Comments on “Extremal Cayley digraphs of finite Abelian groups” [Intercon. Networks 12 (2011), no. 1-2, 125–135]

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Abstract

We comment on the paper “Extremal Cayley digraphs of finite Abelian groups” [Intercon. Networks 12 (2011), no. 1-2, 125–135]. In particular, we give some counterexamples to the results presented there, and provide a correct result for degree two.

1 Introduction

For the description of the problem, its applications, used notation, and the theoretical background, see, e.g. [8, 2, 4, 7].

For some given positive numbers, d (diameter) and k (degree), the authors of [5] consider the following numbers:

- Let $m_*(d,k)$ be the largest positive integer m (number of vertices) such that there exists an m-element finite Abelian group Γ and a k-element generating subset $A \subset \Gamma$ such that $\mathrm{diam}(\mathrm{Cay}(\Gamma, A)) \leq d$.
- Let $m(d,k)$ be the largest positive integer m such that there exists a cyclic group $\mathbb{Z}_m$ and a k-element generating subset $A \subset \mathbb{Z}_m$ such that $\mathrm{diam}(\mathrm{Cay}(\mathbb{Z}_m, A)) \leq d$.

The authors of [5] claim that, for any integer $d \geq 2$, Jia and Hsu [4] proved that

$$m(d,2) = \left\lfloor \frac{d(d+4)}{3} \right\rfloor + 1, \tag{1}$$

but this was proved around ten years before by the authors of [6, 7]. In [6], the following value can be found:

$$m(d,2) = \left\lceil \frac{(d+2)^2}{3} \right\rceil - 1, \qquad (2)$$

which is readily seen to be equivalent to (1). More generally, in Table I of [7] some other optimal values are shown (which minimize the diameter for some fixed number of vertices). Part of the table is shown below with the corresponding generating sets $\{a, b\}$ of the cyclic groups. (The values in boldface correspond to the ones given by (1) or (2).)

$m(d,2)$	d	a	$b(\bmod m)$
$3x^2$	$3x-1$	1	$3x-1$
$3x^2+x$	$3x-1$	1	$3x$
$\mathbf{3x^2+2x}$	$\mathbf{3x-1}$	$\mathbf{1}$	$\mathbf{-3x}$
$3x^2+2x+1$	$3x$	1	$3x+1$
$3x^2+3x+1$	$3x$	1	$3x+2$
$\mathbf{3x^2+4x+1}$	$\mathbf{3x}$	$\mathbf{1}$	$\mathbf{-3x-2}$
$3x^2+4x+2$	$3x+1$	1	$3x+3$
$3x^2+5x+2$	$3x+1$	1	$3x+4$
$\mathbf{3x^2+6x+2}$	$\mathbf{3x+1}$	$\mathbf{1}$	$\mathbf{-3x+4}$
$(=3(x+1)^2-1)$			

Also, as a main result, Mask, Schneider, and Jia [5, Th. 1.1] claimed that, for any d and k,

$$m_*(d,k) = m(d,k). \qquad (3)$$

However, as shown by the counterexamples in the following section, such a result cannot be true even for degree $k = 2$. This is due to an error in the proof of such a theorem. Namely, the first r equalities in [5, Th. 1.1] should be subject to modulo m_j:

$$x_j = \sum_{i=1}^{k} c_i a_{ij} \pmod{m_j} \qquad \text{for } j = 1, 2, \ldots, r.$$

Thus, without this condition, the following equality in [5], which should be modulo $m'_{r-1} = m_{r-1}m_r$, does not necessarily hold.

2 Some counterexamples and a result

In [7] it was shown that for degree $k = |A| = 2$, the minimum diameter d of an Abelian group Γ with m vertices is $d_{\min} = \lceil \sqrt{3m} \rceil - 2$ (see [7, Eq. (9)]). That is,

$$m_*(d,2) \le \left\lfloor \frac{(d+2)^2}{3} \right\rfloor. \qquad (4)$$

In fact the upper bound is attained when $\Gamma = \mathbb{Z}_{3x} \times \mathbb{Z}_x$, with $x \geq 1$, and $A = \{(1,0),(-1,1)\}$, leading to a (2-regular) Cayley digraph with $m = 3x^2$ vertices and diameter $d = 3x - 2$. However, it can be shown that, when $x > 1$, $\operatorname{rank}\Gamma = 2$, so that Γ is not cyclic. In this case, the best result is obtained with the cyclic group $\mathbb{Z}_m$ with $m = \frac{1}{3}(d+2)^2 - 1$ and generating set $A = \{a, b\}$, as shown in the following table.

k	x	$d = 3x - 2$	$m_*(d,2) = 3x^2$	$A \subset \mathbb{Z}_{3x} \times \mathbb{Z}_x$	$m(d,2) = 3x^2 - 1$	$A \subset \mathbb{Z}_m$
2	2	4	12	$\{(1,0),(-1,1)\}$	11	$\{1,3\}$
2	3	7	27	$\{(1,0),(-1,1)\}$	26	$\{1,8\}$
2	4	10	48	$\{(1,0),(-1,1)\}$	47	$\{1,11\}$
2	5	13	75	$\{(1,0),(-1,1)\}$	74	$\{1,14\}$
2	6	16	108	$\{(1,0),(-1,1)\}$	107	$\{1,17\}$

For other values of $m(d,2)$, see [7, Table II] or the results in [3, 1]. In fact, from the results of these papers, and comparing the values of $m(d,2)$ in (2) with the upper bound for $m_*(d,2)$ in (4), one gets the following result for the case of degree $k = 2$:

Proposition 2.1 *For any diameter $d \geq 2$,*

$$m_*(d,2) = \begin{cases} m(d,2)+1, & \text{if } d \equiv 1 \pmod 3, \\ m(d,2), & \text{otherwise.} \end{cases} \tag{5}$$

In the case of the above digraph $\mathrm{Cay}(\mathbb{Z}_{3x} \times \mathbb{Z}_x), \{(1,0),(-1,1)\})$, it can be shown that the two unique vertices at maximum distance $d = 3x - 2$ from the origin are $(2x, x-1)$ and $(x, x-1)$.

Similar counterexamples can be given to prove that the extremal Cayley digraphs with respect to their average distance are not necessarily attainable for cyclic groups ([5, Th. 3.1]).

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