Singularities for a fully non-linear elliptic equation in conformal geometry

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Dedicated to Neil Trudinger on the occasion of his 70th birthday.

Abstract

We construct some radially symmetric solutions of the constant σ_k-equation on $\mathbb{R}^n \setminus \mathbb{R}^p$, which blow up exactly at the submanifold $\mathbb{R}^p \subset \mathbb{R}^n$. These are the basic models to the problem of finding complete metrics of constant σ_k–curvature on a general subdomain of the sphere $\mathbb{S}^n \setminus \Lambda^p$ that blow up exactly at the singular set Λ^p and that are conformal to the canonical metric. More precisely, we look at the case $k = 2$ and $0 < p < \mathfrak{p}_2 := \frac{n-\sqrt{n}-2}{2}$. The main result is the understanding of the precise asymptotics of our solutions near the singularity and their decay away from the singularity. The first aspect will insure the completeness of the metric about the singular locus, whereas the second aspect will guarantee that the model solutions can be locally transplanted to the original metric on $\mathbb{S}^n$, and hence they can be used to deal with the general problem on $\mathbb{S}^n \setminus \Lambda^p$.

Key Words: σ_k–curvature, fully nonlinear elliptic equations, conformal geometry, singular metrics

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1 Introduction

Let (M, g) be a smooth n–dimensional Riemannian manifold. Denote by $Riem$, Ric, R, the Riemannian curvature tensor, the Ricci tensor and the scalar curvature, respectively. Construct the Schouten tensor as

$$A_g = \frac{1}{n-2}\left(Ric_g - \frac{1}{2(n-1)} R_g\, g\right).$$

From the point of view of conformal geometry, we are interested in the study of the Schouten tensor because it contains all the information about the conformal deformations of a given metric. This can be seen from the decomposition

$$Riem \;=\; W + A \owedge g,$$

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where $\owedge$ is the Kulkarni-Nomizu product, and W the Weyl tensor, which is a conformal invariant.

Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of the symmetric endomorphism of TM given by $g^{-1}A_g$. The main object of study of the present paper will be its k-th elementary symmetric function:

$$\sigma_k(g^{-1}A_g) \ := \sum_{i_1<\ldots<i_k} \lambda_{i_1} \ldots \lambda_{i_k}.$$

These σ_k–curvatures, introduced in [37], have received a lot of attention. For instance, $\sigma_1(g^{-1}A_g)$ is given by

$$\sigma_1(g^{-1}A_g) \ = \ \lambda_1 + \ldots + \lambda_n \ = \ \frac{1}{2(n-1)} R\,,$$

and thus it is a multiple of the scalar curvature. So the study of the σ_k–curvatures gives a natural generalization of the Yamabe problem and scalar curvature related questions. Moreover, the sign of the σ_k's has a strong control on the geometry of the manifold. In particular, locally conformally flat metrics with $\sigma_1(g^{-1}A_g), \dots, \sigma_k(g^{-1}A_g) \geq 0$ for some $k > 1$ have (see [19])

$$Ric_g \ \geq \ \frac{(2k-n)(n-1)}{(k-1)} \binom{n}{k}^{-1/k} \left(\sigma_k(g^{-1}A_g)\right)^{1/k} g \tag{1.1}$$

and thus $Ric_g > 0$ when $n < 2k$. Throughout the paper we will assume $2 \leq 2k < n$. Taking advantage of this fact, we introduce the following formalism for the conformal change

$$g_u \ := \ u^{\frac{4k}{n-2k}}\, g,$$

where the conformal factor $u > 0$ is a positive smooth function. Then the constant σ_k–equation for the conformal factor u can be formulated as

$$\sigma_k\big(\, g_u^{-1} A_{g_u}\big) = 2^{-k} \binom{n}{k}. \tag{1.2}$$

We recall that the Schouten tensor of g_u is related to the one of A_g by the conformal transformation law

$$A_{g_u} \ = \ A_g - \frac{2k}{n-2k} u^{-1}\nabla^2 u + \frac{2kn}{(n-2k)^2} u^{-2} du \otimes du - \frac{2k^2}{(n-2k)^2} u^{-2}|du|^2 g,$$

where ∇^2 and $|\cdot|$ are computed with respect to the background metric g. Given a background metric g, we will consider metrics g_u defined as above with the positive smooth conformal factor in the positive cone

$$\Gamma_k^+(g) \ = \ \left\{ u \in C^\infty(M) \,:\, u > 0 \ \text{ and } \ \sigma_1(g_u^{-1}A_{g_u}), \dots, \sigma_k(g_u^{-1}A_{g_u}) > 0 \right\}.$$

For a given matrix A, we define the k-th Newton tensor of the matrix A as

$$T_k(A) = \sigma_k(a) I - \sigma_{k-1}(A) A + \ldots + (-1)^k A^k. \tag{1.3}$$

Note that if A is such that $\sigma_1(A), \dots, \sigma_k(A) > 0$, then T_{k-1} is positive definite.

From the PDE point of view, (1.2) is a fully non-linear elliptic equation of Hessian type, which becomes elliptic (but not necessarily uniformly) in the positive cone. In the case $k = 1$ the complete picture is understood. Indeed, if the background metric g is such that $R_g = 0$, then the constant scalar curvature (or constant σ_1) equation (1.2) for g_u reduces to

$$-\Delta_g u \ = \ \frac{n(n-2)}{4}\, u^{\frac{n+2}{n-2}}. \tag{1.4}$$

With a slightly different formalism for the conformal change, if we set $g_v = v^{-2}|dx|^2$, where $|dx|^2$ represents the Euclidean metric, we can write down the explicit expression for the σ_2-operator

$$2\sigma_2(v) \ = \ \left[(\Delta v)^2 - |D^2 v|^2\right] v^2 - (n-1)\Delta v |\nabla v|^2 v + \frac{n(n-1)}{4} |\nabla v|^4\,,$$

Keeping the same notation for the conformal change, we observe that these type of non-linear equations have an underlying divergence structure, namely

$$m\sigma_m(v) \;=\; v\,\partial_j\left(v_i\,T_{ij}^{m-1}\right) \;-\; n\,T_{ij}^{m-1}\,v_iv_j \;+\; \frac{n-m+1}{2}\,\sigma_{m-1}(v)\,|\nabla v|^2\,, \tag{1.5}$$

where $T^{m-1}=T_{m-1}(g_v^{-1}A_{g_v})$ is the $(m-1)$-th Newton transform of $g_v^{-1}A_{g_v}$. The non-divergence terms are of lower order and indeed, they can be dealt through an inductive process.

The Yamabe problem for the σ_k–curvature equation has been considered in [36, 11]. On $\mathbb{R}^n$, a global positive solution of the constant σ_k–curvature equation (1.2) must be of the form ([4, 5, 6, 26])

$$u(x) \;=\; c(n,k)\left(\frac{a}{1+a^2|x-\bar{x}|^2}\right)^{\frac{n-2k}{2k}}\,,$$

for some real number $a>0$ and some point $\bar{x}\in\mathbb{R}^n$, and hence it comes from the standard metric on $\mathbb{S}^n$ (or its image under a conformal diffeomorphism).

We concentrate now on singular solutions of equation (1.2). In Section 2 we give a small survey on removability/non-removability of such singularities. Then we look at the same problem on $\mathbb{S}^n\backslash\Lambda$, where Λ is a smooth submanifold of $\mathbb{S}^n$ of dimension $p\geq 0$. This is so called the singular k-Yamabe problem. A necessary condition on the dimension of Λ for solvability was shown in [14] and is reviewed in Section 3 here.

In the case where the singular set Λ reduces to a finite number of points ($p=0$), solutions to equation (1.2) have been constructed in [29] and [30]. Here, we deal with the case $p>0$. The strategy of this proof is to first find an approximate solution that can be perturbed in order to produce a suitable solution. This is the content of the forthcoming paper [18]. The main step in the construction of the approximate solution is to construct a model solution in $\mathbb{R}^n\backslash\mathbb{R}^p$, singular along $\mathbb{R}^p$, and that has a very precise decay far from the singularity. This decay allows to transplant it to the original $\mathbb{S}^n\backslash\Lambda$.

The main result in this note is to find radially symmetric solutions of the constant σ_k-equation that blow up exactly at $\mathbb{R}^p\subset\mathbb{R}^n$ with this precise asymptotic behavior, for $k=2$ and $0<p<\mathfrak{p}_2:=\frac{n-\sqrt{n}-2}{2}$ (the reason of this dimension restriction will be explained there). This is the content of Section 4.

2 Local behavior near singularities

First look at an isolated singularity for the constant σ_k–curvature equation:

$$\begin{cases}\sigma_k(u) \;=\; 1 \quad \text{in} \quad B_1\backslash\{0\},\\ u\in\Gamma_k^+, \quad u>0, \quad n>2k.\end{cases} \tag{2.1}$$

In the semilinear case $k=1$, Caffarelli-Gidas-Spruck [2] have given a complete local characterization of isolated singularities. Basically, if u is a positive solution in $B\backslash\{0\}$, then either the singularity is removable or the function has a determined asymptotic behavior

$$\frac{C_1}{|x|^{\frac{n-2}{2}}}\leq u(x)\leq\frac{C_2}{|x|^{\frac{n-2}{2}}} \quad \text{when} \quad |x|\to 0\,. \tag{2.2}$$

In the fully non-linear case ($k>1$), the same classification for solutions of problem (2.1) holds similarly (see [20, 21]), and indeed singular solutions must have an specific asymptotic behavior near the origin. Note that (2.1) is a critical problem. In the subcritical case this classification result is much easier to prove and was shown in [15].

The problem of classification of radial solutions of $\sigma_k(u)=1$ in an annulus was solved by [8], who gave a precise limiting behavior for the solution near the singularity (completeness vs. incompleteness, for instance). Here (2.1) reduces to an ordinary differential equation whose phase portrait may be reasonably well understood (see Proposition 4.1 in Section 4.1 for the explicit calculations).

Now we consider the problem of understanding the local behavior of non-isolated singularities of the constant σ_k-curvature equation, under some capacity conditions on the singular set Λ:

$$\begin{cases} \sigma_k(u) = 1 \quad \text{on} \quad B_1 \backslash \Lambda, \\ u > 0, \quad u \in \Gamma_k^+, \quad n > 2k, \end{cases} \tag{2.3}$$

where $\Lambda \subset B_1$ is a compact subset of the unit ball in $\mathbb{R}^n$.

The classical notion of capacity [1] was introduced to treat singularities of linear and quasilinear PDE. If Λ is a compact subset of $\mathbb{R}^n$, one defines for $k \in \mathbb{N}$ and $q \geq 1$,

$$c_{k,q}(\Lambda) \ := \ \inf \left\{ \|\eta\|_{W^{k,q}}^q \quad : \quad \eta \in \mathcal{C}_0^\infty, \ \eta \geq 1 \text{ on } \Lambda \right\} . \tag{2.4}$$

In particular, for the Laplacian problem (1.4) the Newtonian capacity $c_{1,2}$ is the suitable one (c.f. [9]).

For fully non-linear Hessian equations of the type $\sigma_k(D^2 u)$, [24] has considered a related notion of capacity in terms of potential theory, that is adapted to the new equation. On the other hand, [16] introduced a new concept of capacity that is adapted to the problem (2.3) with its additional structure, and in the spirit of the classical notion (2.4). Since this definition is given inductively and its complicated to write it in general, we just give the σ_2 case.

Definition 2.1. *Let Λ be a compact subset of $\mathbb{R}^n$. For $q \geq 2k$, define*

$$\tilde{c}_{2,q}(\Lambda) = \inf \left\{ \|\eta\|_{L^q}^q + \int |\nabla \eta|^q dx + \int \left| \nabla |\nabla \eta|^2 \right|^{q/3} dx \quad : \quad \eta \in \mathcal{C}_0^\infty, \quad \eta \geq 1 \text{ on } \Lambda \right\}.$$

In the case that $\Lambda \subset \mathbb{R}^n$, one may take only test functions satisfying supp $\eta \subset B_R$ and in this case we write $\tilde{c}(\Lambda, R)$.

Of course, for $k = 1$ all the three definitions agree. However, it is not clear what the relation between the different capacities is, and this is a very interesting problem. In any case, we have:

Lemma 2.2. *For general k:*

1. *If $c_{k,p/k}(\Lambda) = c_{k-1,p/(k-1)}(\Lambda) = \ldots = c_{1,p}(\Lambda) = 0, \quad$ then $\tilde{c}_{k,p}(\Lambda) = 0$.*
2. *If $dim_{\mathcal{H}}(\Lambda) < n - p \quad$ for $\quad n > p > 2k, \quad$ then $\tilde{c}_{k,p}(\Lambda) = 0$.*

Now we are able to show a removability of singularities result:

Theorem 2.1 ([16])**.** *Let $\Lambda \subset B_R \subset \mathbb{R}^n$ be a compact set, $R < 1$, with capacity*

$$\tilde{c}_{k,q}(\Lambda, B_R) = 0$$

for a given $2k < q \leq n$. Let $u \in L^r(B_1)$ for some

$$r \geq \frac{2kn}{n-2k} \quad \text{and} \quad r > \frac{2k^2}{k+1} \left(\frac{q}{q-2k} \right)$$

be a solution of (2.3) with

$$\|u\|_{L^{\frac{2kn}{n-2k}}(B_1)} < \varepsilon$$

for some $\varepsilon > 0$ small enough. Then u belongs to $L^{\tilde{r}}$ for some $\tilde{r} > \frac{2kn}{n-2k}$ in a smaller ball. Also,

$$\|u\|_{L^\infty(B_\rho)} \leq \frac{C}{R^{n/q}} \|u\|_{L^q(B_{2\rho})}$$

for all $q > \frac{2k^2}{k+1}$.

3 Complete metrics on $\mathbb{S}^n$

We are interested now in studying singular sets of complete metrics on $\mathbb{S}^n$ with positive σ_k–curvature, and the topological information they may contain. More precisely, let g be a complete metric on a domain $\Omega \subset S^n$, conformal to the standard metric on the sphere g_c. In [34] (see also chapter VI in [35] for a more detailed discussion), Schoen-Yau proved that if g has positive scalar curvature, then the singular set must be of Hausdorff dimension

$$\dim_{\mathcal{H}}(\partial\Omega) \ < \ \frac{n-2}{2}\,.$$

If one has some additional information on the positivity of σ_k for $k > 1$, then this dimension estimate may be improved, obtaining the following theorem.

Theorem 3.1 ([14]). *Let g be a complete metric on a domain $\Omega \subset S^n$, conformal to g_c, satisfying*

$$\sigma_1(g^{-1}A_g) \geq C_0 > 0 \quad \textit{and} \quad \sigma_2(g^{-1}A_g), \dots, \sigma_k(g^{-1}A_g) \geq 0$$

for some integer $1 \leq k < n/2$. Then

$$\dim_{\mathcal{H}}(\partial\Omega) \ \leq \ \frac{n-2k}{2}\,.$$

If, in addition, for $k > 1$ we have $|R| + |\nabla_g R| \leq c_0$, then

$$\dim_{\mathcal{H}}(\partial\Omega) \ < \ \frac{n-2k}{2}\,. \tag{3.1}$$

In the same paper [34] Schoen and Yau showed that any complete locally conformally flat manifold of positive scalar curvature is conformally equivalent to a subdomain Ω of the sphere. Now, the dimension estimate of Theorem 3.1 will give restrictions on the homotopy and cohomology groups of the original manifold, as stated in [14].

The case $k = 2$ was addressed by Chang-Hang-Yang in [7]. In the general cse, the proof of the theorem above requires a deep understanding of the 'almost divergence' structure (1.5).

We are left to study the case $3 \leq n \leq 2k$. But, looking at the estimate (1.1) an easy argument gives that a singular set for g cannot exist.

The natural question now is to find if (3.1) is sharp. In the scalar curvature case, $k = 1$, [27] constructs a complete metric on $\mathbb{S}^n \backslash \Lambda$, conformal to the standard one g_c, with constant scalar curvature when Λ is a smooth submanifold of dimension $0 < p < \frac{n-2}{2}$. See also [10] when Λ is a smooth submanifold with boundary of the same dimension. In the general $2 < k < n/2$ case, estimate (3.1) does not seem to be optimal. Indeed, in the following we have the explicit calculations for the canonical example $\mathbb{S}^n \setminus \mathbb{S}^p$ with the metric constructed as follows.

By stereographic projection, it is equivalent to work with $\mathbb{R}^n \setminus \mathbb{R}^p$, endowed with coordinates $\mathbb{R}_t \times S^{N-1}_\theta \times \mathbb{R}^p_z$, where $N = n - p$. In these coordinates the Euclidean metric is written in the following way

$$|dx|^2 \ = \ e^{-2t}\big(\, dt^2 \,+\, g_{S^{N-1}}\big) \,+\, \delta_{\alpha\beta}\; dz^\alpha \otimes dz^\beta\,, \tag{3.2}$$

where $\alpha, \beta = 1, \dots, p$. We set now

$$g_u = u^{\frac{4k}{n-2k}}|dx|^2, \quad \text{for} \quad u(t) = \tfrac{4k}{n-2k} e^{\frac{n-2k}{2k}t} v_\infty, \tag{3.3}$$

for some constant $v_\infty > 0$. This metric is conformal to the product $\mathbb{S}^{n-p-1} \times \mathbb{H}^{p+1}$ with its standard metric. The Schouten tensor is diagonal and, modulo a multiplicative constant, it reduces to

$$J_{n,p} \,:=\, -\,dt \otimes \frac{\partial}{\partial t} \,+\, \delta^j_i\; d\theta^i \otimes \frac{\partial}{\partial \theta^i} \,-\, \delta^\beta_\alpha\; dz^\alpha \otimes \frac{\partial}{\partial z^\beta}\,,$$

In particular, we may compute

$$\sigma_k(J_{n,p}) = \sum_{i=0}^{k} \binom{n-p-1}{i}\binom{p+1}{k-i}(-1)^{k-i} =: c_{n,p,k}. \tag{3.4}$$

If we choose v_∞ to be the only positive solution to

$$c_{n,p,k}(v_\infty)^{2k} = \binom{n}{k}(v_\infty)^{\frac{2kn}{n-2k}}, \tag{3.5}$$

then the metric (3.3) satisfies the constant σ_k-curvature equation (1.2).

We now set

$$\mathfrak{p}_k := \sup\{p \geq 0 : \sigma_1(J_{n,p}), \dots, \sigma_k(J_{n,p}) > 0\}, \tag{3.6}$$

so that this u belongs to the positive cone if and only if $p < \mathfrak{p}_k$. We unfortunately do not have an explicit formula for $\mathfrak{p}_k$, except for $k = 1, 2, 3$:

$$\mathfrak{p}_1 := \frac{n-2}{2}, \quad \mathfrak{p}_2 = \frac{n-\sqrt{n}-2}{2}, \quad \mathfrak{p}_3 = \frac{n-2-\sqrt{3n-2}}{2}.$$

However, it was shown in [14] that, fixed $k > 1$,

$$\frac{n}{2} - C_1(k)\sqrt{n} \;\leq\; \mathfrak{p}_k \;<\; \frac{n}{2} - \frac{2+\sqrt{n}}{2},$$

for some constant $C_1(k)$, $n \gg 1$.

4 Construction of singular metrics: an ODE approach

We claim that if Λ is a subset of $\mathbb{S}^n$ which is a closed submanifold of dimension p satisfying $0 < p < \mathfrak{p}_2$, then there exists a complete metric g, conformal to g_c, with positive constant σ_2 curvature, that is singular exactly along Λ. This is the content of the forthcoming paper [18]. We will concentrate in the σ_2 case, but our difficulties for general $k > 2$ are just computational and we conjecture that the results are true in general.

A fundamental step in this kind of constructions consists in finding good model solutions which can then be used to build an accurate approximate solution. The more this *ansatz* will be accurate, the more the following perturbation process has chances to be successfull. Since, up to a blow up, the singular locus will appear as a $\mathbb{R}^n \setminus \mathbb{R}^p$, we are going to investigate the existence of symmetric singular solutions supported on $\mathbb{R}^n \setminus \mathbb{R}^p$ via an ODE analysis.

Let us introduce first some notation. We write $\mathbb{R}^n \setminus \mathbb{R}^p$ as the product $\mathbb{R}_t \times S^{N-1}_\theta \times \mathbb{R}^p_z$, where $N = n-p$, and write the Euclidean metric g in these coordinates as (3.2). In particular, we have that that $A_g = 0$ and the formula for the Schouten tensor reduces to

$$A_{g_u} \;=\; -\frac{2k}{n-2k}u^{-1}\nabla^2 u + \frac{2kn}{(n-2k)^2}u^{-2}du\otimes du - \frac{2k^2}{(n-2k)^2}u^{-2}|du|^2 g\,.$$

(As anticipated, we will specialize this formulæ to the case $k = 2$, in the following). For technical reasons, it is convenient to set

$$B_{g_u} \;:=\; \tfrac{n-2k}{2k}\, u^{\frac{2n}{n-2k}}\, g_u^{-1} A_{g_u}. \tag{4.1}$$

The rotational symmetry of the solutions, will be obtained by imposing $u = u(t)$ in the above formulæ. A straightforward computation gives that the modified tensor (4.1) can be simply written with respect to the background metric (3.2) as

$$\begin{aligned}
\left(B_{g_u}\right)^\beta_\alpha &= \left(-\frac{k}{n-2k}\,\dot{u}^2\right) e^{2t}\,\delta^\beta_\alpha,\\
\left(B_{g_u}\right)^t_t &= \left(\frac{n-k}{n-2k}\dot{u}^2 - \ddot{u}u - \dot{u}u\right) e^{2t},\\
\left(B_{g_u}\right)^j_i &= \left(\dot{u}u - \frac{k}{n-2k}\dot{u}^2\right) e^{2t}\,\delta^i_j\,,
\end{aligned}$$

the other components of B_{g_u} being zero. In particular one can see that if the blow up rate is of the type $u(t) \sim e^{\frac{n-2k}{2k}t}$, as $t \to +\infty$, then the corresponding solution on $\mathbb{R}^n \setminus \mathbb{R}^p$ is complete about the singular locus. Thus, it is natural to set

$$v(t) \ := \ \frac{n-2k}{4k} e^{-\frac{n-2k}{2k}t} u(t)$$

and to look for bounded solutions of the equation

$$\mathcal{M}(v) \ := \ \sigma_k\left(C_v\right) - \binom{n}{k} v^{\frac{2kn}{n-2k}} \quad = \quad 0\,, \tag{4.2}$$

where, after some simplification,

$$C_v \ := \ \lambda\, dt \otimes \frac{\partial}{\partial t} \ + \mu\, \delta_i^j \, d\theta^i \otimes \frac{\partial}{\partial \theta^i} \ + \ \nu\, \delta_\alpha^\beta \ dz^\alpha \otimes \frac{\partial}{\partial z^\beta}\,, \tag{4.3}$$

with

$$\begin{aligned} \lambda &= -\big(\, v^2 + 2a\ddot{v}v - \frac{n-k}{k} a^2 \dot{v}^2\big), \\ \mu &= (v^2 - a^2\dot{v}^2), \\ \nu &= -(v + a\dot{v})^2, \end{aligned} \tag{4.4}$$

for

$$a = \frac{2k}{n-2k}\,. \tag{4.5}$$

In other words, C_v has eigenvalues λ, μ and ν with multiplicities $1, N-1$ and p, respectively.

The case of radial solutions with an isolated singularity at the origin ($p = 0$) was completely described in [7] (see also [29] for a summary of the relevant results). In this case, (4.2) is an integrable ODE and there exists a Hamiltonian function. Although the proof is well known, we repeat it here because it will be useful for the general case $p > 0$.

Proposition 4.1. *Fix $p = 0$. The trajectories of the ODE (4.2) are precisely the level sets of the following Hamiltonian:*

$$H(v, \dot{v}) \ := \ \binom{n}{k} \frac{n-2k}{2kn} \left[(v^2 - a^2\dot{v}^2)^k - v^{\frac{2kn}{n-2k}} \right] \ = \ cst\,. \tag{4.6}$$

Proof. Let

$$D_v \ := \ \lambda\, dt \otimes \frac{\partial}{\partial t} \ + \mu\, \delta_i^j \, d\theta^i \otimes \frac{\partial}{\partial \theta^i} \ + \ \mu\, \delta_\alpha^\beta \ dz^\alpha \otimes \frac{\partial}{\partial z^\beta},$$

for λ and μ as given in (4.4) (note that $p = 0$ here). We can easily calculate

$$\sigma_k(D_v) = \binom{n}{k} \frac{n-2k}{2kn} (v^2 - a^2\dot{v}^2)^{k-1} \left[v^2 - a^2 v\dot{v} \right].$$

In this case, equation (4.2) is a completely integrable ODE. Indeed, multiply the equation

$$\sigma_k(D_v) - \binom{n}{k} v^{\frac{2kn}{n-2k}} = 0$$

by $\dot{v}/v$ and integrate. The result follows immediately. □

Now we go to the general case $p > 0$, and we construct solutions to the ODE (4.2) with the right behavior at $t \to +\infty$ and $t \to -\infty$. On one hand, the corresponding metric should be complete about the singular locus, say $\mathbb{R}^p$, and non complete far away from the singular locus, which allows this solution

to be transplanted into any other manifold. Precisely, this this particular solution is the basic building block that is required to construct a complete singular metric on $\mathbb{S}^n \setminus \mathbb{S}^p$ or, more generally, $\mathbb{S}^n \setminus \Lambda^p$, thus a solution to the σ_k-Yamabe problem.

As we have seen in the previous section, the first solution one finds of the ODE (4.2) is the constant one v_∞ found in (3.5) but this is not the one we are seeking since it does not have the right asymptotics when $t \to -\infty$. The main result of the present note is contained in the following theorem.

Theorem 4.1. *For each $0 < p < \mathfrak{p}_2$ and $n > 4$, there exists a solution u_1 for equation (1.2) of the form*

$$u_1(t) \;=\; \frac{8}{n-4}\, e^{\frac{n-4}{4}t}\, v_1(t)\,,$$

where v_1 satisfies

- *$v_1 > 0$, if $t \in (0, \infty)$.*
- *When $t \to +\infty$,*
$$v_1(t) \to v_\infty > 0\,.$$
- *When $t \to -\infty$,*
$$v_1(t) e^{-\alpha_0 t} \to cst\,,$$
for some $\alpha_0 \in (0, (n-4)/4)$.
- *v_1 is uniformly bounded for all $t \in (0, \infty)$.*

We conclude the first part of this section with a formula that can be used to compute the σ_k curvature in the symmetric situation described above. The proof is very simple and it is left to the reader.

Lemma 4.2. *Setting $D_v := \lambda\, dt \otimes \frac{\partial}{\partial t} + \mu\, \delta_i^j\, d\theta^i \otimes \frac{\partial}{\partial \theta^i}$, one has that*

$$\sigma_k\left(C_v\right) \;=\; \sum_{r=0}^{q} \binom{p}{r}\, \nu^r\, \sigma_{k-r}\big(D_v\big)\,,$$

where $q = \min\{k, p\}$.

4.1 ODE analysis

In the following, we present the proof of Theorem 4.1. We fix $p > 0$. The restriction $p < \mathfrak{p}_k$ appears precisely in the next basic lemma, at the calculation of the equilibrium points. Outside this range, we loose all information about the ODE.

Lemma 4.3. *The ODE (4.2) has two equilibria in the halfplane $\{v \geq 0\}$ of the phase space with coordinates v and $\dot v$ given by*

$$(0,0) \qquad \text{and} \qquad (v_\infty, 0)$$

if and only if the relation between p and n and k is such that $p < \mathfrak{p}_k$. Moreover, the equilibrium $(v_\infty, 0)$ is stable for trajectories that stay in the positive cone. More precisely, the linearization at this equilibrium has, either two negative real eigenvalues, or two complex conjugate eigenvalues with negative real part.

Proof. The first assertion of the lemma is straightforward, since the equilibria are precisely the non-negative constant solutions of the equation $\mathcal{M}(v) = 0$, this is, $v_0 = 0$ and $v_0 = v_\infty$.

For the second assertion, compute the linearization at the point $(v_\infty, 0)$. Then

$$\mathcal{L}(w) := \frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{M}(v_\infty + \epsilon w) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \sigma_k(C_{v_\infty + \epsilon w}) - \binom{n}{k} \frac{2kn}{n-2k} v_\infty^{\frac{2kn}{n-2k}-1} w. \tag{4.7}$$

On the other hand, it is well known that

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \sigma_k(C_{v_\infty+\epsilon w}) = \text{trace}\left(T_{k-1}(C_{v_\infty})\ \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} C_{v_\infty+\epsilon w}\right), \tag{4.8}$$

where $T_k(C)$ is the k-th Newton tensor of the matrix C, defined in (1.3). Note that if C belongs to the positive k-cone, then T_{k-1} is positive definite. Next, we explicitly compute

$$\begin{aligned}
\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \lambda(v_\infty+\epsilon w) &= -2v_\infty w - 2av_\infty \ddot{w},\\
\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mu(v_\infty+\epsilon w) &= 2v_\infty w,\\
\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \nu(v_\infty+\epsilon w) &= -2w_\infty - 2av_\infty \dot{w},
\end{aligned}$$

so from (4.8) we obtain

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \sigma_k(C_{v_\infty+\epsilon w}) = 2v_\infty w\text{trace}\left(T_{k-1}(C_{v_\infty})J_{n,p}\right) - 2av_\infty\text{trace}\left(T_{k-1}(C_{v_\infty})\bar{J}(w)\right), \tag{4.9}$$

where we have defined

$$\bar{J}(w) \ :=\ \ddot{w}\,dt\otimes\frac{\partial}{\partial t}\ +0\,\delta_i^j\,d\theta^i\otimes\frac{\partial}{\partial\theta^i}\ +\ \dot{w}\,\delta_\alpha^\beta\ dz^\alpha\otimes\frac{\partial}{\partial z^\beta}\,.$$

Next, if we substitute

$$\sigma_k(C_{v_\infty}) = \frac{1}{k}\text{trace}\left(T_{k-1}(C_{v_\infty})C_{v_\infty}\right) = \frac{1}{k}(v_\infty)^2\text{trace}\left(T_{k-1}(C_{v_\infty})J_{n,p}\right)$$

into the first term in the right hand side of (4.9) and setting

$$T_{k-1}(C_{v_\infty}) \ =:\ T_{k-1}^{(1)}\,dt\otimes\frac{\partial}{\partial t}\ +T_{k-1}^{(2)}\,\delta_i^j\,d\theta^i\otimes\frac{\partial}{\partial\theta^i}T_{k-1}^{(3)}\,\delta_\alpha^\beta\ dz^\alpha\otimes\frac{\partial}{\partial z^\beta} \tag{4.10}$$

into the last term of (4.9), we obtain

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \sigma_k(C_{v_\infty+\epsilon w}) \ =\ 2kv_\infty^{-1}w\sigma_k(C_{v_\infty}) - 2av_\infty\left(T_{k-1}^{(1)}\ddot{w}+pT_{k-1}^{(3)}\dot{w}\right),$$

Using that v_∞ is a solution we arrive at

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \sigma_k(C_{v_\infty+\epsilon w}) = 2k\binom{n}{k}v_\infty^{\frac{2kn}{n-2k}-1}w - 2av_\infty\left(T_{k-1}^{(1)}\ddot{w}+pT_{k-1}^{(3)}\dot{w}\right).$$

Finally, we can give an explicit expression for the linearization (4.7):

$$\mathcal{L}(w) = -2av_\infty T_{k-1}^{(1)}\ddot{w} - 2av_\infty pT_{k-1}^{(3)}\dot{w} - \frac{(2k)^2}{n-2k}\binom{n}{k}v_\infty^{\frac{2kn}{n-2k}-1}w. \tag{4.11}$$

Now, since we know that the metric given by v_∞ belongs to the positive cone, the coefficients $T^{(1)}$, $T^{(3)}$ as defined in (4.10) are strictly positive. This implies that the coefficients accompanying $w, \dot{w}, \ddot{w}$ in (4.11) are strictly negative, which completes the linear study at the equilibrium $(v_\infty, 0)$. □

In the following, we try to understand the asymptotic behavior at the equilibrium $(0,0)$. We seek solutions for (4.2) that behave like $v_\alpha(t) = e^{\alpha t}$ when $t\to-\infty$, for some $\alpha>0$. First write out the eigenvalues of the matrix C_{v_α} as written in (4.3):

$$\begin{aligned}
\lambda(v_\alpha) &= \left(-1+a^2\alpha^2\right)e^{2\alpha t},\\
\mu(v_\alpha) &= (1-a^2\alpha^2)e^{2\alpha t},\\
\nu(v_\alpha) &= -(1+a\alpha)^2e^{2\alpha t}.
\end{aligned}$$

We set $\beta := a\alpha > 0$ above,

$$\begin{aligned} \lambda_\beta &:= -1 + \beta^2, \\ \mu_\beta &:= (1 - \beta^2), \\ \nu_\beta &:= -(1+\beta)^2, \end{aligned} \tag{4.12}$$

and

$$E_\beta := \lambda_\beta \, dt \otimes \frac{\partial}{\partial t} + \mu_\beta \, \delta_i^j \, d\theta^i \otimes \frac{\partial}{\partial \theta^i} + \nu_\beta \, \delta_\alpha^\beta \, dz^\alpha \otimes \frac{\partial}{\partial z^\beta}.$$

Then, when $t \to -\infty$, equation (4.2) is equivalent to

$$F(\beta) := \sigma_k(E_\beta) = 0. \tag{4.13}$$

But $F(\beta)$ is a polynomial of degree $2k$ in the variable β that could be 'explicitly' computed. Indeed, taking out a factor $\beta + 1$ from the eigenvalues in (4.12), and expanding on the ν_β part as explained in Lemma 4.2:

$$\begin{aligned} F(\beta) &= (\beta+1)^k \sigma_k \left((\beta-1) dt \otimes \frac{\partial}{\partial t} - (\beta-1)\delta_i^j \, d\theta^i \otimes \frac{\partial}{\partial \theta^i} - (1+\beta)\delta_\alpha^\beta \, dz^\alpha \otimes \frac{\partial}{\partial z^\beta} \right) \\ &= (\beta+1)^k \sum_{l=0}^{\min\{p,k\}} \binom{p}{l} (\beta+1)^l (-1)^l \sigma_{k-l} \left((\beta-1) dt \otimes \frac{\partial}{\partial t} - (\beta-1)\delta_i^j \, d\theta^i \otimes \frac{\partial}{\partial \theta^i} \right) \\ &= (\beta+1)^k \sum_{l=0}^{\min\{p,k\}} (\beta+1)^l (\beta-1)^{k-l} (-1)^k \binom{p}{l} \frac{N-2k+2l}{N} \binom{N}{k-l}, \end{aligned}$$

where, for the last equality, we have used that

$$\sigma_{k-l} \left(dt \otimes \frac{\partial}{\partial t} - \delta_i^j \, d\theta^i \otimes \frac{\partial}{\partial \theta^i} \right) = (-1)^{k-l} \binom{N}{k-l} \frac{N-2k+2l}{N}.$$

It is not straightforward to find the roots of the polynomial $F(\beta)$, that is one of our computational difficulties in order to handle this ODE. We can explicitly compute $F(0) = c_{n,p,k} > 0$ for the constant defined in (3.4) because it corresponds to the model example, while $F(1) \geq 0$. We would need to check that there exists at least one root in the interval $(0, 1]$. Note that this is immediately true when $p < k$ because $F(1) = 0$. However, in the general case this is complicated. Due to this and other computational difficulties, we particularize to the case $k = 2$.

4.2 A closer look at σ_2

First we remark that, by a straightforward calculation,

$$\sigma_2(C_v) \leq c_n [\sigma_1(C_v)]^2.$$

Then if σ_2 stays strictly positive along a trajectory, then σ_1 cannot vanish. As a consequence, we must have that either $\sigma_1 > 0$ or $\sigma_1 < 0$ everywhere along that trajectory. Consequently, trajectories that end at the equilibrium point $(v_\infty, 0)$ must lie in the positive cone Γ_2^+ everywhere.

Next, we try to precisely understand the asymptotic behavior of the solution, when it tends to the equilibrium $(0, 0)$, for $t \to -\infty$.

Lemma 4.4. *We let $k = 2$ and $0 < p < \mathfrak{p}_2 = \frac{n-\sqrt{n}-2}{2}$. Then, there exists a trajectory tending to $(0, 0)$, as $t \to -\infty$ with the following asymptotic behavior*

$$v(t) \sim e^{\alpha_0 t}, \quad t \to -\infty,$$

for some $0 < \alpha_0 < \frac{n-4}{4}$.

Proof. We try to find the roots of the polynomial $F(\beta)$ that is defined in (4.13). But after some long computation,

$$F(\beta) = (\beta+1)^2 \left[\binom{p}{2}(\beta+1)^2 + p(N-2)(\beta-1)(\beta+1) + \frac{N-4}{2}(\beta-1)^2 \right],$$

that has roots at

$$\beta = -1, \quad \beta = 1 + 2\frac{-np + 3p \pm \sqrt{4p + 5p^2 - 5pn + pn^2 - p^2 n}}{(n-4)(n-1)}.$$

In the range $0 < p < \mathfrak{p}_2 = \frac{n-\sqrt{n}-2}{2}$ that we are looking at, there exists (at least) one real root $\beta_0 \in (0,1)$. Choosing $\alpha_0 := \beta_0/a$, where a was defined in (4.5), completes the proof of the lemma. □

Finally, equation (4.2) is not completely integrable. However, we are able to relate to the Hamiltonian quantity introduced in (4.6) for the case $p = 0$. As a consequence:

Lemma 4.5. *The trajectories emanating from* $(0,0)$ *found in the previous lemma must tend to the equilibrium point* $(v_\infty, 0)$ *when* $t \to +\infty$*, and are uniformly bounded.*

Proof. We go back to the ODE (4.2) and try to find a suitable Hamiltonian quantity. We remind the reader that the matrix C_v from formula (4.3) is written as

$$C_v := \lambda\, dt \otimes \frac{\partial}{\partial t} + \mu\, \delta_i^j\, d\theta^i \otimes \frac{\partial}{\partial \theta^i} + \nu\, \delta_\alpha^\beta\, dz^\alpha \otimes \frac{\partial}{\partial z^\beta},$$

and the eigenvalues λ, μ, ν, with multiplicities $1, N-1, p$ respectively, are given by (4.4). Our aim is to relate $\sigma_2(C_v)$ to $\sigma_2(D_v)$ for the matrix

$$D_v := \lambda\, dt \otimes \frac{\partial}{\partial t} + \mu\, \delta_i^j\, d\theta^i \otimes \frac{\partial}{\partial \theta^i} + \mu\, \delta_\alpha^\beta\, dz^\alpha \otimes \frac{\partial}{\partial z^\beta},$$

that corresponds to the case $p = 0$ understood in Proposition 4.1. First we split along the third coordinate and replace ν by μ:

$$\begin{aligned}
\sigma_2(C_v) &= \sigma_2\left(\lambda dt \otimes \frac{\partial}{\partial t} + \mu\delta_i^j d\theta^i \otimes \frac{\partial}{\partial \theta^i} + 0\delta_\alpha^\beta dz^\alpha \otimes \frac{\partial}{\partial z^\beta}\right) \\
&\quad + p\nu\sigma_1\left(\lambda dt \otimes \frac{\partial}{\partial t} + \mu\delta_i^j d\theta^i \otimes \frac{\partial}{\partial \theta^i}\right) + \binom{p}{2}\nu^2 \\
&= \sigma_2\left(\lambda dt \otimes \frac{\partial}{\partial t} + \mu\delta_i^j d\theta^i \otimes \frac{\partial}{\partial \theta^i} + \mu\delta_\alpha^\beta dz^\alpha \otimes \frac{\partial}{\partial z^\beta}\right) \\
&\quad + p(\nu-\mu)\sigma_1\left(\lambda dt \otimes \frac{\partial}{\partial t} + \mu\delta_i^j d\theta^i \otimes \frac{\partial}{\partial \theta^i}\right) + \binom{p}{2}(\nu^2 - \mu^2).
\end{aligned}$$

Summarizing,

$$\sigma_2(C_v) = \sigma_2(D_v) + pF(v),$$

where we have defined

$$F(v) := (\nu - \mu)\left[\sigma_1\left(\lambda dt \otimes \frac{\partial}{\partial t} + \mu\delta_i^j dz^i \otimes \frac{\partial}{\partial \theta^i}\right) + \frac{p-1}{2}(\nu+\mu)\right].$$

Substitute this expression into equation (4.2)

$$0 = \sigma_2(C_v) - \binom{n}{2} v^{\frac{4n}{n-4}} = \sigma_2(D_v) + (\nu-\mu)pF(v) - \binom{n}{2} v^{\frac{4n}{n-4}}.$$

As in the proof of Proposition 4.1, we multiply the previous equation by $\dot v/v$ and integrate by parts. The first part has an exact Hamiltonian H. We look for trajectories that tend to $(0,0)$ in the phase space, when $t\to-\infty$ (in this case, $H(v(t),\dot v(t))=\lim_{t\to-\infty}H(v(t),\dot v(t))=H(0,0)=0$). Then we obtain

$$0 \;=\; H(v(t),\dot v(t)) + p\int_{-\infty}^{t}(\nu-\mu)F(v)\dot v/v\,dt \;=\; H(v(t),\dot v(t)) - 2pF_1(t) - 2p\int_{-\infty}^{t}F_2(t)\,dt\,, \tag{4.14}$$

for

$$F_1(t) \;=\; \frac{n-p-2}{4}v^4 - av^2\dot v^2 - \frac{2}{3}a^2v\dot v^3\,,$$

and

$$F_2(t) \;=\; -pa^2v\dot v^3 + (n-2p-1)av^2\dot v^2 - \frac{n-1}{3}a^3\dot v^4\,.$$

Denote, for simplicity, $x=v$, $y=\dot v$, $z=a\dot v$. Then we can consider the new Hamiltonian quantity:

$$\tilde H(x,z) \;:=\; H-2pF_1 \;=\; (n-4)\left[bx^4 - \frac{(n-2p-2)}{4}x^2z^2 + \frac{p}{3}xz^3 + \frac{(n-1)}{8}z^4 - \frac{n-1}{8}x^{\frac{4n}{n-4}}\right],$$

for

$$b \;=\; \tfrac{1}{n-4}\left[\frac{(n-1)(n-4)}{8} - \frac{p(n-p-2)}{2}\right] \;=\; \frac{c_{n,p,2}}{4(n-4)}\,.$$

Note that $b>0$ exactly when $p<\mathfrak{p}_2$. The level sets of $\tilde H$ are closed bounded curves. The maximum is reached precisely at the point $(v_\infty,0)$. On the other hand, there is a branch of the set $\tilde H=0$ which stays in the region $z>0$, tends the origin and it always stays inside the region $\{x\ge|z|\}$.

From (4.14) we can extract some conclusions. We have seen in Lemma 4.4 that there is a trajectory emanating from $(0,0)$ that stays inside $\{x>0, 0<z<x\}$ for a while. We can also check that $F_2(t)\ge0$ as long as the trajectory stays in the region $\{x\ge|z|\}$. Then (4.14) immediately shows that the energy along that trajectory is strictly increasing and must never touch the set $\{\tilde H=0\}$. □

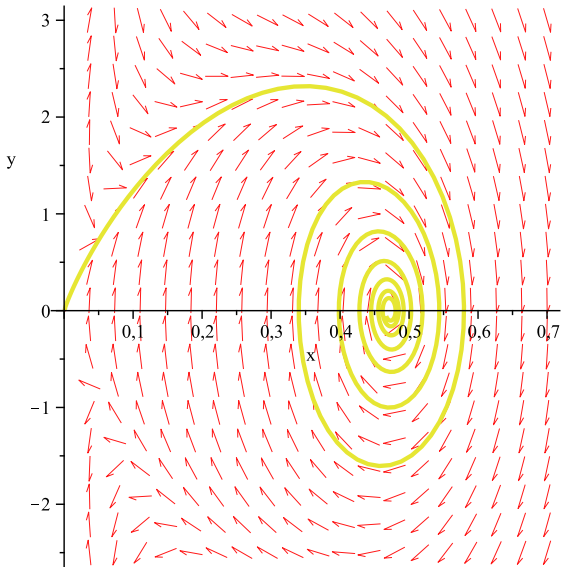

Figure 1: Case $n=200$, $p=2$, $v_\infty=0.4723...$

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