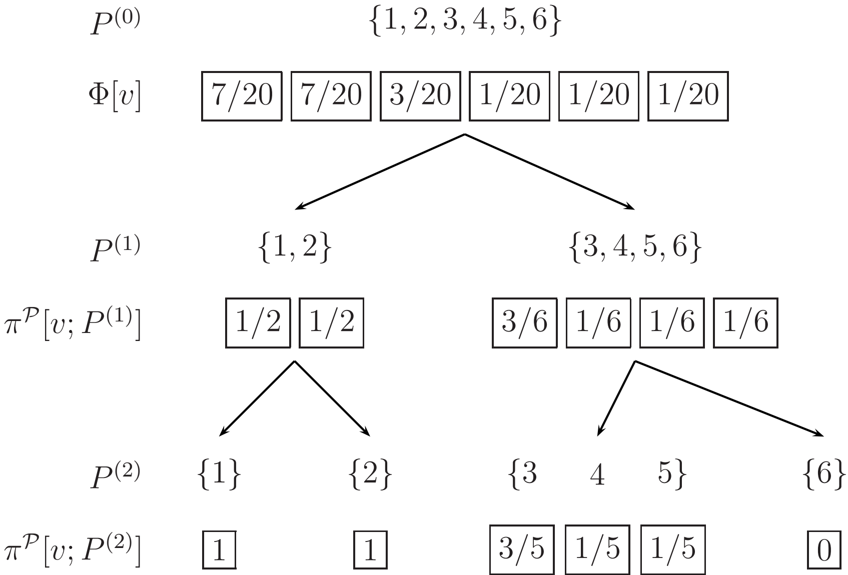

P^(0)
{1, 2, 3, 4, 5, 6}
Φ[v]
7/20 7/20 3/20 1/20 1/20 1/20
P^(1)
{1, 2}
{3, 4, 5, 6}
π^P[v; P^(1)]
1/2 1/2
3/6 1/6 1/6 1/6
P^(2)
{1}
{2}
{3 4 5}
{6}
π^P[v; P^(2)]
1
1
3/5 1/5 1/5
0

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