APPROXIMATE RESULTS FOR RAINBOW LABELINGS

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ABSTRACT. A simple graph $G = (V, E)$ is said to be antimagic if there exists a bijection $f : E \to [1, |E|]$ such that the addition of the values of f on edges incident to every vertex are pairwise different. The graph $G = (V, E)$ is distance antimagic if there exists a bijection $f : V \to [1, |V|]$, such that $\forall x, y \in V$,

$$\sum_{x_i \in N(x)} f(x_i) \neq \sum_{x_j \in N(y)} f(x_j).$$

Using the polynomial method of Alon we prove that there are antimagic injections of any graph G with n vertices and m edges in the interval $[1, 2n + m - 5]$ and, for trees with k base inner vertices, in the interval $[1, m + k]$. In particular, a tree all of whose nonleaves are adjacent to a leaf is antimagic. This gives a partial positive answer to a conjecture by Hartsfield and Ringel.

We also show that there are distance antimagic injections of a graph G with order n and maximum degree Δ in the interval $[1, n + t(n - t)]$, where $t = \min\{\Delta, \lfloor \frac{n}{2} \rfloor\}$, and, for trees with k endvertices, in the interval $[1, 3n - 4k]$. In particular, all trees with $n = 2k$ vertices and no pairs of incident leaves are distance antimagic, a partial solution to a conjecture of Arumugam.

1. INTRODUCTION

In this paper we shall consider two kinds of labelings: antimagic and distance antimagic. The concept of an *antimagic labeling* of a graph was introduced by Hartsfields and Ringel in 1990 [13]. In a terminology introduced later this is a *vertex antimagic edge labeling*, that is, informally, a labeling of edges which has the property that the sum of the adjacent edges is different at every vertex. Although this later terminology is becoming more popular, in this paper we will use just the term "antimagic" since this is shorter and should not lead to any confusion.

More formally, an *edge labeling* of a graph $G = (V, E)$ is a bijection $l : E \to \{1, 2, \ldots, |E|\}$. The *weight* of a vertex v, $wt(v)$, is the sum of the labels of all edges incident with v.

An edge labeling l of G is called *antimagic* if all vertex weights in G are pairwise distinct. A graph G is said to be *antimagic* if it has an antimagic labeling.

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Hartsfield and Ringel [13] showed that path P_m , star S_m, cycle C_m, complete graph K_m, wheel W_m and bipartite graph $K_{2,m}$, $m \geq 3$, are antimagic. They further conjectured

Conjecture 1. *(Harstfield–Ringel, 1990) Every graph $G \neq K_2$ is antimagic.*

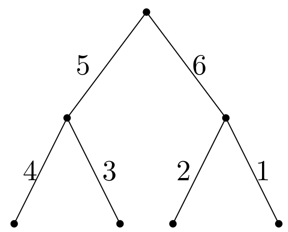

Figure 1. An antimagic labeling of a tree.

Over the period of more than two decades, many attempts have been made to settle the conjecture. While in general the Hartsfield and Ringel conjecture remains open, some partial results are known which support the conjecture. Alon *et al.* [3] used probabilistic methods and some techniques from analytic number theory to show that the conjecture is true for all graphs having minimum degree at least $\Omega(\log |V(G)|)$. They also proved that if G is a graph with order $|V(G)| \geq 4$ and maximum degree $\Delta(G)$, $|V(G)| - 2 \leq \Delta(G) \leq |V(G)| - 1$, then G is antimagic, and that all complete multipartite graphs, except K_2, are antimagic. Cranston [11] proved that every regular bipartite graph (complete or not) is antimagic. Hefetz [14] used the combinatorial nullstellensatz to prove that a graph with 3^k vertices, where k is a positive integer, and admits a K_3-factor, is antimagic. Various papers on the antimagicness of particular classes of graphs have been published, for example, see [9, 18, 19, 21, 22]. For more details on antimagic labeling for particular classes of graphs see the dynamic survey [12], see also [5].

There is now a great wealth of evidence in support of the conjecture. However a full general proof still eludes us. Even the weaker conjecture, that every tree different from K_2, is antimagic, still remains open. The most general result for trees is due to Kaplan, Lev and Roditty [15] who proved that every tree with at most one vertex of degree 2 is antimagic. See also [8, 13] for other results on antimagic trees.

The less well known but closely related type of graph labeling known as the "distance antimagic labeling", or more precisely, the "1-distance vertex antimagic vertex labeling", has been proposed as

Definition 2 (Miller, Rodger, Simantujak, 2003)**.**

A distance antimagic labeling of a graph $G = (V, E)$ is a bijection $f : V \to [1, |V|]$, such that for every pair x, y of vertices,

$$\sum_{u \in N(x)} f(u) \neq \sum_{u \in N(y)} f(u).$$

An obvious necessary condition for G to be distance antimagic is

$$N(x) \neq N(y), \quad \text{for each } x, y \in V(G).$$

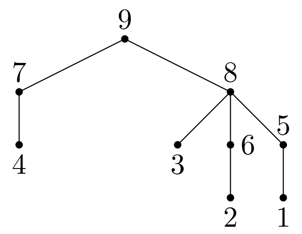

Figure 2. A distance antimagic labeling of a tree.

Paths P_n, cycles C_n, wheels $W_{n\neq 4}$, regular bipartite graphs and some particular caterpilars, are examples of graphs which have been proved to be distance antimagic. Arumugam [4] has conjectured that all trees satisfying the obvious necessary condition are distance antimagic.

Conjecture 3. *(Arumugam, 2012) A tree T is distance antimagic if and only if every vertex is adjacent to at most one leaf.*

In this paper we give approximate results to the two above conjectures, by which we mean that we find upper bounds for the smallest integer such that there is an injection with the corresponding *rainbow property*. An analogous approach has been also considered for other kinds of labelings; see e.g., Bollobás and Pikhurko [6] or Lladó, López and Moragas [16].

We start with distance antimagic graph labeling. For a given class $\mathcal{X}$ of graphs, let $DA(\mathcal{X}, n)$ denote the smallest integer N such that, for each graph $G \in \mathcal{X}$ of order n, there is an injection $f : V \to [1, N]$ such that the sums

$$\sum_{y\in N(x)} f(y),\ x \in V$$

are pairwise distinct. We call such a map a *distance antimagic injection.*

Let $\mathcal{G}$ be the class of all graphs which have no two vertices with the same neighborhood and let $G \in \mathcal{G}$ with order n. For any ordering $v_1, \ldots, v_n$ of the vertices of G the map $f(v_i) = 2^i$ is clearly a distance antimagic injection in G. Thus we have $DA(\mathcal{G}, n) \leq 2^n$. This trivial exponential bound can be reduced for general graphs.

Theorem 4. *Let $\mathcal{G}_\Delta$ be the class of graphs in $\mathcal{G}$ with maximum degree Δ. Then*

$$DA(\mathcal{G}_\Delta, n) \leq n + t(n - t),\ \textit{where}\ t = \min\{\Delta, \lfloor \frac{n}{2} \rfloor\}.$$

A better bound can be obtained for trees.

Theorem 5. *Let $\mathcal{T}_k$ be the class of trees in $\mathcal{G}$ which have k endvertices. Then*

$$DA(\mathcal{T}_k, n) \leq 3n - 4k.$$

In particular, all trees in $\mathcal{T}_k$ with $n = 2k$ vertices are distance antimagic.

The last statement in Theorem 5 implies in particular that, for any arbitrary tree T, the tree obtained from T by attaching one endvertex to each vertex of T is distance antimagic.

Similar definitions can be made for antimagic labeling. Let $A(\mathcal{X}, m)$ denote the smallest integer N such that, for each graph $G \in \mathcal{X}$ with m edges, there is an injection $f : E(G) \to [1, N]$ such that the sums

$$\sum_{yx \in E} f(y),\ x \in V$$

are pairwise distinct. We call such a map f an *antimagic injection.* As in the distance antimagic case, if $e_1, \ldots, e_m$ are the edges of G and $m > 1$ then the map $f(e_i) = 2^i$ is clearly an antimagic injection, so that $A(\mathcal{X}, m) \leq 2^m$. We show

Theorem 6. *Every graph G with $m > 1$ edges and n vertices admits an antimagic injection on $[1, 2n + m - 5]$.*

For trees the upper bound can be reduced. Recall that the base tree of a tree T is obtained from T by removing all its leaves. A vertex of T is said to be an inner vertex of a subtree T' of T if all its neighbours in T belong to T'. In particular, an inner vertex of a base tree is called a base inner vertex.

Theorem 7. *Let $\mathcal{T}_k$ denote the class of trees with k base inner vertices. We have $A(\mathcal{T}_k, m) \leq m + k$.*

In particular, a tree whose base tree has no inner vertices is antimagic.

It follows from the last statement in Theorem 7 that, for an arbitrary tree T, the tree obtained from T by attaching one endvertex to each vertex of T is antimagic.

The proofs of the above theorems use the polynomial method of Alon (Combinatorial Nullstellensatz) which we recall next.

Theorem 8 (Combinatorial Nullstellensatz, Alon (1999) [1])**.** *Let $P(x_1, \ldots, x_k)$ be a polynomial of degree d in $F[x_1, \ldots, x_k]$, (F a field), and let $S_1, \ldots, S_k$ be subsets of F with $|S_i| > d_i \geq 0$ such that $\sum_{i=1}^{k} d_i = d$.*

If the coefficient of the monomial $\prod_{i=1}^{k} {x_i}^{d_i}$ in f is **nonzero**, *then there exists*

$$(s_1, \ldots, s_k) \in S_1 \times \cdots \times S_k$$

such that

$$P(s_1, \ldots, s_k) \neq 0.$$

The proofs of Theorems 4 and 5 are given in Section 2 and Section 3 contains the proofs of Theorems 6 and 7. The paper conlcudes with a section of final remarks.

2. Distance antimagic injections

As mentioned before, the proofs use the polynomial method. For a set $x_1, \ldots, x_n$, we denote by $V(x_1, \ldots, x_n)$ the Vandermonde polynomial

$$V(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

We recall that the Vandermonde polynomial has an expansion of the form

$$V(x_1,\dots,x_n)=\sum_{\sigma\in Sym(n)}(-1)^{sgn(\sigma)}x_{\sigma(1)}^{n-1}x_{\sigma(2)}^{n-2}\cdots x_{\sigma(n)}^{0},$$

and that $V(a_1,\dots,a_n)\neq 0$ if and only if the a_i's are pairwise distinct.

The proof of Theorem 4 is a quite straightforward application of the polynomial method and it is included to illustrate the technique.

Proof. (*of Theorem 4.*) Let $G\in\mathcal{G}_\Delta$ with order n. Let $v_1,\dots,v_n$ be the vertices of G. Let $x_1,\dots,x_n$ be variables and, for each i, define

$$y_i=\sum_{j:v_j\in N(v_i)}x_j.$$

Consider the polynomial $P\in\mathbb{R}[x_1,\dots,x_n]$ defined as

$$P(x_1,\dots,x_n)=V(x_1,\dots,x_n)V(y_1,\dots,y_n).$$

A map $f:V\to\mathbb{N}$ is a distance antimagic injection if and only if

$$P(f(v_1),\dots,f(v_n))\neq 0.$$

Since G contains no two vertices with the same neighborhood, it admits distance antimagic injections, so that P is not the zero polynomial.

On the other hand, considering a term (y_j-y_k), x_i will appear in the term if it is present in exactly one of y_j, y_k. Hence the variable x_i appears at most $t(n-t)$ times in $V(y_1,\dots,y_n)$, where $t=\min\{\Delta,\lfloor n/2\rfloor\}$. Therefore, every monomial

$$x_1^{\alpha_1}\cdots x_n^{\alpha_n}$$

in P with nonzero coefficient satisfies

$$\max\alpha_i\leq(n-1)+t(n-t).$$

It follows from the combinatorial nullstellenstaz that, by choosing

$$S_1=\ldots=S_n=[1,n+t(n-t)],$$

there are $1\leq a_1,\dots,a_n\leq n+\Delta(n-\Delta)$ such that $P(a_1,\dots,a_n)\neq 0$. Thus the assignment $f(v_i)=a_i$ gives a distance antimagic injection. $\square$

The proof of Theorem 5 involves a more efficient use of the polynomial method. We recall that $(V(x_1,\dots,x_n))^2$ has a term of the form

$$x_1^{n-1}\cdots x_n^{n-1},$$

with coefficient $n!$ up to a sign (see e.g. Alon [3].) We next show an analogous result for the fourth power of the Vandermonde polynomial.

Lemma 9. *The coefficient of*

$$x_1^{2(n-1)}\cdots x_n^{2(n-1)},$$

in $(V(x_1,\dots,x_n))^4$ is a sum of squares. In particular, when considered as a polynomial in $\mathbb{R}[x_1,\dots,x_n]$, this coefficient is nonzero.

Proof. The result will follow from the following more precise statement: For each $n \geq 2$, the two monomials

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} \text{ and } x_1^{2(n-1)-\alpha_1} \cdots x_n^{2(n-1)-\alpha_n}$$

have the same coefficient in $(V(x_1, \dots, x_n))^2$.

We prove the above statement by induction on n. The statement clearly holds for $n = 2$: $(V(x_1, x_2))^2 = (x_1 - x_2)^2 = x_1^2 - 2x_1x_2 + x_2^2$.

Let $n > 2$ and write

$$(V(x_1, \dots, x_{n-1}, x_n))^2 = (V(x_1, \dots, x_{n-1}))^2 \prod_{i=1}^{n-1} (x_n - x_i)^2.$$

Fix a monomial

$$M = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \text{ in } (V(x_1, \dots, x_{n-1}, x_n))^2.$$

Let

$$N = x_1^{\beta_1} \cdots x_{n-1}^{\beta_{n-1}} \text{ in } (V(x_1, \dots, x_{n-1})^2 \text{ and } R = x_1^{\gamma_1} \cdots x_n^{\gamma_n} \text{ in } \prod_{i=1}^{n-1} (x_n - x_i)^2$$

be such that $NR = M$.

The exponents $\gamma_1, \dots, \gamma_{n-1}$ all belong to $\{0, 1, 2\}$. Let r and s be the number of these exponents with value 1 and 2 respectively. We have

$$\alpha_n = \gamma_n = 2(n - r - s - 1) + r = 2(n - s - 1) - r,$$

and the coefficient of R in $\prod_{i=1}^{n-1}(x_n - x_i)^2$ is $(-2)^r$. We can write

$$\begin{aligned} 2(n-1) - \alpha_i &= 2(n-2) - \beta_i + (2 - \gamma_i), \ i = 1, \dots, n-1; \\ 2(n-1) - \alpha_n &= 2s + r. \end{aligned}$$

The map

$$\phi : (N, R) \mapsto (x_1^{2(n-2)-\beta_1} \cdots x_{n-1}^{2(n-2)-\beta_{n-1}}, x_1^{2-\gamma_1} \cdots x_{n-1}^{2-\gamma_{n-1}} x_n^{2(n-1)-\gamma_n})$$

is a well defined injection from pairs (N, R) with $NR = M$ to pairs (N', R') with $N'R' = M'$, where $M' = x_1^{2(n-1)-\alpha_1} \cdots x_n^{2(n-1)-\alpha_n}$.

By the induction hypothesis, N and N' have the same coefficient in $(V(x_1, \dots, x_{n-1}))^2$, while the coefficients of R and of R' in $\prod_{i=1}^{n-1}(x_n - x_i)^2$ are both equal to $(-2)^r$. The map ϕ is clearly a bijection. It follows that the coefficients of M and M' in $(V(x_1, \dots, x_{n-1}, x_n))^2$ coincide.

The monomial $x_1^{2(n-1)} \cdots x_n^{2(n-1)}$ in $(V(x_1, \dots, x_{n-1}, x_n))^4$ arises as a product of two monomials in $(V(x_1, \dots, x_{n-1}, x_n))^2$ of the form $x_1^{\alpha_1}, \dots, x_n^{\alpha_n}$ and $x_1^{2(n-1)-\alpha_1}, \dots, x_n^{2(n-1)-\alpha_n}$ which, by the above argument, have the same coefficient in $(V(x_1, \dots, x_{n-1}, x_n))^2$. Hence, its coefficient in $(V(x_1, \dots, x_{n-1}, x_n))^4$ is a sum of squares.

In particular, since the term $x_1^{n-1} \cdots x_n^{n-1}$ in $(V(x_1, \dots, x_n))^2$ has coefficient $n!$ (up to a sign), when the coefficients are taken from $\mathbb{R}$ (or in any field of zero characteristic), the above sum of squares is nonzero. This completes the proof. $\square$

Using Lemma 9 we next prove Theorem 5.

Proof. (*of Theorem 5*) For a subtree $T' \subseteq T$ we denote by

$$I(T') = \{v \in V(T') : N(v) \subset V(T')\},$$

the set of inner vertices in T' (its full neighborhood in T is contained in $V(T')$) and by

$$D(T') = \{v \in V(T') : |N_{T'}(v)| = 1\},$$

the set of endvertices of T'. We also let

$$D^-(T') = \{N_{T'}(v) : v \in D(T')\},$$

denote the set of vertices in T' which are adjacent to some endvertex of T'.

Let $T_0 = T$ and for $i \geq 1$ let $T_i = T_{i-1} - D(T_{i-1})$ be the subtree of T obtained from T_{i-1} by deleting its endvertices. In this way we obtain a monotone decreasing chain

$$T = T_0 \supset T_1 \supset T_2 \supset \cdots \supset T_l,$$

where $V(T_i) = V(T_{i+1}) \cup D(T_i)$ and T_l is the center of T consisting of a single vertex or a single edge. We will define a distance antimagic injection of T level by level starting from T_l. We label the vertices in T_l with $\{1\}$ (if T_l consists of one vertex) or with $\{1, 2\}$ (if it consists of two vertices.)

Suppose f has been defined on $V(T_{i+1})$ satisfying the following three properties:

(i) f is an injection on $V(T_{i+1})$;
(ii) the neighbour sums in

$$S(I(T_{i+1})) = \{S(v) : v \in I(T_{i+1})\}$$

of inner vertices in T_{i+1} are pairwise distinct, where $S(v) = \sum_{u \in N(v)} f(u)$;
(iii) if $v \in D^-(T) \cap V(T_{i+1})$ then $f(v) \notin S(I(T_{i+1}))$.

We will extend f to T_i in two steps by preserving the three above properties.

Let $w_1, \ldots, w_r$ be the vertices in $I(T_i) \setminus I(T_{i+1})$; these are the new inner vertices in T_i, which do have neighbors in $D(T_i)$. We note that, since T has no pair of adjacent leaves, we have

$$r \leq |D(T_i)| \leq |D(T)| = k.$$

For each w_j choose one neighbor $v_j \in N(w_j) \cap D(T_i)$ (Figure 2 intends to illustrate the notation) . Label the vertices in $D(T_i) \setminus \{v_1, \ldots, v_r\}$ with pairwise distinct numbers in the set

$$[1, N_i'] \setminus (f(V(T_{i+1})) \cup S(I(T_{i+1}))), \tag{1}$$

where $N_i' = |V(T_i)| + |I(T_{i+1})|$ (this is possible since this set contains at least $|D(T_i)|$ elements.) In this way, f is still injective and no vertex in $D^-(T)$ has received a value in $S(I(T_{i+1}))$.

Let

$$S'(w_j) = \sum_{u \in N(w_j) \setminus v_j} f(u),\ j = 1, \ldots, r,$$

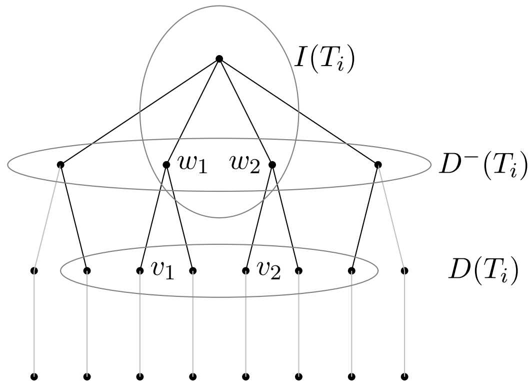

FIGURE 3. Illustration of the notation in the proof of Theorem 5.

and consider the following polynomials in $\mathbb{R}[x_1, \ldots, x_k]$:

$$\begin{aligned}
Q_{1,i} &= V(x_1, \ldots, x_r) \prod_{j=1}^{r} \prod_{u \in V(T_i) \setminus \{v_1, \ldots, v_r\}} (x_j - f(u)); \\
Q_{2,i} &= V(x_1 + S'(w_1), \ldots, x_r + S'(w_r)) \prod_{j=1}^{r} \prod_{u \in I(T_{i+1})} (x_j + S'(w_j) - S(u)); \\
Q_{3,i} &= \prod_{1 \le i < j \le r} (x_i - (S'(w_j) + x_j)) \prod_{j=1}^{k} \prod_{u \in I(T_{i+1})} (x_j - S(u)).
\end{aligned}$$

We observe that

$$\begin{aligned}
V(x_1 + S'(w_1), \ldots, x_r + S'(w_r)) &= V(x_1, \ldots, x_r) + \text{terms of lower order; and} \\
\prod_{1 \le i < j \le r} (x_i - (S'(w_j) + x_j)) &= V(x_1, \ldots, x_r) + \text{terms of lower order.}
\end{aligned}$$

Hence, the polynomial $P_i = Q_{1,i} Q_{2,i} Q_{3,i}$ can be written as

$$P_i = (V(x_1, \ldots, x_r))^3 \prod_{j=1}^{r} x_j^{m_i} + \text{terms of lower order,}$$

where

$$m_i = (|V(T_i)| - r) + 2|I(T_{i+1})|.$$

It follows from Lemma 9 that $(V(x_1, \ldots, x_k))^3$ has a monomial with nonzero coefficient whose largest exponent is at most $2(r-1)$. Hence our polynomial P_i has a term with larger exponent at most

$$N_i = |V(T_i)| + 2|I(T_{i+1})| + r - 2, \tag{2}$$

whose coefficient is nonzero. We note that N_i is larger than the N_i' from (1) defined in the first part of the i–th step.

By Theorem 8 there are $1 \le a_1, \ldots, a_r \le N_i + 1$ such that P_i takes a nonzero value on $(a_1, \ldots, a_k)$. Define $f(u_i) = a_i$, $i = 1, \ldots, r$. Let us check that in this way we have extended f to $V(T_i)$ by preserving properties (i)–(iii).

The way f has been defined on $V(T_i) \setminus \{v_1, \ldots, v_r\}$ and the fact that $Q_{1,i}$ is nonzero on $(a_1, \ldots, a_r)$ ensure that f is injective on $V(T_i)$, yielding property (i). Since no vertex in $D(T_i) \setminus \{v_1, \ldots, v_r\}$ is an inner vertex of T_i, the fact that $Q_{2,i}(a_1, \ldots, a_k) \neq 0$ ensures that the values $S(w_j) = S'(w_j) + a_j$, $1 \leq j \leq r$, are pairwise distinct and different from the values of $S(u)$ for $u \in I(T_{i+1})$. Moreover, if $i = 0$, then the values $S(v_j) = f(w_j)$ are pairwise distinct (by property (i) of f on $V(T_1)$) and different from the values $\{S(u) : u \in I(T_1)\}$ (by property (iii) of f on T_1), which gives property (ii). Finally, the fact that $Q_{3,i}(a_1, \ldots, a_r) \neq 0$ ensures that $\{f(v_j), 1 \leq j \leq r\}$ is disjoint with

$$\{S(u) : u \in I(T_{i+1})\} \cup \{S(w_1), \ldots, S(w_k)\} = \{S(u) : u \in I(T_i) \setminus D\}.$$

Together with the way f has been defined on $V(T_i) \setminus \{v_1, \ldots, v_r\}$, this ensures that the label of every vertex in $D^-(T)$ does not coincide with any neighbor sum, providing property (iii).

For $i \geq 1$, the above procedure produces an injection f on $[1, N_i + 1]$, where N_i is defined in (2). We observe that in the last step, when $i = 0$, by using the notation as in the above procedure, we have $r = k$, $\{w_1, \ldots, w_k\} = D^-(T)$ and $\{v_1, \ldots, v_k\} = D(T)$. In this case the polynomial $Q_{3,0}$ which ensures that the labels given to vertices of $D^-(T)$ do not coincide with neighbor sums, is no longer required since none of the new vertices $v_1, \ldots, v_k$ belongs to $D^-(T)$. Hence the last step can be simplified to just consider the polynomial

$$Q_0 = Q_{1,0}Q_{2,0} = (V(x_1, \ldots, x_k))^2 \prod_{j=1}^{k} x_j^{m_0},$$

where $m_0 = |V(T)| - k + |I(T_1)| = n - k + (n - 2k) = 2n - 3k$. Moreover, the polynomial Q_0 has a term with nonzero coefficient and all the exponents equal to $N_0 = (k-1) + m_0 = 2(n - k) - 1$. Therefore the last extension of f to $V(T_0) = V(T)$ can be performed in the interval $[1, 2(n - k)]$.

When the process finishes at $i = 0$, the properties (i)–(iii) ensure that f is a distance antimagic injection taking values in the interval $[1, \max_i N_i + 1]$. Since $\max_i N_i = \max\{N_1, N_0\}$ and

$$N_1 \leq |V(T_1)| + 2|I(T_2)| + k - 2 \leq (n - k) + 2(n - 2k) + k - 2 = 3n - 4k - 2,$$

we have $\max\{N_1, N_0\} \leq 3n - 4k - 1$. It follows that f can be defined in all cases in the interval $[1, 3n - 4k]$. This completes the proof. $\square$

3. Antimagic injections

The proofs of Theorems 6 and 7 are analogous to the proofs of Theorems 4 and 5 respectively, but they are somewhat simpler.

Proof. (*of Theorem 6.*) Let G be a graph with order n and m edges. Let $v_1, \ldots, v_n$ be the vertices of G and let $e_1, ..., e_m$ be the edges of G. For each vertex v_i denote by $e(v_i)$ the set of edges incident with v_i. Let $x_1, \ldots, x_m$ be variables and, for each $i = 1, \ldots, m$, define

$$y_i = \sum_{j : e_j \in e(v_i)} x_j.$$

Consider the polynomial $P \in \mathbb{R}[x_1, \dots, x_m]$ defined as

$$P(x_1, \dots, x_m) = V(x_1, \dots, x_m)V(y_1, \dots, y_n).$$

A map $f : E \to \mathbb{N}$ is an antimagic injection if and only if

$$P(f(e_1), \dots, f(e_m)) \neq 0.$$

Since G admits antimagic injections, P is not the zero polynomial. We observe that every variable x_i appears in at most *two* different variables y_j (the two endvertices of the corresponding edge). By looking at the expansion of the Vandermonde polynomials, we see that every monomial

$$x_1^{\alpha_1} \cdots x_m^{\alpha_m}$$

in P with nonzero coefficient satisfies $\max \alpha_i < (m-1) + 2(n-2)$. It follows from the combinatorial nullstellenstaz that there are $1 \leq a_1, \dots, a_n \leq 2n + m - 5$ such that $P(a_1, \dots, a_m) \neq 0$. Thus the assignment $f(e_i) = a_i$ gives an antimagic injection. □

We finally prove Theorem 7.

Proof. (*of Theorem 7.*) As mentioned before the proof follows the same lines as the proof of Theorem 5. We use the same notation concerning the set $I(T')$ of inner vertices, the set $D(T')$ of endvertices and $D^-(T')$ the set of vertices adjacent to endvertices of a subtree $T' \subseteq T$. We again define

$$T_0 = T \supset T_1 \supset \cdots \supset T_l.$$

where $T_{i+1} = T_i \setminus D(T_i)$. Suppose that f has been defined on $E(T_{i+1})$ injectively and such that the edgesums $S(v) = \sum_{u \in N(v)} f(uv)$ of the inner vertices of T_{i+1} are pairwise distinct.

Let $w_1, \dots, w_r$ be the vertices in $I(T_i) \setminus I(T_{i+1})$ and, for each w_j choose one neighbor $v_j \in N(w_j) \cap D(T_i)$. Label the edges in $\{uw_j : j = 1, \dots, r, u \in N(w_j) \cap D(T_i) \setminus \{v_j\}$ with pairwise distinct numbers in

$$[1, N_i'] \setminus (f(E(T_{i+1})) \cup S(I(T_{i+1}))),$$

where $N_i' = (|E(T_i)| - r) + |I(T_{i+1})|$ (this is possible since this set contains at least $|D(T_i)| - r$ elements.) In this way, f is still injective.

Let

$$S'(w_j) = \sum_{u \in N(w_j) \setminus v_j} f(uw_j),\ j = 1, \dots, r,$$

and consider the following polynomials in $\mathbb{R}[x_1, \dots, x_r]$:

$$\begin{aligned} Q_{1,i} &= V(x_1, \dots, x_r) \prod_{j=1}^{r} \prod_{e \in E(T_i) \setminus \{v_1w_1, \dots v_rw_r\}} (x_j - f(e)); \\ Q_{2,i} &= V(x_1 + S'(w_1), \dots, x_r + S'(w_r)) \prod_{j=1}^{r} \prod_{u \in I(T_{i+1})} (x_j + S'(w_j) - S(u)). \end{aligned}$$

We can write

$$P_i = Q_{1,i}Q_{2,i} = (V(x_1, \dots, x_r))^2 \prod_{j=1}^{r} x^{m_i} + \text{terms of lower order},$$

where $m_i = |E(T_i)| - r + |I(T_{i+1})|$. Hence P_i has a term

$$x_1^{m_i+r-1} \cdots x_r^{m_i+r-1}$$

with nonzero coefficient. We observe that $N_i = |E(T_i)| + |I(T_{i+1})| - 1 \geq N_i'$. By the combinatorial nullstellensatz, there are $1 \leq a_1, \ldots, a_r \leq N_i + 1$ such that P_i does not vanish in $(a_1, \ldots, a_r)$. By defining $f(v_j w_j) = a_j$ for $j = 1, \ldots, r$ we have extended f to $E(T_i)$ injectively and such that the edgesums of inner vertices of T_i are pairwise distinct. When $i = 0$ we have completed our definition of f taking values on $[1, N_0 + 1]$, where

$$N_0 = |E(T)| + |I(T_1)| - 1 = m + k - 1.$$

This completes the proof. □

4. Final Remarks

Alon's polynomial method is a useful tool for proving the existence of labelings of graphs with some prescribed properties. It has however some limitations and its straight application cannot provide, for instance, a proof of the antimagic conjecture, even for trees. As mentioned in the proof of Theorem 6, a labeling $f : E(T) \to \mathbb{N}$ of a graph G with vertex set $\{v_1, \ldots, v_n\}$ and m edges is antimagic if and only if the polynomial $P(x_1, \ldots, x_m) = V(x_1, \ldots, x_m)V(y_1, \ldots, y_n)$ is nonzero in $(f(v_1), \ldots, f(v_n))$, where the variables y_i are defined as in the proof. For the simple path P_3 with three vertices, this polynomial reduces to $P(x_1, x_2) = -x_1^3x_2 + 2x_1^2x_2^2 - x_1x_2^3$. Direct application of the combinatorial nullstellensatz to this polynomial only ensures that there is an antimagic injection in $\{1, 2, 3\}$, although this path admits an obvious antimagic labeling. The reason is that the method provides the stronger result that *every* set with three elements can be used as values of a labeling, and for this stronger statement the minimum cardinality for P_3 is indeed 3. However one can exploit the generality of the results obtained with the method in other directions. One example is the modular version of Theorem 7. Let us define a *modular* antimagic injection $f : E(G) \to \mathbb{Z}_n$ of a graph as an injection such that the edge sums

$$\{\sum_{u \in N(v)} f(uv) : v \in V(G)\}$$

are pairwise distinct modulo n. In this case we say that T is n–antimagic. It is proved in [15] that every tree with m edges and at most one vertex of degree 2 is $(m+1)$–antimagic whenever m is even. The proof of Theorem 7 provides the following modular version.

Theorem 10. *Let T be a tree with p edges, p a prime, whose base tree has no inner vertices. Then T is p–antimagic.*

Proof. Replacing the field $\mathbb{R}$ in the proof of Theorem 7 by the finite field $\mathbb{F}_p$, p a prime, the coefficient of the monomials which appear in the applications of the combinatorial nullsetellensatz is $r!$ with $r < p$, which is clearly nonzero in $\mathbb{F}_p$. □

We finish by noting that there are simple direct arguments which provide approximate results. For instance, it can be proved by simple induction and the pigeonhole principle that every tree with m edges admits an antimagic injection in $[1, 2m - 1]$. Indeed, by assuming that $T - e$ admits such an antimagic injection for a leave e of T, there are $m - 1$ values already taken by the labels of the edges in $T - e$, and $m - 1$ edgesums which should be avoided for the edgesum of the vertex incident to e in $T - e$ and for the endvertex of e. Thus, if $2m-1$ values are available, at least one of them must lead to an antimagic labeling of T. Such an inductive argument, however, cannot be applied to distance antimagic labelings.

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