

COMPLETION AND DECOMPOSITION OF A CLUTTER INTO REPRESENTABLE MATROIDS

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ABSTRACT. This paper deals with the question of completing a monotone increasing family of subsets Γ of a finite set Ω to obtain the linearly dependent subsets of a family of vectors of a vector space. Specifically, we prove that such vectorial completions of the family of subsets Γ exist and, in addition, we show that the minimal vectorial completions of the family Γ provide a decomposition of the clutter Λ of the inclusion-minimal elements of Γ . The computation of such vectorial decomposition of clutters is also discussed in some cases.

1. INTRODUCTION

A *monotone increasing* family of subsets Γ of a finite set Ω is a collection of subsets of Ω such that any superset of a set in the family Γ also belongs to Γ . All the inclusion-minimal elements of Γ determine a *clutter* Λ , that is, a collection of subsets of Ω none of which is a proper subset of another. Clutters are also known as *antichains*, *Sperner systems* or *simple hypergraphs*.

In this paper we focus our attention on those monotone increasing families of subsets that arise from linear algebra: the collection of the linearly dependent subsets of vectors in a vector space. We say that a clutter Λ is *vectorial* if its elements are the inclusion-minimal linearly dependent subsets of an indexed family of vectors of a vector space.

Vectorial clutters are an important issue in matroid theory. A *matroid* \mathcal{M} is a combinatorial object that provides an axiomatic abstraction of linear dependence on a finite set Ω . The minimal dependent sets of a matroid are called *circuits*. Therefore, the family of circuits of a matroid \mathcal{M} is a clutter. Vectorial clutters are exactly those corresponding to the set of *circuits of representable matroids*.

In some cases it is convenient to use clutters that are either vectorial or are close to being vectorial. Examples of this situation can be found in the context of secret-sharing schemes [3, 5], or in the framework of algebraic combinatorics and commutative algebra [1, 6]. For instance, in the context of secret-sharing schemes, vectorial clutters become a

crucial issue for providing general bounds on the optimal information rate of the scheme, while in the framework of algebraic combinatorics and commutative algebra, they are useful for controlling certain arithmetic properties of either monomial ideals or the face rings of simplicial complexes.

In general, a clutter is far from being vectorial. Therefore it is of interest to determine how it can be transformed into a vectorial clutter. This paper deals with this issue. More specifically, we first define a partial order \leq on the set of all clutters on Ω . Then a vectorial *completion* of a clutter Λ is a vectorial clutter Λ' such that $\Lambda \leq \Lambda'$. We show that these completions exist and that Λ can be recovered from the minimal ones. We speak in this case of a *decomposition* of the clutter Λ .

The structure of the paper is as follows. In Section 2 we recall some definitions and basic facts about clutters and present the problem of the vectorial completion of a clutter. Our main results are gathered in Section 3; namely, we present three theorems concerning vectorial completion and decomposition of clutters (Theorem 4, Theorem 5 and Theorem 6), and we apply them to obtain the decomposition of non-representable matroids into representable matroids (Corollary 7). Finally, Section 4 is devoted to analyzing the computation of such decompositions: first, in Subsection 4.1 we study the vectorial completions and decompositions of clutters on a finite set of size at most seven (Proposition 9); next, in Subsection 4.2 we present the minimal binary completions of the excluded minor of binary matroids (Proposition 12); and we close in Subsection 4.3 by describing the minimal vectorial completions over fields of characteristic two of the non-Fano matroid (Proposition 14).

2. VECTORIAL CLUTTERS AND VECTORIAL COMPLETIONS

In this section we present the definitions and basic facts concerning families of subsets, clutters and vectorial clutters that are used in the paper.

Let Ω be a finite set. A family of subsets Γ of Ω is *monotone increasing* if any superset of a set in Γ must be in Γ ; that is, if $A \in \Gamma$ and $A \subseteq A' \subseteq \Omega$, then $A' \in \Gamma$. A *clutter* of Ω is a collection of subsets Λ of Ω , none of which is a proper subset of another; that is, if $A, A' \in \Lambda$ and $A \subseteq A'$ then $A = A'$.

Observe that if Γ is a monotone increasing family of subsets of Ω , then the collection $\min(\Gamma)$ of its inclusion-minimal elements is a clutter; while if Λ is a clutter on Ω , then $\Lambda^+ = \{A \subseteq \Omega : A_0 \subseteq A \text{ for some } A_0 \in \Lambda\}$ is a monotone increasing family of subsets. Clearly, $\Gamma =$

$(\min(\Gamma))^+$ and $\Lambda = \min(\Lambda^+)$. So a monotone increasing family of subsets Γ is uniquely determined by the clutter $\min(\Gamma)$, while a clutter Λ is uniquely determined by the monotone increasing family Λ^+ .

Let Λ_1, Λ_2 be two clutters on Ω . It is clear that if $\Lambda_1 \subseteq \Lambda_2$ then $\Lambda_1^+ \subseteq \Lambda_2^+$. However, the converse is not true; that is, there exist clutters with $\Lambda_1 \not\subseteq \Lambda_2$ and $\Lambda_1^+ \subseteq \Lambda_2^+$. For instance, on the finite set $\Omega = \{1, 2, 3\}$, let us consider the clutters $\Lambda_1 = \{\{1, 2\}, \{2, 3\}\}$ and $\Lambda_2 = \{\{1\}, \{2, 3\}\}$. Then $\Lambda_1 \not\subseteq \Lambda_2$, while $\Lambda_1^+ = \{\{1, 2\}, \{2, 3\}, \{1, 2, 3\}\} \subseteq \{\{1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\} = \Lambda_2^+$.

This fact leads us to consider a binary relation \leq defined on the set of clutters on Ω . Namely, if Λ_1 and Λ_2 are two clutters on Ω , then we say that $\Lambda_1 \leq \Lambda_2$ if and only if $\Lambda_1^+ \subseteq \Lambda_2^+$. The following lemma will be used several times throughout the paper.

Lemma 1. *Let Ω be a finite set. The following statements hold:*

- (1) *If Λ_1, Λ_2 are two clutters on Ω then, $\Lambda_1 \leq \Lambda_2$ if and only if for all $A_1 \in \Lambda_1$ there exists $A_2 \in \Lambda_2$ such that $A_2 \subseteq A_1$.*
- (2) *The binary relation \leq is a partial order on the set of clutters of Ω .*

Proof The proofs of the statements are a straightforward consequence of the definition of Λ^+ and of the fact that $\Lambda = \min(\Lambda^+)$. \square

There are many interesting families of clutters that can be considered. However, because of their applications, we are interested in those clutters that are *vectorial*.

Let $\Omega = \{x_1, \dots, x_n\}$ be a finite set of n elements. A monotone increasing family Γ of subsets of Ω is said to be a *vectorial family* if there exists an indexed family of not necessarily distinct vectors v_1, \dots, v_n of a \mathbb{K} -vector space such that $\{x_{i_1}, \dots, x_{i_r}\} \in \Gamma$ if, and only if, $\{v_{i_1}, \dots, v_{i_r}\}$ is a linearly dependent multiset of vectors. A clutter Λ on Ω is said to be a *vectorial clutter* if the monotone increasing family Λ^+ is a vectorial family. In such a case we say that the vectors v_1, \dots, v_n provide a \mathbb{K} -*representation* of Λ .

In other words, a monotone increasing family of subsets Γ is vectorial if Γ is the family of the *dependent sets of a representable matroid* \mathcal{M} with ground set Ω ; whereas a clutter Λ is vectorial if the clutter Λ is the set of *circuits of a representable matroid* \mathcal{M} with ground set Ω (definitions and basic facts about matroids are recalled in Subsection 3.3 as no matroid theory is needed until then).

There are clutters on a finite set Ω that are not vectorial (in fact, there are matroids that are not representable matroids). So, a natural question that arises at this point is to determine how to complete a

clutter Λ to obtain a vectorial clutter. In order to look for vectorial completions, it is important to take into account the binary relation \leq rather than the inclusion \subseteq . This is due to the fact that, as the following example shows, there exist clutters Λ such that $\Lambda \not\subseteq \Lambda'$ for any vectorial clutter Λ' .

Example 2. *Let us consider the clutter $\Lambda = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}\}$ on the finite set $\Omega = \{1, 2, 3, 4\}$. Observe that $(\{1, 2\} \cup \{1, 3\}) \setminus \{1\} = \{2, 3\} \subsetneq \{2, 3, 4\}$. Hence it follows that Λ is not a vectorial clutter and, moreover, $\Lambda \not\subseteq \Lambda'$ for any vectorial clutter Λ' . However, we have that $\Lambda \leq \Lambda'$, where Λ' is the vectorial clutter $\Lambda' = \{\{1\}, \{2, 3, 4\}\}$ (a vectorial realization of Λ' is given by the set of vectors $\{v_1, v_2, v_3, v_4\}$ where $v_1 = (0, 0)$, $v_2 = (1, 0)$, $v_3 = (0, 1)$ and $v_4 = (1, 1)$). Furthermore, if Λ'' is the clutter on Ω defined by $\Lambda'' = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, then we have that $\Lambda \leq \Lambda''$ and that the clutter Λ'' is also a vectorial clutter (a vectorial realization of Λ'' is given by the set of vectors $\{w_1, w_2, w_3, w_4\}$ where $w_1 = (1, 1)$, $w_2 = (1, 1)$, $w_3 = (1, 1)$ and $w_4 = (0, 1)$). Notice that now the clutter Λ can be obtained from the vectorial clutters Λ' and Λ'' . Indeed, it is easy to check that the following equality holds $\Lambda = \min(\{A' \cup A'' \text{ where } A' \in \Lambda' \text{ and } A'' \in \Lambda''\})$. Therefore, the vectorial clutters Λ' and Λ'' in some way provide a decomposition of the non-vectorial clutter Λ .*

The above example leads us to the following definition. Let Λ be a clutter on a finite set Ω . A *vectorial completion* of the clutter Λ is a vectorial clutter Λ' on the finite set Ω such that $\Lambda \leq \Lambda'$.

The set of all the vectorial completions of a clutter Λ is denoted by $\text{Vect}(\Lambda)$. Observe that if $\emptyset \in \Lambda$, then $\Lambda = \{\emptyset\}$, and thus $\text{Vect}(\Lambda) = \emptyset$. So, from now on we assume that $\emptyset \notin \Lambda$ if Λ is a clutter. As shown in the next section, this assumption guarantees that $\text{Vect}(\Lambda) \neq \emptyset$ for all clutters and, in addition, we demonstrate that, in the same way as in Example 2, suitable clutters in the non-empty set of the vectorial completions $\text{Vect}(\Lambda)$ provide a decomposition of the clutter Λ .

3. THREE RESULTS ON VECTORIAL COMPLETIONS AND DECOMPOSITIONS

The aim of this section is to present three theoretical results concerning the “decomposition” of a clutter Λ into vectorial clutters $\Lambda_1, \dots, \Lambda_r$. The general case is considered in Theorem 4, while Theorem 5 and Theorem 6 deal with those “decompositions” of Λ whose vectorial components $\Lambda_1, \dots, \Lambda_r$ admit vectorial realizations either over a fixed field \mathbb{K} or over fields having a specific characteristic. The section concludes by applying these theorems to matroids (Corollary 7).

3.1. General case. Let Λ be a clutter on a finite set Ω . Our first result, Theorem 4, states that the set $\text{Vect}(\Lambda)$ of its vectorial completions is a non-empty set and that its minimal elements provide a decomposition of Λ (in the sense that the elements A of the clutter Λ can be obtained from the elements A_i of its minimal vectorial completions $\Lambda_1, \dots, \Lambda_r$). To prove this we will use the following proposition which is a general result about the decomposition of the clutter Λ into clutters of a specific type.

Let us denote by $\text{Clutt}(\Omega)$ the set whose elements are the clutters on the finite set Ω , and for a non-empty subset $X \subseteq \Omega$, let Λ_X be the clutter on Ω defined by $\Lambda_X = \{\{x\} : x \in X\}$.

Proposition 3. *Let Λ be a clutter on a finite set Ω . Let $\Sigma \subseteq \text{Clutt}(\Omega)$ be a collection of clutters on Ω and let $\Sigma(\Lambda) = \{\Lambda' \in \Sigma : \Lambda \leq \Lambda'\}$. Assume that $\Lambda_X \in \Sigma$ for all non-empty subsets X of Ω . Then, $\Sigma(\Lambda) \neq \emptyset$ and*

$\Lambda = \min(\{A_1 \cup \dots \cup A_r : A_i \in \Lambda_i\})$ where $\Lambda_1, \dots, \Lambda_r$ are the minimal elements of the poset $(\Sigma(\Lambda), \leq)$. In particular, $\Lambda \in \Sigma$ if and only if $r = 1$.

Proof Let $n = |\Omega|$ be the size of Ω and let $\Omega = \{x_1, \dots, x_n\}$. On one hand, it is clear that $\Lambda \leq \Lambda_\Omega = \{\{x_1\}, \dots, \{x_n\}\}$. On the other, from our assumption we get that $\Lambda_\Omega \in \Sigma$. Therefore, $\Lambda_\Omega \in \Sigma(\Lambda)$, and so $\Sigma(\Lambda) \neq \emptyset$.

Since the set Ω is a finite set, then $\Sigma(\Lambda)$ is finite. Without loss of generality, we may assume that $\Sigma(\Lambda) = \{\Lambda_1, \dots, \Lambda_r, \dots, \Lambda_m\}$, where $1 \leq r \leq m$ is such that $\Lambda_1, \dots, \Lambda_r$ are the minimal elements of the poset $(\Sigma(\Lambda), \leq)$. Let us denote by Λ_0 the clutter $\Lambda_0 = \min(\{A_1 \cup \dots \cup A_r : A_i \in \Lambda_i\})$. It is necessary to demonstrate the equality $\Lambda = \Lambda_0$. Observe that by using this equality it is easy to prove that $\Lambda \in \Sigma$ if and only if $r = 1$.

So, from now on we are going to prove the equality $\Lambda = \Lambda_0$. In order to do this we use that the binary relation \leq is a partial order (see Lemma 1). Namely, we are going to prove the equality $\Lambda = \Lambda_0$ by proving the two inequalities $\Lambda \leq \Lambda_0$ and $\Lambda_0 \leq \Lambda$.

Let $1 \leq i \leq r$. Since $\Lambda \leq \Lambda_i$, for $A \in \Lambda$, there exist $A_i \in \Lambda_i$ such that $A_i \subseteq A$. Therefore, we obtain that $A_1 \cup \dots \cup A_r \subseteq A$, and hence it follows that $\Lambda \leq \min(\{A_1 \cup \dots \cup A_r : A_i \in \Lambda_i\})$; that is, $\Lambda \leq \Lambda_0$. Therefore, to finish the proof of the proposition we must demonstrate that $\Lambda_0 \leq \Lambda$.

In order to do this, let us consider the clutter Λ'_0 on Ω defined by all the elements of $\Sigma(\Lambda) = \{\Lambda_1, \dots, \Lambda_r, \dots, \Lambda_m\}$, that is, let Λ'_0 be the clutter $\Lambda'_0 = \min(\{A_1 \cup \dots \cup A_m : A_i \in \Lambda_i\})$.

We claim that $\Lambda'_0 = \Lambda_0$. Let us prove our claim. It is clear that if $\{\Lambda_{i_1}, \dots, \Lambda_{i_s}\} \subseteq \{\Lambda_1, \dots, \Lambda_m\}$, then $\Lambda'_0 \leq \min(\{A_{i_1} \cup \dots \cup A_{i_s} : A_{i_j} \in \Lambda_{i_j}\})$. In particular, we obtain that $\Lambda'_0 \leq \Lambda_0$. Next we are going to prove that $\Lambda_0 \leq \Lambda'_0$. So let $A_1 \cup \dots \cup A_r \in \Lambda_0$. Since $\Lambda_1, \dots, \Lambda_r$ are the minimal elements of the poset $(\Sigma(\Lambda), \leq)$, for $j > r$ there exists $\alpha_j \leq r$ such that $\Lambda_{\alpha_j} \leq \Lambda_j$. Therefore, there exists $A'_j \in \Lambda_j$ such that $A'_j \subseteq A_{\alpha_j}$. So we have that $A_1 \cup \dots \cup A_r \cup A'_{r+1} \cup \dots \cup A'_m \subseteq A_1 \cup \dots \cup A_r$, and hence it follows that there exists $C \in \Lambda'_0$ such that $C \subseteq A_1 \cup \dots \cup A_r$. Therefore, by Lemma 1, $\Lambda_0 \leq \Lambda'_0$. This completes the proof of our claim.

Let us consider the set of subsets $\{X_1, \dots, X_t\} = \{X \subseteq \Omega : \Lambda \leq \Lambda_X\}$, (observe that this set is non-empty and so $t \geq 1$ because $\Lambda \leq \Lambda_\Omega$). By assumption $\Lambda_{X_1}, \dots, \Lambda_{X_t} \in \Sigma$. So $\{\Lambda_{X_1}, \dots, \Lambda_{X_t}\} \subseteq \Sigma(\Lambda) = \{\Lambda_1, \dots, \Lambda_m\}$, and hence it follows that $\Lambda_0 = \Lambda'_0 \leq \min(\{A_{X_1} \cup \dots \cup A_{X_t} : A_{X_j} \in \Lambda_{X_j}\})$.

Now the proof of the proposition is completed by showing the inequality $\min(\{A_{X_1} \cup \dots \cup A_{X_t} : A_{X_j} \in \Lambda_{X_j}\}) \leq \Lambda$; that is, we must demonstrate that if $C \in \min(\{A_{X_1} \cup \dots \cup A_{X_t} : A_{X_j} \in \Lambda_{X_j}\})$ then there exists $A \in \Lambda$ such that $A \subseteq C$ (see Lemma 1). So let $C \in \min(\{A_{X_1} \cup \dots \cup A_{X_t} : A_{X_j} \in \Lambda_{X_j}\})$. Then $C = \{x_{\alpha_1}, \dots, x_{\alpha_t}\}$ where $x_{\alpha_j} \in X_j$ for $1 \leq j \leq t$. Assume that $A \not\subseteq C$ for all $A \in \Lambda$. Therefore, if $A \in \Lambda$, then $A \cap (\Omega \setminus C) \neq \emptyset$, and so there exists $x_0 \in \Omega \setminus C$ such that $\{x_0\} \subseteq A$. Hence, by applying Lemma 1 it follows that $\Lambda \leq \Lambda_{\Omega \setminus C}$. So $\Omega \setminus C \in \{X \subseteq \Omega : \Lambda \leq \Lambda_X\}$ and thus $\Omega \setminus C = X_{i_0}$ for a certain $i_0 \in \{1, \dots, t\}$. This leads to a contradiction because $x_{\alpha_{i_0}} \in C \cap X_{i_0}$. Therefore, there exists $A \in \Lambda$ such that $A \subseteq C$. This completes the proof of the proposition. \square

Theorem 4. *Let Λ be a clutter on a finite set Ω . Then, $\text{Vect}(\Lambda) \neq \emptyset$ and $\Lambda = \min(\{A_1 \cup \dots \cup A_r : A_i \in \Lambda_i\})$ where $\Lambda_1, \dots, \Lambda_r$ are the minimal elements of the poset $(\text{Vect}(\Lambda), \leq)$ of the vectorial completions of Λ . In particular, the clutter Λ has a unique minimal vectorial completion if, and only if, the clutter Λ is a vectorial clutter.*

Proof Observe that $\text{Vect}(\Lambda) = \Sigma(\Lambda)$ where $\Sigma \subseteq \text{Clutt}(\Omega)$ is the collection of the vectorial clutters on Ω . Therefore, from Proposition 3, we only must prove that $\Lambda_X \in \Sigma$ if $\emptyset \subsetneq X \subseteq \Omega$; that is, we only must demonstrate that the clutter Λ_X is a vectorial clutter on the finite set Ω if X is a non-empty subset of Ω .

Let $n = |\Omega|$ and let $\Omega = \{x_1, \dots, x_n\}$. Let $\emptyset \subsetneq X \subseteq \Omega$. Without loss of generality, we may assume that $X = \{x_1, \dots, x_r\}$ where $1 \leq r \leq n$. Let \mathbb{K} be a field, and let E be a \mathbb{K} -vector space having

dimension $\dim E \geq n - r$. Let us consider an indexed family of vectors $v_1, \dots, v_n \in E$, where $v_i = 0$ if $1 \leq i \leq r$ and where v_{r+1}, \dots, v_n are linearly independent. Then, it is easy to check that the vectors v_1, \dots, v_n provide a \mathbb{K} -representation of Λ_X . So, Λ_X is a vectorial clutter on Ω . \square

3.2. Completion and decomposition with field restrictions. Observe that the previous theorem, Theorem 4, deals with vectorial completions and decompositions in the case where no field restrictions are assumed. The next theorems, Theorem 5 and Theorem 6, state that similar results occur if we consider only the case in which the vector spaces of the vectorial completions are either over a fixed field or over fields with a specific characteristic. Before stating these theorems, we introduce some notations.

Let \mathbb{K} be a field and let p be a prime number. A vectorial clutter is said to be \mathbb{K} -vectorial if admits a \mathbb{K} -representation, and is said to be p -vectorial if it has an \mathbb{L} -representation for some field \mathbb{L} of characteristic p . For a clutter Λ , let us denote by $\text{Vect}_{\mathbb{K}}(\Lambda)$ the set whose elements are the \mathbb{K} -vectorial completions of Λ , and by $\text{Vect}_p(\Lambda)$ the set whose elements are the p -vectorial completions of Λ ; that is, the elements of $\text{Vect}_{\mathbb{K}}(\Lambda)$ are the \mathbb{K} -vectorial clutters Λ' with $\Lambda \leq \Lambda'$, while the elements of $\text{Vect}_p(\Lambda)$ are the p -vectorial clutters Λ' with $\Lambda \leq \Lambda'$. Observe that $\text{Vect}(\Lambda) = \bigcup_{\mathbb{K}} \text{Vect}_{\mathbb{K}}(\Lambda)$ and that $\text{Vect}(\Lambda) = \bigcup_p \text{Vect}_p(\Lambda)$.

The next theorems state that the sets $\text{Vect}_{\mathbb{K}}(\Lambda)$ and $\text{Vect}_p(\Lambda)$ are non-empty sets and that their minimal elements provide a decomposition of the clutter Λ (in the sense that the elements of Λ can be obtained from these minimal vectorial completions).

Theorem 5. *Let Λ be a clutter on a finite set Ω and let \mathbb{K} be a field. Then, $\text{Vect}_{\mathbb{K}}(\Lambda) \neq \emptyset$ and $\Lambda = \min(\{A_1 \cup \dots \cup A_r : A_i \in \Lambda_i\})$ where $\Lambda_1, \dots, \Lambda_r$ are the minimal elements of the poset $(\text{Vect}_{\mathbb{K}}(\Lambda), \leq)$. In particular, the clutter Λ has a unique minimal vectorial completion over \mathbb{K} if, and only if, the clutter Λ is a \mathbb{K} -vectorial clutter.*

Proof Essentially, the proof of this theorem works like the previous one. We must only take into account that, $\text{Vect}_{\mathbb{K}}(\Lambda) = \Sigma(\Lambda)$ where $\Sigma \subseteq \text{Clutt}(\Omega)$ is the collection of the \mathbb{K} -vectorial clutters on Ω ; and that, for a non-empty subset $X \subseteq \Omega$, the clutter Λ_X is a \mathbb{K} -vectorial clutter. \square

Theorem 6. *Let Λ be a clutter on a finite set Ω and let p be a prime number. Then, $\text{Vect}_p(\Lambda) \neq \emptyset$ and $\Lambda = \min(\{A_1 \cup \dots \cup A_r : A_i \in \Lambda_i\})$ where $\Lambda_1, \dots, \Lambda_r$ are the minimal elements of the poset $(\text{Vect}_p(\Lambda), \leq)$.*

In particular, the clutter Λ has a unique minimal p -vectorial completion if, and only if, the clutter Λ is a p -vectorial clutter.

Proof As before, the proof of this theorem works like the one of Theorem 4. Now we only must bear in mind that, $\text{Vect}_p(\Lambda) = \Sigma(\Lambda)$ where $\Sigma \subseteq \text{Clutt}(\Omega)$ is the collection of the p -vectorial clutters on Ω ; and that, for a non-empty subset $X \subseteq \Omega$, the clutter Λ_X is a p -vectorial clutter. \square

3.3. Completion and decomposition of matroids into representable matroids. Matroids are combinatorial objects that can be axiomatized in terms of their independent sets, bases, circuits, rank function, flats, or hyperplanes (the reader is referred to [4, 7] for general references on matroid theory). Here we present the definition in terms of circuits.

A *matroid* \mathcal{M} is an ordered pair $\mathcal{M} = (\Omega, \mathcal{C})$ consisting of a finite set Ω , called the *ground set* of the matroid, and a clutter \mathcal{C} of non-empty subsets of Ω which satisfies the *weak circuit elimination property*: if C_1 and C_2 are distinct members of \mathcal{C} and $x \in C_1 \cap C_2$, then there is some member C_3 of \mathcal{C} such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{x\}$. The members of the clutter \mathcal{C} are the *circuits* of the matroid \mathcal{M} . We shall often write $\mathcal{C}(\mathcal{M})$ instead of \mathcal{C} . The *dependent sets* of the matroid are the supersets of the circuits, that is, the dependent sets of \mathcal{M} are the members of $\mathcal{C}(\mathcal{M})^+$. Sets that are not dependent are called *independent*.

Observe that since the set of circuits of a matroid is a clutter on the ground set of the matroid, we can consider the partial order induced by \leq on the set of matroids with ground set Ω . Thereby, if \mathcal{M}_1 and \mathcal{M}_2 are two matroids with ground set Ω , then we say that $\mathcal{M}_1 \leq \mathcal{M}_2$ if and only if $\mathcal{C}(\mathcal{M}_1) \leq \mathcal{C}(\mathcal{M}_2)$ where $\mathcal{C}(\mathcal{M}_i)$ is the clutter of the circuits of \mathcal{M}_i . So, $\mathcal{M}_1 \leq \mathcal{M}_2$ if and only if every circuit of \mathcal{M}_1 contains a circuit of \mathcal{M}_2 . In matroid theory this is equivalent to saying that the identity map on Ω is a *weak map* from the matroid \mathcal{M}_1 to the matroid \mathcal{M}_2 (see [4, Proposition 7.3.11]); that is, $\mathcal{M}_1 \leq \mathcal{M}_2$ if \mathcal{M}_1 is above \mathcal{M}_2 in the *weak order*.

A matrix A with entries in a field \mathbb{K} gives rise to a matroid \mathcal{M}_A on its set of columns. The dependent sets of the matroid \mathcal{M}_A are those sets of columns of the matrix A that are linearly dependent as sets of vectors. This matroid is called the *column matroid* of A , and the matrix A is said to represent the matroid. A matroid \mathcal{M} is called *representable* over a field \mathbb{K} if there exists some matrix A with entries in the field \mathbb{K} such that $\mathcal{M} = \mathcal{M}_A$. Therefore, a matroid \mathcal{M} is representable if and only if the clutter $\mathcal{C}(\mathcal{M})$ is vectorial. In addition, a matroid \mathcal{M} is

\mathbb{K} -representable (resp. p -representable) if and only if the clutter $\mathcal{C}(\mathcal{M})$ is \mathbb{K} -representable (resp. p -representable).

In any case, for a given matroid \mathcal{M} , we can now consider its vectorial completions; that is, those representable matroids \mathcal{M}' with $\mathcal{M} \leq \mathcal{M}'$. We shall often write $\text{Vect}(\mathcal{M})$, $\text{Vect}_{\mathbb{K}}(\mathcal{M})$ and $\text{Vect}_p(\mathcal{M})$ instead of $\text{Vect}(\mathcal{C}(\mathcal{M}))$, $\text{Vect}_{\mathbb{K}}(\mathcal{C}(\mathcal{M}))$ and $\text{Vect}_p(\mathcal{C}(\mathcal{M}))$. The following result states that all these three sets of representable matroidal completions of \mathcal{M} are non-empty, and that their minimal elements provide a decomposition of the matroid \mathcal{M} .

Corollary 7. *Let \mathcal{M} be a matroid with ground set Ω . Let \mathbb{K} be a field and let p be a prime number. Then:*

- (1) $\text{Vect}(\mathcal{M}) \neq \emptyset$ and $\mathcal{C}(\mathcal{M}) = \min(\{A_1 \cup \dots \cup A_r : A_i \in \mathcal{C}(\mathcal{M}_i)\})$ where $\mathcal{M}_1, \dots, \mathcal{M}_r$ are the minimal elements of $(\text{Vect}(\mathcal{M}), \leq)$. In particular, the matroid \mathcal{M} has a unique minimal vectorial completion if, and only if, the matroid \mathcal{M} is representable.
- (2) $\text{Vect}_{\mathbb{K}}(\mathcal{M}) \neq \emptyset$ and $\mathcal{C}(\mathcal{M}) = \min(\{A_1 \cup \dots \cup A_r : A_i \in \mathcal{C}(\mathcal{M}_i)\})$ where $\mathcal{M}_1, \dots, \mathcal{M}_r$ are the minimal elements of $(\text{Vect}_{\mathbb{K}}(\mathcal{M}), \leq)$. In particular, the matroid \mathcal{M} has a unique minimal \mathbb{K} -vectorial completion if, and only if, the matroid \mathcal{M} is \mathbb{K} -representable.
- (3) $\text{Vect}_p(\mathcal{M}) \neq \emptyset$ and $\mathcal{C}(\mathcal{M}) = \min(\{A_1 \cup \dots \cup A_r : A_i \in \mathcal{C}(\mathcal{M}_i)\})$ where $\mathcal{M}_1, \dots, \mathcal{M}_r$ are the minimal elements of $(\text{Vect}_p(\mathcal{M}), \leq)$. In particular, the matroid \mathcal{M} has a unique minimal p -vectorial completion if, and only if, the matroid \mathcal{M} is p -representable.

Proof The three statements of the corollary are specializations of the previous theorems. \square

Remark 8. *In this paper we focus on vectorial completions and decompositions, but we could have considered completions and decompositions in some other families of clutters, as long as they include the clutters Λ_X . Indeed, a proof analogous to that of Theorem 4 would give the desired completions and decompositions. For matroids, the ones whose clutter of circuits is of the form Λ_X are the all whose circuits have size 1 or, in other words, the matroids that can be written as a direct sum of loops and coloops. Most of the familiar classes of matroids contain them, as graphic, cographic, regular, algebraic, transversal and cotransversal matroids. Therefore, similar results can be obtained concerning completions and decomposition of matroids into graphic, cographic, regular, algebraic, transversal and cotransversal matroids.*

4. COMPUTING MINIMAL VECTORIAL COMPLETIONS

This section is devoted to the computation of the minimal vectorial completions and decomposition of clutters. The problems under consideration are far from being solved. However, here we present some partial results.

4.1. Vectorial completions of clutters on a finite set of size at most seven. In this subsection we provide a method to obtain the vectorial completions and decompositions of clutters on a finite set of size at most seven (Proposition 9). Our result involves two transformations of clutters: the \mathcal{I} -transformation and the \mathcal{T} -transformation. Both transformations were introduced in [2]. Let us recall them.

Let Λ be a clutter on a finite set Ω . For a subset $X \subseteq \Omega$, we denote by $I_\Lambda(X)$ the intersection of the subsets A in Λ contained in X , that is, $I_\Lambda(X) = \bigcap_A A$ where $A \in \Lambda$ and $A \subseteq X$. We say that a clutter Λ' is an \mathcal{I} -transformation of the clutter Λ if $\Lambda' = \min(\Lambda \cup \{A_1 \cap A_2\})$ where $A_1, A_2 \in \Lambda$ are two different subsets with $I_\Lambda(A_1 \cup A_2) \neq \emptyset$.

The definition of the \mathcal{T} -transformation is more involved. Let Λ be a clutter. We define the elementary transformations $\mathcal{T}^{(1)}(\Lambda)$ and $\mathcal{T}^{(2)}(\Lambda)$ of the clutter Λ as the clutters $\mathcal{T}^{(1)}(\Lambda) = \min(\Lambda \cup \{(A_1 \cup A_2) \setminus \{x\}, \text{ where } A_1, A_2 \in \Lambda \text{ are different and } x \in A_1 \cap A_2\})$ and $\mathcal{T}^{(2)}(\Lambda) = \min(\Lambda \cup \{(A_1 \cup A_2) \setminus I_\Lambda(A_1 \cup A_2), \text{ where } A_1, A_2 \in \Lambda \text{ are different}\})$. Since $\mathcal{T}^{(1)}(\Lambda)$ and $\mathcal{T}^{(2)}(\Lambda)$ are clutters, we can apply the elementary transformations again. Hence, for $(i_1, i_2) \in \{1, 2\} \times \{1, 2\}$ we can consider the clutter $\mathcal{T}^{(i_2)}(\mathcal{T}^{(i_1)}(\Lambda))$. At this point we proceed in a recursive way. Let $r \geq 2$ be a non-negative integer and let $(i_1, \dots, i_r) \in \{1, 2\}^r$ be an r -tuple. Then we define the clutter $\mathcal{T}^{(i_1, \dots, i_r)}(\Lambda)$ by the recursion formula $\mathcal{T}^{(i_1, \dots, i_r)}(\Lambda) = \mathcal{T}^{(i_r)}(\mathcal{T}^{(i_1, \dots, i_{r-1})}(\Lambda))$; that is, $\mathcal{T}^{(i_1, \dots, i_r)}(\Lambda)$ is the elementary transformation $\mathcal{T}^{(i_r)}$ of the clutter $\mathcal{T}^{(i_1, \dots, i_{r-1})}(\Lambda)$. We say that a clutter Λ' is a \mathcal{T} -transformation of the clutter Λ if it is obtained from Λ in this way, that is, if $\Lambda' = \mathcal{T}^{(i_1, \dots, i_r)}(\Lambda)$ for some r -tuple (i_1, \dots, i_r) .

We say that a clutter Λ' is an $(\mathcal{I}, \mathcal{T})$ -transformation of the clutter Λ if Λ' can be obtained from Λ by applying successively \mathcal{I} -transformations or \mathcal{T} -transformations; that is, if there exists a sequence of clutters

$$\Lambda = \Lambda'_0 \rightsquigarrow \Lambda'_1 \rightsquigarrow \dots \rightsquigarrow \Lambda'_r = \Lambda'$$

such that for $i \geq 1$, either Λ'_i is an \mathcal{I} -transformation of Λ'_{i-1} , or Λ'_i is a \mathcal{T} -transformation of Λ'_{i-1} . It is easy to check that if Λ'_i is

an \mathcal{I} -transformation of Λ'_{i-1} then $\Lambda'_{i-1} \leq \Lambda'_i$; the same holds for \mathcal{T} -transformations (see [2, Lemma 7]). Therefore, since \leq is a partial order, if Λ' is an $(\mathcal{I}, \mathcal{T})$ -transformation of Λ , then $\Lambda \leq \Lambda'$.

It is clear that if Λ' is an $(\mathcal{I}, \mathcal{T})$ -transformation of Λ and if Λ'' is an $(\mathcal{I}, \mathcal{T})$ -transformation of Λ' , then Λ'' is an increased $(\mathcal{I}, \mathcal{T})$ -transformation of Λ . A *complete $(\mathcal{I}, \mathcal{T})$ -transformation* of the clutter Λ is an $(\mathcal{I}, \mathcal{T})$ -transformation Λ' of Λ which cannot be increased; that is, an $(\mathcal{I}, \mathcal{T})$ -transformation Λ' of Λ such that Λ' is the unique clutter which can be obtained from Λ by applying \mathcal{I} -transformations or \mathcal{T} -transformations.

The next result states that by using the complete $(\mathcal{I}, \mathcal{T})$ -transformations it is possible to compute all the minimal vectorial completions of a clutter Λ on a finite set Ω of size at most seven.

Proposition 9. *Let Λ be a clutter on a finite set Ω of size $|\Omega| \leq 7$. Then, the minimal vectorial completions of the clutter Λ are the minimal complete $(\mathcal{I}, \mathcal{T})$ -transformations of Λ ; that is, $\min(\text{Vect}(\Lambda)) = \min(\{\Lambda_1, \dots, \Lambda_s\})$ where $\Lambda_1, \dots, \Lambda_s$ are all the complete $(\mathcal{I}, \mathcal{T})$ -transformations of Λ .*

Proof Since $\Lambda_1, \dots, \Lambda_s$ are complete $(\mathcal{I}, \mathcal{T})$ -transformations of Λ , from the discussion above it follows that $\Lambda \leq \Lambda_i$ for $1 \leq i \leq s$.

Next we are going to prove that $\{\Lambda_1, \dots, \Lambda_s\} \subseteq \text{Vect}(\Lambda)$. It is known that if \mathcal{M} is a matroid having fewer than eight elements, then \mathcal{M} is representable (see [4, Proposition 6.4.10]). Therefore, a clutter Λ' on Ω is a vectorial clutter if, and only if, the clutter Λ' is the set of circuits of a matroid \mathcal{M} with ground set Ω . Hence, by applying [2, Proposition 5 and Proposition 8] we get that a clutter Λ' is a vectorial clutter if, and only if, Λ' is the unique clutter which can be obtained from Λ by applying \mathcal{I} -transformations or \mathcal{T} -transformations. Thus it follows that the complete $(\mathcal{I}, \mathcal{T})$ -transformations of Λ are vectorial clutters, that is, $\{\Lambda_1, \dots, \Lambda_s\} \subseteq \text{Vect}(\Lambda)$.

Finally let us show that $\min(\text{Vect}(\Lambda)) = \min(\{\Lambda_1, \dots, \Lambda_s\})$. Recall that since the set Ω has size $|\Omega| \leq 7$, a clutter Λ' on Ω is a vectorial clutter if and only if Λ' is the set of circuits of a matroid \mathcal{M} with ground set Ω . Hence, from [2, Theorem 13] we get that if Λ' is a minimal element of the poset $(\text{Vect}(\Lambda), \leq)$, then there exists a monotone increasing sequence of clutters $\Lambda = \Lambda'_0 \not\geq \Lambda'_1 \not\geq \dots \not\geq \Lambda'_r = \Lambda'$ such that for $i \geq 1$, either Λ'_i is an \mathcal{I} -transformation of Λ'_{i-1} , or Λ'_i is a \mathcal{T} -transformation of Λ'_{i-1} . Hence it follows that $\min(\text{Vect}(\Lambda)) \subseteq \{\Lambda_1, \dots, \Lambda_s\}$. Since $\{\Lambda_1, \dots, \Lambda_s\} \subseteq \text{Vect}(\Lambda)$, the equality $\min(\text{Vect}(\Lambda)) = \min(\{\Lambda_1, \dots, \Lambda_s\})$ holds. \square

We now give two examples to illustrate the above proposition.

Example 10. First let us consider the clutter $\Lambda = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}\}$ on the finite set $\Omega = \{1, 2, 3, 4\}$. In this case only two clutters are obtained by using or by combining \mathcal{I} -transformations and \mathcal{T} -transformations; namely, the clutters $\Lambda_1 = \{\{1\}, \{2, 3, 4\}\}$ and $\Lambda_2 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. Therefore Λ has only two complete $(\mathcal{I}, \mathcal{T})$ -transformations, and so from Proposition 9 it follows that the minimal vectorial completions of the clutter Λ are the minimal elements of $\{\Lambda_1, \Lambda_2\}$. In this case, $\Lambda_1 \not\leq \Lambda_2$ and $\Lambda_2 \not\leq \Lambda_1$, and so $\min(\text{Vect}(\Lambda)) = \{\Lambda_1, \Lambda_2\}$. Observe that now the vectorial decomposition of Λ given in Example 2 can be stated by applying Theorem 4.

Example 11. Now, on the finite set $\Omega = \{1, 2, 3, 4, 5, 6\}$, we consider the clutter $\Lambda = \{\{1, 2, 3\}, \{1, 4, 5\}, \{2, 3, 6\}, \{4, 5, 6\}, \{2, 3, 4, 5\}\}$ (see [8, Example 4.7] for some properties of this clutter). It is a straightforward calculation to check that Λ has seventeen complete $(\mathcal{I}, \mathcal{T})$ -transformations $\Lambda_1, \dots, \Lambda_{17}$. Specifically, by using only \mathcal{I} -transformations we obtain the clutter

$$\Lambda_1 = \{\{2, 3\}, \{4, 5\}\};$$

while the clutters obtained by using only \mathcal{T} -transformations are the clutters

$$\begin{cases} \Lambda_2 = \{\{1, 6\}, \{1, 2, 3\}, \{1, 4, 5\}, \{2, 3, 6\}, \{4, 5, 6\}, \{2, 3, 4, 5\}\}, \\ \Lambda_3 = \{A \subseteq \Omega : |A| = 3\}, \text{ and} \\ \Lambda_4 = \{A \subseteq \{2, 3, 4, 5\} : |A| = 2\} \cup \{\{1, 2, 6\}, \{1, 3, 6\}, \{1, 4, 6\}, \{1, 5, 6\}\}; \end{cases}$$

whereas the new clutters obtained by combining \mathcal{I} -transformations and \mathcal{T} -transformations are the clutters $\Lambda_5, \dots, \Lambda_{17}$, where

$$\left\{ \begin{array}{l} \Lambda_5 = \{\{2, 3\}, \{1, 4, 5\}, \{1, 4, 6\}, \{1, 5, 6\}, \{4, 5, 6\}\}, \\ \Lambda_6 = \{\{2, 3\}, \{1, 6\}, \{1, 4, 5\}, \{4, 5, 6\}\}, \\ \Lambda_7 = \{\{4, 5\}, \{1, 2, 3\}, \{1, 2, 6\}, \{1, 3, 6\}, \{2, 3, 6\}\}, \\ \Lambda_8 = \{\{4, 5\}, \{1, 6\}, \{1, 2, 3\}, \{2, 3, 6\}\}, \\ \Lambda_9 = \{\{1\}, \{6\}, \{2, 3, 4, 5\}\}, \\ \Lambda_{10} = \{\{1\}\} \cup \{A \subseteq \{2, 3, 4, 5, 6\} : |A| = 3\}, \\ \Lambda_{11} = \{\{6\}\} \cup \{A \subseteq \{1, 2, 3, 4, 5\} : |A| = 3\}, \\ \Lambda_{12} = \{\{2\}, \{3\}, \{1, 4, 5\}, \{1, 4, 6\}, \{1, 5, 6\}, \{4, 5, 6\}\}, \\ \Lambda_{13} = \{\{2\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{1, 3, 6\}, \{1, 4, 6\}, \{1, 5, 6\}\}, \\ \Lambda_{14} = \{\{3\}, \{2, 4\}, \{2, 5\}, \{4, 5\}, \{1, 2, 6\}, \{1, 4, 6\}, \{1, 5, 6\}\}, \\ \Lambda_{15} = \{\{4\}, \{5\}, \{1, 2, 3\}, \{1, 2, 6\}, \{1, 3, 6\}, \{2, 3, 6\}\}, \\ \Lambda_{16} = \{\{4\}, \{2, 3\}, \{2, 5\}, \{3, 5\}, \{1, 2, 6\}, \{1, 3, 6\}, \{1, 5, 6\}\}, \text{ and} \\ \Lambda_{17} = \{\{5\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 6\}, \{1, 3, 6\}, \{1, 4, 6\}\}. \end{array} \right.$$

Therefore, by applying Proposition 9 we obtain that the set of the minimal vectorial completions of Λ is $\min(\text{Vect}(\Lambda)) = \min(\{\Lambda_1, \dots, \Lambda_{17}\}) = \{\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_5, \Lambda_7\}$. So, from Theorem 4 we conclude that Λ admits a vectorial decomposition with five components. We remark that the clutters Λ_1 and Λ_2 alone already give a vectorial decomposition of Λ . It is though not always the case that such decompositions exist with two components; for instance, Example 15 from [2] needs at least three components.

4.2. Binary completions and decomposition of non-binary matroids. A *binary matroid* is one that is representable over the finite field $\mathbb{Z}/(2)$. The goal of this subsection is to compute the minimal binary completions of the excluded minor of binary matroids.

The *uniform matroid* $\mathcal{U}_{2,4}$ is the matroid on a ground set Ω such that $|\Omega| = 4$ and with set of circuits $\mathcal{C}(\mathcal{U}_{2,4}) = \{C \subseteq \Omega : |C| = 3\}$. It is well known that the uniform matroid $\mathcal{U}_{2,4}$ is the unique excluded minor for $\mathbb{Z}/(2)$ -representability (see [4, Theorem 9.1.5]).

The uniform matroid $\mathcal{U}_{2,4}$ is \mathbb{K} -representable if and only if $\mathbb{K} \neq \mathbb{Z}/(2)$ (see [4, Proposition 6.5.2]). Therefore, by applying Corollary 7 we obtain that $\min(\text{Vect}(\mathcal{U}_{2,4})) = \{\mathcal{U}_{2,4}\}$; that $\min(\text{Vect}_{\mathbb{K}}(\mathcal{U}_{2,4})) = \{\mathcal{U}_{2,4}\}$ if $\mathbb{K} \neq \mathbb{Z}/(2)$, and that $\min(\text{Vect}_{\mathbb{Z}/(2)}(\mathcal{U}_{2,4}))$ has at least two elements. Our goal is to compute the minimal $\mathbb{Z}/(2)$ -vectorial completions of $\mathcal{U}_{2,4}$.

The following proposition states that the poset $(\text{Vect}_{\mathbb{Z}/(2)}(\mathcal{U}_{2,4}), \leq)$ has six minimal elements.

Proposition 12. *The minimal binary completions of the uniform matroid $\mathcal{U}_{2,4}$ on $\Omega = \{1, 2, 3, 4\}$ are the matroids $\mathcal{M}_{1,2}, \mathcal{M}_{1,3}, \mathcal{M}_{1,4}, \mathcal{M}_{2,3}, \mathcal{M}_{2,4}, \mathcal{M}_{3,4}$ where, if $1 \leq i_1 < i_2 \leq 4$, and if $\{i_3, i_4\} = \{1, 2, 3, 4\} \setminus \{i_1, i_2\}$, then \mathcal{M}_{i_1, i_2} is the matroid with ground set Ω and set of circuits $\mathcal{C}(\mathcal{M}_{i_1, i_2}) = \{\{i_1, i_2\}, \{i_1, i_3, i_4\}, \{i_2, i_3, i_4\}\}$.*

Proof The set of circuits of the uniform matroid $\mathcal{U}_{2,4}$ is $\mathcal{C}(\mathcal{U}_{2,4}) = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$. So $\mathcal{U}_{2,4} \leq \mathcal{M}_{i_1, i_2}$ for $1 \leq i_1 < i_2 \leq 4$. Moreover, the matroid \mathcal{M}_{i_1, i_2} is binary (indeed the vectors $v_{i_1} = v_{i_2} = (1, 0)$, $v_{i_3} = (1, 1)$ and $v_{i_4} = (0, 1)$ provide a $\mathbb{Z}/(2)$ -representation of \mathcal{M}_{i_1, i_2}). Therefore we have that $\{\mathcal{M}_{1,2}, \mathcal{M}_{1,3}, \mathcal{M}_{1,4}, \mathcal{M}_{2,3}, \mathcal{M}_{2,4}, \mathcal{M}_{3,4}\} \subseteq \text{Vect}_{\mathbb{Z}/(2)}(\mathcal{U}_{2,4})$. Observe that $\mathcal{M}_{i_1, i_2} \not\leq \mathcal{M}_{i'_1, i'_2}$ if $(i_1, i_2) \neq (i'_1, i'_2)$. To complete the proof we just need to show that if \mathcal{M} is a binary completion of $\mathcal{U}_{2,4}$ then there exist $1 \leq i_1 < i_2 \leq 4$ such that $\mathcal{M}_{i_1, i_2} \leq \mathcal{M}$.

Let \mathcal{M} be binary completion of $\mathcal{U}_{2,4}$. As $\mathcal{U}_{2,4} \leq \mathcal{M}$, any set of 3 elements is dependent in \mathcal{M} . If every 3-element set of Ω were a circuit of \mathcal{M} , then $\{C \subseteq \Omega : |C| = 3\} \subseteq \mathcal{C}(\mathcal{M})$, hence $\{C \subseteq \Omega : |C| = 3\} = \mathcal{C}(\mathcal{M})$ and so $\mathcal{U}_{2,4} = \mathcal{M}$, which leads us to a contradiction because the matroid \mathcal{M} is binary. Therefore $\{C \subseteq \Omega : |C| = 3\} \subseteq \mathcal{C}(\mathcal{M})^+$ and $\{C \subseteq \Omega : |C| = 3\} \not\subseteq \mathcal{C}(\mathcal{M})$, and so there exists a 2-element set of Ω which is dependent in \mathcal{M} . By symmetry, we can assume that $\{1, 2\}$ is dependent in \mathcal{M} . Then it is straightforward to check that $\mathcal{M}_{1,2} \leq \mathcal{M}$, as needed. \square

Remark 13. *Observe that from this result, and by applying Corollary 7, it follows that the non-binary matroid $\mathcal{U}_{2,4}$ admits a $\mathbb{Z}/(2)$ -vectorial decomposition with six components. However, in this case the decomposition can be achieved with only two components. More concretely, let \mathcal{N}_1 and \mathcal{N}_2 be any two of the minimal binary completions of $\mathcal{U}_{2,4}$. Then it is easy to check that $\mathcal{C}(\mathcal{U}_{2,4}) = \min(\{C_1 \cup C_2 : C_i \in \mathcal{C}(\mathcal{N}_i)\})$.*

4.3. Completions and decomposition of the non-Fano matroid.

In this subsection we describe the minimal 2-vectorial completions of the non-Fano matroid, which is one of the excluded minors of representable matroids over a field of characteristic two (see [4, Proposition 6.5.6]).

Let Ω be the finite set of seven points $\Omega = \{1, 2, 3, 4, 5, 6, 7\}$. The Fano matroid F_7 and the non-Fano matroid F_7^- are the matroids with

ground set Ω and sets of circuits

$$\mathcal{C}(F_7) = \{\{1, 2, 3\}, \{1, 4, 7\}, \{1, 5, 6\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 5\}, \{3, 6, 7\}, \\ \{1, 2, 4, 5\}, \{1, 2, 6, 7\}, \{1, 3, 4, 6\}, \{1, 3, 5, 7\}, \\ \{2, 3, 4, 7\}, \{2, 3, 5, 6\}, \{4, 5, 6, 7\}\};$$

$$\mathcal{C}(F_7^-) = \{\{1, 2, 3\}, \{1, 4, 7\}, \{1, 5, 6\}, \{2, 5, 7\}, \{3, 4, 5\}, \{3, 6, 7\}, \\ \{1, 2, 4, 5\}, \{1, 2, 4, 6\}, \{1, 2, 6, 7\}, \{1, 3, 4, 6\}, \{1, 3, 5, 7\}, \\ \{2, 3, 4, 6\}, \{2, 3, 4, 7\}, \{2, 3, 5, 6\}, \\ \{2, 4, 5, 6\}, \{2, 4, 6, 7\}, \{4, 5, 6, 7\}\}.$$

Geometric representations of these two matroids are given in Figure 1.

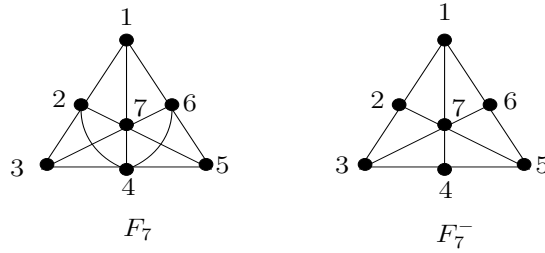


FIGURE 1. Geometric representations of F_7 and F_7^- . The circuits correspond to sets of 3 collinear elements and sets of 4 elements no 3 of them being collinear.

These matroids are representable matroids. Namely F_7 is \mathbb{K} -representable if and only if the characteristic of \mathbb{K} is two; while F_7^- is \mathbb{K} -representable if and only if the characteristic of \mathbb{K} is not two (see [4, Proposition 6.4.8]). Moreover, the non-Fano matroid F_7^- is an excluded minor of 2-representable matroids (see [4, Proposition 6.5.6]). Therefore, by applying Corollary 7 we obtain that $\min(\text{Vect}(F_7^-)) = \{F_7^-\}$; that $\min(\text{Vect}_p(F_7^-)) = \{F_7^-\}$ if $p \neq 2$ is a prime integer; and that $\min(\text{Vect}_2(F_7^-))$ has at least two elements. The next proposition states that $(\text{Vect}_2(F_7^-), \leq)$ has nine minimal elements. The matroids in $\min(\text{Vect}_2(F_7^-))$, except F_7 and up to isomorphism, are depicted in Figure 2.

Proposition 14. *The minimal 2-vectorial completions of the non-Fano matroid F_7^- are the Fano matroid F_7 and the matroids $\mathcal{M}_{1,1}, \mathcal{M}_{1,3}, \mathcal{M}_{1,5}, \mathcal{M}_{1,7}$,*

$\mathcal{M}_{2,0}, \mathcal{M}_{2,2}, \mathcal{M}_{2,4}, \mathcal{M}_{2,6}$ with ground set Ω and sets of circuits:

$$\begin{cases} \mathcal{C}(\mathcal{M}_{1,1}) = \{\{1\}\} \cup \{C \in \mathcal{C}(F_7^-) : C \subseteq \Omega \setminus \{1\}\}, \\ \mathcal{C}(\mathcal{M}_{1,3}) = \{\{3\}\} \cup \{C \in \mathcal{C}(F_7^-) : C \subseteq \Omega \setminus \{3\}\}, \\ \mathcal{C}(\mathcal{M}_{1,5}) = \{\{5\}\} \cup \{C \in \mathcal{C}(F_7^-) : C \subseteq \Omega \setminus \{5\}\}, \\ \mathcal{C}(\mathcal{M}_{1,7}) = \{\{7\}\} \cup \{C \in \mathcal{C}(F_7^-) : C \subseteq \Omega \setminus \{7\}\}, \end{cases}$$

$$\begin{cases} \mathcal{C}(\mathcal{M}_{2,0}) = \{\{1, 3\}, \{1, 5\}, \{1, 7\}, \{3, 5\}, \{3, 7\}, \{5, 7\}\} \cup \\ \quad \cup \{\{1, 2, 4, 6\}, \{2, 3, 4, 6\}, \{2, 4, 5, 6\}, \{2, 4, 6, 7\}\}, \\ \mathcal{C}(\mathcal{M}_{2,2}) = \{\{1, 3\}, \{5, 7\}\} \cup \\ \quad \cup \{A \subseteq \Omega \setminus \{2\} \text{ such that } |A| = 3 \text{ and } \{1, 3\}, \{5, 7\} \not\subseteq A\}, \\ \mathcal{C}(\mathcal{M}_{2,4}) = \{\{1, 7\}, \{3, 5\}\} \cup \\ \quad \cup \{A \subseteq \Omega \setminus \{4\} \text{ such that } |A| = 3 \text{ and } \{1, 7\}, \{3, 5\} \not\subseteq A\}, \\ \mathcal{C}(\mathcal{M}_{2,6}) = \{\{1, 5\}, \{3, 7\}\} \cup \\ \quad \cup \{A \subseteq \Omega \setminus \{6\} \text{ such that } |A| = 3 \text{ and } \{1, 5\}, \{3, 7\} \not\subseteq A\}. \end{cases}$$

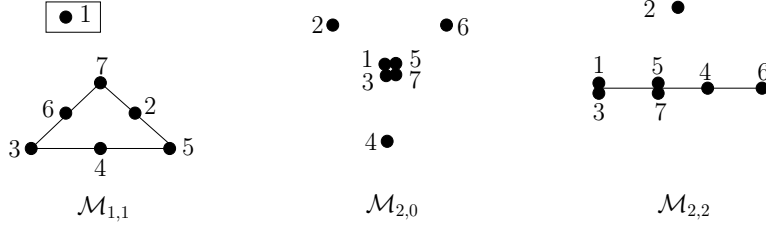


FIGURE 2. Geometric representations of the matroids $\mathcal{M}_{1,1}$, $\mathcal{M}_{2,0}$ and $\mathcal{M}_{2,2}$. Here two elements lying on the same point form a 2-element circuit and an element inside a box is a 1-element circuit.

Proof Let $\Sigma = \{F_7, \mathcal{M}_{1,1}, \mathcal{M}_{1,3}, \mathcal{M}_{1,5}, \mathcal{M}_{1,7}, \mathcal{M}_{2,0}, \mathcal{M}_{2,2}, \mathcal{M}_{2,4}, \mathcal{M}_{2,6}\}$ and take $\mathcal{M} \in \Sigma$. On one hand, by using Lemma 1 it is not hard to check that $F_7^- \leq \mathcal{M}$. On the other, from [4, Proposition 6.4.8] and from the excluded minor characterizations [4, Proposition 6.5.4 and Proposition 6.5.6] we get that there exists a field \mathbb{K} of characteristic two such that \mathcal{M} is \mathbb{K} -representable. So we conclude that if $\mathcal{M} \in \Sigma$ then $F_7^- \leq \mathcal{M}$ and \mathcal{M} is 2-representable, that is, \mathcal{M} is a 2-vectorial completion of the non-Fano matroid.

From the above we have that $\Sigma \subseteq \text{Vect}_2(F_7^-)$. In addition, by using Lemma 1 it is a straightforward proof to check that $\mathcal{M} \not\leq \mathcal{M}'$ if $\mathcal{M}, \mathcal{M}' \in \Sigma$ are different. Hence we have that $\Sigma \subseteq \text{Vect}_2(F_7^-)$ and that two different matroids of Σ are not comparable. Therefore the

proof of the proposition will be completed by showing that if \mathcal{N} is a 2-vectorial completion of F_7^- , then there exists a matroid $\mathcal{M} \in \Sigma$ such that $\mathcal{M} \leq \mathcal{N}$.

So from now on, let \mathcal{N} be a 2-representable matroid with ground set Ω and such that $F_7^- \leq \mathcal{N}$.

We must demonstrate that $\mathcal{M} \leq \mathcal{N}$ for some $\mathcal{M} \in \Sigma$. In order to do this, we distinguish three cases according to the circuits of \mathcal{N} . We systematically use Lemma 1 to compare matroids under the relation \leq .

Case 1: $\{2, 4, 6\} \in \mathcal{C}(\mathcal{N})^+$.

We claim that, in such a case, $F_7 \leq \mathcal{N}$. Indeed, if $\{2, 4, 6\}$ is a dependent set of \mathcal{N} , then there exists a circuit $C_0 \in \mathcal{C}(\mathcal{N})$ such that $C_0 \subseteq \{2, 4, 6\}$, and hence we get that $F_7 \leq \mathcal{N}$ because $\mathcal{C}(F_7) \setminus \{\{2, 4, 6\}\} \subseteq \mathcal{C}(F_7^-)$ and $F_7^- \leq \mathcal{N}$. So, if $\{2, 4, 6\} \in \mathcal{C}(\mathcal{N})^+$, then $F_7 \leq \mathcal{N}$.

Case 2: $\{2, 4, 6\} \notin \mathcal{C}(\mathcal{N})^+$ and \mathcal{N} has a 1-element circuit.

In such a case we are going to prove that there exists $i \in \{1, 3, 5, 7\}$ such that $\mathcal{M}_{1,i} \leq \mathcal{N}$. Let C_0 be a 1-element circuit of \mathcal{N} . Since $\{2, 4, 6\} \notin \mathcal{C}(\mathcal{N})^+$, there exists $i \in \{1, 3, 5, 7\}$ such that $C_0 = \{i\} \in \mathcal{C}(\mathcal{N})$. Then for all $C \in \mathcal{C}(\mathcal{M}_{1,i})$, either $C = \{i\}$, or there exists $C' \in \mathcal{C}(\mathcal{N})$ such that $C' \subseteq C$, (because if $i \notin C \in \mathcal{C}(\mathcal{M}_{1,i})$ then $C \in \mathcal{C}(F_7^-)$ and $F_7^- \leq \mathcal{N}$). Therefore, $\mathcal{M}_{1,i} \leq \mathcal{N}$, as we wanted to prove.

Case 3: $\{2, 4, 6\} \notin \mathcal{C}(\mathcal{N})^+$ and $|C| \geq 2$ for all $C \in \mathcal{C}(\mathcal{N})$.

This is the last case that we must consider. Now, the proof of the proposition will be completed by showing that, in this case, there exists $j \in \{0, 2, 4, 6\}$ such that $\mathcal{M}_{2,j} \leq \mathcal{N}$. In order to prove this we will use some basic matroid theory facts that are recalled in the following. For a subset $S \subseteq \Omega$, let the *closure of S* in \mathcal{N} be $\text{cl}(S) = S \cup \{x \in \Omega : \text{there is } C \in \mathcal{C}(\mathcal{N}) \text{ such that } x \in C \subseteq S \cup \{x\}\}$. Then, the following statements hold:

- (a) If $\{a, b\}$ and $\{a, c\}$ are circuits of \mathcal{N} , then $\{b, c\}$ is also a circuit of \mathcal{N} .
- (b) If $\{a, b\}$ is a circuit of \mathcal{N} then $\text{cl}(\{a, c\}) = \text{cl}(\{b, c\})$ for all $c \in \Omega$.
- (c) If $y \in \text{cl}(S)$, then $\text{cl}(S \cup \{y\}) = \text{cl}(S)$.
- (d) If $T \subseteq \text{cl}(S)$ and $T \cup \{x\} \in \mathcal{C}(\mathcal{N})$, then $x \in \text{cl}(S)$.
- (e) Every subset of $\text{cl}(S)$ with more than $|S|$ elements is dependent.

The first statement is an immediate application of the weak circuit elimination property. To prove the other four statements, let us interpret the closure operator when \mathcal{N} is the column matroid of the matrix A (which in fact is the only case that we need here). Recall that, in such a case, circuits correspond to minimal sets of linearly dependent columns, and thus the closure of a set S consists of the columns of A that are in the linear span of the columns corresponding to S . Therefore, the properties (b)–(e) are clear from properties of linear dependence. This completes the proof of the five statements.

Hereafter, we finalize the proof of the proposition.

We are assuming that \mathcal{N} is a 2-vectorial completion of F_7^- such that $\{2, 4, 6\}$ is not a dependent set of \mathcal{N} and that no singleton is a circuit of \mathcal{N} (that is, \mathcal{N} is *loopless*). In such a case we are going to prove that $\mathcal{M}_{2,j} \leq \mathcal{N}$ for some $j \in \{0, 2, 4, 6\}$. We proceed by three steps.

Step 1. There exists a circuit $C \in \mathcal{C}(\mathcal{N})$ with $|C| = 2$.

Proof. Let us assume that \mathcal{N} has no circuit of size 2. We have then that every 3-element circuit of F_7^- is also a circuit of \mathcal{N} because $F_7^- \leq \mathcal{N}$. At this point observe that if $X \subseteq \Omega$ is a subset with $|X| \geq 3$, then either $X \subseteq C'$ or $C' \subseteq X$ for some $C' \in \mathcal{C}(F_7^-)$. Therefore, if $\mathcal{C}(F_7^-) \subseteq \mathcal{C}(\mathcal{N})$ then $\mathcal{C}(F_7^-) = \mathcal{C}(\mathcal{N})$ and thus $F_7^- = \mathcal{N}$ which leads us to a contradiction because the matroid \mathcal{N} is 2-representable. Hence it follows that $\mathcal{C}(F_7^-) \not\subseteq \mathcal{C}(\mathcal{N})$ and so, since $F_7^- \leq \mathcal{N}$, some 4-element circuit of F_7^- must properly contain a 3-element circuit C_0 of \mathcal{N} . Up to symmetry, there are three possibilities for this circuit C_0 of \mathcal{N} : $\{1, 2, 4\}$, $\{1, 3, 4\}$ and $\{1, 3, 5\}$. In order to obtain a contradiction, we analyze each one of the different situations that may occur.

First assume that $C_0 = \{1, 2, 4\} \in \mathcal{C}(\mathcal{N})$. Consider $L = \text{cl}(\{1, 2\})$, which by property (c) equals $\text{cl}(\{1, 4\})$ (all closures are taken in \mathcal{N}). Now the circuit $\{1, 2, 3\}$ forces 3 to be in L (recall that every 3-element circuit of F_7^- is also a circuit of \mathcal{N}); similarly, the circuit $\{1, 4, 7\}$ forces 7 to belong to L . But now the circuit $\{3, 6, 7\}$ gives that 6 is in L . Thus $\{2, 4, 6\} \subseteq L = \text{cl}(\{1, 2\})$ and hence $\{2, 4, 6\}$ is dependent by property (e), which is impossible as we are assuming that $\{2, 4, 6\}$ is independent.

Next assume that $C_0 = \{1, 3, 4\} \in \mathcal{C}(\mathcal{N})$. In such a case we have that $\{1, 2, 3\}, \{1, 3, 4\} \in \mathcal{C}(\mathcal{N})$ and so, from the weak circuit elimination property, it follows that $\{1, 2, 4\} \in \mathcal{C}(\mathcal{N})$. At this point, a contradiction is obtained by applying the previous case.

Now assume that $C_0 = \{1, 3, 5\}$. So we have $\{1, 2, 3\}, \{1, 3, 5\} \in \mathcal{C}(\mathcal{N})$. Hence, from the weak circuit elimination property we get that

$\{1, 2, 5\} \in \mathcal{C}(\mathcal{N})$. Thereby $\{1, 2, 6\} \in \mathcal{C}(\mathcal{N})$ because $\{1, 5, 6\} \in \mathcal{C}(\mathcal{N})$. In this case a contradiction is obtained by applying the first case to $C'_0 = \{1, 2, 6\}$.

This completes the proof of the first step.

Step 2. There exists a circuit $C \in \mathcal{C}(\mathcal{N})$ with $|C| = 2$ and $C \subseteq \{1, 3, 5, 7\}$.

Proof. Assume that no two elements of $\{1, 3, 5, 7\}$ form a circuit of \mathcal{N} . On one hand, $\{2, 4, 6\}$ is not a dependent set of \mathcal{N} . On the other, from the previous step \mathcal{N} has some 2-element circuit. Therefore, by symmetry, we may assume that $\{1, 2\} \in \mathcal{C}(\mathcal{N})$. We have that $\{1, 4\}$ is independent, as otherwise property (a) would imply that $\{2, 4\}$ is a circuit; similarly, we get that $\{2, 7\}$ is also independent. Consider $L' = \text{cl}(\{1, 4\})$; as $\{1, 2\} \in \mathcal{C}(\mathcal{N})$, the element 2 belongs to L' . The circuit $\{1, 4, 7\}$ of F_7^- forces 7 to be in L' because $F_7^- \leq \mathcal{N}$. Since $\{2, 5, 7\} \in \mathcal{C}(F_7^-)$ and $F_7^- \leq \mathcal{N}$, but $\{2, 7\} \notin \mathcal{C}(\mathcal{N})$, there exists $C \in \mathcal{C}(\mathcal{N})$ such that $5 \in C \subseteq \{2, 5, 7\}$, and hence property (d) gives that $5 \in L'$ because $2, 7 \in L'$. Similarly, as $\{1, 5, 6\} \in \mathcal{C}(F_7^-)$ and we are assuming that $\{1, 5\} \notin \mathcal{C}(\mathcal{N})$, we deduce that 6 belongs to L' . But now $\{2, 4, 6\} \subseteq L' = \text{cl}(\{1, 4\})$ and thus it is dependent by property (e), which is not possible.

Step 3. There exists $j \in \{0, 2, 4, 6\}$ such that $\mathcal{M}_{2,j} \leq \mathcal{N}$.

Proof. Recall that we are assuming that $\{2, 4, 6\} \notin \mathcal{C}(\mathcal{N})^+$ and that \mathcal{N} is loopless. From the previous step, $\{1, 3, 5, 7\}$ contains a 2-element circuit. By symmetry, assume $\{1, 3\} \in \mathcal{C}(\mathcal{N})$. If both $\{1, 5\}$ and $\{1, 7\}$ are also circuits, then by property (a) every 2-element subset of $\{1, 3, 5, 7\}$ is a circuit of \mathcal{N} and one easily checks $\mathcal{M}_{2,0} \leq \mathcal{N}$ because $F_7^- \leq \mathcal{N}$.

So let us assume that $\{1, 3\} \in \mathcal{C}(\mathcal{N})$ but $\{1, 5\} \notin \mathcal{C}(\mathcal{N})$, and thus by property (a) we get that $\{3, 5\} \notin \mathcal{C}(\mathcal{N})$. Now consider $L'' = \text{cl}(\{1, 5\})$, that equals $\text{cl}(\{3, 5\})$ by property (b). As $\{1, 5, 6\}$ and $\{3, 4, 5\}$ are dependent in \mathcal{N} , the elements 6 and 4 belong to L'' . Observe that $2 \notin L''$ because if so the set $\{2, 4, 6\}$ would be dependent in \mathcal{N} by property (e).

Now observe that it cannot be that both $\{1, 4\}$ and $\{3, 6\}$ are circuits of \mathcal{N} , since if that were the case, applying property (a) twice we would get that $\{4, 6\} \in \mathcal{C}(\mathcal{N})$, contrary to $\{2, 4, 6\}$ being independent in \mathcal{N} .

Assume by symmetry that $\{1, 4\} \notin \mathcal{C}(\mathcal{N})$. Then, as $\{1, 4, 7\} \in \mathcal{C}(\mathcal{N})^+$, there is a circuit C of \mathcal{N} such that $7 \in C \subseteq \{1, 4, 7\}$. Therefore, since $1, 4 \in L''$, the element 7 also belongs to L'' by property (d). Thus,

L'' contains all elements except 2. Now consider the set $\{2, 5, 7\} \in \mathcal{C}(\mathcal{N})^+$; because $2 \notin L''$, it is forced that $\{5, 7\} \in \mathcal{C}(\mathcal{N})$. So we have that $\{1, 3\}$ and $\{5, 7\}$ are circuits of \mathcal{N} and that $L'' = \text{cl}(\{1, 5\}) = \Omega \setminus \{2\}$. Hence we conclude that $\mathcal{M}_{2,2} \leq \mathcal{N}$, as we wanted to prove.

This step completes the proof of the third case, and so the proof of the proposition. \square

Remark 15. *Since $(\text{Vect}_2(F_7^-), \leq)$ has nine minimal elements, from Corollary 7 we conclude that the non-Fano matroid F_7^- admits a 2-vectorial decomposition with nine components. However, as in the case of $\mathcal{U}_{2,4}$, it is possible to obtain a decomposition of F_7^- by using only some of the minimal completions obtained in Proposition 14. Namely, if \mathcal{N}_1 and \mathcal{N}_2 are two such minimal completions, then*

$$\mathcal{C}(F_7^-) = \min(\{C_1 \cup C_2 : C_i \in \mathcal{C}(\mathcal{N}_i)\})$$

if and only if either \mathcal{N}_1 or \mathcal{N}_2 is the Fano matroid F_7 .

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