Denumerants of 3-numerical semigroups

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Abstract. Denumerants of numerical semigroups are known to be difficult to obtain, even with small embedding dimension of the semigroups. In this work we give some results on denumerants of 3-semigroups $S = \langle a, b, c\rangle$. The time efficiency of the resulting algorithms range from $O(1)$ to $O(c)$. Closed expressions are obtained under certain conditions.

Key words: Denumerant, L-shapes, numerical semigroup, factorization.

1 Introduction

Let $\mathbb{N}_0$ be the set of non negative integers. Given a set $A = \{a_1, \ldots, a_n\} \subset \mathbb{N}_0$, $\gcd(a_1, \ldots, a_n) = 1$, the *n-numerical semigroup* $S = S(A)$ generated by A is defined by $\langle a_1, \ldots, a_m\rangle = \{x_1a_1 + \cdots + x_na_n : \; x_1, \ldots, x_n \in \mathbb{N}_0\}$. If A is a minimal set of generators then S has *embedding dimension* $\mathrm{e}(S) = n$. An element $m \in S$ has a *factorization* $(t_1, \ldots, t_n)$ in S if $m = t_1a_1 + \cdots + t_na_n$. The set of factorizations of m in S is denoted by $\mathcal{F}(m, S) = \{(x_1, \ldots, x_n) \in \mathbb{N}_0^n : \; x_1a_1 + \cdots + x_na_n = m\}$. The *denumerant* of m in S is the cardinality $\mathrm{d}(m, S) = |\mathcal{F}(m, S)|$.

In this work we give some results on denumerants of generic elements of embedding dimension three numerical semigroups $S = \langle a, b, c\rangle$. Algorithms of time–cost ranging from $O(1)$ to $O(c)$ (in the worst case) are also derived. We use *minimum distance diagrams* related to S as a main tool. In particular, we use the main results given in [1].

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2 Tools

From now on we denote the equivalence class of n modulo N as $[n]_N$. Given $m \in S \setminus \{0\}$, the *Apéry set* of m in S is $\mathrm{Ap}(m,S) = \{s \in S : \; s - m \notin S\}$. This set encodes many properties of semigroup. It can be shown that $|\mathrm{Ap}(m,S)| = m$ and $\mathrm{Ap}(m,S) = \{l_0, \ldots, l_{m-1}\}$ with $l_k \in [k]_m$ for $0 \le k < m$.

Minimum Distance Diagrams, $\mathcal{H}$, of numerical semigroups are used to study many distance–related and distribution of the semigroup elements. The minimum distance diagrams related to $S = \langle a_1, \ldots, a_n \rangle$ are sets of cardinality $|\mathcal{H}| = a_n$ of unitary cubes $[\![i_1, \ldots, i_{n-1}]\!] = [i_1, i_1+1] \times \cdots \times [i_{n-1}, i_{n-1}+1]$ in the first orthant of $\mathbb{R}^{n-1}$ with coordinates $(i_1, \ldots, i_{n-1}) \in \mathbb{N}_0^{n-1}$ representing factorizations of the elements of $\mathrm{Ap}(a_n, S)$, where $[s,t] = \{r \in \mathbb{R} : \; s \le r \le t\}$. Each element $l_k \in \mathrm{Ap}(a_n, S)$ is represented by exactly one cube $[\![i_1, \ldots, i_{n-1}]\!]$ in $\mathcal{H}$ with $i_1 a_1 + \cdots + i_{n-1} a_{n-1} = l_k$. Given any $m \in S$, with $m \in [l_k]_{a_n}$, we call the *basic coordinates* of m with respect to $\mathcal{H}$ to $(x_1, \ldots, x_{n-1}) \in \mathbb{N}_0^{n-1}$ if $[\![x_1, \ldots, x_{n-1}]\!] \in \mathcal{H}$ and $x_1 a_1 + \cdots + x_{n-1} a_{n-1} = l_k$. We also call the *basic factorization* of m in S with respecto to $\mathcal{H}$ to $(x_1, \ldots, x_{n-1}, \frac{m - l_k}{a_n}) \in \mathcal{F}(m,S)$.

These diagrams were used in [1] to study some aspects on factorization and catenary degree of 3-numerical semigroups $S = \langle a, b, c \rangle$. In particular, we recall here some nomenclature and results. Diagrams related to S are L-shaped or rectangles (considered as degenerated L-shapes with $wy = 0$) and are denoted by $\mathcal{H} = \mathrm{L}(l, h, w, y)$, where these entries are the lengths of the sides $0 \le w < l$, $0 \le y < h$, $\gcd(l, h, w, y) = 1$ and $lh - wy = c$. The diagram $\mathcal{H}$ tessellates $\mathbb{R}^2$ by translation through the vectors $\boldsymbol{u} = (l, -y)$ and $\boldsymbol{v} = (-w, h)$. See [2,3] for more details.

For the parameters $\delta = (la - yb)/c$ and $\theta = (hb - wa)/c$, we have $\delta \ge 0$, $\theta \ge 0$, $\delta + \theta > 0$, $\delta + y > 0$ and $\theta + w > 0$ whenever $\mathcal{H} = \mathrm{L}(l, h, w, y)$ is a minimum distance diagram of $S = \langle a, b, c \rangle$.

Theorem 1 ([1, Th. 2]). *Given $\mathcal{H} = \mathrm{L}(l, h, w, y)$ a minimum distance diagram of $S = \langle a, b, c \rangle$ and $m \in S$, let (x_0, y_0, z_0) be the basic factorization of m in S respect to $\mathcal{H}$, then*

$$\mathrm{d}(m,S) = 1 + \left\lfloor \frac{z_0}{\delta + \theta} \right\rfloor + \sum_{k=0}^{\left\lfloor \frac{z_0}{\delta+\theta} \right\rfloor} (S_k + T_k),$$

with

$$S_k = \begin{cases} \left\lfloor \frac{y_0 + k(h-y)}{y} \right\rfloor & \text{if } \delta = 0, \\ \left\lfloor \frac{z_0 - k(\delta+\theta)}{\delta} \right\rfloor & \text{if } y = 0, \\ \min\{ \left\lfloor \frac{y_0 + k(h-y)}{y} \right\rfloor, \left\lfloor \frac{z_0 - k(\delta+\theta)}{\delta} \right\rfloor \} & \text{if } \delta y \neq 0, \end{cases}$$

and

$$T_k = \begin{cases} \left\lfloor \frac{x_0+k(l-w)}{w} \right\rfloor & \textit{if } \theta = 0, \\ \left\lfloor \frac{z_0-k(\delta+\theta)}{\theta} \right\rfloor & \textit{if } w = 0, \\ \min\{ \left\lfloor \frac{x_0+k(l-w)}{w} \right\rfloor , \left\lfloor \frac{z_0-k(\delta+\theta)}{\theta} \right\rfloor \} & \textit{if } \theta w \neq 0. \end{cases}$$

This theorem does not give an efficient algorithm for computing denumerants. The reason is not the expression of minima appearing in S_k when $\delta y \neq 0$ or in T_k when $\theta w \neq 0$, these minima can be left out for $k \geq \left\lceil \frac{yz_0-\delta y_0}{\delta(h-y)+y(\delta+\theta)} \right\rceil$ in S_k when $\delta y \neq 0$ and for $k \geq \left\lceil \frac{wz_0-\theta x_0}{\theta(l-w)+w(\delta+\theta)} \right\rceil$ in T_k when $\theta w \neq 0$. The reason is the expression of the sum. Essentially we have to compute sums like

$$\sum_{k=0}^{N} \left\lfloor \frac{\alpha \pm k\beta}{q} \right\rfloor = \sum_{k=0}^{N} (\overline{\alpha} \pm k\overline{\beta}) + \sum_{k=0}^{N} \left\lfloor \frac{s \pm kt}{q} \right\rfloor ,$$

for $\alpha, \beta, \overline{\alpha}, \overline{\beta}, q \in \mathbb{N}$, with $q \neq 0$, $\alpha = \overline{\alpha}q + s$, $0 \leq s < q$, and $\beta = \overline{\beta}q + t$, $0 \leq t < q$.

3 Trying $\sum_{k=0}^{N} \left\lfloor \frac{s \pm kt}{q} \right\rfloor$

The efficiency of computing denumerants using these sums is compromised because of the number of terms to be added N, $O(N) = O(z_0)$. The coordinate z_0 of a basic factorization can become very large. Thus, we need to save add up all those terms.

In this section we try the sum $\sum_{k=0}^{N} \left\lfloor \frac{s+kt}{q} \right\rfloor$ with $0 \leq s, t < q$. The sum with the minus sign can be solved using an analogous method by symmetry. The idea of the method we propose here is making this addition as a Lebesgue–like discrete integration. Let us consider the graph of the function $f(x) = \left\lfloor \frac{s+tx}{q} \right\rfloor$ and pay attention to the constant parts of the graph. Let us denote the interval $I_m = [x_m, x_{m+1})$ where de function f has constant value m, that is $x_m = \frac{mq-s}{t}$ (if $t = 0$ the entire sum equals zero). Then, some main properties hold:

(1) There are $M + 1$ different intervals, with $M = \left\lfloor \frac{s+Nt}{q} \right\rfloor$. Each interval I_m has length $x_{n+1} - x_m = \frac{q}{t}$ except, possibly, I_0 and/or I_M.
(2) $\left\lfloor \frac{q}{t} \right\rfloor \leq |I_m \cap \mathbb{N}| \leq \left\lceil \frac{q}{t} \right\rceil$ except, possibly, I_0 and/or I_M.
(3) Those intervals I_m with $x_m \in \mathbb{N}_0$ are iS–type intervals. Intervals with $|I_m \cap \mathbb{N}| = \left\lceil \frac{q}{t} \right\rceil$ are mS–type intervals. Intervals iS are also mS except, possibly, I_M. There can be mS–intervals that are not iS–intervals.
(4) The behaviour of intervals at the same relative position between two consecutive iS–intervals is the same. There are $t - 1$ intervals between two

consecutive iS–intervals. The behaviour of intervals I_m is periodic of period t. Depending on the values s,t and q, this period is not the minimum one.

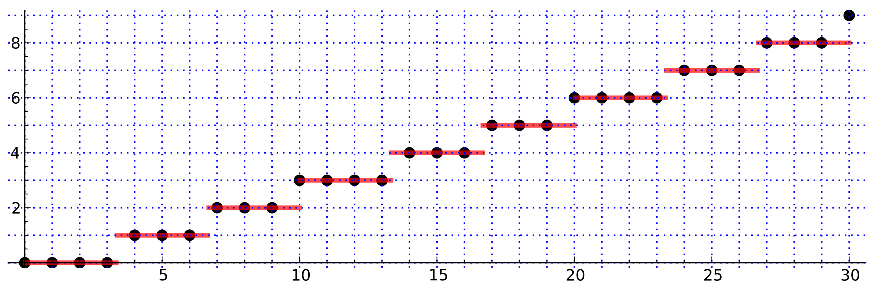

Fig. 1. Graph of f with $s = 0$, $t = 3$, $q = 10$

Figure 1 shows the graph of f with parameters $s = 0$, $t = 3$ and $q = 10$. Note the 3–periodic behaviour of the intervals I_m, $m \in \{0, \ldots, 9\}$. In this case, only the iS–intervals contain $\lceil \frac{q}{t} \rceil = 4$ integral values, except I_M (with $M = 9$) for $N = 30$ that is only one integral point $I_9 = \{9\}$.

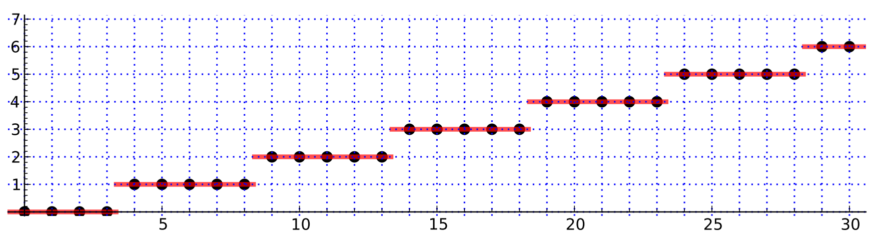

Fig. 2. Graph of f with $s = 5$, $t = 3$, $q = 15$

We study two main cases, $\gcd(t, q) > 1$ and $\gcd(t, q) = 1$. Figure 1 is the simple behaviour of the second case. The first case can be studied for $t \mid q$ and $t \nmid q$. When $t \mid q$, all intervals have the same behavior except, possibly, the first and the last ones; this is the case of Figure 2. Note that all intervals are of type mS and there is no iS–intervals, essentially the existence of iS–intervals depends on the value s. When $t \nmid q$ and $\gcd(t, q) = g > 1$, depending on the value of s appear iS–intervals and the period is not t but t/g; Figure 3 shows this case.

Depending on the behavior of intervals I_m the value of the sum can be arranged in an almost closed expression that sometimes results in a closed expression. The following results take into account all cases (that are more than the examples appearing in the three figures mentioned above).

From now on set $M = \left\lfloor \frac{s+Nt}{q} \right\rfloor$ and $x_M = \frac{Mq-s}{t}$.

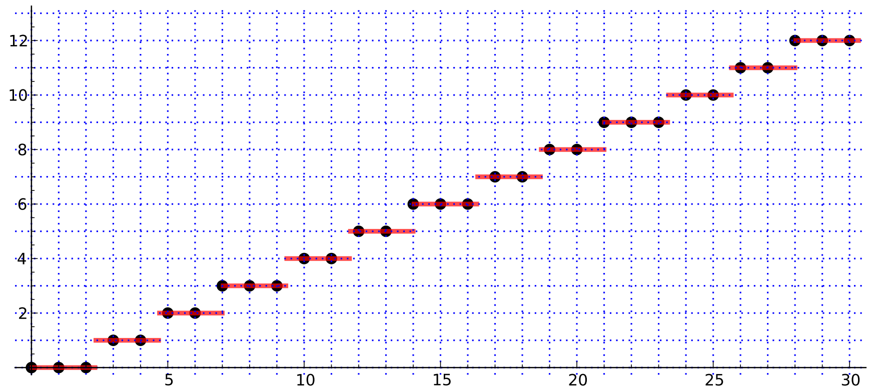

Fig. 3. Graph of f with $s=0$, $t=6$, $q=14$

Theorem 2 (Case $t \mid q$). *Assume $q=\overline{q}t$. Then,*

$$\sum_{k=0}^{N}\left\lfloor\frac{s+kt}{q}\right\rfloor=\overline{q}\frac{M(M-1)}{2}+M(N-\lceil x_M\rceil+1).$$

Theorem 2 is a simple case for applying the Lebesgue–like discrete summation. Thi theorem give closed expressions for denumerants.

Now we give the cases $t \nmid q$ with $\gcd(t,q)=1$ or $\gcd(t,q)=g>1$. To describe it we must define another type of intervals. Define $\hat{s}$ and $\hat{q}$ by

$$q=\overline{q}t+\hat{q},\ s=\overline{s}t+\hat{s},\quad 0\leq\hat{s},\hat{q}<t.$$

We say that I_m is an hS–interval if $(\hat{s}-m\hat{q})(\mathrm{mod}\ t)<\hat{q}$. We denote the sets $I=\{i_1,\ldots,i_\alpha\}$, $J=\{j_1,\ldots,j_n\}$, $K=\{k_1,\ldots,k_v\}$ of ordered indices of all hS-type intervals taken from some region to be defined. We also denote the sums $S_I=\sum_{l=1}^{\alpha}i_l$, $S_J=\sum_{l=1}^{n}j_l$, $S_K=\sum_{l=1}^{v}k_l$, $S=\overline{q}\frac{(M-1)M}{2}+M(N-\lceil x_M\rceil+1)$ and $\mathbb{S}=\sum_{k=0}^{N}\left\lfloor\frac{s+kt}{q}\right\rfloor$.

Theorem 3 (Case $t\nmid q$ and $\gcd(t,q)=1$). *Set $m_0\equiv q^{-1}s(\mathrm{mod}\ t)$ with $m_0\in\{0,\ldots,t-1\}$ and $u=\left\lfloor\frac{M-m_0-1}{t}\right\rfloor$.*

(a) If $m_0=0$ and $u\leq 0$, or $m_0=M$, take $K\subset[1,M-1]$. Then $\mathbb{S}=S+S_K$.
(b) If $m_0=0$ and $u>0$, take $J\subset[0,t-1]$ and $K\subset[ut,M-1]$. Then $\mathbb{S}=S+uS_J+nt\frac{(u-1)u}{2}+S_K$.
(c) If $0<m_0<M$ and $m_0+t\geq M$, take $I\subset[1,m_0-1]$ and $K\subset[m_0,M-1]$. Then $\mathbb{S}=S+S_I+S_K$.
(d) If $0<m_0<M$ and $m_0+t<M$, take the sets $I\subset[1,m_0-1]$ and $J\subset[m_0,m_0+t-1]$. Take also $K\subset[m_0+ut,M-1]$ whenever $u>0$ and $K=\emptyset$ otherwise (thus $S_K=0$). Then $\mathbb{S}=S+S_I+uS_J+nt\frac{(u-1)u}{2}+S_K$.

Theorem 4 (Case $t \nmid q$ and $\gcd(t,q) = g > 1$). *Set $\tilde{t} = t/g$. Take $J \subset [0, \tilde{t}-1]$. Set $m_0 = j_1$ and $u = \left\lfloor \frac{M-m_0-1}{\tilde{t}} \right\rfloor$.*

(a) If $m_0 \geq M$, then $\mathbb{S} = S$.
(b) If $0 \leq m_0 < M$, take $K \subset [m_0 + u\tilde{t}, M-1]$. Then $\mathbb{S} = S + uS_J + n\tilde{t}\frac{(u-1)u}{2} + S_K$.

Theorems 3 and 4 do not give closed expressions for denumerants because of the computation of the sets of indices. This task has order $O(t)$ in the worst case. However, the resulting algorithm is efficient. Similar results can be obtained to compute the sum $\sum_{k=0}^{N} \lfloor \frac{s-kt}{q} \rfloor$.

4 Application example

Each 3-semigroup has its own denumerant's information encoded in the related L-shapes. As an example, let us consider for instance $S = \langle 121, 1111, 2323\rangle$ with related L-shape $\mathrm{L}(101, 23, 0, 11)$. Thus, $\delta = 0$ and $\theta = 25553$.

This semigroup belongs to the general case $\delta = w = 0$, according with Theorem 1 (there are exactly five generic cases $\{\delta = 0, w = 0\}$, $\{\delta = 0, w \neq 0\}$, $\{\theta = 0, y = 0\}$, $\{\theta = 0, y \neq 0\}$ and $\{\delta \neq 0, \theta \neq 0\}$). Let us consider the case $\{\delta = 0, w = 0\}$ (then $y\theta \neq 0$). Assume $S = \langle a, b, c\rangle$ and $\mathcal{H} = \mathrm{L}(l, h, 0, y)$ belongs to this case. Then, the denumerant is given by

$$\mathrm{d}(m, S) = 1 + \left\lfloor \frac{z_0}{\theta} \right\rfloor + \sum_{k=0}^{\lfloor \frac{z_0}{\theta} \rfloor} \left(\left\lfloor \frac{y_0 + k(h-y)}{y} \right\rfloor + \left\lfloor \frac{z_0 - k\theta}{\theta} \right\rfloor \right),$$

where (x_0, y_0, z_0) is the basic factorization of $m \in S$ with respect to $\mathcal{H}$. Set $y_0 = \overline{y_0} y + \hat{y_0}$ and $h - y = \overline{n} y + \hat{n}$ with $0 \leq \hat{y_0}, \hat{n} < y$. Then,

$$\sum_{k=0}^{\lambda_0} \left\lfloor \frac{y_0 + k(h-y)}{y} \right\rfloor = (1+\lambda_0)[\overline{y_0} + \frac{1}{2}\lambda_0 \hat{n}] + \sum_{k=0}^{\lambda_0} \left\lfloor \frac{\hat{y_0} + k\hat{n}}{y} \right\rfloor,$$

with $\lambda_0 = \lfloor \frac{z_0}{\theta} \rfloor$. Similarly,

$$\sum_{k=0}^{\lambda_0} \left\lfloor \frac{z_0 - k\theta}{\theta} \right\rfloor = \sum_{k=0}^{\lambda_0} (\lambda_0 - k) = \frac{1}{2}\lambda_0(1+\lambda_0).$$

Therefore, the denumerant is given by

$$\mathrm{d}(m, S) = (1+\lambda_0)[1 + \overline{y_0} + \frac{1}{2}\lambda_0(1+\hat{n})] + \sum_{k=0}^{\lambda_0} \left\lfloor \frac{\hat{y_0} + k\hat{n}}{y} \right\rfloor.$$

For $S = \langle 121, 1111, 2323\rangle$, let us take the family of elements of $m_t \in S$ given by the basic factorization $(x_0, y_0, z_0) = (1, 1, t)$, i.e. $m_t = 1232 + 2323t$. Then, in this case, the parameters are $\overline{y_0} = 0$, $\hat{y_0} = 1$, $\overline{n} = 1$, $\hat{n} = 1$, $y = 11$ and $\lambda_0 = \lfloor \frac{t}{11} \rfloor$. Thus, the denumerant is

$$\mathrm{d}(m_t, S) = (1 + \lfloor \frac{t}{11} \rfloor)^2 + \sum_{k=0}^{\lfloor \frac{t}{11} \rfloor} \left\lfloor \frac{1+k}{11} \right\rfloor .$$

Applying the results given in Section 2, we can take large values for t. For instance, when $t = 10^{10^3}$, the denumerant $\mathrm{d}(m_t, \langle 121, 1111, 2323 \rangle)$ is

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864012021036814423737256323704185088152361139470083848659000
183554825819370705756877060672971703726568721241271063550138
370539920526103078475761125038281445045861486374199657633604
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550828038327091991019818305448508303470070548482347871522933
63588331442400219780
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