

HAMILTON-JACOBI THEORY IN MULTISYMPLECTIC CLASSICAL FIELD THEORIES

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Abstract

The geometric framework for the Hamilton-Jacobi theory developed in [14, 17, 39] is extended for multisymplectic first-order classical field theories. The Hamilton-Jacobi problem is stated for the Lagrangian and the Hamiltonian formalisms of these theories as a particular case of a more general problem, and the classical Hamilton-Jacobi equation for field theories is recovered from this geometrical setting. Particular and complete solutions to these problems are defined and characterized in several equivalent ways in both formalisms, and the equivalence between them is proved. The use of distributions in jet bundles that represent the solutions to the field equations is the fundamental tool in this formulation. Some examples are analyzed and, in particular, the Hamilton-Jacobi equation for non-autonomous mechanical systems is obtained as a special case of our results.

Key words: *Classical field theories; Hamilton-Jacobi equations; Multisymplectic manifolds*

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1 Introduction

The Hamilton-Jacobi theory, as it is classically presented in the textbooks and works on analytical mechanics, is a way to integrate Hamilton equations (that is, a system of first-order ordinary differential equations), which consists in giving an appropriate canonical transformation leading the system to equilibrium [3, 33, 55]. This transformation is constructed from its generating function which, in this method, is obtained as the solution to a partial differential equation: the so-called Hamilton-Jacobi equation. This method is based on a famous contribution from Hamilton on geometric optics, where he showed that the propagation of wavefronts is characterized by a function (the characteristic function) which is the solution to a first-order partial differential equation called *eikonal equation*, which is related to the Hamilton-Jacobi equation. Thus, from a purely mathematical perspective, the Hamilton-Jacobi theory is a very important example showing the deep connection between systems of first-order ordinary differential equations and first-order partial differential equations [54]. The Hamilton-Jacobi equation appears

also when short-wave approximations for the solutions of wave-type (hyperbolic) equations are considered. In this way, from a physical point of view, being a classical equation, it is also very close to the Schrödinger equation of quantum mechanics, since from a complete solution to the Hamilton-Jacobi equation, we are able to reconstruct an approximate solution to the Schrödinger equation [27, 48] and thus it allows to establish an approach within classical theory of the notions of wave function and state in quantum theory.

For all these reasons, Hamilton-Jacobi theory is a matter of continuous interest, and it was studied and generalized also in other classical ambients; in particular, for constrained systems arising from singular Lagrangians (gauge theories) [21] or also for higher-order dynamics [20].

Furthermore, in the last decades, great efforts have been done in understanding physical systems from a geometric perspective. Concerning to geometric mechanics, the intrinsic formulation of Hamilton-Jacobi equation is also clear and can be found in [1, 46, 47]. In addition, in [14] a generic geometric framework for the Hamilton-Jacobi theory was formulated both in the Lagrangian and the Hamiltonian formalisms, for autonomous and non-autonomous mechanics, recovering the usual Hamilton-Jacobi equation as a special case in this generalized framework. In particular, it is shown that the existence of constants of motion helps to solve the Hamilton-Jacobi problem, which can be regarded as a way to describe the dynamics on the phase space of the system in terms of a family of vector fields on a submanifold of it. The basic ideas of this generalization of the Hamilton-Jacobi formalism are similar to those outlined in [35].

These geometric frameworks have been used by other authors to develop the Hamilton-Jacobi theory in many different situations in mechanics. For instance, the statement and applications of the Hamilton-Jacobi method for non-holonomic and holonomic mechanical systems is done in [8, 16, 40, 32, 49, 50], the geometric treatment of the theory for dynamical systems described by singular Lagrangians is analyzed in [37, 42, 43], the application to control theory is given in [6, 61, 62], and the generalization for higher-order dynamical systems is established in [18, 19]. Moreover, the Hamilton-Jacobi theory has been extended for mechanical systems which are described using more general geometrical frameworks, such as Lie algebroids [4, 36], almost-Poisson manifolds [44], and fiber bundles in general [17], and the relationship between the Hamilton-Jacobi equation and some other geometric structures in mechanics are analyzed in [7, 15]. Finally, the geometric discretization of the Hamilton-Jacobi equation is also considered in [5, 51].

The extension of the Hamilton-Jacobi theory for first-order classical field theories has been developed for different covariant formulations (k -symplectic and k -cosymplectic) in the Hamiltonian formalism [41, 45] and also for the non-covariant Hamiltonian formulation (Cauchy data space) [10]. A first quick approach to state the Hamilton-Jacobi equation for the most general framework (the multisymplectic one) was outlined in [39], also in the Hamiltonian formalism. Furthermore, using a different approach involving connections, the theory has been generalized to higher-order field theories [59] and also for partial differential equations in general [60].

The aim of this paper is to complete these previous developments; that is, to use the guidelines stated in the aforementioned references on the Hamilton-Jacobi theory in geometric mechanics in order to give a complete description of this theory for the multisymplectic formalism of first-order classical field theories, both in the Lagrangian and the Hamiltonian formalisms, and showing the equivalence between them. Our standpoint is [39] and, in particular, some of our results are a development of the ideas pointed out in this reference. As a fundamental difference with these previous works, we consider the sections which are solutions of the field equations as integral sections of integrable distributions in the corresponding phase spaces (jet bundles and bundles of forms) where the equations are defined, and we represent these distributions by means of (classes of) multivector fields in general [25, 26]. This allows us to adapt the

geometric models for the Hamilton-Jacobi problem in mechanics given in [14, 17] to the present case.

The paper is organized as follows: Section 2 is a short review on multisymplectic geometry, jet bundles, and multivector fields and their relation with integrable distributions, which is given in order to do the paper self-contained. The main results are presented in Sections 3 and 4, where first the generalized Hamilton-Jacobi problem, and later the standard Hamilton-Jacobi problem are stated in the Lagrangian and Hamiltonian formalisms, and the equivalence between both formalisms is analyzed. In these Sections, the particular and complete solutions of the Hamilton-Jacobi equations are introduced and interpreted geometrically. In Section 5, some examples are studied; in particular, non-autonomous dynamical systems as the particular case of a field theory with 1-dimensional base manifold, quadratic Lagrangian densities, and the problem of minimal surfaces in dimension three. Finally, the conclusions and further research are presented in Section 6, where the comparison and differences between our model and the aforementioned previous works are also discussed.

All the manifolds are real, second countable and C^∞ . The maps and the structures are assumed to be C^∞ . Sum over crossed repeated indices is understood.

2 Geometrical background

2.1 Multisymplectic geometry

In this section we give a short review on multisymplectic geometry and some particular submanifolds of a multisymplectic manifold (see [11, 12, 22] for details).

Let M be an m -dimensional smooth manifold. A *multisymplectic k -form* in M is a closed k -form $\omega \in \Omega^k(M)$ which, in addition, is 1-nondegenerate, that is, for every $p \in M$, $i(X_p)\omega_p = 0$ if, and only if, $X_p = 0$, where $X_p \in T_pM$. If ω is closed and 1-degenerate, it is called a *premultisymplectic k -form*. A manifold endowed with a (pre)multisymplectic form is called a *(pre)multisymplectic manifold of order k* .

Observe that a necessary condition for a k -form to be 1-nondegenerate is $1 < k \leq \dim M$.

Given a symplectic manifold, we have a natural definition of “orthogonality” in terms of the symplectic form. This definition can be generalized to multisymplectic manifolds, bearing in mind that there are several levels of orthogonality to be considered.

Definition 1. *Let (M, ω) be a multisymplectic manifold of order k , and $F \subseteq TM$ a vector subbundle. The l th orthogonal complement of F , with $1 \leq l < k$ is the subbundle $F^{\perp, l} \subseteq TM$ defined as*

$$F^{\perp, l} = \{(p, u_p) \in TM \mid \omega_p(u_p, v_1, \dots, v_l) = 0 \text{ for every } (p, v_i) \in F\} .$$

Definition 2. *A subbundle $F \subset TM$ is called l -isotropic if $F \subseteq F^{\perp, l}$, l -coisotropic if $F^{\perp, l} \subseteq F$, and l -Lagrangian if $F = F^{\perp, l}$, for $1 \leq l < k$.*

Bearing in mind this last Definition, one can generalize the concepts of l -isotropic, l -coisotropic and l -Lagrangian subbundles to immersed submanifolds as follows.

Definition 3. *Let (M, ω) be a multisymplectic manifold of order k , and $N \hookrightarrow M$ a submanifold with canonical embedding $i: N \hookrightarrow M$. Let us consider the subbundle $Ti(TN) \subseteq TM$. Then, N is a l -isotropic (immersed) submanifold (resp., l -coisotropic submanifold, l -Lagrangian submanifold) if $Ti(TN)$ is a l -isotropic (resp., l -coisotropic, l -Lagrangian) subbundle.*

Finally, one has the following characterization of isotropic submanifolds of maximum order.

Lemma 1. *A submanifold $i: N \hookrightarrow M$ is $(k-1)$ -isotropic if, and only if, $i^*\omega = 0$.*

2.2 First-order jet bundles

In this section we give a short review on jet bundles: definition, some canonical structures and the concept of “dual bundle” (see [57] for details).

Definition and local coordinates. Prolongation of sections. Holonomy.

Let M be an orientable m -dimensional smooth manifold with fixed volume form $\eta \in \Omega^m(M)$, and let $E \xrightarrow{\pi} M$ be a bundle with $\dim E = m + n$. The *first-order jet bundle* of the projection π , $J^1\pi$, is the manifold of the 1-jets of local sections $\phi \in \Gamma(\pi)$; that is, equivalence classes of local sections of π by the relation of equality on every first-order partial derivative. A point in $J^1\pi$ is denoted by $j_x^1\phi$, where $x \in M$ and $\phi \in \Gamma(\pi)$ is a representative of the equivalence class. The manifold $J^1\pi$ is endowed with the following natural projections

$$\begin{array}{ccc} \pi^1: J^1\pi & \longrightarrow & E \\ j_x^1\phi & \longmapsto & \phi(x) \end{array} \quad ; \quad \begin{array}{ccc} \bar{\pi}^1 = \pi \circ \pi^1: J^1\pi & \longrightarrow & M \\ j_x^1\phi & \longmapsto & x \end{array} .$$

The fibers $(\pi^1)^{-1}(u) \subseteq J^1\pi$, with $u \in E$, are denoted $J_u^1\pi$.

Local coordinates in $J^1\pi$ are introduced as follows: let (x^i) , $1 \leq i \leq m$, be local coordinates in M such that $\eta = dx^1 \wedge \dots \wedge dx^m$, and (x^i, u^α) , $1 \leq \alpha \leq n$, local coordinates in E adapted to the bundle structure. Let $\phi \in \Gamma(\pi)$ be a section with coordinate expression $\phi(x^i) = (x^i, \phi^\alpha(x^i))$. Then, local coordinates in $J^1\pi$ are $(x^i, u^\alpha, u_i^\alpha)$, with $1 \leq i \leq m$ and $1 \leq \alpha \leq n$, where

$$u^\alpha = \phi^\alpha \quad ; \quad u_i^\alpha = \frac{\partial \phi^\alpha}{\partial x^i} .$$

Using these coordinates, the local expressions of the natural projections are

$$\pi^1(x^i, u^\alpha, u_i^\alpha) = (x^i, u^\alpha) \quad ; \quad \bar{\pi}^1(x^i, u^\alpha, u_i^\alpha) = (x^i) .$$

If $\phi \in \Gamma(\pi)$ is a section, we denote the *prolongation* of ϕ to $J^1\pi$ by $j^1\phi \in \Gamma(\bar{\pi}^1)$. In the natural coordinates of $J^1\pi$, if $\phi(x^i) = (x^i, \phi^\alpha(x^i))$, the prolongation of ϕ is given by

$$j^1\phi(x^i) = \left(x^i, \phi^\alpha, \frac{\partial \phi^\alpha}{\partial x^i} \right) .$$

Definition 4. *A section $\psi \in \Gamma(\bar{\pi}^1)$ is holonomic if $j^1(\pi^1 \circ \psi) = \psi$, that is, if there exists a section $\phi = \pi^1 \circ \psi \in \Gamma(\pi)$ such that ψ is the prolongation of ϕ to $J^1\pi$.*

In natural coordinates, if $\psi \in \Gamma(\bar{\pi}^1)$ is given by $\psi(x^i) = (x^i, \psi^\alpha, \psi_i^\alpha)$, then the condition for ψ to be holonomic gives the system of partial differential equations

$$\psi_i^\alpha = \frac{\partial \psi^\alpha}{\partial x^i}, \quad 1 \leq i \leq m, \quad 1 \leq \alpha \leq n, \quad (1)$$

An alternative characterization of holonomic sections is given in terms of the *canonical structure form* of $J^1\pi$.

Definition 5. The canonical structure form of $J^1\pi$ is the 1-form θ in $J^1\pi$ with values in $V(\pi)$ defined by

$$\theta_{j_x^1\phi}(v) = (d_{\phi(x)}^V\phi)(T_{j_x^1\phi}\pi^1(v)),$$

where $v \in T_{j_x^1\phi}J^1\pi$ and $d_{\phi(x)}^V\phi$ is the vertical differential of ϕ at $\phi(x) \in E$, and is defined as the map $d_{\phi(x)}^V\phi: T_{\phi(x)}E \rightarrow T_{\phi(x)}E$ such that $d_{\phi(x)}^V\phi = T_{\phi(x)}\text{Id} - T_{\phi(x)}(\phi \circ \pi)$.

Proposition 1. A section $\psi \in \Gamma(\bar{\pi}^1)$ is holonomic if, and only if, $\psi^*\theta = 0$.

Dual bundles

Let us consider the dual space of $J^1\pi$ as an affine bundle over E , which is the set of affine maps from $J_u^1\pi$ to $(\Lambda^m T^*M)_{\pi(u)}$, with $u \in E$, that is, the set

$$\bigcup_{u \in E} \text{Aff}(J_u^1\pi, (\Lambda^m(T^*M))_{\pi(u)}).$$

From [13] we know that this set is a manifold diffeomorphic to the smooth vector bundle of π -semibasic m -forms over E , $\Lambda_2^m(T^*E)$. This bundle is called the *extended dual jet bundle* of π , and we have the following canonical projections

$$\begin{array}{ccc} \pi_E: \Lambda_2^m(T^*E) & \longrightarrow & E \\ (u, \omega_u) & \longmapsto & u \end{array} \quad ; \quad \begin{array}{ccc} \bar{\pi}_E: \Lambda_2^m(T^*E) & \longrightarrow & M \\ (u, \omega_u) & \longmapsto & \pi(u) \end{array} .$$

Since $\Lambda_2^m(T^*E)$ is a bundle of forms, it is endowed with some canonical forms. First, the *Liouville m -form*, or *tautological m -form*, is the form $\Theta \in \Omega^m(\Lambda_2^m(T^*E))$ defined by

$$\Theta(\omega)(X_1, \dots, X_m) = \omega(T_\omega\pi_E(X_1), \dots, T_\omega\pi_E(X_m)),$$

where $\omega \in \Lambda_2^m(T^*E)$, and $X_1, \dots, X_m \in T_\omega(\Lambda_2^m(T^*E))$. As usual, this form satisfies the property $\xi^*\Theta = \xi$ for every $\xi \in \Omega^m(E)$. From this, the *Liouville $(m+1)$ -form*, or *canonical multisymplectic $(m+1)$ -form*, is the form $\Omega = -d\Theta \in \Omega^{m+1}(\Lambda_2^m(T^*E))$.

Local coordinates in $\Lambda_2^m(T^*E)$ are constructed as follows: let (x^i) be local coordinates in M , and (x^i, u^α) coordinates in E adapted to the bundle structure. Then, local coordinates in $\Lambda_2^m(T^*E)$ are $(x^i, u^\alpha, p, p_\alpha^i)$, where $1 \leq i \leq m$, $1 \leq \alpha \leq n$. In these coordinates, the canonical projections have the following local expressions

$$\pi_E(x^i, u^\alpha, p, p_\alpha^i) = (x^i, u^\alpha) \quad ; \quad \bar{\pi}_E(x^i, u^\alpha, p, p_\alpha^i) = (x^i).$$

On the other hand, the Liouville m and $(m+1)$ -forms have the following local expressions

$$\Theta = pd^m x + p_\alpha^i du^\alpha \wedge d^{m-1} x_i \quad ; \quad \Omega = -dp \wedge d^m x - dp_\alpha^i \wedge du^\alpha \wedge d^{m-1} x_i, \quad (2)$$

where $d^m x = dx^1 \wedge \dots \wedge dx^m$ and $d^{m-1} x_i = i(\partial/\partial x^i)d^m x$. It is clear from this coordinate expression that Ω is a multisymplectic form on $\Lambda_2^m(T^*E)$.

As $\Lambda_2^m(T^*E)$ is, in fact, a vector bundle over E , we can consider its quotient by any vector subbundle. The *reduced dual jet bundle* of π , denoted $J^1\pi^*$, is the quotient of the extended dual jet bundle, $\Lambda_2^m(T^*E)$, by constant affine transformations along the fibers of π^1 , and is diffeomorphic to the quotient bundle $\Lambda_2^m(T^*E)/\Lambda_1^m(T^*E)$. The natural quotient map is $\mu: \Lambda_2^m(T^*E) \rightarrow J^1\pi^*$.

It can be proved that $J^1\pi^*$ may be endowed with the structure of a smooth manifold and, moreover, $\mu: \Lambda_2^m(T^*E) \rightarrow J^1\pi^*$ is a smooth vector bundle of rank 1. In addition, we have the canonical projections $\pi_E^r: J^1\pi^* \rightarrow E$ and $\bar{\pi}_E^r: J^1\pi^* \rightarrow M$.

Finally, adapted coordinates (x^i, u^α) in E induce coordinates $(x^i, u^\alpha, p_\alpha^i)$ in $J^1\pi^*$ such that the coordinate expression of the natural quotient map is

$$\mu(x^i, u^\alpha, p, p_\alpha^i) = (x^i, u^\alpha, p_\alpha^i),$$

where $(x^i, u^\alpha, p, p_\alpha^i)$ are the induced coordinates in $\Lambda_2^m(\mathbb{T}^*E)$. In these coordinates, the natural projections are given by

$$\pi_E^r(x^i, u^\alpha, p_\alpha^i) = (x^i, u^\alpha) \quad ; \quad \bar{\pi}_E^r(x^i, u^\alpha, p_\alpha^i) = (x^i).$$

2.3 Multivector fields

In this section we give a short review on multivector fields and their relation with integrable distributions (see [25] for details).

Locally decomposable multivector fields. Integrability conditions

A *multivector field of degree k* , or *k -multivector field*, on a m -dimensional smooth manifold M is a section of the bundle $\Lambda^k(TM) \rightarrow M$, that is, a skew-symmetric contravariant tensor. The set of all multivector fields of degree k in M is denoted $\mathfrak{X}^k(M)$.

In general, given a k -multivector field $\mathcal{X} \in \mathfrak{X}^k(M)$, for every $p \in M$ there exists an open neighborhood $U_p \subseteq M$ and $X_1, \dots, X_r \in \mathfrak{X}(U_p)$ such that

$$\mathcal{X} = \sum_{1 \leq i_1 < \dots < i_k \leq r} f^{i_1 \dots i_k} X_{i_1} \wedge \dots \wedge X_{i_k},$$

with $f^{i_1 \dots i_k} \in C^\infty(U_p)$ and $k \leq r \leq \dim M$. If for every p we have $r = k$, that is, there exists an open neighborhood $U_p \subseteq M$ and $X_1, \dots, X_k \in \mathfrak{X}(U_p)$ such that $\mathcal{X} = X_1 \wedge \dots \wedge X_k$ on U_p , then we say that the multivector field \mathcal{X} is *locally decomposable*.

Let \mathcal{D} be a k -dimensional distribution in M , that is, a k -dimensional subbundle of TM . It is clear that sections of $\Lambda^k\mathcal{D} \rightarrow M$ are k -multivector fields in M , and that the existence of a non-vanishing global section of $\Lambda^k\mathcal{D} \rightarrow M$ is equivalent to the orientability of the distribution \mathcal{D} . Then, we say that a non-vanishing multivector field $\mathcal{X} \in \mathfrak{X}^k(M)$ and a k -dimensional distribution $\mathcal{D} \subset TM$ are *locally associated* if there exists a connected open set $U \subseteq M$ such that $\mathcal{X}|_U$ is a section of $\Lambda^k\mathcal{D}|_U$.

As a consequence of this we can introduce an equivalence relation on the set of non-vanishing k -multivector fields in M as follows: two k -multivector fields $\mathcal{X}, \mathcal{X}' \in \mathfrak{X}^k(M)$ are related if, and only if, they are both locally associated, on the same connected open set $U \subseteq M$, with the same distribution \mathcal{D} . In addition, in this case there exists a non-vanishing function $f \in C^\infty(U)$ such that $\mathcal{X}' = f\mathcal{X}$ on U . The equivalence classes of this quotient set will be denoted by $\{\mathcal{X}\}_U$. Then, one can prove that there is a bijective correspondence between the set of k -dimensional orientable distributions $\mathcal{D} \subseteq TM$ and set of equivalence classes $\{\mathcal{X}\}_M$ of non-vanishing, locally decomposable k -multivector fields in M .

If $\mathcal{X} \in \mathfrak{X}^k(M)$ is a non-vanishing, locally decomposable k -multivector field and $U \subseteq M$ is a connected open set, then the distribution associated to the equivalence class $\{\mathcal{X}\}_U$ will be denoted by $\mathcal{D}_U(\mathcal{X})$. If $U = M$, then we write simply $\mathcal{D}(\mathcal{X})$.

A non-vanishing, locally decomposable multivector field $\mathcal{X} \in \mathfrak{X}^k(M)$ is said to be *integrable* (resp. *involutive*) if its associated distribution $\mathcal{D}_U(\mathcal{X})$ is integrable (resp. involutive). It is clear then that if $\mathcal{X} \in \mathfrak{X}^k(M)$ is integrable (resp. involutive), then so is every other in its equivalence

class $\{\mathcal{X}\}$, and all of them have the same integral manifolds. Moreover, *Frobenius theorem* allows us to state that a non-vanishing and locally decomposable multivector field is integrable if, and only if, it is involutive. Nevertheless, in many applications we have locally decomposable multivector fields $\mathcal{X} \in \mathfrak{X}^k(M)$ which are not integrable in M , but integrable in a submanifold of M . A (local) algorithm for finding this submanifold has been developed [25].

Multivector fields in fiber bundles and jet bundles. Holonomy condition

We are interested in the particular situation of a fiber bundle and, more precisely, of jet bundles.

Let $\pi: E \rightarrow M$ be a fiber bundle, with $\dim M = m$ and $\dim E = m + n$. A multivector field $\mathcal{X} \in \mathfrak{X}^m(M)$ is said to be π -transverse if at every point $u \in E$ we have $(i(\mathcal{X})(\pi^*\omega))_y \neq 0$ for every $\omega \in \Omega^m(M)$ satisfying $\omega(\pi(y)) \neq 0$. It can be proved that if \mathcal{X} is integrable, then the π -transverse condition is equivalent to requiring the integral manifolds of \mathcal{X} to be local sections of π . In this case, if $\phi: U \subseteq M \rightarrow E$ is a local section with $\phi(x) = u$ and $\phi(U)$ is the integral manifold of \mathcal{X} , then $T_u(\text{Im } \phi) = \mathcal{D}_u(\mathcal{X})$.

Now, let us consider the first-order jet bundle of π , $J^1\pi$. A multivector field $\mathcal{X} \in \mathfrak{X}^m(J^1\pi)$ is *holonomic* if \mathcal{X} is integrable, $\bar{\pi}^1$ -transverse, and the integral sections of \mathcal{X} are holonomic.

As in mechanics, the holonomy of a multivector field may be characterized using the geometry of $J^1\pi$. First, a π^1 -transverse and locally decomposable multivector field $\mathcal{X} \in \mathfrak{X}^m(J^1\pi)$ is said to be *semi-holonomic*, or a *SOPDE multivector field* if, and only if, $i(\theta)\mathcal{X} = 0$, where θ is the canonical structure form in $J^1\pi$. Then, the relation between integrable, holonomic and semi-holonomic multivector fields in $J^1\pi$ is given by the following result from [25].

Theorem 1. *A multivector field $\mathcal{X} \in \mathfrak{X}^m(J^1\pi)$ is holonomic if, and only if, it is integrable and semi-holonomic.*

In natural coordinates, let $\mathcal{X} \in \mathfrak{X}^m(J^1\pi)$ be a locally decomposable and $\bar{\pi}^1$ -transverse multivector field. From the results in [25], this multivector field \mathcal{X} may be chosen to have the following coordinate expression

$$\mathcal{X} = f \bigwedge_{j=1}^m X_j = f \bigwedge_{j=1}^m \left(\frac{\partial}{\partial x^j} + f_j^\alpha \frac{\partial}{\partial u^\alpha} + F_{j,i}^\alpha \frac{\partial}{\partial u_i^\alpha} \right),$$

with f being a non-vanishing local function. Then, the condition for \mathcal{X} to be semi-holonomic gives the mn equations $f_j^\alpha = u_j^\alpha$. In addition, from the results in [25], we know that the necessary and sufficient condition for a locally decomposable multivector field to be integrable is that its associated distribution is involutive, which is equivalent to requiring the $m(m-1)/2$ conditions $[X_j, X_k] = 0$, with $1 \leq j < k \leq m$. In coordinates, these gives the following system of $nm(m^2-1)/2$ partial differential equations for the component functions $F_{j,i}^\alpha$

$$F_{j,k}^\alpha - F_{k,j}^\alpha = 0 \quad ; \quad \frac{\partial F_{k,i}^\alpha}{\partial x^j} + u_j^\beta \frac{\partial F_{k,i}^\alpha}{\partial u^\beta} + F_{j,l}^\beta \frac{\partial F_{k,i}^\alpha}{\partial u_l^\beta} - \frac{\partial F_{j,i}^\alpha}{\partial x^k} - u_k^\beta \frac{\partial F_{j,i}^\alpha}{\partial u^\beta} - F_{k,l}^\beta \frac{\partial F_{j,i}^\alpha}{\partial u_l^\beta} = 0. \quad (3)$$

Remark. It is important to point out that a locally decomposable, $\bar{\pi}^1$ -transverse and semi-holonomic multivector field \mathcal{X} may not be holonomic, since the SOPDE condition is not a sufficient nor necessary condition for the multivector field to be integrable. On the other hand, the integrability of a multivector field does not imply that the integral sections are holonomic: as in mechanics, a multivector field may admit integral sections through every point in $J^1\pi$, but these integral sections may not be projectable to the base manifold. \diamond

Relation with jet fields

Let $\pi: E \rightarrow M$ be a fiber bundle, with $\dim M = m$ and $\dim E = m + n$, and $J^1\pi$ the first-order jet bundle of π .

Definition 6. A jet field in E is a global section of the projection $\pi^1: J^1\pi \rightarrow E$.

It is proved in [57] that there is a bijective correspondence between jet fields in E and connections $\nabla \in \Lambda_1^1(T^*E) \otimes \mathfrak{X}(E)$. Therefore, there is a bijective correspondence between jet fields in E and distributions in E . We denote $\mathcal{D}(\Psi)$ the unique distribution in E associated to the jet field $\Psi \in \Gamma(\pi^1)$. This enables us to give the following definitions.

Definition 7. Let $\Psi \in \Gamma(\pi^1)$ be a jet field and $\mathcal{D}(\Psi)$ its associated distribution in E .

1. Ψ is said to be orientable if, and only if, $\mathcal{D}(\Psi)$ is an orientable distribution in E . In particular, if M is orientable, then every jet field is also orientable.
2. Ψ is said to be integrable if, and only if, $\mathcal{D}(\Psi)$ is an integrable distribution.
3. A section $\phi \in \Gamma(\pi)$ is an integral section of Ψ if, and only if, $\Psi \circ \phi = j^1\phi$. In particular, Ψ is integrable if, and only if, it admits integral sections through every point of E .

With these notations, the relation between multivector fields and jet fields is given by the following result, stated in [25].

Theorem 2. There is a bijective correspondence between the set of orientable jet fields $\Psi \in \Gamma(\pi^1)$ and the set of equivalence classes of locally decomposable and π -transverse multivector fields $\{\mathcal{X}\} \subseteq \mathfrak{X}^m(E)$. They are characterized by the fact that $\mathcal{D}(\Psi) = \mathcal{D}(\mathcal{X})$. In addition, the orientable jet field Ψ is integrable if, and only if, so is every \mathcal{X} in the equivalence class.

3 The Hamilton-Jacobi problem in the Lagrangian formalism

The geometrical setting for the Lagrangian formalism for multisymplectic field theories is the following (see, for instance, [2, 24, 25, 28, 30, 31, 38, 53] for more details). Let $\pi: E \rightarrow M$ be a fiber bundle modeling the configuration space, where M is a m -dimensional orientable smooth manifold with fixed volume form $\eta \in \Omega^m(M)$, and $\dim E = m + n$. Let $\mathcal{L} \in \Omega^m(J^1\pi)$ be a Lagrangian density containing the physical information of the theory, that is, a $\bar{\pi}^1$ -semibasic m -form. We denote $L \in C^\infty(J^1\pi)$ the function satisfying $\mathcal{L} = L(\bar{\pi}^1)^*\eta$, which we call the Lagrangian function associated to \mathcal{L} and η . Using the canonical vertical endomorphism $\nu \in \Gamma(T^*J^1\pi \otimes_{J^1\pi} TM \otimes_{J^1\pi} V(\pi^1))$, the Cartan forms $\Theta_{\mathcal{L}} = i(\nu)d\mathcal{L} + \mathcal{L} \in \Omega^m(J^1\pi)$ and $\Omega_{\mathcal{L}} = -d\Theta_{\mathcal{L}} \in \Omega^{m+1}(J^1\pi)$ are constructed, with coordinate expressions

$$\Theta_{\mathcal{L}} = \frac{\partial L}{\partial u_i^\alpha} du^\alpha \wedge d^{m-1}x_i - \left(\frac{\partial L}{\partial u_i^\alpha} u_i^\alpha - L \right) d^m x, \quad (4)$$

$$\begin{aligned} \Omega_{\mathcal{L}} = & \frac{\partial^2 L}{\partial u_i^\alpha \partial u^\beta} du^\alpha \wedge du^\beta \wedge d^{m-1}x_i + \frac{\partial^2 L}{\partial u_i^\alpha \partial u_j^\beta} du^\alpha \wedge du_j^\beta \wedge d^{m-1}x_i \\ & + \left(u_i^\alpha \frac{\partial^2 L}{\partial u_i^\alpha \partial u^\beta} - \frac{\partial L}{\partial u^\beta} \right) du^\beta \wedge d^m x + u_i^\alpha \frac{\partial^2 L}{\partial u_i^\alpha \partial u_j^\beta} du_j^\beta \wedge d^m x. \end{aligned} \quad (5)$$

Then, the Lagrangian problem for first-order classical field theories is the following: to find a m -dimensional, $\bar{\pi}^1$ -transverse and integrable distribution $\mathcal{D}_{\mathcal{L}}$ in $J^1\pi$ such that the integral

sections of $\mathcal{D}_{\mathcal{L}}$ are prolongations of sections $\phi \in \Gamma(\pi)$ satisfying

$$(j^1\phi)^* i(X)\Omega_{\mathcal{L}} = 0, \text{ for every } X \in \mathfrak{X}(J^1\pi). \quad (6)$$

If the Lagrangian density is regular, then the Cartan $(m+1)$ -form $\Omega_{\mathcal{L}}$ is multisymplectic, and then there exists such a distribution, although it is not necessarily integrable. In the following we assume that the Lagrangian density \mathcal{L} is regular, and that the distribution $\mathcal{D}_{\mathcal{L}}$ is, in addition, integrable.

From the results in Section 2.3, this distribution $\mathcal{D}_{\mathcal{L}}$ is associated with a class of holonomic multivector fields $\{\mathcal{X}_{\mathcal{L}}\} \subseteq \mathfrak{X}^m(J^1\pi)$ satisfying the equation

$$i(\mathcal{X}_{\mathcal{L}})\Omega_{\mathcal{L}} = 0, \text{ for every } \mathcal{X}_{\mathcal{L}} \in \{\mathcal{X}_{\mathcal{L}}\}. \quad (7)$$

The same comments apply in the regular case: if the Lagrangian density is regular, then there exists a class of multivector fields $\{\mathcal{X}_{\mathcal{L}}\} \subseteq \mathfrak{X}^m(J^1\pi)$ solution to equation (7) which is $\bar{\pi}^1$ -transverse and SOPDE, but not necessarily integrable. In the following we assume that the Lagrangian density is regular and that every multivector field in the class is, in addition, integrable. This class is denoted by $\{\mathcal{X}_{\mathcal{L}}\}$ along this work.

3.1 The generalized Lagrangian Hamilton-Jacobi problem

Following the patterns in [14], we first state a generalized version of the Hamilton-Jacobi problem in the Lagrangian formalism.

Definition 8. *The generalized Lagrangian Hamilton-Jacobi problem consists in finding a jet field $\Psi \in \Gamma(\pi^1)$ and a m -dimensional and integrable distribution \mathcal{D} in E such that if $\gamma \in \Gamma(\pi)$ is an integral section of \mathcal{D} , then $\Psi \circ \gamma \in \Gamma(\bar{\pi}^1)$ is an integral section of $\mathcal{D}_{\mathcal{L}}$, that is,*

$$T_u \text{Im}(\gamma) = \mathcal{D}_u \quad \forall u \in \text{Im}(\gamma) \implies T_{\bar{u}} \text{Im}(\Psi \circ \gamma) = (\mathcal{D}_{\mathcal{L}})_{\bar{u}} \quad \forall \bar{u} \in \text{Im}(\Psi \circ \gamma). \quad (8)$$

From the results in Section 2.3, since both \mathcal{D} and $\mathcal{D}_{\mathcal{L}}$ are associated with their corresponding classes of multivector fields, the problem can be stated equivalently in terms of multivector fields as the search of a jet field $\Psi \in \Gamma(\pi^1)$ and a class of locally decomposable and integrable multivector fields $\{\mathcal{X}\} \subseteq \mathfrak{X}^m(E)$ such that if $\gamma \in \Gamma(\pi)$ is an integral section of every multivector field $\mathcal{X} \in \{\mathcal{X}\}$ then $\Psi \circ \gamma \in \Gamma(\bar{\pi}^1)$ is an integral section of every multivector field $\mathcal{X}_{\mathcal{L}} \in \{\mathcal{X}_{\mathcal{L}}\}$ solution to equation (7), that is,

$$\mathcal{X} \circ \gamma = \Lambda^m \dot{\gamma} \quad \forall \mathcal{X} \in \{\mathcal{X}\} \implies \mathcal{X}_{\mathcal{L}} \circ (\Psi \circ \gamma) = \Lambda^m (\overline{\Psi \circ \gamma}) \quad \forall \mathcal{X}_{\mathcal{L}} \in \{\mathcal{X}_{\mathcal{L}}\}, \quad (9)$$

where $\Lambda^m \dot{\gamma}: M \rightarrow \Lambda^m(TE)$ denotes the canonical lift of γ to $\Lambda^m(TE)$. In the following we denote by $\{\mathcal{X}\} \subseteq \mathfrak{X}^m(E)$ the class of locally decomposable and integrable multivector fields associated with the integrable distribution \mathcal{D} in E . The diagram illustrating this equivalent formulation of the generalized Lagrangian Hamilton-Jacobi problem is the following

The diagram is a commutative diagram with the following nodes and arrows:

- Top node: $\Lambda^m(TJ^1\pi)$
- Middle node: $J^1\pi$
- Bottom-left node: M
- Bottom-middle node: E
- Bottom-right node: $\Lambda^m(TE)$

Arrows and their labels:

- A vertical arrow from $J^1\pi$ to $\Lambda^m(TJ^1\pi)$ labeled $\mathcal{X}_{\mathcal{L}}$.
- A vertical arrow from $J^1\pi$ to E labeled π^1 .
- A vertical arrow from E to $J^1\pi$ labeled Ψ .
- A horizontal arrow from M to E labeled γ .
- A horizontal arrow from E to $\Lambda^m(TE)$ labeled \mathcal{X} .
- A curved arrow from M to $\Lambda^m(TE)$ labeled $\Lambda^m \dot{\gamma}$.
- A curved arrow from M to $\Lambda^m(TJ^1\pi)$ labeled $\Lambda^m(\overline{\Psi \circ \gamma})$.
- A diagonal arrow from M to $J^1\pi$ labeled $\Psi \circ \gamma$.

where the interpretation is: *if the lower diagram formed by γ , \mathcal{X} and $\Lambda^m \dot{\gamma}$ is commutative for every $\mathcal{X} \in \{\mathcal{X}\}$, then the upper diagram formed by $\Psi \circ \gamma$, $\mathcal{X}_{\mathcal{L}}$ and $\Lambda^m(\overline{\Psi \circ \gamma})$ is also commutative for every $\mathcal{X}_{\mathcal{L}} \in \{\mathcal{X}_{\mathcal{L}}\}$.*

Remark. Since the section $\Psi \circ \gamma \in \Gamma(\bar{\pi}^1)$ is an integral section of $\mathcal{D}_{\mathcal{L}}$ (or, equivalently, an integral section of the associated class of holonomic multivector fields), in particular it must satisfy equation (6), that is,

$$(\Psi \circ \gamma)^* i(X)\Omega_{\mathcal{L}} = 0, \text{ for every } X \in \mathfrak{X}(J^1\pi).$$

Nevertheless, observe that the action of the m -form $(\Psi \circ \gamma)^* i(X)\Omega_{\mathcal{L}} \in \Omega^m(M)$ on m tangent vectors $v_i \in T_x M$, with $x \in M$, is defined as

$$((\Psi \circ \gamma)^* i(X)\Omega_{\mathcal{L}})_x(v_1, \dots, v_m) = (\Omega_{\mathcal{L}})_{\Psi(\gamma(x))}(X(\Psi(\gamma(x))), T_x(\Psi \circ \gamma)(v_1), \dots, T_x(\Psi \circ \gamma)(v_m)),$$

from where we observe that $X(\Psi(\gamma(x))) \in T_{\Psi(\gamma(x))} \text{Im}(\Psi \circ \gamma) \subset T_{\Psi(\gamma(x))} \text{Im}(\Psi)$, that is, the vector field X is tangent to the submanifold $\text{Im}(\Psi) \hookrightarrow J^1\pi$. Therefore, in this particular situation, equation (6) is equivalent to

$$(\Psi \circ \gamma)^* i(X)\Omega_{\mathcal{L}} = 0, \text{ for every } X \in \mathfrak{X}(J^1\pi) \text{ tangent to } \text{Im}(\Psi). \quad (10)$$

◇

Remark. Since every integral section of the distribution $\mathcal{D}_{\mathcal{L}}$ is the prolongation of a section of π , this holds, in particular, for the section $\Psi \circ \gamma$, and we have $\Psi \circ \gamma = j^1\phi$ for some $\phi \in \Gamma(\pi)$. Now, composing this last equality with the natural projection π^1 , we obtain $\gamma = \phi$. Then, replacing ϕ by γ in the previous expression, we have $\Psi \circ \gamma = j^1\gamma$, from where we deduce that if \mathcal{D} is an integrable distribution, then the jet field $\Psi \in \Gamma(\pi^1)$ is integrable, and every integral section of \mathcal{D} is an integral section of Ψ . Moreover, this enables us to reformulate the generalized Lagrangian Hamilton-Jacobi problem as follows:

The generalized Lagrangian Hamilton-Jacobi problem consists in finding an integrable jet field $\Psi \in \Gamma(\pi^1)$ such that if $\gamma \in \Gamma(\pi)$ is an integral section of Ψ , then $j^1\gamma \in \Gamma(\bar{\pi}^1)$ is an integral section of $\mathcal{D}_{\mathcal{L}}$.

Nevertheless, we stick to the statement in Definition 8, or the equivalent formulation given in terms of multivector fields, in order to give several equivalent conditions to being a solution to the generalized Lagrangian Hamilton-Jacobi problem. ◇

It is clear from this last remark that the distribution \mathcal{D} in E , the jet field $\Psi \in \Gamma(\pi^1)$ and the distribution $\mathcal{D}_{\mathcal{L}}$ in $J^1\pi$ are closely related. In fact, we have the following result.

Proposition 2. *The jet field $\Psi \in \Gamma(\pi^1)$ and the distribution \mathcal{D} in E satisfy condition (8) if, and only if, \mathcal{D} and $\mathcal{D}_{\mathcal{L}}$ are Ψ -related, that is, for every $\mathcal{X}_{\mathcal{L}} \in \{\mathcal{X}_{\mathcal{L}}\}$ (resp., for every $\mathcal{X} \in \{\mathcal{X}\}$) there exists $\mathcal{X} \in \{\mathcal{X}\}$ (resp., $\mathcal{X}_{\mathcal{L}} \in \{\mathcal{X}_{\mathcal{L}}\}$) such that $\mathcal{X}_{\mathcal{L}} \circ \Psi = \Lambda^m T\Psi \circ \mathcal{X}$.*

Proof. We prove this result in terms of the associated classes of multivector fields. Let $\gamma \in \Gamma(\pi)$ be an integral section of \mathcal{D} , which is equivalent to γ being an integral section of every $\mathcal{X} \in \{\mathcal{X}\}$, and let $\mathcal{X}_{\mathcal{L}} \in \{\mathcal{X}_{\mathcal{L}}\}$ be a representative of the equivalence class. Then we have

$$\mathcal{X}_{\mathcal{L}} \circ \Psi \circ \gamma = \Lambda^m(\overline{\Psi \circ \gamma}) = \Lambda^m T\Psi \circ \Lambda^m \dot{\gamma} = \Lambda^m T\Psi \circ \mathcal{X} \circ \gamma,$$

where the multivector field $\mathcal{X} \in \{\mathcal{X}\}$ in the last equality exists since γ is an integral section of \mathcal{D} . Then, since \mathcal{X} is integrable, it admits integral sections through every point in E , and therefore we have proved that for every $\mathcal{X}_{\mathcal{L}} \in \{\mathcal{X}_{\mathcal{L}}\}$ there exists $\mathcal{X} \in \{\mathcal{X}\}$ such that $\mathcal{X}_{\mathcal{L}} \circ \Psi = \Lambda^m T\Psi \circ \mathcal{X}$.

Reversing this reasoning we prove that for every $\mathcal{X} \in \{\mathcal{X}\}$ there exists $\mathcal{X}_{\mathcal{L}} \in \{\mathcal{X}_{\mathcal{L}}\}$ such that $\mathcal{X}_{\mathcal{L}} \circ \Psi = \Lambda^m \mathbb{T}\Psi \circ \mathcal{X}$. Therefore, the distributions \mathcal{D} and $\mathcal{D}_{\mathcal{L}}$ are Ψ -related.

Conversely, let us suppose that \mathcal{D} and $\mathcal{D}_{\mathcal{L}}$ are Ψ -related. Then, if $\gamma \in \Gamma(\pi)$ is an integral section of \mathcal{D} and $\mathcal{X}_{\mathcal{L}} \in \{\mathcal{X}_{\mathcal{L}}\}$ is a representative of the equivalence class, we have

$$\mathcal{X}_{\mathcal{L}} \circ \Psi \circ \gamma = \Lambda^m \mathbb{T}\Psi \circ \mathcal{X} \circ \gamma = \Lambda^m \mathbb{T}\Psi \circ \Lambda^m \dot{\gamma} = \Lambda^m (\overline{\Psi \circ \gamma}),$$

where $\mathcal{X} \in \{\mathcal{X}\}$ in the first equality is any multivector field in $\{\mathcal{X}\}$ which is Ψ -related to the given $\mathcal{X}_{\mathcal{L}}$. That is, the jet field Ψ and the distribution \mathcal{D} satisfy condition (8). \blacksquare

A straightforward consequence of this last result is the following.

Corollary 1. *If the jet field $\Psi \in \Gamma(\pi^1)$ and the distribution \mathcal{D} in E satisfy condition (8), then for every $\mathcal{X} \in \{\mathcal{X}\}$ there exists a multivector field $\mathcal{X}_{\mathcal{L}} \in \{\mathcal{X}_{\mathcal{L}}\}$ such that \mathcal{X} is given by*

$$\mathcal{X} = \Lambda^m \mathbb{T}\pi^1 \circ \mathcal{X}_{\mathcal{L}} \circ \Psi.$$

Proof. Let $\mathcal{X} \in \{\mathcal{X}\}$ be an arbitrary representative of the equivalence class. From Proposition 2 we know that if Ψ and \mathcal{D} satisfy condition (8), then there exists a multivector field $\mathcal{X}_{\mathcal{L}} \in \{\mathcal{X}_{\mathcal{L}}\}$ which is Ψ -related to the given \mathcal{X} , that is, $\mathcal{X}_{\mathcal{L}} \circ \Psi = \Lambda^m \mathbb{T}\Psi \circ \mathcal{X}$. Then, composing both sides of this last equality with the map $\Lambda^m \mathbb{T}\pi^1: \Lambda^m(\mathbb{T}J^1\pi) \rightarrow \Lambda^m(\mathbb{T}E)$, and bearing in mind that $\Psi \in \Gamma(\pi)$, we obtain

$$\Lambda^m \mathbb{T}\pi^1 \circ \mathcal{X}_{\mathcal{L}} \circ \Psi = \Lambda^m \mathbb{T}\pi^1 \circ \Lambda^m \mathbb{T}\Psi \circ \mathcal{X} = \Lambda^m \mathbb{T}(\pi^1 \circ \Psi) \circ \mathcal{X} = \mathcal{X}.$$

\blacksquare

That is, every multivector field $\mathcal{X} \in \{\mathcal{X}\}$ is completely determined by the jet field $\Psi \in \Gamma(\pi^1)$ and some multivector field $\mathcal{X}_{\mathcal{L}} \in \{\mathcal{X}_{\mathcal{L}}\}$, and it is called the *multivector field associated to Ψ and $\mathcal{X}_{\mathcal{L}}$* . The diagram illustrating this situation is the following:

$$\begin{array}{ccc} J^1\pi & \xrightarrow{\mathcal{X}_{\mathcal{L}}} & \Lambda^m(\mathbb{T}J^1\pi) \\ \Psi \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \pi^1 & & \Lambda^m \mathbb{T}\pi^1 \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \Lambda^m \mathbb{T}\Psi \\ E & \xrightarrow{\mathcal{X}} & \Lambda^m(\mathbb{T}E) \end{array}$$

In particular, the distribution \mathcal{D} is completely determined by the jet field $\Psi \in \Gamma(\pi^1)$ and the distribution $\mathcal{D}_{\mathcal{L}}$ as $\mathcal{D}_{\pi^1(\bar{u})} = \mathbb{T}_{\bar{u}}\pi^1((\mathcal{D}_{\mathcal{L}})_{\bar{u}})$ for every $\bar{u} \in \text{Im}(\Psi) \hookrightarrow J^1\pi$, or $\mathcal{D} = \mathbb{T}\pi^1(\mathcal{D}_{\mathcal{L}}|_{\text{Im}(\Psi)})$, and it is called the *distribution associated to Ψ* .

Remark. From Corollary 1 we deduce that if the jet field Ψ is integrable, then so is every multivector field $\mathcal{X} \in \{\mathcal{X}\}$, and hence the distribution \mathcal{D} . Moreover, taking into account the remark in page 11, we deduce that the jet field Ψ is integrable if, and only if, the distribution \mathcal{D} is integrable, and they have the same integral sections. Hence, Ψ and \mathcal{D} are associated in the sense of [25] (Section 2.3), that is, they define the same horizontal subbundle of $\mathbb{T}E$. \diamond

Taking into account Corollary 1 it is clear that the search for a jet field $\Gamma(\pi^1)$ and a distribution \mathcal{D} in E satisfying condition (8) is equivalent to the search of a jet field Ψ satisfying the same condition with the associated distribution $\mathbb{T}\pi^1(\mathcal{D}_{\mathcal{L}}|_{\text{Im}(\Psi)}) \subseteq \mathbb{T}E$. Therefore, we can give the following definition.

Definition 9. A solution to the generalized Lagrangian Hamilton-Jacobi problem is an integrable jet field $\Psi \in \Gamma(\pi^1)$ such that if $\gamma \in \Gamma(\pi)$ is an integral section of the m -dimensional and integrable distribution $\mathbb{T}\pi^1(\mathcal{D}_{\mathcal{L}}|_{\text{Im}(\Psi)})$ in E , then $\Psi \circ \gamma \in \Gamma(\bar{\pi}^1)$ is an integral section of $\mathcal{D}_{\mathcal{L}}$.

Now we state the following characterizations for a jet field to be a solution to the generalized Lagrangian Hamilton-Jacobi problem.

Proposition 3. Let $\Psi \in \Gamma(\pi^1)$ be an integrable jet field. Then, the following statements are equivalent.

1. Ψ is a solution to the generalized Lagrangian Hamilton-Jacobi problem.
2. The distribution $\mathcal{D}_{\mathcal{L}}$ in $J^1\pi$ is tangent to the submanifold $\text{Im}(\Psi) \hookrightarrow J^1\pi$, that is, $(\mathcal{D}_{\mathcal{L}})_{\bar{u}} \subseteq \mathbb{T}_{\bar{u}}\text{Im}(\Psi)$ for every $\bar{u} \in \text{Im}(\Psi)$.
3. Ψ satisfies the equation

$$\gamma^* i(Y)(\Psi^* \Omega_{\mathcal{L}}) = 0, \text{ for every } Y \in \mathfrak{X}(E),$$

where $\gamma \in \Gamma(\pi)$ is an integral section of the associated distribution $\mathbb{T}\pi^1(\mathcal{D}_{\mathcal{L}}|_{\text{Im}(\Psi)})$.

Proof.

(1 \iff 2) Assume that $\Psi \in \Gamma(\pi^1)$ is a solution to the generalized Lagrangian Hamilton-Jacobi problem, and let $\gamma \in \Gamma(\pi)$ be an integral section of the integrable distribution $\mathcal{D} = \mathbb{T}\pi^1(\mathcal{D}_{\mathcal{L}}|_{\text{Im}(\Psi)})$. Then, since Ψ and \mathcal{D} satisfy condition (8) and it is clear that $\mathbb{T}_{\bar{u}}\text{Im}(\Psi \circ \gamma) \subseteq \mathbb{T}_{\bar{u}}\text{Im}(\Psi)$ for every $\bar{u} \in \text{Im}(\Psi \circ \gamma)$, we have

$$(\mathcal{D}_{\mathcal{L}})_{\bar{u}} = \mathbb{T}_{\bar{u}}\text{Im}(\Psi \circ \gamma) \subseteq \mathbb{T}_{\bar{u}}\text{Im}(\Psi) \quad \forall \bar{u} \in \text{Im}(\Psi \circ \gamma).$$

Finally, since Ψ is integrable, for every $\bar{u} \in \text{Im}(\Psi)$ there exists an integral section $\gamma \in \Gamma(\pi)$ such that $\bar{u} \in \text{Im}(\Psi \circ \gamma)$, and therefore we have proved $(\mathcal{D}_{\mathcal{L}})_{\bar{u}} \subseteq \mathbb{T}_{\bar{u}}\text{Im}(\Psi)$ for every $\bar{u} \in \text{Im}(\Psi)$.

For the converse, assume that the distribution $\mathcal{D}_{\mathcal{L}}$ in $J^1\pi$ is tangent to the submanifold $\text{Im}(\Psi) \hookrightarrow J^1\pi$, that is, we have $(\mathcal{D}_{\mathcal{L}})_{\bar{u}} \subseteq \mathbb{T}_{\bar{u}}\text{Im}(\Psi)$ for every $\bar{u} \in \text{Im}(\Psi)$, which is equivalent to $(\mathcal{D}_{\mathcal{L}})_{\Psi(u)} \subseteq \mathbb{T}_{\Psi(u)}\text{Im}(\Psi)$ for every $u \in E$. We deduce from this that for every $w \in (\mathcal{D}_{\mathcal{L}})_{\Psi(u)}$ there exists a $v_u \in \mathbb{T}_u E$ such that $w = \mathbb{T}_u \Psi(v_u)$. Hence, for every $u \in E$ we define a m -dimensional subspace $\mathcal{D}_u \subseteq \mathbb{T}_u E$ as follows

$$\mathcal{D}_u = \{v \in \mathbb{T}_u E \mid w = \mathbb{T}_u \Psi(v), w \in (\mathcal{D}_{\mathcal{L}})_{\Psi(u)}\}.$$

Then, we define the m -dimensional distribution \mathcal{D} in E as $\mathcal{D} = \bigcup_{u \in E} \mathcal{D}_u$. It is clear that \mathcal{D} is a smooth and integrable distribution, since it satisfies $\mathcal{D} = \mathbb{T}\pi^1(\mathcal{D}_{\mathcal{L}}|_{\text{Im}(\Psi)})$, and both $\mathcal{D}_{\mathcal{L}}$ and Ψ are smooth and integrable. Now, let $\gamma \in \Gamma(\pi)$ be an integral section of \mathcal{D} . Then, by definition of \mathcal{D} , the condition for γ to be an integral section of \mathcal{D} gives

$$\mathbb{T}_u \text{Im}(\gamma) = \mathbb{T}_{\Psi(u)} \pi^1((\mathcal{D}_{\mathcal{L}})_{\Psi(u)}) \quad \forall u \in \text{Im}(\gamma).$$

Composing this equality with the map $\mathbb{T}_u \Psi: \mathbb{T}_u E \rightarrow \mathbb{T}_{\Psi(u)} J^1\pi$, and bearing in mind that $\Psi \circ \pi^1 = \text{Id}_{\text{Im}(\Psi)}$, we have

$$(\mathcal{D}_{\mathcal{L}})_{\Psi(u)} = \mathbb{T}_u \Psi(\mathbb{T}_u \text{Im}(\gamma)) = \mathbb{T}_{\Psi(u)} \text{Im}(\Psi \circ \gamma) \quad \forall u \in \text{Im}(\gamma),$$

which is clearly equivalent to

$$(\mathcal{D}_{\mathcal{L}})_{\bar{u}} = \mathbb{T}_{\bar{u}} \text{Im}(\Psi \circ \gamma) \quad \forall \bar{u} \in \text{Im}(\Psi \circ \gamma),$$

that is, to condition (8). Thus, Ψ is a solution to the generalized Lagrangian Hamilton-Jacobi problem.

(1 \iff 3) Let $\Psi \in \Gamma(\pi^1)$ be a jet field solution to the generalized Lagrangian Hamilton-Jacobi problem, and $\gamma \in \Gamma(\pi)$ an integral section of $\mathbb{T}\pi^1(\mathcal{D}_{\mathcal{L}}|_{\text{Im}(\Psi)})$. Then the section $\Psi \circ \gamma \in \Gamma(\bar{\pi}^1)$ is an integral section of $\mathcal{D}_{\mathcal{L}}$. In particular, $\Psi \circ \gamma$ is a solution to equation (6), that is,

$$(\Psi \circ \gamma)^* i(X)\Omega_{\mathcal{L}} = 0, \text{ for every } X \in \mathfrak{X}(J^1\pi).$$

Calculating, we have

$$(\Psi \circ \gamma)^* i(X)\Omega_{\mathcal{L}} = \gamma^*(\Psi^* i(X)\Omega_{\mathcal{L}}) = \gamma^* i(Y)\Psi^*\Omega_{\mathcal{L}},$$

where $Y \in \mathfrak{X}(E)$ is a vector field Ψ -related with X . Nevertheless, since equation (6) holds for every vector field in $J^1\pi$, we have proved

$$\gamma^* i(Y)(\Psi^*\Omega_{\mathcal{L}}) = 0, \text{ for every } Y \in \mathfrak{X}(E).$$

For the converse, let $\gamma \in \Gamma(\pi)$ be an integral section of the distribution $\mathbb{T}\pi^1(\mathcal{D}_{\mathcal{L}}|_{\text{Im}(\Psi)})$, and $\Psi \in \Gamma(\pi^1)$ an integrable jet field. By the hypothesis we have that $\gamma^* i(Y)(\Psi^*\Omega_{\mathcal{L}}) = 0$ for every $Y \in \mathfrak{X}(E)$. Then, computing, we have

$$\gamma^* i(Y)(\Psi^*\Omega_{\mathcal{L}}) = \gamma^*(\Psi^* i(X)\Omega_{\mathcal{L}}) = (\Psi \circ \gamma)^* i(X)\Omega_{\mathcal{L}},$$

where $X \in \mathfrak{X}(J^1\pi)$ is a vector field Ψ -related to Y . In particular, X is tangent to $\text{Im}(\Psi) \hookrightarrow J^1\pi$ by construction, and we have proved that $\Psi \circ \gamma \in \Gamma(\bar{\pi}^1)$ is a solution to equation

$$(\Psi \circ \gamma)^* i(X)\Omega_{\mathcal{L}} = 0, \text{ for every } X \in \mathfrak{X}(J^1\pi) \text{ tangent to } \text{Im}(\Psi),$$

that is, to equation (10), which is equivalent to equation (6) for a section of the form $\Psi \circ \gamma$. Therefore, the section $\Psi \circ \gamma \in \Gamma(\bar{\pi}^1)$ is an integral section of $\mathcal{D}_{\mathcal{L}}$, and therefore Ψ is a solution to the generalized Lagrangian Hamilton-Jacobi problem. \blacksquare

A straightforward consequence of the above result is the following.

Corollary 2. *Let $\Psi \in \Gamma(\pi^1)$ be an integrable jet field solution to the generalized Lagrangian Hamilton-Jacobi problem. Then the integral sections of $\mathcal{D}_{\mathcal{L}}$ with boundary conditions in $\text{Im}(\Psi)$ project to the integral sections of $\mathcal{D} = \mathbb{T}\pi^1(\mathcal{D}_{\mathcal{L}}|_{\text{Im}(\Psi)})$.*

Proof. Let $\Psi \in \Gamma(\pi^1)$ be a solution to the generalized Lagrangian Hamilton-Jacobi problem, and $\psi_{\mathcal{L}} \in \Gamma(\bar{\pi}^1)$ an integral section of $\mathcal{D}_{\mathcal{L}}$ with boundary conditions in $\text{Im}(\Psi)$. Then, since the distribution $\mathcal{D}_{\mathcal{L}}$ is tangent to $\text{Im}(\Psi)$ by Proposition 3, we have that $\text{Im}(\psi_{\mathcal{L}}) \subseteq \text{Im}(\Psi)$, and hence

$$\mathbb{T}_u \text{Im}(\pi^1 \circ \psi_{\mathcal{L}}) = \mathbb{T}_{\Psi(u)} \pi^1(\mathbb{T}_{\Psi(u)} \text{Im}(\psi_{\mathcal{L}})) = \mathbb{T}_{\Psi(u)} \pi^1((\mathcal{D}_{\mathcal{L}})_{\Psi(u)}) = \mathcal{D}_u,$$

where we have used that $\text{Im}(\psi_{\mathcal{L}}) \subseteq \text{Im}(\Psi)$ and $\Psi \circ \pi^1 = \text{Id}|_{\text{Im}(\Psi)}$. \blacksquare

Coordinate expression. Let (x^i) , $1 \leq i \leq m$, be local coordinates in M such that $\eta = d^m x = dx^1 \wedge \dots \wedge dx^m$, and (x^i, u^α) , $1 \leq \alpha \leq n$ local coordinates in E adapted to the bundle structure. Then, the induced coordinates in $J^1\pi$ are $(x^i, u^\alpha, u_i^\alpha)$, which coincide with the local coordinates adapted to the bundle structure $\pi^1: J^1\pi \rightarrow E$. In these coordinates, a jet field $\Psi \in \Gamma(\pi^1)$ is given locally by $\Psi(x^i, u^\alpha) = (x^i, u^\alpha, \psi_i^\alpha)$, where $\psi_i^\alpha(x^i, u^\alpha)$ are local smooth functions on E .

Let us compute the local condition for a jet field $\Psi \in \Gamma(\pi^1)$ to be a solution to the generalized Lagrangian Hamilton-Jacobi problem. From Proposition 3 we know that this is equivalent to require the distribution $\mathcal{D}_{\mathcal{L}}$ in $J^1\pi$ to be tangent to the submanifold $\text{Im}(\Psi) \hookrightarrow J^1\pi$, or, in terms of the class of multivector fields $\{\mathcal{X}_{\mathcal{L}}\} \subseteq \mathfrak{X}^m(J^1\pi)$ associated to $\mathcal{D}_{\mathcal{L}}$, to require every multivector field in the class to be tangent to $\text{Im}(\Psi)$. From [25] we know that a representative $\mathcal{X}_{\mathcal{L}} \in \{\mathcal{X}_{\mathcal{L}}\}$ which is locally decomposable, $\bar{\pi}^1$ -transverse and semi-holonomic may be chosen to have the following coordinate expression

$$\mathcal{X}_{\mathcal{L}} = \bigwedge_{j=1}^m \left(\frac{\partial}{\partial x^j} + u_j^\alpha \frac{\partial}{\partial u^\alpha} + F_{j,i}^\alpha \frac{\partial}{\partial u_i^\alpha} \right),$$

where the functions $F_{j,i}^\alpha$ are the solutions to the Euler-Lagrange equations

$$\frac{\partial L}{\partial u^\alpha} - \frac{\partial^2 L}{\partial u_i^\alpha \partial x^i} - u_i^\beta \frac{\partial^2 L}{\partial u_i^\alpha \partial u^\beta} - F_{j,i}^\beta \frac{\partial^2 L}{\partial u_i^\alpha \partial u_j^\beta} = 0, \quad (11)$$

in addition to the integrability conditions (3) (if necessary). Observe that every multivector field in the class $\{\mathcal{X}_{\mathcal{L}}\}$ is obtained by multiplying this representative by an arbitrary non-vanishing function $f \in C^\infty(J^1\pi)$. Then, bearing in mind that the submanifold $\text{Im}(\Psi) \hookrightarrow J^1\pi$ is locally defined by the mn constraints $\psi_k^\beta - u_k^\beta = 0$, the condition for this particular $\mathcal{X}_{\mathcal{L}}$ to be tangent to $\text{Im}(\Psi)$ gives the following partial differential equations

$$\frac{\partial \psi_k^\beta}{\partial x^j} + u_j^\alpha \frac{\partial \psi_k^\beta}{\partial u^\alpha} - F_{j,k}^\beta \Big|_{\text{Im}(\Psi)} = 0. \quad (12)$$

This is a system of nm^2 partial differential equations with nm unknown functions ψ_k^β , that is, we have more equations than unknown functions.

Remark. Recall that the n equations (11) do not enable us to determinate all the m^2n coefficient functions $F_{i,j}^\alpha$, and, in general, there are $n(m^2 - 1)$ arbitrary functions. Therefore, equations (12) may fix not only the coefficients ψ_k^β of the jet field Ψ , but also some of the remaining functions $F_{i,j}^\alpha$ of the Euler-Lagrange multivector fields which are solutions to the field equation (7). In this way, we have a system of m^2n partial differential equations with $n(m^2 + m - 1)$ unknown functions. Note that, even in the most favorable cases, there still are $n(m - 1)$ arbitrary functions to be determined, which may be fixed by the integrability condition (3) or not. \diamond

Remark. On time-dependent mechanics, that is, for $m = 1$ we obtain exactly n partial differential equations and n unknown functions, since there are no arbitrary functions on the vector field solution to the Lagrangian dynamical equation. \diamond

3.2 The Lagrangian Hamilton-Jacobi problem

As in mechanics (see [14, 18]), to solve the generalized Lagrangian Hamilton-Jacobi problem is, in general, a very difficult task, since it amounts to find $(m + n)$ -dimensional submanifolds of $J^1\pi$ such that the m -dimensional distribution $\mathcal{D}_{\mathcal{L}}$ is tangent to them. Because of this, we impose an additional condition on the jet field $\Psi \in \Gamma(\pi^1)$ in order to consider a less general problem.

Definition 10. *The Lagrangian Hamilton-Jacobi problem consists in finding a jet field $\Psi \in \Gamma(\pi^1)$ solution to the generalized Lagrangian Hamilton-Jacobi problem satisfying that $\Psi^*\Omega_{\mathcal{L}} = 0$. Such a jet field is called a solution to the Lagrangian Hamilton-Jacobi problem.*

With this new assumption we can state the following result, which is a straightforward consequence of Proposition 3, Corollary 2 and the results in [12].

Proposition 4. *Let $\Psi \in \Gamma(\pi^1)$ be an integrable jet field satisfying $\Psi^*\Omega_{\mathcal{L}} = 0$. Then the following statements are equivalent:*

1. Ψ is a solution to the Lagrangian Hamilton-Jacobi problem.
2. The submanifold $\text{Im}(\Psi) \hookrightarrow J^1\pi$ is m -Lagrangian and the distribution $\mathcal{D}_{\mathcal{L}}$ is tangent to it.
3. The integral sections of $\mathcal{D}_{\mathcal{L}}$ with boundary conditions in $\text{Im}(\Psi)$ project onto the integral sections of $\mathcal{D} = \mathbb{T}\pi^1(\mathcal{D}_{\mathcal{L}}|_{\text{Im}(\Psi)})$.

Coordinate expression. In coordinates, we have

$$\begin{aligned} \Psi^*\Omega_{\mathcal{L}} = & \left(\frac{\partial^2 L}{\partial u_i^\alpha \partial u^\beta} + \frac{\partial^2 L}{\partial u_i^\alpha \partial u_k^\delta} \frac{\partial \psi_k^\delta}{\partial u^\beta} \right) du^\alpha \wedge du^\beta \wedge d^{m-1}x_i \\ & + \left(\frac{\partial^2 L}{\partial u_i^\alpha \partial u_k^\beta} \frac{\partial \psi_k^\beta}{\partial x^i} + \psi_i^\beta \frac{\partial^2 L}{\partial u_i^\beta \partial u^\alpha} - \frac{\partial L}{\partial u^\alpha} + \psi_i^\delta \frac{\partial^2 L}{\partial u_i^\delta \partial u_k^\beta} \frac{\partial \psi_k^\beta}{\partial u^\alpha} \right) du^\alpha \wedge d^m x. \end{aligned}$$

Hence, the condition $\Psi^*\Omega_{\mathcal{L}} = 0$ gives the following system of $n(1 + m(n-1))$ partial differential equations

$$\frac{\partial^2 L}{\partial u_i^\alpha \partial u^\beta} + \frac{\partial^2 L}{\partial u_i^\alpha \partial u_k^\delta} \frac{\partial \psi_k^\delta}{\partial u^\beta} = 0 \quad ; \quad \frac{\partial^2 L}{\partial u_i^\alpha \partial u_k^\beta} \frac{\partial \psi_k^\beta}{\partial x^i} + \psi_i^\beta \frac{\partial^2 L}{\partial u_i^\beta \partial u^\alpha} - \frac{\partial L}{\partial u^\alpha} + \psi_i^\delta \frac{\partial^2 L}{\partial u_i^\delta \partial u_k^\beta} \frac{\partial \psi_k^\beta}{\partial u^\alpha} = 0,$$

where $1 \leq i \leq m$, $1 \leq \alpha < \beta \leq n$ in the first set, and $1 \leq \alpha \leq n$ in the second. These two sets of equations may be combined to obtain the following system

$$\frac{\partial^2 L}{\partial u_i^\alpha \partial u^\beta} + \frac{\partial^2 L}{\partial u_i^\alpha \partial u_k^\delta} \frac{\partial \psi_k^\delta}{\partial u^\beta} = 0 \quad ; \quad \frac{\partial^2 L}{\partial u_i^\alpha \partial u_k^\beta} \frac{\partial \psi_k^\beta}{\partial x^i} - \frac{\partial L}{\partial u^\alpha} = 0. \quad (13)$$

On the other hand, observe that since $\Psi^*\Omega_{\mathcal{L}} = -d(\Psi^*\Theta_{\mathcal{L}})$, the condition $\Psi^*\Omega_{\mathcal{L}} = 0$ is equivalent to requiring the m -form $\Psi^*\Theta_{\mathcal{L}} \in \Omega^m(E)$ to be closed. In particular, using Poincaré's Lemma, the m -form $\Psi^*\Theta_{\mathcal{L}}$ is locally exact, that is, there exists a $(m-1)$ -form $\omega \in \Omega^{m-1}(U)$, with $U \subseteq E$ an open set, such that $d\omega = \Psi^*\Theta_{\mathcal{L}}$. Moreover, since $\Theta_{\mathcal{L}}$ is π^1 -semibasic, so is $\Psi^*\Theta_{\mathcal{L}}$, and therefore ω must be π -semibasic. In coordinates, bearing in mind the coordinate expression (4) of the Cartan m -form, we obtain

$$\Psi^*\Theta_{\mathcal{L}} = \frac{\partial L}{\partial u_i^\alpha} du^\alpha \wedge d^{m-1}x_i - \left(\frac{\partial L}{\partial u_i^\alpha} \psi_i^\alpha - L \right) d^m x.$$

In addition, the coordinate expression for a generic π -semibasic local $(m-1)$ -form ω in E is

$$\omega = W^i d^{m-1}x_i,$$

where $W^i \in C^\infty(E)$ are local functions. From this we deduce the local expression of the m -form $d\omega$, which is

$$d\omega = \sum_{i=1}^m \frac{\partial W^i}{\partial x^i} d^m x + \frac{\partial W^i}{\partial u^\alpha} du^\alpha \wedge d^{m-1}x_i.$$

Finally, requiring $d\omega = \Psi^*\Theta_{\mathcal{L}}$, we obtain

$$\sum_{i=1}^m \frac{\partial W^i}{\partial x^i} + \psi_i^\alpha \frac{\partial L}{\partial u_i^\alpha} \Big|_{\text{Im}(\Psi)} - L(x^i, u^\alpha, \psi_i^\alpha) = 0 \quad ; \quad \frac{\partial W^i}{\partial u^\alpha} = \frac{\partial L}{\partial u_i^\alpha} \Big|_{\text{Im}(\Psi)},$$

which may be combined to give the equation

$$\sum_{i=1}^m \frac{\partial W^i}{\partial x^i} + \psi_i^\alpha \frac{\partial W^i}{\partial u^\alpha} - L(x^i, u^\alpha, \psi_i^\alpha) = 0, \quad (14)$$

which is the Hamilton-Jacobi equation in the Lagrangian formalism.

3.3 Complete solutions

In the above Sections we stated the Hamilton-Jacobi problem in the Lagrangian formalism, and a jet field $\Psi \in \Gamma(\pi^1)$ solution to this problem gives a particular solution to the Lagrangian problem in the form of a submanifold the phase space $J^1\pi$. Nevertheless, this is not a complete solution to the Lagrangian problem, since only the integral sections of the distribution $\mathcal{D}_{\mathcal{L}}$ with boundary conditions in $\text{Im}(\Psi)$ can be recovered from the solution to the Hamilton-Jacobi problem.

Hence, in order to obtain a complete solution to the problem, we need to endow the phase space $J^1\pi$ with a foliation such that every leaf is the image set of a jet field solution to the Lagrangian Hamilton-Jacobi problem. The precise definition is:

Definition 11. *A complete solution to the Lagrangian Hamilton-Jacobi problem is a local diffeomorphism $\Phi: U \times E \rightarrow J^1\pi$, with $U \subseteq \mathbb{R}^{mn}$ an open set, such that for every $\lambda \in U$, the map $\Psi_{\lambda}(\bullet) \equiv \Phi(\lambda, \bullet): E \rightarrow J^1\pi$ is a jet field in E solution to the Lagrangian Hamilton-Jacobi problem.*

Remark. An alternative, but equivalent, definition of a complete solution consists in giving the full set of jet fields $\{\Psi_{\lambda} \in \Gamma(\pi^1) \mid \lambda \in U \subseteq \mathbb{R}^{mn}\}$ depending on mn parameters, instead of the local diffeomorphism Φ . \diamond

From the definition we deduce that a complete solution endows the Lagrangian phase space $J^1\pi$ with a foliation transverse to the fibers such that every leaf has dimension $m+n$ and the distribution $\mathcal{D}_{\mathcal{L}}$ is tangent to it.

It follows from this last comment that a complete solution to the Lagrangian Hamilton-Jacobi problem enables us to recover every integral section of the distribution $\mathcal{D}_{\mathcal{L}}$ solution to the Lagrangian problem, that is, we can recover every section solution to the Euler-Lagrange equations for classical field theories. In particular, let Φ be a complete solution, and let us consider the following set of distributions in E :

$$\left\{ \mathcal{D}_{\lambda} = \text{T}\pi^1(\mathcal{D}_{\mathcal{L}}|_{\text{Im}(\Psi_{\lambda})}) \subseteq \text{T}E \mid \lambda \in U \subseteq \mathbb{R}^{mn} \right\},$$

where $\Psi_{\lambda}(\bullet) \equiv \Phi(\lambda, \bullet)$. Then, the integral sections of \mathcal{D}_{λ} , for different values of the parameter $\lambda \in U$, provide all the integral sections of the distribution $\mathcal{D}_{\mathcal{L}}$ solution to the Lagrangian problem. Indeed, let $j_x^1\phi \in J^1\pi$ be a point, and let us denote $u = \phi(x) = \pi^1(j_x^1\phi)$. Then, since Φ is a complete solution, there exists $\lambda_o \in U$ such that $\Phi(\lambda_o, u) \equiv \Psi_{\lambda_o}(u) = j_x^1\phi$, and the integral sections of \mathcal{D}_{λ_o} through u , composed with Ψ_{λ_o} , give the integral sections of $\mathcal{D}_{\mathcal{L}}$ through $j_x^1\phi$.

4 The Hamilton-Jacobi problem in the Hamiltonian formalism

As in the Lagrangian formalism, the configuration bundle in the Hamiltonian formulation for multisymplectic classical field theories is a bundle $\pi: E \rightarrow M$, where M is a m -dimensional orientable manifold with fixed volume form $\eta \in \Omega^m(M)$, and $\dim E = m+n$. Two phase spaces are considered in this formulation: the extended multimomentum bundle $\Lambda_2^m(\text{T}^*E)$ and the restricted multimomentum bundle $J^1\pi^*$ introduced in Section 2.2.

The physical information is given in terms of a *Hamiltonian section* $h \in \Gamma(\mu)$, which is specified by a *local Hamiltonian function* $H \in C^\infty(J^1\pi^*)$, that is, we have $h(x^i, u^\alpha, p_\alpha^i) = (x^i, u^\alpha, -H, p_\alpha^i)$. Then, from the canonical forms in $\Lambda_2^m(\text{T}^*E)$, and using this Hamiltonian section, the Hamilton-Cartan forms $\Theta_h = h^*\Theta \in \Omega^m(J^1\pi^*)$ and $\Omega_h = h^*\Omega = -d\Theta_h \in \Omega^{m+1}(J^1\pi^*)$ are constructed, with coordinate expressions

$$\Theta_h = p_\alpha^i du^\alpha \wedge d^{m-1}x_i - H d^m x \quad ; \quad \Omega_h = -dp_\alpha^i \wedge du^\alpha \wedge d^{m-1}x_i + dH \wedge d^m x.$$

Then, the Hamiltonian problem for a first-order classical field theory is the following: *to find m -dimensional, $\bar{\pi}_E^r$ -transverse and integrable distribution \mathcal{D}_h in $J^1\pi^*$ such that the integral sections $\psi_h \in \Gamma(\bar{\pi}_E^r)$ of \mathcal{D}_h are solutions to the field equation*

$$\psi_h^* i(X)\Omega_h = 0, \text{ for every } X \in \mathfrak{X}(J^1\pi^*). \quad (15)$$

Contrary to the Lagrangian formalism for classical field theories, the $(m+1)$ -form $\Omega_h \in \Omega^{m+1}(J^1\pi^*)$ is multisymplectic regardless of the Hamiltonian section $h \in \Gamma(\mu)$ provided, in the same way that it occurs in Classical Mechanics in the Lagrangian and Hamiltonian formalisms. Therefore, there exists such a distribution \mathcal{D}_h , although it is not necessarily integrable. In the following we assume that the distribution \mathcal{D}_h is integrable.

From the results in Section 2.3, this distribution \mathcal{D}_h is associated with a class of integrable and $\bar{\pi}_E^r$ -transverse multivector fields $\{\mathcal{X}_h\} \subseteq \mathfrak{X}^m(J^1\pi^*)$ satisfying

$$i(\mathcal{X}_h)\Omega_h = 0, \text{ for every } \mathcal{X}_h \in \{\mathcal{X}_h\}. \quad (16)$$

Same comments apply: since the $(m+1)$ -form Ω_h is 1-nondegenerate a $\bar{\pi}_E^r$ -transverse solution to equation (16) does exist, but it may not be integrable. In the following we assume that every multivector field in the class $\{\mathcal{X}_h\}$ is integrable.

(For more details on the Hamiltonian formalism of field theories see, for instance, [13, 26, 31, 34, 38, 53]).

4.1 The generalized Hamiltonian Hamilton-Jacobi problem

Following the patterns in [14] and in previous Sections, we first state a generalized version of the Hamilton-Jacobi problem in the Hamiltonian formalism.

Definition 12. *The generalized Hamiltonian Hamilton-Jacobi problem consists in finding a section $s \in \Gamma(\pi_E^r)$ and a m -dimensional and integrable distribution \mathcal{D} in E such that if $\gamma \in \Gamma(\pi)$ is an integral section of \mathcal{D} , then $s \circ \gamma \in \Gamma(\bar{\pi}_E^r)$ is an integral section of \mathcal{D}_h , that is,*

$$T_u \text{Im}(\gamma) = \mathcal{D}_u \quad \forall u \in \text{Im}(\gamma) \implies T_{[\omega]} \text{Im}(s \circ \gamma) = (\mathcal{D}_h)_{[\omega]} \quad \forall [\omega] \in \text{Im}(s \circ \gamma). \quad (17)$$

As in the Lagrangian formulation stated in Section 3, from the results in Section 2.3 we know that both \mathcal{D} and \mathcal{D}_h are associated with their corresponding classes of multivector fields. Thus, we can state the generalized Hamiltonian Hamilton-Jacobi problem in an equivalent way in terms of multivector fields as the search of a section $s \in \Gamma(\pi_E^r)$ and a class of locally decomposable and integrable multivector fields $\{\mathcal{X}\} \subseteq \mathfrak{X}^m(E)$ such that if $\gamma \in \Gamma(\pi)$ is an integral section of every multivector field $\mathcal{X} \in \{\mathcal{X}\}$, then $s \circ \gamma \in \Gamma(\bar{\pi}_E^r)$ is an integral section of every multivector field $\mathcal{X}_h \in \{\mathcal{X}_h\}$ solution to equation (16), that is,

$$\mathcal{X} \circ \gamma = \Lambda^m \dot{\gamma} \quad \forall \mathcal{X} \in \{\mathcal{X}\} \implies \mathcal{X}_h \circ s \circ \gamma = \Lambda^m(\overline{s \circ \gamma}) \quad \forall \mathcal{X}_h \in \{\mathcal{X}_h\}. \quad (18)$$

Again, as in the Lagrangian formalism we have the following diagram illustrating this equivalent formulation of the generalized Hamiltonian Hamilton-Jacobi problem

$$\begin{array}{ccccc}
& & \Lambda^m(TJ^1\pi^*) & & \\
& \nearrow \Lambda^m(\overline{s \circ \gamma}) & \uparrow \mathcal{X}_h & & \\
& & J^1\pi^* & & \\
& \nearrow s \circ \gamma & \uparrow s & \downarrow \pi^1 & \\
M & \xrightarrow{\gamma} & E & \xrightarrow{\mathcal{X}} & \Lambda^m(TE) \\
& \searrow \Lambda^m \dot{\gamma} & & &
\end{array}$$

where the interpretation is the same as in the corresponding diagram for the Lagrangian formalism: *if the lower diagram formed by γ , \mathcal{X} and $\Lambda^m \dot{\gamma}$ is commutative for every $\mathcal{X} \in \{\mathcal{X}\}$, then the upper diagram formed by $s \circ \gamma$, \mathcal{X}_h and $\Lambda^m(\overline{s \circ \gamma})$ is also commutative for every $\mathcal{X}_h \in \{\mathcal{X}_h\}$.*

Remark. Analogously to the Lagrangian formulation, the section $s \circ \gamma \in \Gamma(\bar{\pi}_E^r)$ is an integral section of the distribution \mathcal{D}_h solution to the Hamiltonian problem and, therefore, it is a solution to the equation (15), that is,

$$(s \circ \gamma)^* i(X)\Omega_h = 0, \text{ for every } X \in \mathfrak{X}(J^1\pi^*).$$

Then, bearing in mind that the action of the m -form $(s \circ \gamma)^* i(X)\Omega_h \in \Omega^m(M)$ on m tangent vectors $v_i \in T_x M$ ($x \in M$) is defined as

$$((s \circ \gamma)^* i(X)\Omega_h)_x(v_1, \dots, v_m) = (\Omega_h)_{s(\gamma(x))}(X(s(\gamma(x))), T_x(s \circ \gamma)(v_1), \dots, T_x(s \circ \gamma)(v_m)),$$

from where we observe that $X(s(\gamma(x))) \in T_{s(\gamma(x))} \text{Im}(s \circ \gamma) \subset T_{s(\gamma(x))} \text{Im}(s)$, that is, the vector field X is tangent to the submanifold $\text{Im}(s) \hookrightarrow J^1\pi^*$. Hence, we conclude that in this case equation (15) is equivalent to

$$(s \circ \gamma)^* i(X)\Omega_h = 0, \text{ for every } X \in \mathfrak{X}(J^1\pi^*) \text{ tangent to } \text{Im}(s).$$

◇

It is clear from the Definition that the distribution \mathcal{D} in E , the section $s \in \Gamma(\pi_E^r)$ and the Hamiltonian distribution \mathcal{D}_h in $J^1\pi^*$ are closely related. In fact, we have the following result, which is the analogous to Proposition 2 in the Hamiltonian formalism.

Proposition 5. *The section $s \in \Gamma(\pi_E^r)$ and the distribution \mathcal{D} in E satisfy condition (17) if, and only if, \mathcal{D} and \mathcal{D}_h are s -related.*

Proof. This proof follows exactly the same patterns as the proof of Proposition 5. ■

As in the Lagrangian formalism, a straightforward consequence of this last result is the following.

Corollary 3. *If the section $s \in \Gamma(\pi_E^r)$ and the distribution \mathcal{D} in E satisfy condition (17), then for every $\mathcal{X} \in \{\mathcal{X}\}$ there exists a multivector field $\mathcal{X}_h \in \{\mathcal{X}_h\}$ such that \mathcal{X} is given by*

$$\mathcal{X} = \Lambda^m T\pi_E^r \circ \mathcal{X}_h \circ s.$$

Proof. The proof follows the same patterns as in the proof of Corollary 1. ■

That is, the class of integrable multivector fields $\{\mathcal{X}\} \subseteq \mathfrak{X}^m(E)$ is completely determined by the section $s \in \Gamma(\pi_E^r)$ and the class of multivector fields $\{\mathcal{X}_h\} \subseteq \mathfrak{X}^m(J^1\pi^*)$ solution to the equation (16), and every multivector field $\mathcal{X} = \Lambda^m T\pi_E^r \circ \mathcal{X}_h \circ s \in \{\mathcal{X}\}$ is called the *multivector field associated to s and \mathcal{X}_h* . The diagram which illustrates this situation is the following:

$$\begin{array}{ccc} J^1\pi^* & \xrightarrow{\mathcal{X}_h} & \Lambda^m(TJ^1\pi^*) \\ \left. \begin{array}{c} \uparrow \\ s \\ \downarrow \end{array} \right\} \pi_E^r & & \left. \begin{array}{c} \uparrow \\ \Lambda^m T\pi_E^r \\ \downarrow \end{array} \right\} \Lambda^m T s \\ E & \xrightarrow{\mathcal{X}} & \Lambda^m(TE) \end{array}$$

As a consequence, the integrable distribution \mathcal{D} in E is completely determined by the section $s \in \Gamma(\pi_E^r)$ and the distribution \mathcal{D}_h solution to the Hamiltonian problem as $\mathcal{D}_{\pi_E^r([\omega])} = \mathbb{T}_{[\omega]}\pi_E^r((\mathcal{D}_h)_{[\omega]})$ for every $[\omega] \in J^1\pi^*$, or $\mathcal{D} = \mathbb{T}\pi_E^r(\mathcal{D}_h|_{\text{Im}(s)})$, and it is called the *distribution associated to s* .

From Corollary 3 we deduce that the search for a section $s \in \Gamma(\pi_E^r)$ and a distribution \mathcal{D} in E satisfying condition (17) is equivalent to the search of a section $s \in \Gamma(\pi_E^r)$ such that condition (17) is satisfied with the associated distribution $\mathbb{T}\pi_E^r(\mathcal{D}_h|_{\text{Im}(s)})$. Therefore, we can give the following definition.

Definition 13. *A solution to the generalized Hamiltonian Hamilton-Jacobi problem is a section $s \in \Gamma(\pi_E^r)$ such that if $\gamma \in \Gamma(\pi)$ is an integral section of the m -dimensional and integrable distribution $\mathbb{T}\pi_E^r(\mathcal{D}_h|_{\text{Im}(s)})$ in E , then $s \circ \gamma \in \Gamma(\pi_E^r)$ is an integral section of \mathcal{D}_h .*

Proposition 6. *Let $s \in \Gamma(\pi_E^r)$ be a section. Then, the following conditions are equivalent.*

1. *s is a solution to the generalized Hamiltonian Hamilton-Jacobi problem.*
2. *The distribution \mathcal{D}_h in $J^1\pi^*$ is tangent to the submanifold $\text{Im}(s) \hookrightarrow J^1\pi^*$, that is, $(\mathcal{D}_h)_{[\omega]} \subseteq \mathbb{T}_{[\omega]}\text{Im}(s)$ for every $[\omega] \in \text{Im}(s)$.*
3. *s satisfies the equation*

$$\gamma^* i(Y)d(h \circ s) = 0, \text{ for every } Y \in \mathfrak{X}(E),$$

where $\gamma \in \Gamma(\pi)$ is an integral section of the associated distribution $\mathbb{T}\pi_E^r(\mathcal{D}_h|_{\text{Im}(s)})$.

Proof. This proof follows the same patterns as the proof of Proposition 3, bearing in mind the properties of the tautological m -form $\Theta \in \Omega^m(\Lambda_2^m(\mathbb{T}^*E))$, that is, we have $\omega^*\Theta = \omega$ for every $\omega \in \Omega^m(E)$. Because of this, we have

$$s^*\Omega_h = s^*(h^*\Omega) = (h \circ s)^*\Omega = (h \circ s)^*(-d\Theta) = -d(h \circ s)^*\Theta = -d(h \circ s), \quad (19)$$

and therefore the equation

$$(s \circ \gamma)^* i(X)\Omega_h = 0, \text{ for every } X \in \mathfrak{X}(J^1\pi^*),$$

gives rise to equation

$$\gamma^* i(Y)d(h \circ s) = 0, \text{ for every } Y \in \mathfrak{X}(E). \quad \blacksquare$$

As in the Lagrangian formalism, a consequence of Proposition 6 is the following result.

Corollary 4. *Let $s \in \Gamma(\pi_E^r)$ be a section solution to the generalized Hamiltonian Hamilton-Jacobi problem. Then the integral sections of \mathcal{D}_h with boundary conditions in $\text{Im}(s)$ project to the integral sections of $\mathcal{D} = \mathbb{T}\pi_E^r(\mathcal{D}_h|_{\text{Im}(s)})$.*

Proof. This proof is analogous to the proof of Corollary 2. \blacksquare

Coordinate expression. Let (x^i) , $1 \leq i \leq m$, be local coordinates in M such that $\eta = d^m x = dx^1 \wedge \dots \wedge dx^m$, and (x^i, u^α) , $1 \leq \alpha \leq n$ local coordinates in E adapted to the bundle structure. Then, the induced coordinates in $J^1\pi^*$ are $(x^i, u^\alpha, p_\alpha^i)$, which coincide with the local coordinates adapted to the bundle structure $\pi_E^r: J^1\pi^* \rightarrow E$. In these coordinates, a section $s \in \Gamma(\pi_E^r)$ is given locally by $s(x^i, u^\alpha) = (x^i, u^\alpha, s_\alpha^i(x^i, u^\alpha))$, where $s_\alpha^i(x^i, u^\alpha)$ are local smooth functions on E .

Let us compute the local condition for a section $s \in \Gamma(\pi_E^r)$ to be a solution to the generalized Hamiltonian Hamilton-Jacobi problem. From Proposition 6 we know that this is equivalent to require the distribution \mathcal{D}_h solution to the Hamiltonian problem to be tangent to the submanifold $\text{Im}(s) \hookrightarrow J^1\pi^*$, or, in terms of the associated class of multivector fields $\{\mathcal{X}_h\} \subseteq \mathfrak{X}^m(J^1\pi^*)$, to require every multivector field in the class to be tangent to $\text{Im}(s)$. From [53] we know that a representative $\mathcal{X}_h \in \{\mathcal{X}_h\}$ which is locally decomposable and $\bar{\pi}_E^r$ -transverse may be chosen to have the following coordinate expression

$$\mathcal{X}_h = \bigwedge_{j=1}^m \left(\frac{\partial}{\partial x^j} + \frac{\partial H}{\partial p_\alpha^j} \frac{\partial}{\partial u^\alpha} + G_{\alpha,j}^i \frac{\partial}{\partial p_\alpha^i} \right),$$

where the functions $G_{\alpha,j}^i$ are the solutions to the equations

$$\sum_{i=1}^m G_{\alpha,i}^i = -\frac{\partial H}{\partial u^\alpha}, \quad (20)$$

in addition to the integrability conditions (if necessary), which in this case give the following system of $nm(m^2 - 1)/2$ partial differential equations for the component functions $G_{\alpha,j}^i$

$$\begin{aligned} \frac{\partial^2 H}{\partial x^j \partial p_\alpha^k} + \frac{\partial H}{\partial p_\beta^j} \frac{\partial^2 H}{\partial u^\beta \partial p_\alpha^k} + G_{\beta,j}^l \frac{\partial^2 H}{\partial p_\beta^l \partial p_\alpha^k} - \frac{\partial^2 H}{\partial x^k \partial p_\alpha^j} - \frac{\partial H}{\partial p_\beta^k} \frac{\partial^2 H}{\partial u^\beta \partial p_\alpha^j} - G_{\beta,k}^l \frac{\partial^2 H}{\partial p_\beta^l \partial p_\alpha^j} &= 0, \\ \frac{\partial G_{\alpha,k}^i}{\partial x^j} + \frac{\partial H}{\partial p_\beta^j} \frac{\partial G_{\alpha,k}^i}{\partial u^\beta} + G_{\beta,j}^l \frac{\partial G_{\alpha,k}^i}{\partial p_\beta^l} - \frac{\partial G_{\alpha,j}^i}{\partial x^k} - \frac{\partial H}{\partial p_\beta^k} \frac{\partial G_{\alpha,j}^i}{\partial u^\beta} - G_{\beta,k}^l \frac{\partial G_{\alpha,j}^i}{\partial p_\beta^l} &= 0. \end{aligned} \quad (21)$$

Then, bearing in mind that the submanifold $\text{Im}(s) \hookrightarrow J^1\pi^*$ is locally defined by the mn constraints $s_\beta^k - p_\beta^k = 0$, the condition for this representative of the class to be tangent to $\text{Im}(s)$ gives the following partial differential equations

$$\left. \frac{\partial s_\beta^k}{\partial x^j} + \frac{\partial H}{\partial p_\alpha^j} \frac{\partial s_\beta^k}{\partial u^\alpha} - G_{\beta,j}^k \right|_{\text{Im}(s)} = 0. \quad (22)$$

This is a system of nm^2 partial differential equations with nm unknown functions s_β^k , that is, we have more equations than unknown functions.

Remark. Recall that equations (20) do not determinate uniquely all the coefficient functions $G_{\alpha,j}^i$, since there are m^2n unknown functions $G_{\alpha,j}^i$ and we have only n equations, which implies that, in general, there are $n(m^2 - 1)$ arbitrary functions. Therefore, equations (22) could enable us to fix some of the arbitrary functions $G_{\alpha,j}^i$ of the Hamiltonian multivector fields solution to the field equation (16). From this point of view, equations (22) are a system of m^2n partial differential equations with $n(m^2 + m - 1)$ unknown functions. Note that in the most favorable cases there still are $n(m - 1)$ arbitrary functions to be determined, which may be fixed by the integrability condition (21) or not. \diamond

Remark. On time-dependent mechanics, that is, for $m = 1$ we obtain exactly n partial differential equations and n unknown functions, since there are no arbitrary functions on the vector field solution to the Hamiltonian dynamical equation. \diamond

4.2 The Hamiltonian Hamilton-Jacobi problem

As in mechanics (see [14, 18]), to solve the generalized Hamiltonian Hamilton-Jacobi problem is, in general, a very difficult task, since it amounts to find mn -codimensional submanifolds of $J^1\pi^*$ such that the m -dimensional distribution \mathcal{D}_h is tangent to them. For this reason we require an additional condition to the section $s \in \Gamma(\pi_E^r)$, and thus we consider a more particular problem.

Definition 14. *The Hamiltonian Hamilton-Jacobi problem consists in finding a section $s \in \Gamma(\pi_E^r)$ solution to the generalized Hamiltonian Hamilton-Jacobi problem such that $s^*\Omega_h = 0$. Such a section is called a solution to the Hamiltonian Hamilton-Jacobi problem.*

Remark. Bearing in mind the properties of the tautological m -form $\Theta \in \Omega^m(\Lambda_2^m(\mathbb{T}^*E))$, and the calculations in (19), the condition $s^*\Omega_h = 0$ is equivalent to the closedness of the m -form $h \circ s \in \Omega^m(E)$. \diamond

With this new assumption we can state the following result, which is a straightforward consequence of Proposition 6, Corollary 4 and the results on isotropic submanifold of multisymplectic manifolds in [12].

Proposition 7. *Let $s \in \Gamma(\pi_E^r)$ be a section satisfying $s^*\Omega_h = 0$. Then the following statements are equivalent:*

1. *s is a solution to the Hamiltonian Hamilton-Jacobi problem.*
2. *The submanifold $\text{Im}(s) \hookrightarrow J^1\pi^*$ is m -Lagrangian and the distribution \mathcal{D}_h solution to the Hamiltonian problem is tangent to it.*
3. *The integral sections of \mathcal{D}_h with boundary conditions in $\text{Im}(s)$ project onto the integral sections of $\mathcal{D} = \mathbb{T}\pi_E^r(\mathcal{D}_h|_{\text{Im}(s)})$.*

Coordinate expression. In coordinates, we have

$$s^*\Omega_h = -d(h \circ s) = \left(\frac{\partial H}{\partial u^\alpha} + \frac{\partial H}{\partial p_\beta^j} \frac{\partial s_\beta^j}{\partial u^\alpha} + \sum_{i=1}^m \frac{\partial s_\alpha^i}{\partial x^i} \right) du^\alpha \wedge d^m x + \frac{\partial s_\alpha^i}{\partial u^\beta} du^\alpha \wedge du^\beta \wedge d^{m-1} x_i.$$

Hence, the condition $s^*\Omega_h = 0$ or, equivalently, $d(h \circ s) = 0$, gives the following $n(1 + m(n - 1))$ partial differential equations

$$\frac{\partial H}{\partial u^\alpha} + \frac{\partial H}{\partial p_\beta^j} \frac{\partial s_\beta^j}{\partial u^\alpha} + \sum_{i=1}^m \frac{\partial s_\alpha^i}{\partial x^i} = 0 \quad ; \quad \frac{\partial s_\alpha^i}{\partial u^\beta} - \frac{\partial s_\beta^i}{\partial u^\alpha} = 0, \quad (23)$$

where $1 \leq \alpha \leq n$ in the first set, and $1 \leq \alpha < \beta \leq n$ in the second. Therefore, a section $s \in \Gamma(\pi_E^r)$ solution to the Hamiltonian Hamilton-Jacobi problem must satisfy the $m^2n + mn(n - 1) + n$ partial differential equations (22) and (23).

Remark. Since equations (23) are the analogous to equations (13) in the Hamiltonian formalism we deduce that, in general, equations (23) may not be locally linear independent among themselves or together with (22). In particular, we may have less equations than the $m^2n + mn(n - 1) + n$ given by (22) and (23). \diamond

Now we recover the classic Hamilton-Jacobi equation for first-order classical field theories. Since the π_E^r -semibasic m -form $h \circ s$ is closed, by Poincaré's Lemma it is locally exact, that is, there exists a π -semibasic $(m - 1)$ -form $\omega \in \Omega^{m-1}(U)$, with $U \subseteq E$ an open set, such that $d\omega = h \circ s$. In coordinates, a semibasic $(m - 1)$ -form defined in an open set of E is given by

$$\omega = W^i d^{m-1} x_i,$$

with $W^i \in C^\infty(E)$ being local functions. From this, the m -form $d\omega$ is given by

$$d\omega = \sum_{i=1}^m \frac{\partial W^i}{\partial x^i} d^m x + \frac{\partial W^i}{\partial u^\alpha} du^\alpha \wedge d^{m-1} x_i.$$

Hence, requiring the identity $d\omega = h \circ s$ to hold, we obtain

$$-H(x^i, u^\alpha, s_\alpha^i) = \sum_{i=1}^m \frac{\partial W^i}{\partial x^i} \quad ; \quad \frac{\partial W^i}{\partial u^\alpha} = s_\alpha^i,$$

from where the classic Hamilton-Jacobi equation is deduced

$$\sum_{i=1}^m \frac{\partial W^i}{\partial x^i} + H\left(x^i, u^\alpha, \frac{\partial W^i}{\partial u^\alpha}\right) = 0. \quad (24)$$

4.3 Complete solutions

As in the Lagrangian formalism stated in Section 3, in the previous Sections we stated the Hamiltonian formulation of the Hamilton-Jacobi problem for first-order classical field theories. As we have proved previously, a section $s \in \Gamma(\pi_E^r)$ solution to the Hamilton-Jacobi problem gives rise to a particular set of solutions to the Hamiltonian problem in terms of a submanifold of phase space $J^1\pi^*$. Observe, however, that this section of the bundle $J^1\pi^* \rightarrow E$ is not a complete solution to the Hamiltonian problem, since only the integral sections of \mathcal{D}_h with boundary conditions in $\text{Im}(s)$ can be recovered from the solution to the Hamilton-Jacobi problem.

Therefore, a complete solution to the problem is given in terms of a foliation of the phase space $J^1\pi^*$ such that every leaf is the image set of a section solution to the Hamiltonian Hamilton-Jacobi problem.

Definition 15. *A complete solution to the Hamiltonian Hamilton-Jacobi problem is a local diffeomorphism $\Phi: U \times E \rightarrow J^1\pi^*$, with $U \subseteq \mathbb{R}^{mn}$ an open set, such that for every $\lambda \in U$, the map $s_\lambda(\bullet) \equiv \Phi(\lambda, \bullet): E \rightarrow J^1\pi^*$ is a section of the projection π_E^r which is a solution to the Hamiltonian Hamilton-Jacobi problem.*

As in the Lagrangian formalism, it follows from the definition that a complete solution provides the manifold $J^1\pi^*$ with a foliation transverse to the fibers satisfying that every leaf is $(m+n)$ -dimensional and the distribution \mathcal{D}_h is tangent to it.

Moreover, from a complete solution to the Hamiltonian Hamilton-Jacobi problem we can recover every section solution to the Hamilton-De Donder-Weyl equations. In particular, let Φ be a complete solution, and let us consider the following set of distributions in E

$$\left\{ \mathcal{D}_\lambda = \text{T}\pi_E^r(\mathcal{D}_h|_{\text{Im}(s_\lambda)}) \subset \text{T}E \mid \lambda \in U \subseteq \mathbb{R}^{mn} \right\},$$

where $s_\lambda(\bullet) \equiv \Phi(\lambda, \bullet)$. Then, the integral sections of \mathcal{D}_λ , for different values of $\lambda \in U$, provide all the integral sections of the distribution \mathcal{D}_h solution to the Hamiltonian problem. Indeed, let $[\omega] \in J^1\pi^*$ be a point, and let $u = \pi_E^r([\omega])$ be its projection to E . Then, since Φ is a complete solution, there exists $\lambda_o \in U$ such that $\Phi(\lambda_o, u) \equiv s_{\lambda_o}(u) = [\omega]$, and the integral sections of \mathcal{D}_{λ_o} through u , lifted to $J^1\pi^*$ by s_{λ_o} , give the integral sections of \mathcal{D}_h through $[\omega]$.

4.4 Equivalence with the Lagrangian formalism

In previous Sections we have stated the (generalized) Hamilton-Jacobi problem for multisymplectic classical field theories in both the Lagrangian and Hamiltonian formalisms. In the following we establish a bijective relation between the solution of the Hamilton-Jacobi problem in both formulations in terms of the restricted Legendre map.

First of all, recall that since the Cartan m -form $\Theta_{\mathcal{L}} \in \Omega^m(J^1\pi)$ is π^1 -semibasic, we can give the following definition.

Definition 16. *The extended Legendre map associated with the Lagrangian $\mathcal{L} \in \Omega^m(J^1\pi)$ is the bundle map $\widetilde{\mathcal{FL}}: J^1\pi \rightarrow \Lambda_2^m(\mathbb{T}^*E)$ over E defined as follows:*

$$(\Theta_{\mathcal{L}}(j_x^1\phi))(Y_1(j_x^1\phi), \dots, Y_m(j_x^1\phi)) = (\widetilde{\mathcal{FL}}(j_x^1\phi))((\mathbb{T}_{j_x^1\phi}\pi^1 Y_1)(\phi(x)), \dots, (\mathbb{T}_{j_x^1\phi}\pi^1 Y_m)(\phi(x))),$$

where $Y_i \in \mathfrak{X}(J^1\pi)$, and therefore $\mathbb{T}\pi^1 Y_i \in \mathfrak{X}(E)$.

This map verifies $\pi_E \circ \widetilde{\mathcal{FL}} = \pi^1$, that is, $\widetilde{\mathcal{FL}}$ is a bundle morphism over E . Furthermore, we have that $\widetilde{\mathcal{FL}}^* \Theta = \Theta_{\mathcal{L}}$ and $\widetilde{\mathcal{FL}}^* \Omega = \Omega_{\mathcal{L}}$. Then, bearing in mind the coordinate expressions (2) of the tautological m -form $\Theta \in \Omega^m(\Lambda_2^m(\mathbb{T}^*E))$, and (4) of the Cartan m -form $\Theta_{\mathcal{L}} \in \Omega^m(J^1\pi)$, the coordinate expression of the extended Legendre map is

$$\widetilde{\mathcal{FL}}^* x^i = x^i \quad ; \quad \widetilde{\mathcal{FL}}^* u^\alpha = u^\alpha \quad ; \quad \widetilde{\mathcal{FL}}^* p_\alpha^i = \frac{\partial L}{\partial u_i^\alpha} \quad ; \quad \widetilde{\mathcal{FL}}^* p = L - u_i^\alpha \frac{\partial L}{\partial u_i^\alpha}. \quad (25)$$

The composition of the extended Legendre map $\widetilde{\mathcal{FL}}: J^1\pi \rightarrow \Lambda_2^m(\mathbb{T}^*E)$ with the natural quotient map $\mu: \Lambda_2^m(\mathbb{T}^*E) \rightarrow J^1\pi^*$ gives rise to a bundle morphism $\mu \circ \widetilde{\mathcal{FL}}: J^1\pi \rightarrow J^1\pi^*$, which leads to the following definition.

Definition 17. *The restricted Legendre map associated to the Lagrangian density $\mathcal{L} \in \Omega^m(J^1\pi)$ is the bundle morphism $\mathcal{FL}: J^1\pi \rightarrow J^1\pi^*$ over E defined as $\mathcal{FL} = \mu \circ \widetilde{\mathcal{FL}}$.*

In the natural coordinates of $J^1\pi^*$, the local expression of the restricted Legendre map is

$$\mathcal{FL}^* x^i = x^i \quad ; \quad \mathcal{FL}^* u^\alpha = u^\alpha \quad ; \quad \mathcal{FL}^* p_\alpha^i = \frac{\partial L}{\partial u_i^\alpha}.$$

As for the extended Legendre map, the map $\mathcal{FL}: J^1\pi \rightarrow J^1\pi^*$ satisfies $\pi_E^r \circ \mathcal{FL} = \pi^1$, $\mathcal{FL}^* \Theta_h = \Theta_{\mathcal{L}}$ and $\mathcal{FL}^* \Omega_h = \Omega_{\mathcal{L}}$. Moreover, the Lagrangian density $\mathcal{L} \in \Omega^m(J^1\pi)$ is regular if, and only if, the restricted Legendre map is a local diffeomorphism, and \mathcal{L} is said to be *hyperregular* if \mathcal{FL} is a global diffeomorphism. Furthermore, in the hyperregular case the Hamiltonian section $h \in \Gamma(\mu)$ is defined by $h = \widetilde{\mathcal{FL}} \circ \mathcal{FL}^{-1}$ (or by restriction on the corresponding open sets where \mathcal{FL} is a diffeomorphism in the regular case).

Finally, to close the review on the properties of the Legendre maps, we have the following fundamental result from [53].

Theorem 3. *Let $(J^1\pi, \mathcal{L})$ be a Lagrangian field theory with $\mathcal{L} \in \Omega^m(J^1\pi)$ a hyperregular Lagrangian density, and $(J^1\pi^*, \Omega_h)$ the associated Hamiltonian field theory.*

1. *If $\phi \in \Gamma(\pi)$ is a solution to equation (6), then the section $\psi_h = \mathcal{FL} \circ j^1\phi \in \Gamma(\bar{\pi}_E^r)$ is a solution to equation (15).*
2. *Conversely, if $\psi_h \in \Gamma(\bar{\pi}_E^r)$ is a solution to equation (15), then the section $\phi = \pi_E^r \circ \psi_h \in \Gamma(\pi)$ is a solution to equation (6).*

In particular, Theorem 3 states that the distributions $\mathcal{D}_{\mathcal{L}}$ and \mathcal{D}_h solutions to the Lagrangian and Hamiltonian problems, respectively, are \mathcal{FL} -related and, moreover, it establishes a bijective correspondence between the integral sections of both distributions.

Now we can state the equivalence between the solutions of the (generalized) Lagrangian and Hamiltonian Hamilton-Jacobi problems in terms of the restricted Legendre map. First we need the following technical results.

Lemma 2. Let $E_1 \xrightarrow{\pi_1} M$ and $E_2 \xrightarrow{\pi_2} M$ be two fiber bundles, $F: E_1 \rightarrow E_2$ a fiber bundle morphism, and two F -related k -multivector fields $\mathcal{X}_1 \in \mathfrak{X}^k(E_1)$ and $\mathcal{X}_2 \in \mathfrak{X}^k(E_2)$. If $s_1 \in \Gamma(\pi_1)$ is a section of π_1 and we define a section of π_2 as $s_2 = F \circ s_1 \in \Gamma(\pi_2)$, then

$$\Lambda^k \mathbb{T}\pi_1 \circ \mathcal{X}_1 \circ s_1 = \Lambda^k \mathbb{T}\pi_2 \circ \mathcal{X}_2 \circ s_2 \in \mathfrak{X}^k(M).$$

Proof. As $F: E_1 \rightarrow E_2$ is a fiber bundle morphism (that is, $\pi_1 = \pi_2 \circ F$), and \mathcal{X}_1 and \mathcal{X}_2 are F -related (that is, $\Lambda^k \mathbb{T}F \circ \mathcal{X}_1 = \mathcal{X}_2 \circ F$), we have the following commutative diagram

$$\begin{array}{ccc} \Lambda^k \mathbb{T}E_1 & \xrightarrow{\Lambda^k \mathbb{T}F} & \Lambda^k \mathbb{T}E_2 \\ \mathcal{X}_1 \uparrow & & \uparrow \mathcal{X}_2 \\ E_1 & \xrightarrow{F} & E_2 \\ & \searrow \pi_1 & \swarrow \pi_2 \\ & M & \end{array}$$

Then we have

$$\begin{aligned} \Lambda^k \mathbb{T}\pi_1 \circ \mathcal{X}_1 \circ s_1 &= \Lambda^k \mathbb{T}(\pi_2 \circ F) \circ \mathcal{X}_1 \circ s_1 = \Lambda^k \mathbb{T}\pi_2 \circ \Lambda^k \mathbb{T}F \circ \mathcal{X}_1 \circ s_1 \\ &= \Lambda^k \mathbb{T}\pi_2 \circ \mathcal{X}_2 \circ F \circ s_1 = \Lambda^k \mathbb{T}\pi_2 \circ \mathcal{X}_2 \circ s_2. \end{aligned}$$

■

Lemma 3. Let $E_1 \xrightarrow{\pi_1} M$ and $E_2 \xrightarrow{\pi_2} M$ be two fiber bundles, $F: E_1 \rightarrow E_2$ a fiber bundle morphism, and two F -related k -dimensional distributions \mathcal{D}_1 in E_1 and \mathcal{D}_2 in E_2 . If $s_1 \in \Gamma(\pi_1)$ is a section of π_1 and we define a section of π_2 as $s_2 = F \circ s_1 \in \Gamma(\pi_2)$, then

$$\mathbb{T}\pi_1(\mathcal{D}_1|_{\text{Im}(s_1)}) = \mathbb{T}\pi_2(\mathcal{D}_2|_{\text{Im}(s_2)}).$$

Proof. Since \mathcal{D}_1 and \mathcal{D}_2 are F -related, it suffices to apply Lemma 2 to every pair of F -related multivector fields in the classes of multivector fields associated to \mathcal{D}_1 and \mathcal{D}_2 . ■

Finally, the equivalence Theorem is the following.

Theorem 4. Let $(J^1\pi, \mathcal{L})$ be a Lagrangian field theory with $\mathcal{L} \in \Omega^m(J^1\pi)$ a hyperregular Lagrangian density, and $(J^1\pi^*, \Omega_h)$ its associated Hamiltonian field theory.

1. If $\Psi \in \Gamma(\pi^1)$ is an integrable jet field solution to the (generalized) Lagrangian Hamilton-Jacobi problem, then the section $s = \mathcal{F}\mathcal{L} \circ \Psi \in \Gamma(\pi_E^r)$ is a solution to the (generalized) Hamiltonian Hamilton-Jacobi problem.
2. Conversely, if $s \in \Gamma(\pi_E^r)$ is a solution to the (generalized) Hamiltonian Hamilton-Jacobi problem, then the jet field $\Psi = \mathcal{F}\mathcal{L}^{-1} \circ s \in \Gamma(\pi^1)$ is a solution to the (generalized) Lagrangian Hamilton-Jacobi problem.

Proof. This proof follows the patterns of Theorem 3 in [14] and Theorem 1 in [18], now in terms of distributions.

In particular, assume that $\Psi \in \Gamma(\pi^1)$ is a solution to the generalized Lagrangian Hamilton-Jacobi problem. First, if $\mathcal{D} = \mathbb{T}\pi^1(\mathcal{D}_{\mathcal{L}}|_{\text{Im}(\Psi)})$, $\bar{\mathcal{D}} = \mathbb{T}\pi_E^r(\mathcal{D}_h|_{\text{Im}(s)})$ are the integrable distributions associated to Ψ and $s = \mathcal{F}\mathcal{L} \circ \Psi \in \Gamma(\pi_E^r)$, respectively, then by Lemma 3 we have $\mathcal{D} = \bar{\mathcal{D}}$. Hence, both distributions are denoted by \mathcal{D} .

Now we prove that $s = \mathcal{FL} \circ \Psi \in \Gamma(\pi_E^r)$ is a solution to the generalized Hamiltonian Hamilton-Jacobi problem. Let $\gamma \in \Gamma(\pi)$ be an integral section of \mathcal{D} . Computing, we have

$$\begin{aligned} \gamma^* i(Y)d(h \circ s) &= \gamma^* i(Y)d(h \circ s) = \gamma^* i(Y)(s^* \Omega_h) \\ &= \gamma^* i(Y)((\mathcal{FL} \circ \Psi)^* \Omega_h) = \gamma^* i(Y)\Psi^*(\mathcal{FL}^* \Omega_h) \\ &= \gamma^* i(Y)(\Psi^* \Omega_{\mathcal{L}}). \end{aligned}$$

Hence, since this last expression vanishes for every $Y \in \mathfrak{X}(E)$ by the hypothesis and Proposition 3, we have proved

$$\gamma^* i(Y)d(h \circ s) = 0, \text{ for every } Y \in \mathfrak{X}(E),$$

which, using Proposition 6, is equivalent to $s = \mathcal{FL} \circ \Psi$ being a solution to the generalized Hamiltonian Hamilton-Jacobi problem.

Now we require, in addition, $\Psi^* \Omega_{\mathcal{L}} = 0$, that is, Ψ is a solution to the Lagrangian Hamilton-Jacobi problem. Then we have

$$s^* \Omega_h = (\mathcal{FL} \circ \Psi)^* \Omega_h = \Psi^*(\mathcal{FL}^* \Omega_h) = \Psi^* \Omega_{\mathcal{L}} = 0,$$

and therefore $s = \mathcal{FL} \circ \Psi$ is a solution to the Hamiltonian Hamilton-Jacobi problem.

The converse is proved analogously, but using \mathcal{FL}^{-1} instead of \mathcal{FL} . ■

5 Examples

5.1 Non-autonomous dynamical systems

Let us consider the case of non-autonomous dynamical systems, that is, the base manifold M is 1-dimensional, and in particular we assume that $M = \mathbb{R}$ with the canonical volume form $\eta \in \Omega^1(\mathbb{R})$. Let us consider a first-order non-autonomous dynamical system with n degrees of freedom, and let $\pi: E \rightarrow \mathbb{R}$ be the configuration bundle for this system, with $\dim E = n+1$. The dynamical information is enclosed on a Lagrangian density $\mathcal{L} \in \Omega^1(J^1\pi)$, which is a $\bar{\pi}^1$ -semibasic 1-form. Because of this, let us denote by $L \in C^\infty(J^1\pi)$ the function satisfying $\mathcal{L} = L \cdot (\bar{\pi}^1)^* \eta$.

Remark. Observe that since M is 1-dimensional, it is either diffeomorphic to the real line \mathbb{R} or the unit circle \mathbb{S}^1 . The only difference for our calculations is that, contrary to the case of the unit circle, \mathbb{R} has a canonical global atlas with a distinguished coordinate. Nevertheless, both manifolds are orientable and parallelizable, and therefore all the calculations remain the same in the unit circle despite the absence of a global atlas. ◇

Local coordinates along this Section are denoted in the usual way for non-autonomous dynamical systems. In particular, let (t) denote the global coordinate in \mathbb{R} , and let (t, q^A) , $1 \leq A \leq n$, be local coordinates in E adapted to the bundle structure. Then, the induced local coordinates in $J^1\pi$, T^*E and $J^1\pi^*$ are denoted (t, q^A, v^A) , (t, q^A, p, p_A) and (t, q^A, p_A) , respectively.

Lagrangian formalism

The Lagrangian problem for first-order non-autonomous dynamical systems consists in finding a class of $\bar{\pi}^1$ -transverse and semi-holonomic vector fields $\{X_{\mathcal{L}}\} \subseteq \mathfrak{X}(J^1\pi)$ satisfying the field equation (7), that is,

$$i(X_{\mathcal{L}})\Omega_{\mathcal{L}} = 0, \text{ for every } X_{\mathcal{L}} \in \{X_{\mathcal{L}}\}.$$

If the Lagrangian density is regular, the Cartan 2-form $\Omega \in \Omega^2(J^1\pi)$ has maximal rank $2n$ on $J^1\pi$, and therefore there exists a class of vector fields $\{X_{\mathcal{L}}\}$ solution to the above equation which is holonomic. Hence, in the following we assume that the Lagrangian density is regular.

Remark. Note that the integrability is assured without further assumptions in this setting, since every vector field defined on a manifold is always integrable. \diamond

The local expressions of the Cartan 1-form $\Theta_{\mathcal{L}} \in \Omega^1(J^1\pi)$ is

$$\Theta_{\mathcal{L}} = \frac{\partial L}{\partial v^A} dq^A - \left(v^A \frac{\partial L}{\partial v^A} - L \right) dt, \quad (26)$$

from where we deduce the coordinate expression of the Cartan 2-form, which is

$$\begin{aligned} \Omega_{\mathcal{L}} = & \frac{\partial^2 L}{\partial v^A \partial q^B} dq^A \wedge dq^B + \frac{\partial^2 L}{\partial v^A \partial v^B} dq^A \wedge dv^B \\ & + \left(v^A \frac{\partial^2 L}{\partial v^A \partial q^B} - \frac{\partial L}{\partial q^B} \right) dq^B \wedge dt + v^A \frac{\partial^2 L}{\partial v^A \partial v^B} dv^B \wedge dt. \end{aligned} \quad (27)$$

Thus, a representative of the equivalence class $\{X_{\mathcal{L}}\} \subseteq \mathfrak{X}(J^1\pi)$ of $\bar{\pi}^1$ -transverse vector fields solution to the above dynamical equation is given in coordinates by

$$X_{\mathcal{L}} = f \left(\frac{\partial}{\partial t} + v^A \frac{\partial}{\partial q^A} + F^A \frac{\partial}{\partial v^A} \right), \quad (28)$$

where f is a non-vanishing local function, and the n functions F^A are the unique solutions to the Euler-Lagrange equations

$$\frac{\partial L}{\partial q^B} - \frac{\partial^2 L}{\partial t \partial v^B} - v^A \frac{\partial^2 L}{\partial q^A \partial v^B} - F^A \frac{\partial^2 L}{\partial v^A \partial v^B} = 0.$$

Observe that the vector fields in the class $\{X_{\mathcal{L}}\} \subseteq \mathfrak{X}(J^1\pi)$ are completely determined (except for the non-vanishing function f). In the following we take $f = 1$ as a representative of the equivalence class to state the Lagrangian Hamilton-Jacobi problem.

For the generalized Lagrangian Hamilton-Jacobi problem, all the Definitions and results stated in Section 3.1 remain without changes, except for Corollary 2, where “boundary conditions” should be replaced by “initial conditions”. Note that every mention to the integrability of the jet field is redundant, since the associated distribution in this case is 1-dimensional, and therefore integrable without further assumptions.

In coordinates, let $\Psi \in \Gamma(\pi^1)$ be a jet field locally given by $\Psi(t, q^A) = (t, q^A, \psi^A(t, q^A))$, with $\psi^A \in C^\infty(E)$ being local functions. Then, from Proposition 3 we know that Ψ is a solution to the generalized Lagrangian Hamilton-Jacobi problem if, and only if, every vector field in the class $\{X_{\mathcal{L}}\}$ is tangent to the submanifold $\text{Im}(\Psi) \hookrightarrow J^1\pi$, which is locally defined by the constraints $v^A - \psi^A = 0$, $1 \leq A \leq n$. Then, the conditions $L(X_{\mathcal{L}})(v^A - \psi^A) = 0$ give rise to the following system of n partial differential equations

$$F^A \Big|_{\text{Im}(\Psi)} - \frac{\partial \psi^A}{\partial t} - \psi^B \frac{\partial \psi^A}{\partial q^B} = 0.$$

Now, in order to state the Lagrangian Hamilton-Jacobi problem, we must require in addition the jet field $\Psi \in \Gamma(\pi^1)$ to satisfy the condition $\Psi^* \Omega_{\mathcal{L}} = 0$. In this situation, Definition 10 remains unchanged, but the statement of Proposition 4 must be changed as follows.

Proposition 8. *Let $\Psi \in \Gamma(\pi^1)$ be a jet field satisfying $\Psi^*\Omega_{\mathcal{L}} = 0$. Then the following statements are equivalent:*

1. Ψ is a solution to the Lagrangian Hamilton-Jacobi problem.
2. The submanifold $\text{Im}(\Psi) \hookrightarrow J^1\pi$ is Lagrangian with respect to the presymplectic form $\Omega_{\mathcal{L}}$ and every vector field $X_{\mathcal{L}} \in \{X_{\mathcal{L}}\}$ is tangent to it.
3. The integral curves of $X_{\mathcal{L}} \in \{X_{\mathcal{L}}\}$ with initial conditions in $\text{Im}(\Psi)$ project onto the integral curves of $X = \mathbb{T}\pi^1 \circ X_{\mathcal{L}} \circ \Psi$.

Bearing in mind the coordinate expression (27) of the Cartan 2-form, the pull-back $\Psi^*\Omega_{\mathcal{L}} \in \Omega^2(E)$ has the following coordinate expression

$$\begin{aligned} \Psi^*\Omega_{\mathcal{L}} &= \left(\frac{\partial^2 L}{\partial v^A \partial q^B} + \frac{\partial^2 L}{\partial v^A \partial v^C} \frac{\partial \psi^C}{\partial q^B} \right) dq^A \wedge dq^B \\ &+ \left(\frac{\partial^2 L}{\partial v^B \partial v^A} \frac{\partial \psi^A}{\partial t} + \psi^A \frac{\partial^2 L}{\partial v^A \partial q^B} - \frac{\partial L}{\partial q^B} + \psi^A \frac{\partial^2 L}{\partial v^A \partial v^C} \frac{\partial \psi^C}{\partial q^B} \right) dq^B \wedge dt. \end{aligned}$$

Hence, the condition $\Psi^*\Omega_{\mathcal{L}} = 0$ gives the following system of partial differential equations

$$\frac{\partial^2 L}{\partial v^A \partial q^B} + \frac{\partial^2 L}{\partial v^A \partial v^C} \frac{\partial \psi^C}{\partial q^B} = 0 \quad ; \quad \frac{\partial^2 L}{\partial v^B \partial v^A} \frac{\partial \psi^A}{\partial t} + \psi^A \frac{\partial^2 L}{\partial v^A \partial q^B} - \frac{\partial L}{\partial q^B} + \psi^A \frac{\partial^2 L}{\partial v^A \partial v^C} \frac{\partial \psi^C}{\partial q^B} = 0,$$

which may be combined to obtain equations (13) in this setting, that is, the following system of n^2 partial differential equations

$$\frac{\partial^2 L}{\partial v^A \partial q^B} + \frac{\partial \psi^C}{\partial q^B} \frac{\partial^2 L}{\partial v^A \partial v^C} = 0 \quad ; \quad \frac{\partial^2 L}{\partial v^B \partial v^A} \frac{\partial \psi^A}{\partial t} - \frac{\partial L}{\partial q^B} = 0.$$

Finally, we state the Hamilton-Jacobi equation in the Lagrangian formalism. Since the jet field $\Psi \in \Gamma(\pi^1)$ is a solution to the Lagrangian Hamilton-Jacobi problem, we have that $d(\Psi^*\Theta_{\mathcal{L}}) = 0$. Thus, there exists a local function $W \in C^\infty(E)$ such that $dW = \Psi^*\Theta_{\mathcal{L}}$. In coordinates, the 1-form $\Psi^*\Theta_{\mathcal{L}}$ is given by

$$\Psi^*\Theta_{\mathcal{L}} = \frac{\partial L}{\partial v^A} dq^A - \left(\psi^A \frac{\partial L}{\partial v^A} - L(t, q^A, \psi^A) \right) dt.$$

Hence, requiring $\Psi^*\Theta_{\mathcal{L}} = dW$, we obtain

$$\frac{\partial W}{\partial t} = -\psi^A \frac{\partial L}{\partial v^A} + L(t, q^A, \psi^A) \quad ; \quad \frac{\partial W}{\partial q^A} = \frac{\partial L}{\partial v^A},$$

which may be combined in the following single equation

$$\frac{\partial W}{\partial t} + \psi^A \frac{\partial W}{\partial q^A} - L(t, q^A, \psi^A) = 0.$$

This is the usual Hamilton-Jacobi equation for a first-order non-autonomous Lagrangian dynamical system.

Hamiltonian formalism

Now we state the Hamiltonian formulation of the Hamilton-Jacobi problem in this setting. In the natural coordinates of $J^1\pi^*$, the coordinate expression of the restricted Legendre map $\mathcal{FL}: J^1\pi \rightarrow J^1\pi^*$ in this setting is the following

$$\mathcal{FL}^*p_A = \frac{\partial L}{\partial v^A},$$

from where we deduce the coordinate expression of the extended Legendre map, which is

$$\widetilde{\mathcal{FL}}^*p_A = \frac{\partial L}{\partial v^A} \quad ; \quad \widetilde{\mathcal{FL}}^*p = L - v^A \frac{\partial L}{\partial v^A}.$$

Recall that since $\mathcal{L} \in \Omega^1(J^1\pi)$ is assumed to be regular, the restricted Legendre map is a local diffeomorphism. For simplicity, in the following we assume that \mathcal{L} is hyperregular, and thus \mathcal{FL} is a global diffeomorphism (the regular but not hyperregular case is recovered by restriction on the open sets where the restricted Legendre map is a diffeomorphism). Let $h = \widetilde{\mathcal{FL}} \circ \mathcal{FL}^{-1} \in \Gamma(\mu)$ be the Hamiltonian section associated to the Lagrangian density \mathcal{L} , and $H \in C^\infty(J^1\pi)$ a local Hamiltonian function which specifies this Hamiltonian section h .

Using the Hamiltonian section $h \in \Gamma(\mu)$ we define the Hamilton-Cartan forms $\Theta_h = h^*\Theta \in \Omega^1(J^1\pi^*)$ and $\Omega_h = h^*\Omega \in \Omega^2(J^1\pi^*)$, with coordinate expressions

$$\Theta_h = p_A dq^A - H dt \quad ; \quad \Omega_h = dq^A \wedge dp_A + dH \wedge dt. \quad (29)$$

The Hamiltonian problem for first-order non-autonomous dynamical systems consists in finding a class of $\bar{\pi}_E^r$ -transverse vector fields $\{X_h\} \subseteq \mathfrak{X}(J^1\pi^*)$ satisfying equation (16), that is,

$$i(X_h)\Omega_h = 0, \quad \text{for every } X_h \in \{X_h\}.$$

As in the general setting described at the beginning of Section 4, in this formulation the 2-form Ω_h in $J^1\pi^*$ has maximal rank $2n$ regardless of the Hamiltonian section $h \in \Gamma(\mu)$. Therefore, there always exists such a class of vector fields.

Remark. As in the Lagrangian formulation described previously, the integrability is assured without further assumptions in this setting. \diamond

Then, a representative of the equivalence class $\{X_h\} \subseteq \mathfrak{X}(J^1\pi^*)$ of $\bar{\pi}_E^r$ -transverse vector fields solution to the above dynamical equation is given in coordinates by

$$X_h = f \left(\frac{\partial}{\partial t} + \frac{\partial H}{\partial p_A} \frac{\partial}{\partial q^A} - \frac{\partial H}{\partial q^A} \frac{\partial}{\partial p_A} \right).$$

As in the Lagrangian formulation, note that the vector fields in the class $\{X_h\} \subseteq \mathfrak{X}(J^1\pi^*)$ are completely determined (again, except for the non-vanishing function f). In the following we take $f = 1$ as a representative of the equivalence class to state the Hamiltonian Hamilton-Jacobi problem.

For the generalized Hamiltonian Hamilton-Jacobi problem, all the Definitions and results stated in Section 4.1 remain unchanged, except for Corollary 4, where “boundary conditions” should be replaced by “initial conditions”.

In coordinates, let $s \in \Gamma(\pi_E^r)$ be a section locally given by $s(t, q^A) = (t, q^A, s_A(t, q^A))$, where $s_A \in C^\infty(E)$ are local functions. Then, from Proposition 6 we know that s is a solution to the generalized Hamiltonian Hamilton-Jacobi problem if, and only if, every vector field in the class

$\{X_h\}$ is tangent to the submanifold $\text{Im}(s) \hookrightarrow J^1\pi^*$. Since this submanifold is defined locally by the constraints $p_A - s_A = 0$, $1 \leq A \leq n$, we must check if the conditions $L(X_h)(p_A - s_A) = 0$ hold, which give rise the following system of n partial differential equations

$$\frac{\partial H}{\partial q^A} + \frac{\partial s_A}{\partial t} + \frac{\partial H}{\partial p_B} \frac{\partial s_A}{\partial q^B} = 0, \quad (\text{on } \text{Im}(s)).$$

Now, in order to state the Hamiltonian Hamilton-Jacobi problem, we must require in addition the section $s \in \Gamma(\pi_E^r)$ to satisfy the condition $s^*\Omega_h = 0$. In this case, the statement in Definition 14 and the remark that follows remain unchanged, but the statement of Proposition 7 must be changed as follows

Proposition 9. *Let $s \in \Gamma(\pi_E^r)$ be a section satisfying $s^*\Omega_h = 0$. Then the following statements are equivalent:*

1. *s is a solution to the Hamiltonian Hamilton-Jacobi problem.*
2. *The submanifold $\text{Im}(s) \hookrightarrow J^1\pi^*$ is Lagrangian with respect to the presymplectic form Ω_h and every vector field $X_h \in \{X_h\}$ is tangent to it.*
3. *The integral curves of $X_h \in \{X_h\}$ with initial conditions in $\text{Im}(s)$ project onto the integral curves of $X = \mathbb{T}\pi_E^r \circ X_h \circ s$.*

In coordinates, bearing in mind the coordinate expression (29) of the Hamilton-Cartan 2-form Ω_h , the pull-back $s^*\Omega_h \in \Omega^2(E)$ has the following local expression

$$s^*\Omega_h = \left(\frac{\partial s_A}{\partial t} + \frac{\partial H}{\partial q^A} + \frac{\partial H}{\partial p_B} \frac{\partial s_B}{\partial q^A} \right) dq^A \wedge dt + \frac{\partial s_A}{\partial q^B} dq^A \wedge dq^B.$$

Hence, the condition $s^*\Omega_h = 0$ gives equations (23), which in this case correspond to the following system of n^2 partial differential equations

$$\frac{\partial s_A}{\partial t} + \frac{\partial H}{\partial q^A} + \frac{\partial H}{\partial p_B} \frac{\partial s_B}{\partial q^A} \quad ; \quad \frac{\partial s_A}{\partial q^B} - \frac{\partial s_B}{\partial q^A} = 0.$$

Finally, we state the Hamilton-Jacobi equation in the Hamiltonian formalism. Since the section $s \in \Gamma(\pi_E^r)$ is a solution to the Lagrangian Hamilton-Jacobi problem, we have that $h \circ s \in \Omega^1(E)$ is a closed form. Therefore, using Poincaré's Lemma, there exists a local function $W \in C^\infty(E)$ such that $dW = h \circ s$. Bearing in mind that the 1-form $h \circ s \in \Omega^1(E)$ is given in coordinates by

$$h \circ s = s_A dq^A - (H \circ s) dt,$$

the condition $h \circ s = dW$ gives the following partial differential equations

$$\frac{\partial W}{\partial t} = -H(t, q^A, s_A) \quad ; \quad \frac{\partial W}{\partial q^A} = s_A,$$

which may be combined to obtain the following single equation

$$\frac{\partial W}{\partial t} + H\left(t, q^A, \frac{\partial W}{\partial q^A}\right) = 0.$$

This is the classic Hamilton-Jacobi equation for a first-order non-autonomous dynamical system with n degrees of freedom and Hamiltonian function H .

5.2 Quadratic Lagrangian densities

Let us consider a classical field theory with n fields depending on m independent variables given in terms of a quadratic Lagrangian density, and let $\pi: E \rightarrow M$ be the configuration bundle, with M being a m -dimensional orientable smooth manifold with fixed volume form $\eta \in \Omega^m(M)$, and $\dim E = m + n$. Most of the quadratic Lagrangian field theories can be modeled as follows [56, 29]: let us assume that the bundle $\pi: E \rightarrow M$ is trivial (that is, $E = M \times Q$, with $\dim Q = n$), so that $\pi^1: J^1\pi \rightarrow E$ is a vector bundle. Let g be a pseudo-Riemannian metric in this vector bundle, Γ a connection of the projection π^1 and $V \in C^\infty(E)$ a potential function. Then a quadratic Lagrangian density $\mathcal{L} \in \Omega^1(J^1\pi)$ is given by

$$\mathcal{L}(j_x^1\phi) = \left(\frac{1}{2} g(j_x^1\phi - (\Gamma \circ \pi^1)(j_x^1\phi), j_x^1\phi - (\Gamma \circ \pi^1)(j_x^1\phi)) + ((\pi^1)^*V)(j_x^1\phi) \right) (\pi^1)^*\eta.$$

In the natural coordinates $(x^i, u^\alpha, u_i^\alpha)$ of $J^1\pi$, the m -form \mathcal{L} has the following local expression

$$\mathcal{L}(x^i, u^\alpha, u_i^\alpha) = \left(\frac{1}{2} g_{\alpha\beta}^{ij} (u_i^\alpha - \Gamma_i^\alpha)(u_j^\beta - \Gamma_j^\beta) + V(x^i, u^\alpha) \right) d^m x,$$

where $g_{\alpha\beta}^{ij} = g_{\alpha\beta}^{ij}(x^i, u^\alpha)$ are the coefficients of the metric, which moreover satisfy $g_{\alpha\beta}^{ij} = g_{\beta\alpha}^{ji}$ due to the symmetry assumption, and $\Gamma_i^\alpha = \Gamma_i^\alpha(x^i)$ are the component functions of the connection.

In order to simplify the problem, we assume that the connection Γ is integrable. As a consequence, there exist natural charts of coordinates in $J^1\pi$ such that $\Gamma_i^\alpha = 0$, which implies that the coordinate expression of the Lagrangian density \mathcal{L} reduces to

$$\mathcal{L}(x^i, u^\alpha, u_i^\alpha) = \left(\frac{1}{2} g_{\alpha\beta}^{ij} u_i^\alpha u_j^\beta + V(x^i, u^\alpha) \right) d^m x \equiv L \cdot d^m x. \quad (30)$$

Observe that the Hessian matrix of the Lagrangian function L associated with \mathcal{L} and η coincides with the matrix of the coefficients of g , that is,

$$\left(\frac{\partial^2 L}{\partial u_i^\alpha \partial u_j^\beta} \right) = \left(g_{\alpha\beta}^{ij} \right),$$

and therefore the Lagrangian density \mathcal{L} is regular since g is a pseudo-Riemannian metric, and in particular non-degenerate.

Lagrangian formalism

The local expression of the Cartan m -form $\Theta_{\mathcal{L}} \in \Omega^m(J^1\pi)$ obtained from the Lagrangian density given locally by (30) is

$$\Theta_{\mathcal{L}} = g_{\alpha\beta}^{ij} u_j^\beta du^\alpha \wedge d^{m-1}x_i - \left(\frac{1}{2} g_{\alpha\beta}^{ij} u_i^\alpha u_j^\beta - V(x^i, u^\alpha) \right) d^m x, \quad (31)$$

from where we obtain the coordinate expression of the Cartan $(m+1)$ -form, which is

$$\begin{aligned} \Omega_{\mathcal{L}} = & -\frac{\partial g_{\alpha\beta}^{ij}}{\partial u^\delta} u_j^\beta du^\delta \wedge du^\alpha \wedge d^{m-1}x_i - g_{\alpha\beta}^{ij} du_j^\beta \wedge du^\alpha \wedge d^{m-1}x_i + g_{\alpha\beta}^{ij} u_i^\alpha du_j^\beta \wedge d^m x \\ & + \left(\frac{\partial g_{\delta\beta}^{ij}}{\partial x^i} u_j^\beta + \frac{1}{2} \frac{\partial g_{\alpha\beta}^{ij}}{\partial u^\delta} u_i^\alpha u_j^\beta - \frac{\partial V}{\partial u^\delta} \right) du^\delta \wedge d^m x. \end{aligned} \quad (32)$$

Hence, a representative of the class of holonomic multivector fields $\{\mathcal{X}_{\mathcal{L}}\} \subseteq \mathfrak{X}^m(J^1\pi)$ solution to the Lagrangian field equation (7) is given in coordinates by

$$\mathcal{X}_{\mathcal{L}} = \bigwedge_{j=1}^m \left(\frac{\partial}{\partial x^j} + u_j^\alpha \frac{\partial}{\partial u^\alpha} + F_{j,i}^\alpha \frac{\partial}{\partial u_i^\alpha} \right),$$

where the functions $F_{j,i}^\alpha$ are solutions to the Euler-Lagrange equations for multivector fields (11), which in this case are

$$\frac{1}{2} \frac{\partial g_{\delta\beta}^{ij}}{\partial u^\alpha} u_i^\delta u_j^\beta + \frac{\partial V}{\partial u^\alpha} - \sum_{i=1}^m \frac{\partial g_{\alpha\beta}^{ij}}{\partial x^i} u_j^\beta - \frac{\partial g_{\alpha\delta}^{ij}}{\partial u^\beta} u_i^\beta u_j^\delta - F_{j,i}^\beta g_{\alpha\beta}^{ij} = 0, \quad (33)$$

in addition to the integrability conditions (3) (if necessary).

First we state the generalized version of the Hamilton-Jacobi problem. Let $\Psi \in \Gamma(\pi^1)$ be a jet field locally given by $\Psi(x^i, u^\alpha) = (x^i, u^\alpha, \psi_i^\alpha(x^i, u^\alpha))$, where $\psi_i^\alpha \in C^\infty(E)$ are local functions. Then, from Proposition 3 we know that Ψ is a solution to the generalized Lagrangian Hamilton-Jacobi problem if, and only if, every multivector field $\mathcal{X}_{\mathcal{L}} \in \{\mathcal{X}_{\mathcal{L}}\}$ is tangent to the submanifold $\text{Im}(\Psi) \hookrightarrow J^1\pi$. Bearing in mind that $\text{Im}(\Psi)$ is defined locally by the constraints $u_k^\alpha - \psi_k^\alpha = 0$, the tangency condition gives rise to equations (12), where now the component functions $F_{j,i}^\alpha$ are solutions to the n equations (33).

In order to state the equations of the Lagrangian Hamilton-Jacobi problem, we must require in addition the jet field $\Psi \in \Gamma(\pi^1)$ to satisfy the condition $\Psi^*\Omega_{\mathcal{L}} = 0$. Bearing in mind the coordinate expression (32) of the Cartan $(m+1)$ -form $\Omega_{\mathcal{L}}$, the pull-back $\Psi^*\Omega_{\mathcal{L}} \in \Omega^{m+1}(E)$ is given in coordinates by

$$\begin{aligned} \Psi^*\Omega_{\mathcal{L}} = & - \left(\frac{\partial g_{\alpha\beta}^{ij}}{\partial u^\delta} \psi_j^\beta + g_{\alpha\beta}^{ij} \frac{\partial \psi_j^\beta}{\partial u^\delta} \right) du^\delta \wedge d^\alpha \wedge d^{m-1}x_i \\ & + \left(g_{\delta\beta}^{ij} \frac{\partial \psi_j^\beta}{\partial x^i} + g_{\alpha\beta}^{ij} \psi_i^\alpha \frac{\partial \psi_j^\beta}{\partial u^\delta} + \frac{\partial g_{\delta\beta}^{ij}}{\partial x^i} \psi_j^\beta + \frac{1}{2} \frac{\partial g_{\alpha\beta}^{ij}}{\partial u^\delta} \psi_i^\alpha \psi_j^\beta - \frac{\partial V}{\partial u^\delta} \right) du^\delta \wedge d^m x. \end{aligned}$$

Hence, the condition $\Psi^*\Omega_{\mathcal{L}} = 0$ gives the following system of partial differential equations

$$\frac{\partial g_{\alpha\beta}^{ij}}{\partial u^\delta} \psi_j^\beta + g_{\alpha\beta}^{ij} \frac{\partial \psi_j^\beta}{\partial u^\delta} = 0 \quad ; \quad g_{\delta\beta}^{ij} \frac{\partial \psi_j^\beta}{\partial x^i} + g_{\alpha\beta}^{ij} \psi_i^\alpha \frac{\partial \psi_j^\beta}{\partial u^\delta} + \frac{\partial g_{\delta\beta}^{ij}}{\partial x^i} \psi_j^\beta + \frac{1}{2} \frac{\partial g_{\alpha\beta}^{ij}}{\partial u^\delta} \psi_i^\alpha \psi_j^\beta - \frac{\partial V}{\partial u^\delta} = 0,$$

which may be combined to obtain equations (13), that is, the following system of $n(1+m(n-1))$ partial differential equations

$$\frac{\partial g_{\alpha\beta}^{ij}}{\partial u^\delta} \psi_j^\beta + g_{\alpha\beta}^{ij} \frac{\partial \psi_j^\beta}{\partial u^\delta} = 0 \quad ; \quad g_{\delta\beta}^{ij} \frac{\partial \psi_j^\beta}{\partial x^i} + \frac{\partial g_{\delta\beta}^{ij}}{\partial x^i} \psi_j^\beta - \frac{1}{2} \frac{\partial g_{\alpha\beta}^{ij}}{\partial u^\delta} \psi_i^\alpha \psi_j^\beta - \frac{\partial V}{\partial u^\delta} = 0.$$

Finally, we deduce the Lagrangian Hamilton-Jacobi equation. Since the jet field $\Psi \in \Gamma(\pi^1)$ is a solution to the Lagrangian Hamilton-Jacobi problem, we have that $\Psi^*\Theta_{\mathcal{L}} \in \Omega^m(E)$ is a closed form, and thus there exists a local π -semibasic $(m-1)$ -form $\omega \in \Omega^{m-1}(E)$ such that $d\omega = \Psi^*\Theta_{\mathcal{L}}$. In coordinates, bearing in mind the coordinate expression (31) of the Cartan m -form, we obtain

$$\Psi^*\Theta_{\mathcal{L}} = g_{\alpha\beta}^{ij} \psi_j^\beta du^\alpha \wedge d^{m-1}x_i - \left(\frac{1}{2} g_{\alpha\beta}^{ij} \psi_i^\alpha \psi_j^\beta - V(x^i, u^\alpha) \right) d^m x.$$

Then, if the $(m-1)$ -form ω is given locally by $\omega = W^i d^{m-1}x_i$, the condition $d\omega = \Psi^*\Theta_{\mathcal{L}}$ gives rise to the following equations

$$\sum_{i=1}^m \frac{\partial W^i}{\partial x^i} + \frac{1}{2} g_{\alpha\beta}^{ij} \psi_i^\alpha \psi_j^\beta - V(x^i, u^\alpha) = 0 \quad ; \quad \frac{\partial W^i}{\partial u^\alpha} = g_{\alpha\beta}^{ij} \psi_j^\beta,$$

which may be combined to give the classic Hamilton-Jacobi equation in the Lagrangian formalism (14), which in this example is

$$\sum_{i=1}^m \frac{\partial W^i}{\partial x^i} + \frac{1}{2} \tilde{g}_{ij}^{\alpha\beta} \frac{\partial W^i}{\partial u^\alpha} \frac{\partial W^j}{\partial u^\beta} - V(x^i, u^\alpha) = 0. \quad (34)$$

where $\tilde{g}_{ij}^{\alpha\beta}$ denote the coefficients of the inverse matrix of $(g_{\alpha\beta}^{ij})$, which exists since we assume g to be non-degenerate.

Hamiltonian formalism

Now we state the Hamiltonian formalism for the Hamilton-Jacobi problem. First, let us compute the coordinate expression of the Legendre maps associated to the Lagrangian density given by (30). In the natural coordinates $(x^i, u^\alpha, p_\alpha^i)$ of $J^1\pi^*$, the restricted Legendre map $\mathcal{FL}: J^1\pi \rightarrow J^1\pi^*$ has the following coordinate expression

$$\mathcal{FL}^* p_\alpha^i = g_{\alpha\beta}^{ij} u_j^\beta,$$

from where the local expression of the extended Legendre map is straightforwardly deduced as

$$\widetilde{\mathcal{FL}}^* p_\alpha^i = g_{\alpha\beta}^{ij} u_j^\beta \quad ; \quad \widetilde{\mathcal{FL}}^* p = V(x^i, u^\alpha) - \frac{1}{2} g_{\alpha\beta}^{ij} u_i^\alpha u_j^\beta,$$

Using the coordinate expression of both Legendre maps we obtain a local Hamiltonian function associated to the Lagrangian density \mathcal{L} , whose coordinate expression is

$$H(x^i, u^\alpha, p_\alpha^i) = \frac{1}{2} \tilde{g}_{ij}^{\alpha\beta} p_\alpha^i p_\beta^j - V(x^i, u^\alpha),$$

where, as before, $(\tilde{g}_{ij}^{\alpha\beta})$ is the inverse matrix of $(g_{\alpha\beta}^{ij})$.

Using the Hamiltonian section $h \in \Gamma(\mu)$ specified by this local Hamiltonian function we define the Hamilton-Cartan forms $\Theta_h = h^*\Theta \in \Omega^m(J^1\pi^*)$, $\Omega_h = -d\Theta_h \in \Omega^{m+1}(J^1\pi^*)$, whose coordinate expressions are

$$\Theta_h = p_\alpha^i du^\alpha \wedge d^{m-1}x_i - \left(\frac{1}{2} \tilde{g}_{ij}^{\alpha\beta} p_\alpha^i p_\beta^j - V(x^i, u^\alpha) \right) d^m x, \quad (35)$$

$$\Omega_h = -dp_\alpha^i \wedge du^\alpha \wedge d^{m-1}x_i + \left(\frac{1}{2} \frac{\partial \tilde{g}_{ij}^{\alpha\beta}}{\partial u^\delta} p_\alpha^i p_\beta^j - \frac{\partial V}{\partial u^\delta} \right) du^\delta \wedge d^m x + \tilde{g}_{ij}^{\alpha\beta} p_\alpha^i dp_\beta^j \wedge d^m x. \quad (36)$$

Then, a representative of the class of $\bar{\pi}_E^T$ -transverse multivector fields $\{\mathcal{X}_h\} \subseteq \mathfrak{X}^m(J^1\pi^*)$ which are solutions to the Hamiltonian field equation (16) is given in coordinates by

$$\mathcal{X}_h = \bigwedge_{j=1}^m \left(\frac{\partial}{\partial x^j} + \tilde{g}_{ij}^{\alpha\beta} p_\alpha^i \frac{\partial}{\partial u^\beta} + G_{\alpha,j}^i \frac{\partial}{\partial p_\alpha^i} \right), \quad (37)$$

with the functions $G_{\alpha,j}^i$ satisfying the Hamilton-De Donder-Weyl equations (20), which in this case are

$$\sum_{i=1}^m G_{\delta,i}^i = - \left(\frac{1}{2} \frac{\partial \tilde{g}_{ij}^{\alpha\beta}}{\partial u^\delta} p_\alpha^i p_\beta^j - \frac{\partial V}{\partial u^\delta} \right), \quad (38)$$

in addition to the integrability conditions (21) (if necessary).

Let us state the generalized Hamiltonian Hamilton-Jacobi problem for this field theory. Let $s \in \Gamma(\pi_E^r)$ be a section given locally by $s(x^i, u^\alpha) = (x^i, u^\alpha, s_\alpha^i(x^i, u^\alpha))$, where $s_\alpha^i \in C^\infty(E)$ are local functions. By Proposition 6, the section s is a solution to the generalized Hamiltonian Hamilton-Jacobi problem if, and only if, the multivector field \mathcal{X}_h given in coordinates by (37) is tangent to the submanifold $\text{Im}(s) \hookrightarrow J^1\pi^*$ defined locally by the constraints $p_\alpha^i - s_\alpha^i = 0$. Computing, the tangency condition gives rise to equations (22), that is,

$$\frac{\partial s_\delta^k}{\partial x^j} + \tilde{g}_{ij}^{\alpha\beta} p_\alpha^i \frac{\partial s_\delta^k}{\partial u^\beta} - G_{\delta,j}^k \Big|_{\text{Im}(s)} = 0.$$

Now, in order to state the Hamiltonian Hamilton-Jacobi problem, we must require in addition the section $s \in \Gamma(\pi_E^r)$ to satisfy the condition $s^*\Omega_h = 0$ or, equivalently, $d(h \circ s) = 0$. Bearing in mind the coordinate expression (35) of the Hamilton-Cartan $(m+1)$ -form Ω_h , the pull-back $s^*\Omega_h \in \Omega^{m+1}(E)$ is given locally by

$$s^*\Omega_h = \left(\frac{\partial s_\delta^i}{\partial x^i} + \frac{1}{2} \frac{\partial \tilde{g}_{ij}^{\alpha\beta}}{\partial u^\delta} s_\alpha^i s_\beta^j - \frac{\partial V}{\partial u^\delta} + \tilde{g}_{ij}^{\alpha\beta} s_\alpha^i \frac{\partial s_\beta^j}{\partial u^\delta} \right) du^\delta \wedge d^m x - \frac{\partial s_\alpha^i}{\partial u^\beta} du^\beta \wedge du^\alpha \wedge d^{m-1} x_i.$$

Hence, the condition $s^*\Omega_h = 0$ gives equations (23) for this problem, that is,

$$\frac{\partial s_\delta^i}{\partial x^i} + \frac{1}{2} \frac{\partial \tilde{g}_{ij}^{\alpha\beta}}{\partial u^\delta} s_\alpha^i s_\beta^j - \frac{\partial V}{\partial u^\delta} + \tilde{g}_{ij}^{\alpha\beta} s_\alpha^i \frac{\partial s_\beta^j}{\partial u^\delta} = 0 \quad ; \quad \frac{\partial s_\alpha^i}{\partial u^\beta} - \frac{\partial s_\beta^i}{\partial u^\alpha} = 0.$$

Finally, we compute the Hamilton-Jacobi equation in the Hamiltonian formalism. Since the section $s \in \Gamma(\pi_E^r)$ is a solution to the Hamiltonian Hamilton-Jacobi problem, we have that $h \circ s \in \Omega^m(E)$ is a closed form. Hence, there exists a local π -semibasic $(m-1)$ -form $\omega \in \Omega^{m-1}(E)$ such that $d\omega = h \circ s$. In coordinates, the local expression of the m -form $h \circ s$ is

$$h \circ s = s_\alpha^i du^\alpha \wedge d^{m-1} x_i - \left(\frac{1}{2} \tilde{g}_{ij}^{\alpha\beta} s_\alpha^i s_\beta^j - V(x^i, u^\alpha) \right) d^m x,$$

Then, if the $(m-1)$ -form ω is given locally by $\omega = W^i d^{m-1} x_i$, the condition $d\omega = h \circ s$ gives rise to the following equations

$$\sum_{i=1}^m \frac{\partial W^i}{\partial x^i} + \frac{1}{2} \tilde{g}_{ij}^{\alpha\beta} s_\alpha^i s_\beta^j - V(x^i, u^\alpha) = 0 \quad ; \quad \frac{\partial W^i}{\partial u^\alpha} = s_\alpha^i,$$

which may be combined to give the classic Hamilton-Jacobi equation in the Hamiltonian formalism (24), and it coincides with the equation (34) obtained in the Lagrangian formulation, that is,

$$\sum_{i=1}^m \frac{\partial W^i}{\partial x^i} + \frac{1}{2} \tilde{g}_{ij}^{\alpha\beta} \frac{\partial W^i}{\partial u^\alpha} \frac{\partial W^j}{\partial u^\beta} - V(x^i, u^\alpha) = 0.$$

5.3 Minimal surfaces in dimension three

Let us consider the following problem: we look for smooth maps $\phi: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that the set $\text{graph}(\phi) \subseteq \mathbb{R}^3$, which is a surface in \mathbb{R}^3 , has minimal area and satisfies certain boundary conditions. This problem can be modeled as a first-order classical field theory with base manifold $M = \mathbb{R}^2$ and 1-dimensional fibers, that is, $E = \mathbb{R}^2 \times \mathbb{R}$. Let (x, y) be the global coordinates in $M = \mathbb{R}^2$, and (x, y, u) the global coordinates in $E = \mathbb{R}^3$. In these coordinates, the canonical volume form $\eta \in \Omega^2(\mathbb{R}^2)$ is given by $\eta = dx \wedge dy$. Then, in the natural coordinates (x, y, u, u_1, u_2) of $J^1\pi$, the Lagrangian density $\mathcal{L} \in \Omega^2(J^1\pi)$ for this field theory is

$$\mathcal{L} = \sqrt{1 + u_1^2 + u_2^2} dx \wedge dy. \quad (39)$$

Observe that \mathcal{L} is a regular Lagrangian density, since the Hessian matrix of the Lagrangian function $L = \sqrt{1 + u_1^2 + u_2^2}$ associated with \mathcal{L} and η is

$$\left(\frac{\partial^2 L}{\partial u_i \partial u_j} \right) = \frac{1}{\sqrt{(1 + u_1^2 + u_2^2)^3}} \begin{pmatrix} u_2^2 + 1 & -u_1 u_2 \\ -u_1 u_2 & u_1^2 + 1 \end{pmatrix},$$

which has determinant

$$\det \left(\frac{\partial^2 L}{\partial u_i \partial u_j} \right) = \frac{1}{(1 + u_1^2 + u_2^2)^2} \neq 0.$$

Lagrangian formalism

The local expression of the Cartan 2-form $\Theta_{\mathcal{L}} \in \Omega^2(J^1\pi)$ is

$$\Theta_{\mathcal{L}} = \frac{u_1}{\sqrt{1 + u_1^2 + u_2^2}} du \wedge dy - \frac{u_2}{\sqrt{1 + u_1^2 + u_2^2}} du \wedge dx + \frac{1}{\sqrt{1 + u_1^2 + u_2^2}} dx \wedge dy, \quad (40)$$

from where we deduce the following coordinate expression for the Cartan 3-form $\Omega_{\mathcal{L}} \in \Omega^3(J^1\pi)$

$$\begin{aligned} \Omega_{\mathcal{L}} = & -\frac{u_2^2 + 1}{\sqrt{(1 + u_1^2 + u_2^2)^3}} du_1 \wedge du \wedge dy + \frac{u_1 u_2}{\sqrt{(1 + u_1^2 + u_2^2)^3}} du_2 \wedge du \wedge dy \\ & - \frac{u_1 u_2}{\sqrt{(1 + u_1^2 + u_2^2)^3}} du_1 \wedge du \wedge dx + \frac{u_1^2 + 1}{\sqrt{(1 + u_1^2 + u_2^2)^3}} du_2 \wedge du \wedge dx \\ & + \frac{u_1}{\sqrt{(1 + u_1^2 + u_2^2)^3}} du_1 \wedge dx \wedge dy + \frac{u_2}{\sqrt{(1 + u_1^2 + u_2^2)^3}} du_2 \wedge dx \wedge dy. \end{aligned} \quad (41)$$

Thus, a multivector field $\mathcal{X}_{\mathcal{L}} \in \mathfrak{X}^2(J^1\pi)$ solution to the Lagrangian field equation (7) is given in coordinates by

$$\mathcal{X}_{\mathcal{L}} = f X_1 \wedge X_2 = f \left(\frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + F_{1,1} \frac{\partial}{\partial u_1} + F_{1,2} \frac{\partial}{\partial u_2} \right) \wedge \left(\frac{\partial}{\partial y} + u_2 \frac{\partial}{\partial u} + F_{2,1} \frac{\partial}{\partial u_1} + F_{2,2} \frac{\partial}{\partial u_2} \right),$$

where $f \in C^\infty(J^1\pi)$ is a non-vanishing function, and the functions $F_{i,j}$ satisfy the Euler-Lagrange equations for multivector fields (11), which in this case reduce to the following single equation

$$(1 + u_2^2)F_{1,1} - u_1 u_2 (F_{1,2} + F_{2,1}) + (1 + u_1^2)F_{2,2} = 0. \quad (42)$$

In addition, since we need the multivector fields solution to the field equation to be integrable in order to give a suitable Hamilton-Jacobi formulation, we must require equations (3) to hold.

In our case, since $\mathcal{X}_{\mathcal{L}} = X_1 \wedge X_2$, we must require $[X_1, X_2] = 0$. Computing, we have

$$\begin{aligned} [X_1, X_2] &= (F_{1,2} - F_{2,1}) \frac{\partial}{\partial u} \\ &+ \left(\frac{\partial F_{2,1}}{\partial x} + u_1 \frac{\partial F_{2,1}}{\partial u} + F_{1,1} \frac{\partial F_{2,1}}{\partial u_1} + F_{1,2} \frac{\partial F_{2,1}}{\partial u_2} \right. \\ &\quad \left. - \frac{\partial F_{1,1}}{\partial y} - u_2 \frac{\partial F_{1,1}}{\partial u} - F_{2,1} \frac{\partial F_{1,1}}{\partial u_1} - F_{2,2} \frac{\partial F_{1,1}}{\partial u_2} \right) \frac{\partial}{\partial u_1} \\ &+ \left(\frac{\partial F_{2,2}}{\partial x} + u_1 \frac{\partial F_{2,2}}{\partial u} + F_{1,1} \frac{\partial F_{2,2}}{\partial u_1} + F_{1,2} \frac{\partial F_{2,2}}{\partial u_2} \right. \\ &\quad \left. - \frac{\partial F_{1,2}}{\partial y} - u_2 \frac{\partial F_{1,2}}{\partial u} - F_{2,1} \frac{\partial F_{1,2}}{\partial u_1} - F_{2,2} \frac{\partial F_{1,2}}{\partial u_2} \right) \frac{\partial}{\partial u_2} \end{aligned}$$

Then, requiring this last expression to vanish, we have that the multivector fields $\mathcal{X}_{\mathcal{L}}$ which are solutions to the field equation are integrable if, and only if, the following 3 equations hold

$$\begin{aligned} F_{1,2} - F_{2,1} &= 0, \\ \frac{\partial F_{2,1}}{\partial x} + u_1 \frac{\partial F_{2,1}}{\partial u} + F_{1,1} \frac{\partial F_{2,1}}{\partial u_1} + F_{1,2} \frac{\partial F_{2,1}}{\partial u_2} - \frac{\partial F_{1,1}}{\partial y} - u_2 \frac{\partial F_{1,1}}{\partial u} - F_{2,1} \frac{\partial F_{1,1}}{\partial u_1} - F_{2,2} \frac{\partial F_{1,1}}{\partial u_2} &= 0, \\ \frac{\partial F_{2,2}}{\partial x} + u_1 \frac{\partial F_{2,2}}{\partial u} + F_{1,1} \frac{\partial F_{2,2}}{\partial u_1} + F_{1,2} \frac{\partial F_{2,2}}{\partial u_2} - \frac{\partial F_{1,2}}{\partial y} - u_2 \frac{\partial F_{1,2}}{\partial u} - F_{2,1} \frac{\partial F_{1,2}}{\partial u_1} - F_{2,2} \frac{\partial F_{1,2}}{\partial u_2} &= 0. \end{aligned} \tag{43}$$

A particular solution to equations (43) is given by

$$F_{1,1} = \frac{\partial u_1}{\partial x} \quad ; \quad F_{1,2} = F_{2,1} = \frac{\partial u_1}{\partial y} = \frac{\partial u_2}{\partial x} \quad ; \quad F_{2,2} = \frac{\partial u_2}{\partial y}.$$

Moreover, one can easily check that these functions are also a solution to the Euler-Lagrange equation (42). Therefore, a particular holonomic multivector field solution to the field equation is given in coordinates by

$$\mathcal{X}_{\mathcal{L}} = f \left(\frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + \frac{\partial u_1}{\partial x} \frac{\partial}{\partial u_1} + \frac{\partial u_2}{\partial x} \frac{\partial}{\partial u_2} \right) \wedge \left(\frac{\partial}{\partial y} + u_2 \frac{\partial}{\partial u} + \frac{\partial u_1}{\partial y} \frac{\partial}{\partial u_1} + \frac{\partial u_2}{\partial y} \frac{\partial}{\partial u_2} \right). \tag{44}$$

In the following we use the particular solution given by (44) with $f = 1$ to state the Lagrangian Hamilton-Jacobi problem.

First we state the generalized version of the Hamilton-Jacobi problem. Let $\Psi \in \Gamma(\pi^1)$ be a jet field locally given by $\Psi(x, y, u) = (x, y, u, \psi_1(x, y, u), \psi_2(x, y, u))$, with $\psi_1, \psi_2 \in C^\infty(\mathbb{R}^3)$ being local functions. Then, from Proposition 3 we know that Ψ is a solution to the generalized Lagrangian Hamilton-Jacobi problem if, and only if, the Euler-Lagrange multivector field $\mathcal{X}_{\mathcal{L}}$ given locally by (44) is tangent to the submanifold $\text{Im}(\Psi) \hookrightarrow J^1\pi$, which is locally defined by the constraints $u_j - \psi_j = 0$, $j = 1, 2$. Then, from the conditions $L(X_i)(u_j - \psi_j) = 0$ we obtain the following systems of 4 partial differential equations

$$\psi_1 \frac{\partial \psi_1}{\partial u} = 0 \quad ; \quad \psi_1 \frac{\partial \psi_2}{\partial u} = 0 \quad ; \quad \psi_2 \frac{\partial \psi_1}{\partial u} = 0 \quad ; \quad \psi_2 \frac{\partial \psi_2}{\partial u} = 0,$$

which admits the following set of solutions

$$\{\psi_1(x, y, u) = f_1(x, y), \psi_2(x, y, u) = f_2(x, y)\}, \tag{45}$$

where $f_1, f_2 \in C^\infty(\mathbb{R}^3)$ are functions depending only on the coordinates of the base manifold $M = \mathbb{R}^2$, that is, they are constant along the fibers of the bundle $\mathbb{R}^3 \rightarrow \mathbb{R}^2$. Observe that both functions may be constant, and even vanish everywhere.

In order to obtain the equations of the Lagrangian Hamilton-Jacobi problem, we require in addition the jet field $\Psi \in \Gamma(\pi^1)$ to satisfy the condition $\Psi^*\Omega_{\mathcal{L}} = 0$. Bearing in mind the coordinate expression of the Cartan 3-form $\Omega_{\mathcal{L}} \in \Omega^3(J^1\pi)$ given in (41), the pull-back $\Psi^*\Omega_{\mathcal{L}} \in \Omega^3(\mathbb{R}^3)$ by a jet field $\Psi(x, y, u) = (x, y, u, \psi_1, \psi_2)$ gives

$$\Psi^*\Omega_{\mathcal{L}} = \frac{1}{\sqrt{(1 + \psi_1^2 + \psi_2^2)^3}} \left((\psi_2^2 + 1) \frac{\partial \psi_1}{\partial x} - \psi_1 \psi_2 \left(\frac{\partial \psi_2}{\partial x} + \frac{\partial \psi_1}{\partial y} \right) + (\psi_1^2 + 1) \frac{\partial \psi_2}{\partial y} \right) du \wedge dx \wedge dy.$$

Therefore, $\Psi^*\Omega_{\mathcal{L}} = 0$ if, and only if, the following partial differential equation holds

$$(\psi_2^2 + 1) \frac{\partial \psi_1}{\partial x} - \psi_1 \psi_2 \left(\frac{\partial \psi_2}{\partial x} + \frac{\partial \psi_1}{\partial y} \right) + (\psi_1^2 + 1) \frac{\partial \psi_2}{\partial y} = 0.$$

It is easy to check that, from the functions in the set (45) of solutions to the generalized Lagrangian Hamilton-Jacobi problem, the following functions are solutions to the Lagrangian Hamilton-Jacobi problem:

$$\{\psi_1 = \bar{f}(y), \psi_2 = f(x)\}$$

with $f, \bar{f} \in C^\infty(E)$ being functions depending only on the coordinate x and y , respectively (with possibly one or both of them vanishing). In addition, when $\psi_i = g(x, y)$, we do not obtain a closed formula for ψ_j in terms of the function g , but there may be functions satisfying the arising partial differential equation.

Finally, we state the Hamilton-Jacobi equation in the Lagrangian formalism. Since the jet field $\Psi \in \Gamma(\pi^1)$ is a solution to the Lagrangian Hamilton-Jacobi problem, we have that $d(\Psi^*\Theta_{\mathcal{L}}) = 0$. Thus, there exists a 1-form $\omega \in \Omega^1(E)$ given locally by $\omega = W^1 dy - W^2 dx$ such that $d\omega = \Psi^*\Theta_{\mathcal{L}}$. The pull-back of the Cartan 2-form $\Theta_{\mathcal{L}}$ by Ψ gives in coordinates

$$\Psi^*\Theta_{\mathcal{L}} = \frac{1}{\sqrt{1 + \psi_1^2 + \psi_2^2}} (\psi_1 du \wedge dy - \psi_2 du \wedge dx + dx \wedge dy).$$

Hence, requiring $\Psi^*\Theta_{\mathcal{L}} = d\omega$, we obtain

$$\frac{\partial W^1}{\partial x} + \frac{\partial W^2}{\partial y} = \frac{1}{\sqrt{1 + \psi_1^2 + \psi_2^2}} \quad ; \quad \frac{\partial W^1}{\partial u} = \frac{\psi_1}{\sqrt{1 + \psi_1^2 + \psi_2^2}} \quad ; \quad \frac{\partial W^2}{\partial u} = \frac{\psi_2}{\sqrt{1 + \psi_1^2 + \psi_2^2}},$$

which may be combined in the following single equation

$$\frac{\partial W^1}{\partial x} + \frac{\partial W^2}{\partial y} = \sqrt{1 - \left(\frac{\partial W^1}{\partial u} \right)^2 - \left(\frac{\partial W^2}{\partial u} \right)^2}. \quad (46)$$

This is the Hamilton-Jacobi equation for this field theory.

Hamiltonian formalism

Now we state the Hamiltonian formulation of the Hamilton-Jacobi problem for this field theory. In the natural coordinates (x, y, u, p^1, p^2) of $J^1\pi^*$, the restricted Legendre map $\mathcal{FL}: J^1\pi \rightarrow J^1\pi^*$ associated to the Lagrangian density \mathcal{L} given by (39) has the following coordinate expression

$$\mathcal{FL}^* p^1 = \frac{u_1}{\sqrt{1 + u_1^2 + u_2^2}} \quad ; \quad \mathcal{FL}^* p^2 = \frac{u_2}{\sqrt{1 + u_1^2 + u_2^2}}.$$

From this last expression we deduce the coordinate expression of the extended Legendre map, which is

$$\widetilde{\mathcal{FL}}^* p^1 = \frac{u_1}{\sqrt{1 + u_1^2 + u_2^2}} \quad ; \quad \widetilde{\mathcal{FL}}^* p^2 = \frac{u_2}{\sqrt{1 + u_1^2 + u_2^2}} \quad ; \quad \widetilde{\mathcal{FL}}^* p = -\frac{1}{\sqrt{1 + u_1^2 + u_2^2}},$$

as well as the coordinate expression of the (local) inverse map $\mathcal{FL}^{-1}: J^1\pi^* \rightarrow J^1\pi$

$$(\mathcal{FL}^{-1})^*u_1 = \frac{p^1}{\sqrt{1 - (p^1)^2 - (p^2)^2}} \quad ; \quad (\mathcal{FL}^{-1})^*u_2 = \frac{p^2}{\sqrt{1 - (p^1)^2 - (p^2)^2}}.$$

Observe that the inverse Legendre map is not defined on the points of $J^1\pi^*$ satisfying $(p^1)^2 + (p^2)^2 = 1$.

The local Hamiltonian function associated to the Lagrangian formulation is then given by

$$H(x, y, u, p^1, p^2) = -\sqrt{1 - (p^1)^2 - (p^2)^2}.$$

Using the Hamiltonian section $h \in \Gamma(\mu)$ specified by this local Hamiltonian function we define the Hamilton-Cartan forms $\Theta_h = h^*\Theta \in \Omega^2(J^1\pi^*)$, $\Omega_h = h^*\Omega \in \Omega^3(J^1\pi^*)$, whose coordinate expressions are

$$\Theta_h = p^1 du \wedge dy - p^2 du \wedge dx + \sqrt{1 - (p^1)^2 - (p^2)^2} dx \wedge dy, \quad (47)$$

$$\begin{aligned} \Omega_h &= -dp^1 \wedge du \wedge dy + dp^2 \wedge du \wedge dx \\ &+ \frac{1}{\sqrt{1 - (p^1)^2 - (p^2)^2}} (p^1 dp^1 \wedge dx \wedge dy + p^2 dp^2 \wedge dx \wedge dy). \end{aligned} \quad (48)$$

Then, a locally decomposable 2-vector field $\mathcal{X}_h \in \mathfrak{X}^2(J^1\pi^*)$ solution to the field equation $i(\mathcal{X}_h)\Omega_h = 0$ is locally given by

$$\begin{aligned} \mathcal{X}_h &= f \left(\frac{\partial}{\partial x} + \frac{p^1}{\sqrt{1 - (p^1)^2 - (p^2)^2}} \frac{\partial}{\partial u} + G_1^1 \frac{\partial}{\partial p^1} + G_1^2 \frac{\partial}{\partial p^2} \right) \\ &\wedge \left(\frac{\partial}{\partial y} + \frac{p^2}{\sqrt{1 - (p^1)^2 - (p^2)^2}} \frac{\partial}{\partial u} + G_2^1 \frac{\partial}{\partial p^1} + G_2^2 \frac{\partial}{\partial p^2} \right), \end{aligned}$$

with the functions G_i^j satisfying the Hamilton-De Donder-Weyl equations (20), which in this case reduce to the following single equation

$$G_1^1 + G_2^2 = 0. \quad (49)$$

Following the same procedure given in the Lagrangian formalism, an integrability condition must be required to this multivector field. From [23] we know that a particular choice of a locally decomposable and integrable multivector field solution to the field equation is given in coordinates by

$$\begin{aligned} X_h &= f \left(\frac{\partial}{\partial x} + \frac{p^1}{\sqrt{1 - (p^1)^2 - (p^2)^2}} \frac{\partial}{\partial u} + \frac{\partial p^1}{\partial x} \frac{\partial}{\partial p^1} + \frac{\partial p^2}{\partial x} \frac{\partial}{\partial p^2} \right) \\ &\wedge \left(\frac{\partial}{\partial y} + \frac{p^2}{\sqrt{1 - (p^1)^2 - (p^2)^2}} \frac{\partial}{\partial u} + \frac{\partial p^1}{\partial y} \frac{\partial}{\partial p^1} + \frac{\partial p^2}{\partial y} \frac{\partial}{\partial p^2} \right), \end{aligned} \quad (50)$$

As in the Lagrangian formalism, in the following we use the particular solution given by (50) with $f = 1$ to state the Hamiltonian Hamilton-Jacobi problem.

In order to state the generalized Hamiltonian Hamilton-Jacobi problem, let $s \in \Gamma(\pi_{\mathbb{R}^3}^T)$ be a section given in coordinates by $s(x, y, u) = (x, y, u, s^1(x, y, u), s^2(x, y, u))$, where $s^1, s^2 \in C^\infty(\mathbb{R}^3)$ are local functions. By Proposition 6, the section s is a solution to the generalized Hamiltonian Hamilton-Jacobi problem if, and only if, the multivector field \mathcal{X}_h given in coordinates by (50)

is tangent to the submanifold $\text{Im}(s) \hookrightarrow J^1\pi^*$ defined locally by the constraints $p^j - s^j = 0$, $j = 1, 2$. Then the tangency of \mathcal{X}_h along the submanifold $\text{Im}(s)$ gives the following system of 4 partial differential equations.

$$\begin{aligned} \frac{s^1}{\sqrt{1 - (s^1)^2 - (s^2)^2}} \frac{\partial s^1}{\partial u} = 0 & \quad ; \quad \frac{s^1}{\sqrt{1 - (s^1)^2 - (s^2)^2}} \frac{\partial s^2}{\partial u} = 0, \\ \frac{s^2}{\sqrt{1 - (s^1)^2 - (s^2)^2}} \frac{\partial s^1}{\partial u} = 0 & \quad ; \quad \frac{s^2}{\sqrt{1 - (s^1)^2 - (s^2)^2}} \frac{\partial s^2}{\partial u} = 0. \end{aligned}$$

This system of partial differential equations admits the following set of local solutions

$$\{s^1(x, y, u) = f^1(x, y), s^2(x, y, u) = f^2(x, y)\}, \quad (51)$$

where $f^1, f^2 \in C^\infty(\mathbb{R}^3)$ are functions depending only on the coordinates (x, y) of the base manifold $M = \mathbb{R}^2$, and satisfying $(f^1)^2 + (f^2)^2 \neq 1$.

Now, to obtain the equation of the Hamiltonian Hamilton-Jacobi problem, we require in addition that the section $s \in \Gamma(\pi_{\mathbb{R}^3}^r)$ satisfies the condition $s^*\Omega_h = 0$ or, equivalently, we require the 2-form $h \circ s \in \Omega^2(E)$ to be closed. In coordinates, bearing in mind the coordinate expression (48) of the Hamilton-Cartan 3-form $\Omega_h \in \Omega^3(J^1\pi^*)$, we have

$$s^*\Omega_h = \left(\frac{\partial s^1}{\partial x} + \frac{\partial s^2}{\partial y} \right) du \wedge dx \wedge dy,$$

from where the condition $s^*\Omega_h = 0$ is locally equivalent to the equation:

$$\frac{\partial s^1}{\partial x} + \frac{\partial s^2}{\partial y} = 0.$$

This additional equation restricts the set of solutions (51) to the following:

$$\{s^1 = \bar{f}(y), s^2 = f(x)\}.$$

As in the Lagrangian Hamilton-Jacobi problem, $f, \bar{f} \in C^\infty(E)$ are functions depending only on the coordinate x and y , respectively, and they may vanish. In addition, when $s^i = g(x, y)$, we do not obtain a closed formula for s^j in terms of the function g , but there may be functions satisfying the arising partial differential equation. Recall that both f, \bar{f} must satisfy $f^2 + \bar{f}^2 \neq 1$.

Finally, we state the Hamilton-Jacobi equation in the Hamiltonian formalism. Since the section $s \in \Gamma(\pi_{\mathbb{R}^3}^r)$ is a solution to the Hamiltonian Hamilton-Jacobi problem, the form $h \circ s \in \Omega^2(E)$ is closed. Thus, there exists a 1-form $\omega \in \Omega^1(E)$ given locally by $\omega = W^1 dy - W^2 dx$ such that $d\omega = h \circ s$. In coordinates, the form $h \circ s = s^*\Theta_h$ is given by

$$h \circ s = s^1 du \wedge dy - s^2 du \wedge dx + \sqrt{1 - (s^1)^2 - (s^2)^2} dx \wedge dy.$$

Thus, requiring $h \circ s = d\omega$, we obtain

$$\frac{\partial W^1}{\partial x} + \frac{\partial W^2}{\partial y} = \sqrt{1 - (s^1)^2 - (s^2)^2} \quad ; \quad \frac{\partial W^1}{\partial u} = s^1 \quad ; \quad \frac{\partial W^2}{\partial u} = s^2,$$

which may be combined to obtain equation (46), that is,

$$\frac{\partial W^1}{\partial x} + \frac{\partial W^2}{\partial y} = \sqrt{1 - \left(\frac{\partial W^1}{\partial u} \right)^2 - \left(\frac{\partial W^2}{\partial u} \right)^2}.$$

6 Conclusions and further research

Starting from the geometric Hamilton-Jacobi theory developed mainly in [14, 17] for mechanical systems and using the results given in [39] as standpoint, we have stated a geometric framework for first-order classical field theories described in the multisymplectic setting.

The theory has been developed for the Lagrangian and the Hamiltonian formalisms. In both cases, first we have stated the so-called *generalized Hamilton-Jacobi problem*, which is the most natural one in this geometrical ambient, and hence we have defined from it the standard *Hamilton-Jacobi problem*. Particular solutions to these problems are defined and characterized in several equivalent ways and, in particular, one of these characterizations for the standard case in the Hamiltonian formalism, when written in natural coordinates, leads the classical Hamilton-Jacobi equation for field theories. After that, the definition and geometric characterization of complete solutions is also given and different features about them are discussed. Finally, the equivalence between the Lagrangian and the Hamiltonian Hamilton-Jacobi problems is also proved.

It is important to point out that this generalization of the theory has been achieved using distributions in the jet bundles and multimomentum bundles where the Lagrangian and Hamiltonian formalisms of multisymplectic classical field theories are developed. These are integrable distributions whose integral sections are the solutions to the Lagrangian and Hamiltonian field equations, and they are represented, in general, by means of equivalence classes of multivector fields. This choice has enabled us to give a construction of the Hamilton-Jacobi theory in a very natural way. Thus, our model is different from that given by L. Vitagliano in [59] for higher-order field theories, who uses connections as the main geometrical tool and a unified formalism to describe the Lagrangian and Hamiltonian formulations at once.

We have analyzed several examples. First, non-autonomous mechanical systems can be considered as a special situation of field theories, and hence the Hamilton-Jacobi equation for these systems has been recovered from our model as a particular case. Second, we have applied our results to obtain the Hamilton-Jacobi equation for field theories described by quadratic affine Lagrangians. Finally, we have written this equation for a more particular example: minimal surfaces in dimension three.

As further research, we believe that our geometric framework for the Hamilton-Jacobi theory can be extended to higher-order field theories using the formulations in [9, 52], thus generalizing the results of [18, 19] for higher-order mechanics and giving a different but equivalent perspective to that of [59] for this kind of theories.

A very relevant application of the Hamilton-Jacobi theory for first-order field theories would be to the Palatini approach of General Relativity and, once the extension to higher-order field theory is made, also to the Einstein-Hilbert Lagrangian approach, as well as to other gravitational theories. As a previous step, a suitable multisymplectic description of these gravitational models must be done and, although there are some recent attempts to do it [58], more work in this way is necessary and research in this way is in progress.

Another interesting question in the ambient of the Hamilton-Jacobi theory is the existence of conserved quantities (or conservation laws), and the integrability of the system. It is known (see [14] for more details) that, in the case of mechanics, for a dynamical system with n degrees of freedom, the existence of complete solutions to the generalized Hamilton-Jacobi problem is associated with the local existence of families of n functions which are constants of motion. From a geometrical perspective, a complete solution is a foliation in the fiber bundle which represents the phase space of the system, which is transverse to the fibers, and such that the dynamical vector field is tangent to the leaves of this foliation. Then, these leaves are locally the level

sets of the functions which are constants of motion. In addition, a complete solution to the Hamilton-Jacobi problem corresponds to a Lagrangian foliation (with respect to the symplectic structure, canonical or not, which the phase space is endowed with), and thus the constants of motion are in involution and the system is completely integrable.

In the case of field theories, from a geometrical point of view, the situation is quite similar: as it is defined in Sections 3.3 and 4.3, complete solutions to the generalized Hamilton-Jacobi problem endow the jet and multimomentum bundles with foliations which are transverse to the fibers and such that their leaves contain the image of the sections solution to the field equations. When we consider just the Hamilton-Jacobi problem, these are m -Lagrangian foliations (with respect to the corresponding multisymplectic structures, in the sense defined in [12]). Nevertheless, up to our knowledge, the notion of “integrability” is not clearly stated in these cases. Furthermore, although the leaves of these foliations can be also locally defined as level sets of families of functions, how to associate these functions with conservation laws in field theories must be investigated and, even if the foliation is m -Lagrangian, these functions are not said “to be in involution” because, although there are several attempts to define unambiguously a Poisson bracket (for functions) in covariant field theories (that is, in multisymplectic geometry), this problem is not solved in a completely satisfactory way. The discussion on all these topics is also under research.

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