ON THE DIAMETER OF RANDOM PLANAR GRAPHS

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ABSTRACT. We show that the diameter $\mathrm{diam}(G_n)$ of a random labelled connected planar graph with n vertices is asymptotically almost surely of order $n^{1/4}$, in the sense that there exists a constant $c > 0$ such that

$$P(\mathrm{diam}(G_n) \in (n^{1/4-\epsilon}, n^{1/4+\epsilon})) \geq 1 - \exp(-n^{c\epsilon})$$

for ϵ small enough and $n \geq n_0(\epsilon)$. We prove similar statements for 2-connected and 3-connected planar graphs and maps.

1. Introduction

A map is a connected planar graph with a given embedding in the plane. The diameter of random maps has attracted a lot of attention since the pioneering work by Chassaing and Schaeffer [7] on the radius $r(Q_n)$ of random quadrangulations with n vertices, where they show that $r(Q_n)$ rescaled by $n^{1/4}$ converges as $n \to \infty$ to an explicit continuous distribution related to the Brownian snake. This convergence was shown to hold for large families of planar maps [20, 22], and it was conjectured that random maps of size n rescaled by $n^{1/4}$ converge in some sense to a continuum object, the *Brownian map* [21, 12]. In recent years, several properties of the limiting object have been obtained [13, 23], and the convergence result was proved very recently independently by Miermont and Le Gall [24, 14]. At the combinatorial level, the two-point function of quadrangulations has surprisingly a simple exact expression, a beautiful result found in [5] that allows one to derive easily the limit distribution, rescaled by $n^{1/4}$, of the distance between two randomly chosen vertices in a random quadrangulation. In contrast, little is known about the profile of random *unembedded* connected planar graphs, even if it is strongly believed that the results should be similar as in the embedded case.

Our main result in this paper is a large deviation statement for the diameter, which strongly supports the belief that $n^{1/4}$ is the right scaling order. We say that a property A, defined for all values n of a parameter, holds asymptotically almost surely, a.a.s. for short, if

$$P(A) \to 1, \qquad \text{as } n \to \infty.$$

In this paper we need a certain rate of convergence of the probabilities. Suppose property A depends on a real number $\epsilon > 0$, usually very small. Then we say that A holds a.a.s. with exponential rate if there is a constant $c > 0$, such that for every ϵ small enough there exists an integer $n_0(\epsilon)$ so that

$$P(\text{not } A) \leq e^{-n^{c\epsilon}} \qquad \text{for all } n \geq n_0(\epsilon). \tag{1}$$

The diameter of a graph (or map) G is denoted by $\mathrm{diam}(G)$. The main results proved in this paper are the following.

Theorem 1.1. *The diameter of a random connected labelled planar graph with n vertices is in the interval $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$ a.a.s. with exponential rate.*

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Theorem 1.2. *Let $1 < \mu < 3$. The diameter of a random connected labelled planar graph with n vertices and $\lfloor \mu n \rfloor$ edges is in the interval $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$ a.a.s. with exponential rate.*

These are the first results obtained on the diameter of random planar graphs. They give the right order of magnitude and show the connection to the well-studied problem of the radius of random quadrangulations. It is still open and seems technically very involved to show a limit distribution for the profile or radius of a random connected planar graph rescaled by $n^{1/4}$. We remark that the diameter is a notoriously difficult parameter in this context, since one cannot use the power of the Erdős-Rényi model, where edges are drawn *independently*. Other extremal parameters that have been analyzed recently in random planar graphs using analytic techniques are the size of the largest k-connected component [18, 26] and the maximum vertex degree [9, 10].

The results for planar graphs contrast with the so-called "subcritical" graph families, such as trees, outerplanar graphs, and series-parallel graphs, where the diameter is in the interval $(n^{1/2-\epsilon}, n^{1/2+\epsilon})$ a.a.s. with exponential rate; see Section 6 at the end of the article.

Let us give a brief sketch of the proof. Recall that a graph is k-connected if one needs to delete at least k vertices to disconnect it (2-connected graphs are assumed to be loopless, 3-connected graphs are assumed to be loopless and simple). First we prove the result for planar maps via quadrangulations, using a bijection with labelled trees by Schaeffer that keeps track of a distance parameter. Then we prove the result for 2-connected maps using the fact that a random map has a large 2-connected core with non-negligible probability. A similar argument allows us to extend the result to 3-connected maps, which proves it also for 3-connected planar graphs, since by Whitney's theorem they have a unique embedding in the sphere. We then reverse the previous arguments and go first to 2-connected and then to connected planar graphs, but this is not straightforward. One difficulty is that the largest 3-connected component of a random 2-connected graph does not have the typical ratio between number of edges and number of vertices, and this is why we must study maps with a given weight at vertices, so as to adjust the ratio between edges and vertices. In addition, we must show that there is a 3-connected component of size $n^{1-\epsilon}$ a.a.s. with exponential rate, and similarly for 2-connected components. Finally, we must show that the height of the tree associated to the decomposition of a 2-connected graph into 3-connected components is at most n^{ϵ}, and similarly for the tree of the decomposition of a connected graph into 2-connected components.

2. Preliminaries

In this section we recall first some easy inequalities given by generating functions. Then we describe the chain of correspondences and decompositions that will allow us to carry large deviation estimates for the diameter, starting from quadrangulations (and labelled trees associated to them) and all the way down to connected planar graphs. In the sequel, the diameter of a graph G (whether a tree, a planar graph or a map) is denoted $\mathrm{diam}(G)$.

2.1. **Saddle bounds and exponentially small tails.**

Let $f(z) = \sum_n f_n z^n$ be a series with nonnegative coefficients and let $x > 0$ be a value such that $f(x)$ converges; in particular x is at most the radius of convergence ρ. Then we have the following elementary inequality for $n \geq 0$:

$$f_n \leq f(x) x^{-n}. \tag{2}$$

When minimized over x, this inequality is called *saddle-point bound.*

A bivariate version yields a lemma that will be used several times; it provides a simple criterion to ensure that the distribution of a parameter has an exponentially fast decaying tail. First let us give some terminology. A *weighted combinatorial class* is a class of combinatorial objects (such as graphs, trees or maps) $\mathcal{A} = \cup_n \mathcal{A}_n$ endowed with a weight-function $w : \mathcal{A} \mapsto \mathbb{R}_+$. We write $|\alpha| = n$ if $\alpha \in \mathcal{A}_n$. The *weighted distribution* in size n is the unique distribution on $\mathcal{A}_n$ proportional to the weight: $P(\alpha) \propto w(\alpha)$ for every $\alpha \in \mathcal{A}_n$.

Lemma 2.1. *Let $\mathcal{A} = \cup_n \mathcal{A}_n$ be a weighted combinatorial class, $\chi : \mathcal{A} \to \mathcal{N}$ a parameter on $\mathcal{A}$, and let $A(z,u) = \sum_{\alpha \in \mathcal{A}} w(\alpha) z^{|\alpha|} u^{\chi(\alpha)}$. Let $\rho > 0$ be the dominant singularity of $A(z,1)$, and let*

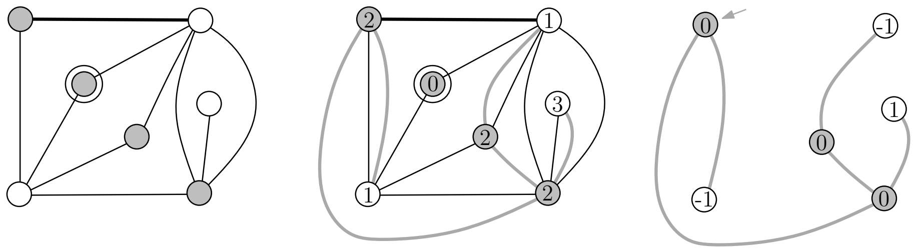

FIGURE 1. Left: A bicolored quadrangulation with a marked vertex (surrounded) and a marked edge (bolder). Right: the associated bicolored labelled tree.

$A_n = [z^n]A(z)$. *Assume that, for some* $\alpha > 0$,

$$A_n = \Omega(n^{-\alpha}\rho^{-n}).$$

Assume also that there exists $u_0 > 1$ *such that* $A(\rho, u_0)$ *converges.*

Then $\chi(R_n) \leq n^\epsilon$ *a.a.s. with exponential rate.*

Proof. We have $P(\chi(R_n) = k) = [z^n u^k]A(z,u)/[z^n]A(z,1)$. A bivariate version of (2) ensures that $[z^n u^k]A(z,u) \leq A(\rho,u_0)\rho^{-n}u_0^{-k} = O(\rho^{-n}e^{-ck})$, where $c = \log(u_0)$. Hence $P(\chi(R_n) = k) = O(n^\alpha e^{-ck})$. This directly implies that $\chi(R_n) \leq n^\epsilon$ a.a.s. with exponential rate. □

2.2. **Maps.** A *planar map* (shortly called a map here) is a connected unlabelled graph embedded in the oriented sphere up to isotopic deformation. Loops and multiple edges are allowed. A *rooted map* is a map where an edge is marked and oriented. The face to the left of the root is called the *outer face*; this face is taken as the infinite face in plane representations (e.g. in Figure 1, left part). A *quadrangulation* is a map where all faces have degree 4. Notice that an isthmus contributes twice to the degree of a face.

2.2.1. *Labelled trees and quadrangulations.* We recall Schaeffer's bijection (itself a reformulation of an earlier bijection by Cori and Vauquelin [8]) between labelled trees and quadrangulations. A *rooted plane tree* is a rooted map with a unique face. A *labelled tree* is a rooted plane tree with an integer label $\ell_v \in \mathbb{Z}$ on each vertex v so that the labels of the end-points of each edge $e = (v, v')$ satisfy $|\ell_v - \ell_{v'}| \leq 1$, and such that the root vertex has label 0. The minimal (resp. maximal) label in the tree is denoted ℓ_{min} (resp. ℓ_{max}). A *bicolored* labelled tree is a labelled tree endowed with a 2-coloring of the vertices (in black and white) such that vertices of odd labels are of one color and vertices of even labels are of the other color. Such a tree is called *black-rooted* (resp. *white-rooted*) if the root-vertex is black (resp. white). A *bicolored quadrangulation* is a quadrangulation endowed with a 2-coloring of its vertices (in black and white) such that adjacent vertices have different colors. Such a 2-coloring is unique once the color of a given vertex is specified. A rooted quadrangulation will be assumed to be endowed with the unique 2-coloring such that the root-vertex is black.

Theorem 2.2 (Schaeffer [27], Chapuy, Marcus, Schaeffer [6])**.** *Bicolored quadrangulations with a marked vertex and a marked edge are in bijection with bicolored labelled trees. Each face of a bicolored quadrangulation* Q *corresponds to an edge in the associated bicolored labelled tree* τ*. Each non-marked vertex* v *of* Q *corresponds to a vertex* v *of the same color in* τ*, such that* $\ell_v - \ell_{min} + 1$ *gives the distance from* v *to* v_0 *in* Q*.*

An example is shown in Figure 1; see [6] for a detailed description of the bijection. Define the *label-span* of τ as the quantity $L(\tau) = \ell_{max}(\tau) - \ell_{min}(\tau)$. It follows from the bijection in Theorem 2.2 that $L(\tau) + 1$ is the radius of Q centered at v_0. Hence

$$L(\tau) + 1 \leq \mathrm{diam}(Q) \leq 2L(\tau) + 2. \quad (3)$$

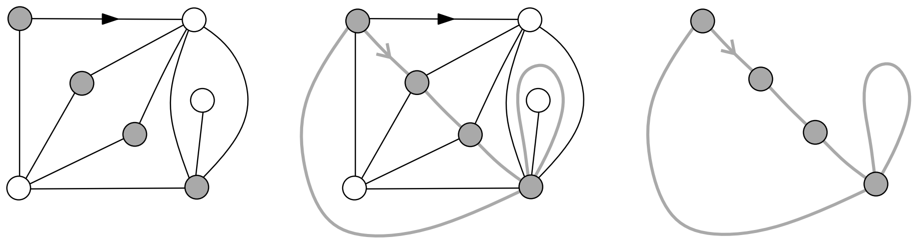

FIGURE 2. Left: A rooted quadrangulation. Right: the associated rooted map.

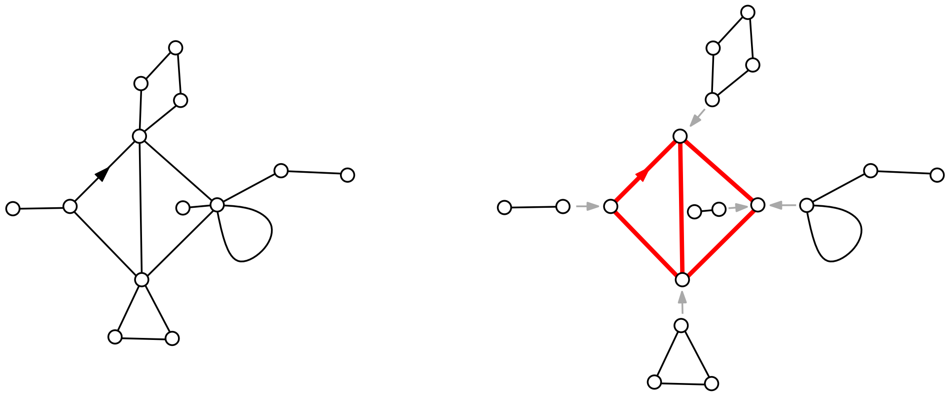

FIGURE 3. A rooted map is obtained from a 2-connected map (the core) where at each corner a rooted map is possibly inserted.

2.2.2. *Quadrangulations and maps.* We recall a classical bijection between rooted quadrangulations with n faces (and thus $n+2$ vertices) and rooted maps with n edges. Starting from Q endowed with its canonical 2-coloring, add in each face a new edge connecting the two diagonally opposed black vertices. Return the rooted map M formed by the newly added edges and the black vertices, rooted at the edge corresponding to the root-face of Q, and with same root-vertex as Q; see Figure 2. Conversely, to obtain Q from M, add a new white vertex v_f inside each face f of M and add new edges from v_f to every corner around f; then delete all edges from M, and take as root-edge of Q the one corresponding to the incidence root-vertex/outer-face in M. Clearly, under this bijection, vertices of a map correspond to black vertices of the associated quadrangulation, and faces correspond to white vertices. Let M be a rooted map with n edges and let Q be the associated rooted quadrangulation (with $n+2$ vertices). Every path $b_1 b_2 \dots b_k$ in M yields a path $b_1 w_1 b_2 \dots w_{k-1} b_k$ in Q, where w_i is the white vertex corresponding to the face to the left of (b_i, b_{i+1}). Hence $\mathrm{diam}(Q) \leq 2\,\mathrm{diam}(M)$. Let $x = b_1 w_1 b_2 w_2 \dots b_k = y$ be a path in Q, where the b_i are black and the w_i are white. Let f_i be the face in M corresponding to b_i. Then we can find a path in M between x and y of length at most $k + \deg(f_1) + \cdots + \deg(f_k)$. Therefore, calling $\Delta(M)$ the maximal face-degree in M, we obtain $\mathrm{diam}(M) \leq \mathrm{diam}(Q) \cdot \Delta(M)$. We thus obtain the following inequalities that we use for estimating the diameter of random maps from estimates of the diameter of random quadrangulations:

$$\mathrm{diam}(Q)/2 \leq \mathrm{diam}(M) \leq \mathrm{diam}(Q) \cdot \Delta(M). \tag{4}$$

2.2.3. *The 2-connected core of a map.* It is convenient here to consider the map consisting of a single loop as 2-connected (all 2-connected maps with at least two edges are loopless). As described by Tutte in [28], a rooted map M is obtained by taking a rooted 2-connected map C, called the *core* of M, and then inserting at each corner i of C an arbitrary rooted map M_i; see Figure 3. The maps M_i are called the *pieces* of M. The following inequalities will be used to estimate the diameter of random rooted 2-connected maps from estimates of the diameter of random rooted

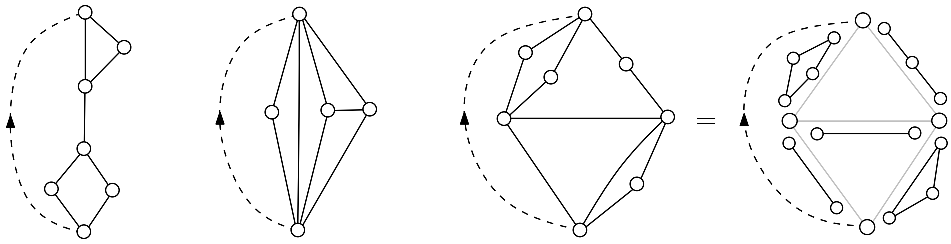

FIGURE 4. (a) A network made of 3 networks assembled in series. (b) A network made of 3 networks (one of which is an edge) assembled in parallel. (c) A network with a 3-connected core (which is a K_4) where each edge is substituted by a network.

maps:

$$\mathrm{diam}(C) \leq \mathrm{diam}(M) \leq \mathrm{diam}(C) + 2 \cdot \max_i(\mathrm{diam}(M_i)). \tag{5}$$

The first inequality is trivial, and the second one follows from the fact that a diametral path in M either stays in a single piece, or it connects two different pieces while traversing edges of C.

2.2.4. *The 3-connected core of a 2-connected map.* A *plane network* is a map M with two marked vertices in the outer face, called the *poles* of M —the 0-pole and the ∞-pole— such that adding an edge e between these two vertices yields a rooted 2-connected map, called the *completed map* of the network. Conversely a plane network is just obtained from a 2-connected map with at least two edges by deleting the root-edge, the origin and end of the root-edge being distinguished respectively as the 0-pole and the ∞-pole. A *polyhedral network* is a plane network such that the poles are not adjacent and such that the completed map is 3-connected. As shown by Tutte [28] (see Figure 4), a plane network C is either a series or parallel composition of plane networks, or it is obtained from a polyhedral network T where each edge e is possibly substituted by a plane network C_e, identifying the end-points of e with those of the root of C_e. In that case T is called the *3-connected core* of C and the components C_e are called the *pieces* of C. Calling d_e the degree of the root face of C_e, we obtain the following inequalities, which will be used to get a diameter estimate for random 3-connected maps from a diameter estimate for random 2-connected maps:

$$\mathrm{diam}(T) \leq \mathrm{diam}(C) \leq \mathrm{diam}(T) \cdot \max_{e\in T}(d_e) + 2\max_{e\in T}(\mathrm{diam}(C_e)). \tag{6}$$

The first inequality is trivial. The second one follows from the fact that a diametral path P in C starts in a piece, ends in a piece, and in between it passes by vertices $v_1, \ldots, v_k$ of T such that for $1 \leq i < k$, v_i and v_{i+1} are adjacent in T —let $e = \{v_i, v_{i+1}\}$— and P travels in the piece C_e to reach v_{i+1} from v_i; since P is geodesic, its length in C_e is bounded by the distance from v_i to v_{i+1}, which is clearly bounded by d_e.

2.3. **Planar graphs.** By a theorem of Whitney, a 3-connected planar graph has a unique embedding on the oriented sphere. Hence 3-connected planar maps are equivalent to 3-connected planar graphs. Once we have an estimate for the diameter of random 3-connected maps, hence also for random 3-connected planar graphs, we can carry such an estimate up to random connected planar graphs, using a well known decomposition of a connected planar graph into 3-connected components, via a decomposition into 2-connected components. We now describe these decompositions and give inequalities relating the diameter of a graph to the diameters of its components.

2.3.1. *Decomposing a connected planar graph into 2-connected components.* There is a well-known decomposition of a graph into 2-connected components [25, 29]. Given a connected graph C, a *block* of C is a maximal 2-connected subgraph of C. The set of blocks of C is denoted by $\mathfrak{B}(C)$. A vertex $v \in C$ is said to be *incident* to a block $B \in \mathfrak{B}(C)$ if v belongs to B. The *Bv-tree* is the

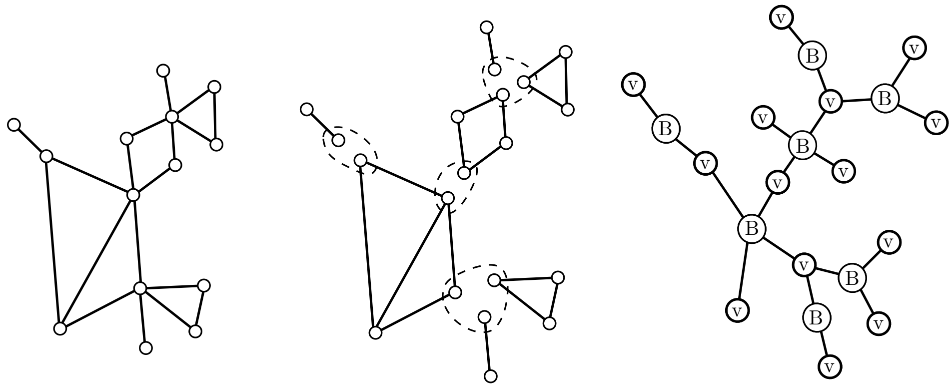

FIGURE 5. Decomposition of a connected graph into blocks, and the associated Bv-tree.

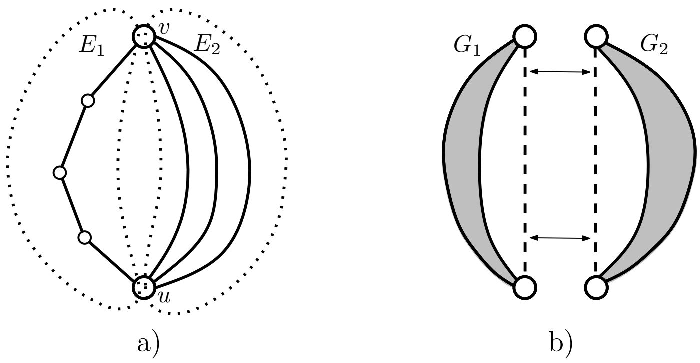

FIGURE 6. (a) Example of a split-candidate. (b) Splitting a graph along a virtual edge.

bipartite graph $\tau(C)$ with vertex-set $V(C) \cup \mathfrak{B}(C)$, and edge-set given by the incidences between the vertices and the blocks of C; see Figure 5. It is easy to see that $\tau(C)$ is actually a tree.

We will use the following inequalities to get a diameter estimate for random connected planar graphs from a diameter estimate for random 2-connected planar graphs. For a 2-connected planar graph G, with Bv-tree τ and $B_1, \ldots, B_k$, we have:

$$\max_i(\mathrm{diam}(B_i)) \leq \mathrm{diam}(G) \leq \max_i(\mathrm{diam}(B_i)) \cdot \mathrm{diam}(\tau). \tag{7}$$

The first inequality is trivial. The second inequality follows from the fact that a diametral path in G induces a path P in τ of length at most $\mathrm{diam}(\tau)$, and the length "used" by each block B along P is at most $\mathrm{diam}(B)$.

2.3.2. *Decomposing a 2-connected planar graph into 3-connected components.* In this section we recall Tutte's decomposition of a 2-connected graph into 3-connected components [28]. First, we define connectivity modulo a pair of vertices. Let G be a 2-connected graph (possibly with multiple edges) and $\{u, v\}$ a pair of vertices of G. Then G is said to be *connected modulo* $[u, v]$ if u and v are not adjacent and if $G \backslash \{u, v\}$ is connected.

Consider a 2-separator E_1, E_2 of a 2-connected graph G, with u, v the corresponding separating vertex-pair. Then E_1, E_2 is called a *split-candidate*, denoted by $\{E_1, E_2, u, v\}$, if $G[E_1]$ is connected modulo $[u, v]$ and $G[E_2]$ is 2-connected. Figure 6(a) gives an example of a split-candidate, where $G[E_1]$ is connected modulo $[u, v]$ but not 2-connected, while $G[E_2]$ is 2-connected but not connected modulo $[u, v]$.

As described below, split-candidates make it possible to decompose completely a 2-connected graph into 3-connected components. We consider here only 2-connected graphs with at least three

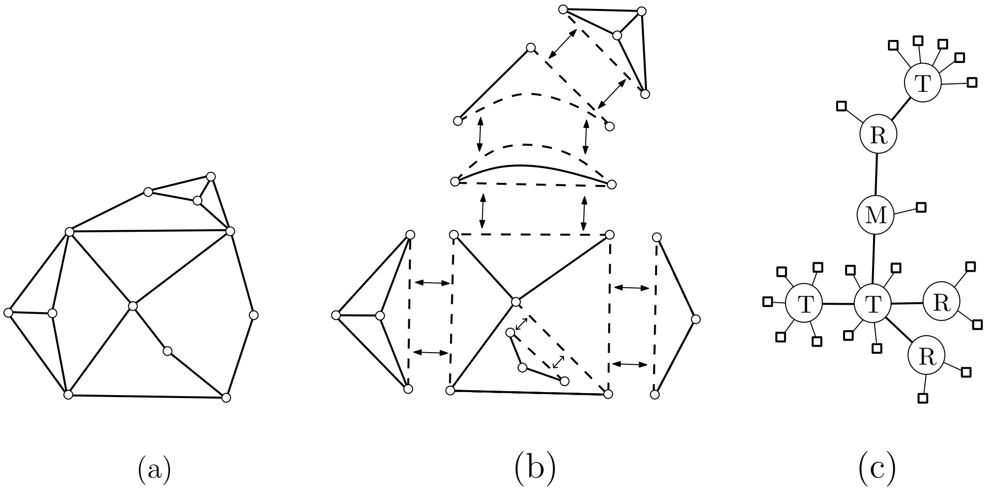

FIGURE 7. (a) A 2-connected graph, (b) decomposed into bricks. (c) The associated RMT-tree.

edges (graphs with less edges are degenerated for this decomposition). Given a split-candidate $S = \{E_1, E_2, u, v\}$ in a 2-connected graph G (see Figure 6(b)), the corresponding *split operation* is defined as follows, see Figure 6(b):

- an edge e, called a *virtual edge*, is added between u and v,
- the graph $G[E_1]$ is separated from the graph $G[E_2]$ by cutting along the edge e.

Such a split operation yields two graphs G_1 and G_2, which correspond respectively to $G[E_1]$ and $G[E_2]$, together with e as a real edge; see Figure 6(d). The graphs G_1 and G_2 are said to be *matched* by the virtual edge e. It is easily checked that G_1 and G_2 are 2-connected (and have at least three edges). The splitting process can be repeated until no split-candidate remains.

As shown by Tutte in [29], the structure resulting from the split operations is independent of the order in which they are performed. It is a collection of graphs, called the *bricks* of G, which are articulated around virtual edges; see Figure 7(b). By definition of the decomposition, each brick has no split-candidate; Tutte shows that such graphs are either multiedge-graphs (M-bricks) or ring-graphs (R-bricks), or 3-connected graphs with at least four vertices (T-bricks).

The *RMT-tree* of G is the graph $\tau(G)$ whose inner nodes correspond to the bricks of G, and the edges between such vertices correspond to the virtual edges of G (each virtual edge matches two bricks); additionally the leaves of $\tau(G)$ correspond to the real (not virtual) edges of G; see Figure 7. The graph $\tau(G)$ is indeed a tree [29]. By maximality of the decomposition, it is easily checked that $\tau(G)$ has no two adjacent R-bricks nor two adjacent M-bricks.

We will use the following inequalities to get a diameter estimate for random 2-connected planar graphs from a diameter estimate for random 3-connected planar graphs (which are equivalent to random 3-connected maps, by Whitney's theorem). For a 2-connected planar graph G, with RMT-tree τ, bricks $B_1, \ldots, B_k$, and $\mathcal{E}_{\text{virt}}$ as set of pairs of vertices of G connected by a virtual edge, we have:

$$\max_i(\text{diam}(B_i)) \leq \text{diam}(G) \leq \max_i(\text{diam}(B_i)) \cdot \text{diam}(\tau) \cdot \max_{(u,v)\in\mathcal{E}_{\text{virt}}}\text{Dist}_G(u,v). \tag{8}$$

The first inequality is trivial. The second inequality follows from the following facts:

- a diametral path P_G in G induces a path P in τ (of length at most $\text{diam}(\tau)$),
- for a brick B traversed by P_G (B corresponds to a vertex of τ that lies on P), the path P_G induces a path $P_B = (v_0, \ldots, v_k)$ in B, where each edge $\{v_i, v_{i+1}\}$ is either a virtual edge or a real edge of G.
- the length of P_G "used" when traversing an edge $e = \{v_i, v_{i+1}\} \in P_B$ is at most the distance between v_i and v_{i+1} in G.

Hence the length of P_G "used by B" is at most $\mathrm{diam}(B)\cdot\max_{(u,v)\in\mathcal{E}_{\mathrm{virt}}}\mathrm{Dist}(u,v)$, so that the total length of P_G is given by the second inequality.

3. Diameter estimates for families of maps

In this section we consider families of maps, starting with quadrangulations and ending with 3-connected maps. In each case we show that for a random map G of size n in such a family, we have $\mathrm{diam}(G)\in(n^{1/4-\epsilon},n^{1/4+\epsilon})$ a.a.s. with exponential rate, where the size parameter n is typically the number of edges or the number of faces. In order to carry such estimates to planar graphs we need to show that such concentration properties hold more generally in a weighted setting. More precisely, if a combinatorial class $\mathcal{G}=\cup_n\mathcal{G}_n$ (each $\gamma\in\mathcal{G}$ has a size $|\gamma|\in\mathbb{N}$, and the set of objects of $\mathcal{G}$ of size n is denoted $\mathcal{G}_n$) has an additional weight-function $w(\cdot)$, then the *generating function* of $\mathcal{G}$ is

$$G(z)=\sum_{\alpha\in\mathcal{G}}w(\alpha)z^{|\alpha|},$$

and the *weighted* probability distribution in size n assigns to each map $G\in\mathcal{G}_n$ the probability

$$\mathbb{P}(G)=\frac{w(G)}{C_n},\quad\text{with } C_n=\sum_{G\in\mathcal{G}_n}w(G).$$

Typically, for planar maps and planar graphs, the weight will be of the form $w(G)=x^{\chi(G)}$, with x a fixed positive real value and χ a parameter such as the number of vertices; in that case the terminology will be "a random map of size n with weight x at vertices".

3.1. Quadrangulations.

From Schaeffer's bijection in Section 2.2.1 it is easy to show large deviation results for the diameter of a quadrangulation. The basic idea, originating in [7], is that the typical depth k of a vertex in the tree is $n^{1/2}$, and the typical discrepancy of the labels along a branch is $k^{1/2}=n^{1/4}$. We use a fundamental result from [11], namely that under very general conditions the height of a random tree of size n from a given family is in $(n^{1/2-\epsilon},n^{1/2+\epsilon})$ a.a.s. with exponential rate.

Let $y(z)=\sum_{\tau\in\mathcal{T}}z^{|\tau|}w(\tau)$ be the weighted generating function of some combinatorial class $\mathcal{T}$ (typically $\mathcal{T}$ is a class of rooted trees), and denote by ρ the radius of convergence of $y(z)$, assumed to be strictly positive. Assume $y\equiv y(z)$ satisfies an equation of the form

$$y=F(z,y), \tag{9}$$

with $F(z,y)$ a bivariate function with nonnegative coefficients, nonlinear in y, analytic around $(0,0)$, such that $F(0,0)=0$ and $F(0,y)=0$. By the non-linearity of (9) with respect to y, $y(\rho)$ is finite; let $\tau=y(\rho)$. Equation (9) is called *admissible* if $F(z,y)$ is analytic at (ρ,τ). A *height-parameter* for (9) is a nonnegative integer parameter ξ for structures in $\mathcal{T}$ such that $y_h(z)=\sum_{\tau\in\mathcal{T},\xi(\tau)\le h}w(\tau)z^{|\tau|}$ satisfies

$$y_{h+1}(z)=F(z,y_h(z))\ \text{ for } h\ge 0,\quad y_0=0.$$

Lemma 3.1 (Theorem 3.1 in [11])**.** *Let $\mathcal{T}$ be a combinatorial class endowed with a weight-function $w(\cdot)$ so that the corresponding weighted generating function $y(z)$ satisfies an equation of the form* (9)*, and such that* (9) *is admissible.*

Let ξ be a height-parameter for (9) *and let T_n be taken at random in $\mathcal{T}_n$ under the weighted distribution in size n. Then $\xi(T_n)\in(n^{1/2-\epsilon},n^{1/2+\epsilon})$ a.a.s. with exponential rate*[1].

The next proposition is proved as a warm up, what we will need is a weighted version that is more technical to prove.

Proposition 3.2. *The diameter of a random rooted quadrangulation with n faces is, a.a.s. with exponential rate, in the interval $(n^{1/4-\epsilon},n^{1/4+\epsilon})$.*

[1]The authors of [11] prove the result for rooted plane trees, then they say that all the arguments in the proof hold for any system of the form $y=z\phi(y)$. The arguments hold even more generally for any admissible system of the form $y=F(z,y)$.

Proof. When the number of black vertices is not taken into account, the statement of Theorem 2.2 simplifies: it gives a 1-to-2 correspondence between labelled trees having n edges and rooted pointed quadrangulations having n faces; once again for a vertex v of a labelled tree τ, the quantity $\ell_v - \ell_{\min}$ gives the distance of v from the pointed vertex in the associated quadrangulation. According to (3), we just have to show that, for a uniformly random labelled tree τ with n vertices, $L(\tau) = \ell_{\max} - \ell_{\min}$ is in $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$ a.a.s. with exponential rate. Since the label either increases by 1, stays equal, or decreases by 1 along each edge (going away from the root), the series $T(z)$ of labelled trees counted according to vertices satisfies

$$T(z) = \frac{z}{1 - 3T(z)},$$

and the usual height of the tree is a height-parameter for this equation. The equation is clearly admissible (the singularity is at $1/12$ and $T(1/12) = 1/6$), hence by Lemma 3.1 the height is in $(n^{1/2-\epsilon}, n^{1/2+\epsilon})$ a.a.s. with exponential rate. So in a random labelled tree there is a.a.s. with exponential rate a path B of length $k = n^{1/2-\epsilon}$ starting from the root. The labels along B form a random walk with increments $+1$, 0, -1, each with probability $1/3$. Classically the maximum of such a walk is at least $k^{1/2-\epsilon}$ (which is at least $n^{1/4-\epsilon}$) a.a.s. with exponential rate. Hence the label of the vertex v on B at which the maximum occurs is at least the label of the root-vertex plus $n^{1/4-\epsilon}$, so $\ell_{max} \geq n^{1/4-\epsilon}$ a.a.s. with exponential rate. Since $\ell_{min} \leq 0$, this proves the lower bound.

For the upper bound (already proved in [7]), since the height is at most $n^{1/2+\epsilon}$ a.a.s. with exponential rate, the same is true for the depth k of a random vertex v in a random labelled tree of size n. The labels along the path from the root to v form a random walk of length k, the maximum of which is at most $k^{1/2+\epsilon}$ a.a.s. with exponential rate. Hence $|\ell(v)| \leq n^{(1/2+\epsilon)^2}$ a.a.s. with exponential rate, so the same holds for the property $|\ell(v)| \leq n^{1/4+\epsilon}$. Since multiplying by n keeps the probability of failure exponentially small, the property $\{\forall v \in Q,\ |\ell(v)| \leq n^{1/4+\epsilon}\}$ is true a.a.s. with exponential rate. This completes the proof. □

The next theorem generalizes Proposition 3.2 to the weighted case, which is needed later on. The analytical part of the proof is more delicate since the system specifying weighted labelled trees needs two lines, and has to be transformed to a one-line equation in order to apply Lemma 3.1.

Theorem 3.3. *Let $0 < a < b$. The diameter of a random rooted quadrangulation with n faces and weight x at black vertices is, a.a.s. with exponential rate, in the interval $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$, uniformly over $x \in [a, b]$.*

Proof. A bicolored labelled tree is called *black-rooted* (resp. white-rooted) if the root-vertex is black (resp. white). In a bicolored labelled tree the *white-black* depth of a vertex v is defined as the number of edges going from a white to a black vertex on the path from the root-vertex to v, and the *white-black height* is defined as the maximum of the white-black depth over all vertices. We use here a decomposition of a bicolored labelled tree into monocolored components (the components are obtained by removing the bicolored edges), each such component being a plane tree. Let $f(z)$ (resp. $g(z)$) be the weighted generating function of black-rooted (resp. white-rooted) bicolored labelled trees, where z marks the number of vertices, and where each tree τ with i black vertices has weight $w(\tau) = x^i$. Let $T(z)$ be the series counting rooted plane trees according to edges, $T(z) = 1/(1 - zT(z))$. A tree counted by $f(z)$ is made of a monochromatic component (a rooted plane tree) where in each corner one might insert a sequence of trees counted by $g(z)$; in addition each time one inserts a tree counted by $g(z)$ one has to choose if the label increases or decreases along the corresponding black-white edge. Since a rooted plane tree with k edges has $2k + 1$ corners and $k + 1$ vertices, we obtain

$$f(z) = \frac{xz}{1 - 2g(z)} T\left(\frac{xz}{(1 - 2g(z))^2}\right).$$

Similarly

$$g(z) = \frac{z}{1 - 2f(z)} T\left(\frac{z}{(1 - 2f(z))^2}\right).$$

Hence the series $y = f(z)$ satisfies the equation $y = F(z, y)$, where $F(z, y)$ is specified by the system of equations

$$\begin{aligned} F(z,y) &= \frac{xz}{1-2G(z,y)} T\left(\frac{xz}{(1-2G(z,y))^2}\right), \\ G(z,y) &= \frac{z}{1-2y} T\left(\frac{z}{(1-2y)^2}\right). \end{aligned} \tag{10}$$

In addition, the white-black height is a height-parameter for this system.

Claim. *The system* (10) *is admissible.*

Proof of the claim. Let ρ be the singularity of $f(z)$ and $\tau = f(\rho)$. Let us prove first that $G(z, y)$ is analytic at (ρ, τ). Note that $\tau < 1/2$, otherwise there would be $z_0 \le \rho$ such that $f(z_0) = 1/2$, in which case $g(z)$ (and $f(z)$ as well) would diverge to ∞ as $z \to z_0^-$, contradicting the fact that $f(z)$ converges for $0 \le |z| \le \rho$. The other possible cause of singularity is $\rho/(1-2\tau)^2$ being a singularity of $T(z)$. We use the symbol $\succeq$ for coefficient-domination, i.e., $A(z) \succeq B(z)$ if $[z^n]A(z) \ge [z^n]B(z)$ for all $n \ge 0$. Clearly we have

$$f(z) \succeq 2xzg(z), \qquad g'(z) \succeq 2zf'(z)T'\left(\frac{z}{(1-2f(z))^2}\right),$$

hence

$$f'(z) \succeq 4xz^2 f'(z)T'\left(\frac{z}{(1-2f(z))^2}\right).$$

As a consequence,

$$T'\left(\frac{z}{(1-2f(z))^2}\right) \le \frac{1}{4xz^2}, \quad \text{as } z \to \rho^-.$$

Since $T'(u)$ diverges at its singularity $1/4$, we have $\rho/(1-2\tau)^2 \ne 1/4$, otherwise there would be the contradiction that the left-hand side diverges whereas the right-hand side, which is larger, converges as $z \to \rho^-$. Hence T is analytic at $\rho/(1-2\tau)^2$, which ensures that $G(z, y)$ is analytic at (ρ, τ). One proves similarly that $F(z, y)$ is also analytic at (ρ, τ). △

The claim, combined with Lemma 3.1, ensures that the white-black height of a random black-rooted bicolored labelled tree with n edges and weight x at black vertices ($x \in [a, b]$) is in $(n^{1/2-\epsilon}, n^{1/2+\epsilon})$ a.a.s. with exponential rate. In addition, the chain of calculations in [11] to prove Lemma 3.1 is easily seen to be uniform in $x \in [a, b]$. A similar analysis ensures that the *white-black* height of a random white-rooted bicolored labelled tree with n edges and weight x at black vertices is in $(n^{1/2-\epsilon}, n^{1/2+\epsilon})$ a.a.s. with exponential rate. Hence, overall, the white-black height of a random bicolored tree (either black-rooted or white-rooted) with n edges and weight x at black vertices is in $(n^{1/2-\epsilon}, n^{1/2+\epsilon})$ a.a.s. with exponential rate.

Now the proof can be concluded in a similar way as in Proposition 3.2. Define the bicolored depth of a vertex v from the root as the number of bicolored edges on the path from the root to v, and define the bicolored height as the maximum of the bicolored depth over all vertices in the tree. Note that the bicolored depth $d(v)$ and the white-black depth $d'(v)$ of a vertex v satisfy the inequalities $2d'(v) \le d(v) \le 2d'(v) + 2$, so the bicolored height is in $(n^{1/2-\epsilon}, n^{1/2+\epsilon})$ a.a.s. with exponential rate, uniformly over $x \in [a, b]$. Similarly as in Proposition 3.2, this ensures that $\ell_{max} - \ell_{min}$ is in $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$ a.a.s. with exponential rate. And the uniformity over $x \in [a, b]$ follows from the uniformity over $x \in [a, b]$ for the height.

Finally, using the bijection of Theorem 2.2, the property that $\ell_{max} - \ell_{min}$ is in $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$ a.a.s. with exponential rate is transferred to the property that the diameter of a random quadrangulation with n faces (with a marked vertex and a marked edge) and weight x at each black vertex is in $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$ a.a.s. with exponential rate. There is however a last subtlety to deal with, namely that in the bijection from bicolored labelled trees to quadrangulations with a marked vertex and a marked edge, the number of black vertices in the tree corresponds either to the number of black vertices or to the number of black vertices plus one in the associated quadrangulation. So the weighted distribution (weight x at black vertices) on bicolored labelled trees with n edges is not exactly transported to the weighted distribution (weight x at black vertices) on rooted pointed quadrangulations with n faces. However, since the inaccuracy on the number of black vertices in the quadrangulation is by at most one, the transported weighted distribution is biased by at most x, so the large deviation result also holds under the (perfectly) weighted distribution for quadrangulations [2]. □

3.2. **Maps.** We use here the bijection of Section 2.2.2 to get a diameter estimate for random maps from a diameter estimate for random quadrangulations. First we need the following lemma.

Lemma 3.4. *Let $M(z,u)$ be the generating function of rooted maps, where z marks the number of edges, u marks the degree of the outer face, and with weight x at each vertex. Let ρ be the radius of convergence of $M(z,1)$ (note that ρ depends on x). Then there is $u_0 > 1$ such that $M(\rho,u_0)$ converges. In addition for $0 < a < b$, the value of u_0 can be chosen uniformly over $x \in [a,b]$, and $M(\rho,u_0)$ is uniformly bounded over $x \in [a,b]$.*

Proof. The result follows easily from a bijection by Bouttier, Di Francesco and Guitter between vertex-pointed planar maps and a certain family of decorated trees called *mobiles*, such that each face of degree i in the map corresponds to a (black) vertex of degree i in the mobile. Thanks to this bijection, the generating function $M^\circ(z,u)$ of rooted maps with a secondary marked vertex (where again z marks the number of edges and u marks the root-face degree) equals the generating function of rooted mobiles where z marks half the total degree of (black) vertices and u marks the root-vertex degree. Since mobiles (as rooted trees) satisfy an explicit decomposition at the root, the series $M^\circ(z,u)$ is easily shown to have, for any $x > 0$, a square-root singular development of the form

$$M^\circ(z,u) = a(z,u) - b(z,u)\sqrt{1 - z/\rho},$$

valid in a neighborhood of $(\rho, 1)$, with $a(z,u)$ and $b(z,u)$ analytic in the parameters z, u, x. Hence the statement holds for $M^\circ(z,u)$. Since $M^\circ(z,u)$ dominates $M(z,u)$ coefficient-wise, the statement also holds for $M(z,u)$. □

Theorem 3.5. *Let $0 < a < b$. The diameter of a random rooted map with n edges and weight x at the vertices is in the interval $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$ a.a.s. with exponential rate, uniformly over $x \in [a,b]$.*

Proof. The first important observation is that the bijection of Section 2.2.2 transports the weighted (weight x at black vertices) distribution on rooted quadrangulations with n faces to the weighted (weight x at vertices) distribution on rooted maps with n edges. Let M be a random rooted map with n edges and let Q be the associated rooted quadrangulation (with $n+2$ vertices). Since $\mathrm{diam}(Q) \leq 2\,\mathrm{diam}(M)$, the diameter of M is at least $n^{1/4-\epsilon}$ a.a.s. with exponential rate. The upper bound is proved from the inequality $\mathrm{diam}(M) \leq \mathrm{diam}(Q) \cdot \Delta(M)$, where $\Delta(M)$ is the maximal face degree in M. Together with Lemma 2.1, Lemma 3.4 ensures that the root-face degree $\delta(M)$ in a random rooted planar map with n edges and weight x at vertices has exponentially fast decaying tail. The probability distribution of $\delta(M)$ is the same if M is bi-rooted (i.e., has two roots that are possibly equal, the root-face being the face incident to the primary root). When exchanging the secondary root with the primary root, the root-face can be seen as a face f taken at random under the distribution $P(f) = \deg(f)/(2n)$. Thus $\delta(M)$ is distributed as the degree of the (random) face f. Hence

$$P(\Delta(M) \geq k) \leq \frac{2n}{k} P(\delta(M) \geq k),$$

so that $\Delta(M) \leq n^\epsilon$ a.a.s. with exponential rate. We conclude from (4) that the diameter of M is at most $n^{1/4+\epsilon}$ a.a.s. with exponential rate. The uniformity in $x \in [a,b]$ follows from the uniformity in $x \in [a,b]$ in Theorem 3.3 and Lemma 3.4. □

[2]The color of the pointed vertex would be a delicate issue if we were trying to prove an explicit limit distribution (instead of large deviation results) for the diameter.

3.3. **2-connected maps.** Let $x > 0$. Denote by $M(z)$ (resp. $C(z)$) the weighted generating function of rooted connected (resp. 2-connected) maps according to edges and with weight x at non-root vertices. Since a core with k edges has $2k$ corners where to insert rooted maps, this decomposition yields

$$M(z) = \sum_{n\geq 0} z^n \sum_{\tau\in\mathcal{C}_n} M(z)^{2n} = C(H(z)), \quad \text{where } H(z) = z(1+M(z))^2. \tag{11}$$

An important property of the core-decomposition is that it preserves the distribution with weight x at vertices. Precisely, let M be a random rooted map with n edges and weight x at vertices. Let C be the core of M and let k be its size. Let $M_1, \ldots, M_{2k}$ be the pieces of M, and $n_1, \ldots, n_{2k}$ their sizes. Then, conditioned to having size k, C is a random rooted 2-connected map with k edges and weight x at vertices; and conditioned to having size n_i, the ith piece M_i is a random rooted map with n_i edges and weight x at vertices.

Lemma 3.6. *Let $0 < a < b$, and let $x \in [a, b]$. Let ρ be the radius of convergence of $z \mapsto M(z)$ ($M(z)$ gives weight x to vertices). Following* [2], *define*

$$\alpha = \frac{H(\rho)}{\rho H'(\rho)}.$$

Let $n \geq 0$, and let M be a random rooted map with n edges and weight x at vertices. Let $X_n = |C|$ be the size of the core of M, and let $M_1, \ldots, M_{2|C|}$ be the pieces of M. Then

$$P\left(X_n = \lfloor \alpha n \rfloor,\ \max(|M_i|) \leq n^{3/4}\right) \sim P\left(X_n = \lfloor \alpha n \rfloor\right) = \Theta(n^{-2/3})$$

uniformly over $x \in [a, b]$.

Proof. The statement $P(X_n = \lfloor \alpha n \rfloor) = \Theta(n^{-2/3})$ uniformly over $x \in [a, b]$ is proved in [2]. So what we have to prove is that $P(X_n = \lfloor \alpha n \rfloor,\ \max(|M_i|) > n^{3/4}) = o(n^{-2/3})$ uniformly over $x \in [a, b]$.

Claim. *Given a fixed $\delta > 0$, we have for $i > n^{2/3+\delta}$*

$$P(X_n = \lfloor \alpha n \rfloor, \quad |M_1| = i) = O(\exp(-n^{\delta/2})).$$

Proof of the claim. Let a_m be the number of rooted maps and c_m the number of rooted 2-connected maps with m edges. As proved in [28], these numbers have the well known asymptotic estimates $a_m \sim c\rho^{-m}m^{-5/2}$, $c_m \sim c'\sigma^{-m}m^{-5/2}$. Equation (11) implies

$$P(X_n = k) = c_k \frac{[z^n]H(z)^k}{a_n}.$$

It is proved in [15, Theorem 1 (iii)-(b)], (and the bounds are easily checked to hold uniformly over $x \in [a, b]$) that for $k \geq \alpha n + n^{2/3+\delta}$,

$$[z^n]H(z)^k = O(\sigma^k \rho^{-n} \exp(-n^\delta)). \tag{12}$$

Let $k_0 = \lfloor \alpha n \rfloor$ and let $n^{2/3+\delta} < i \leq n - k_0$. We have

$$\begin{aligned} P(X_n = k_0,\ |M_1| = i) &= c_{k_0} \frac{a_i [z^{n-i}] z^{k_0} (1+M(z))^{2k_0-1}}{a_n} \\ &\leq c_{k_0} \frac{a_i [z^{n-i}] H(z)^{k_0}}{a_n} = O(\sigma^{-k_0} \rho^{n-i} [z^{n-i}] H(z)^{k_0}). \end{aligned}$$

Since $k_0/(n-i) \geq \alpha(1 + i/n)$, we have $k_0 \geq \alpha(n-i) + \alpha(n-i)^{2/3+\delta}$, so (12) ensures that

$$[z^{n-i}]H(z)^{k_0} = O(\sigma^{k_0} \rho^{-n+i} \exp(-(n-i)^\delta)).$$

Hence, for $i > n^{2/3+\delta}$,

$$P(X_n = k_0,\ |M_1| = i) = O(\exp(-(n-i)^\delta)),$$

so that $P(X_n = k_0,\ |M_1| = i) = O(\exp(-n^{\delta/2}))$. △

The claim implies that $P(X_n = \lfloor \alpha n \rfloor,\ |M_1| > n^{2/3+\delta}) = O(n\exp(-n^{\delta/2}))$, and by symmetry the same estimate holds for each piece M_i. As a consequence $P(X_n = \lfloor \alpha n \rfloor,\ \mathrm{Max}(|M_i|) > n^{2/3+\delta}) = O(n^2\exp(-n^{\delta/2})) = O(\exp(-n^{\delta/3}))$. Hence

$$P(X_n = \lfloor \alpha n \rfloor,\ \mathrm{Max}(|M_i|) \leq n^{2/3+\delta}) \sim P(X_n = \lfloor \alpha n \rfloor) = \Theta(n^{-2/3}).$$

This concludes the proof, taking $\delta = 3/4 - 2/3 = 1/12$. □

In [2] the authors show that $n^{2/3}P(X_n = \lfloor \alpha n \rfloor)$ converges; they even prove that $(X_n - \alpha n)/n^{2/3}$ converges in law. Lemma 3.6 just makes sure that the asymptotic estimate of $P(X_n = \lfloor \alpha n \rfloor)$ is the same under the additional condition that all pieces are of size at most $n^{3/4}$ (more generally, under the condition that all pieces are of size at most $n^{2/3+\delta}$, for any $\delta > 0$). A closely related result proved in [15] is that, for any fixed $\delta > 0$, there is a.a.s. no piece of size larger than $n^{2/3+\delta}$ provided the core has size larger than $n^{2/3+\delta}$.

Theorem 3.7. *For $0 < a < b$, the diameter of a random rooted 2-connected map with n edges and weight x at vertices is, a.a.s. with exponential rate, in the interval $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$, uniformly over $x \in [a, b]$.*

Proof. Let M be a rooted map with n edges and weight x at vertices. Denote by C the core of M and by $(M_i)_{i\in[1..2|C|]}$ the pieces of M. Since the event $\{|C| = \lfloor \alpha n \rfloor\}$ has polynomially small probability (order $\Theta(n^{-2/3})$, as shown in [2]), and since the event $\mathrm{diam}(M) \leq n^{1/4+\epsilon}$ holds a.a.s. with exponential rate, the event $\mathrm{diam}(M) \leq n^{1/4+\epsilon}$, knowing that $|C| = \lfloor \alpha n \rfloor$, also holds a.a.s. with exponential rate. Since $\mathrm{diam}(M) \geq \mathrm{diam}(C)$, we conclude that for C a random 2-connected map with $\lfloor \alpha n \rfloor$ edges and weight x at vertices, $\mathrm{diam}(C) \leq n^{1/4+\epsilon}$ a.a.s. with exponential rate. Of course the same holds for C a random rooted 2-connected map with n edges and weight x at vertices. This yields the a.a.s. upper bound on $\mathrm{diam}(C)$.

To prove the lower bound, we use Lemma 3.6, which ensures that the event

$$\{|C| = \lfloor \alpha n \rfloor,\ \max(|M_i|) \leq n^{3/4}\}$$

occurs with polynomially small probability, precisely $\Theta(n^{-2/3})$. We claim that, under the condition that $\max(|M_i|) \leq n^{3/4}$, then $\max(\mathrm{diam}(M_i)) \leq n^{1/5}$ a.a.s. (in n) with exponential rate. Indeed, consider a piece M_i of size n_i. When $n_i \leq n^{1/5}$, $\mathrm{diam}(M_i) \leq n^{1/5}$ trivially. Moreover, Theorem 3.5 implies that, for $\delta > 0$ small enough, $P(\mathrm{diam}(M_i) > n_i^{1/4+\delta}) \leq \exp(-n_i^{c\delta})$ for some $c > 0$. Hence when $n^{1/5} \leq n_i \leq n^{3/4}$, $P(\mathrm{diam}(M_i) > n^{3/4(1/4+\delta)}) \leq \exp(-n^{c\delta/5})$, and we can take δ small enough so that $3/4(1/4 + \delta) \leq 1/5$. Hence, when $n_i \leq n^{3/4}$, the event $\mathrm{diam}(M_i) > n^{1/5}$ has exponentially small probability in n (meaning, in $O(\exp(-n^{\alpha})$ for some $\alpha > 0$), and the same holds for $\max(\mathrm{diam}(M_i))$. Hence

$$\mathbb{P}(\{|C| = \lfloor \alpha n \rfloor,\ \max(\mathrm{diam}(M_i) \leq n^{1/5}\}) \sim \mathbb{P}(\{|C| = \lfloor \alpha n \rfloor\}) = \Theta(n^{-2/3}).$$

In other words the event $\{|C| = \lfloor \alpha n \rfloor,\ \max(\mathrm{diam}(M_i) \leq n^{1/5}\}$ occurs with polynomially small probability. In that case, since $\mathrm{diam}(C) \geq \mathrm{diam}(M) - 2\max(\mathrm{diam}(M_i))$, and since the event $\mathrm{diam}(M) < n^{1/4-\epsilon}$ occurs a.a.s. with exponential rate, we conclude that $\mathrm{diam}(C) \geq n^{1/4-\epsilon} - 2n^{1/5}$ holds a.a.s. with exponential rate under the event $\mathcal{E} = \{|C| = \lfloor \alpha n \rfloor,\ \max(\mathrm{diam}(M_i) \leq n^{1/5}\}$. Since $\mathcal{E}$ occurs with probability $\Theta(n^{-2/3})$ and since $n^{1/5} = o(n^{1/4-\epsilon})$ for ϵ small enough, we conclude (similarly as in the proof of Theorem 3.7) that for C a random 2-connected map with $\lfloor \alpha n \rfloor$ edges and weight x at vertices, we have $\mathrm{diam}(C) \geq n^{1/4-\epsilon}$ a.a.s. with exponential rate. The same holds for C a random rooted 2-connected with n edges and weight x at vertices.

The uniformity in $x \in [a, b]$ of the bounds follows from the uniformity in x in Theorem 3.5 and Lemma 3.6. □

3.4. **3-connected maps.** In the following we assume 3-connected maps (and 3-connected planar graphs) to have at least 4 vertices, so the smallest 3-connected planar graph is K_4. We use here the plane network decomposition (Section 2.2.4) to carry the diameter concentration property from 2-connected to 3-connected maps. For $x > 0$, call $N(z)$ (resp. $\widehat{N}(z)$) the weighted generating functions —weight x at vertices not incident to the root-edge— of plane networks (resp. plane networks with a 3-connected core), where z marks the number of edges. Note that $N(z)$ is very close to the generating function $C(z)$ of rooted 2-connected maps with weight x at non-root vertices and with z marking the number of edges:

$$C(z) = z + xz + xzN(z),$$

where the first two terms in the right-hand side stand for the two 2-connected maps with a single edge, either a loop or a link between two distinct vertices. Call $T(z)$ the weighted generating function of rooted 3-connected maps, with weight x at vertices not incident to the root-edge, and with z marking the number of non-root edges. Clearly, the weighted generating function $S(z)$ of plane networks decomposable as a sequence of plane networks satisfies $S(z) = (N(z)-S(z))xN(z)$, hence $S(z) = xN(z)^2/(1 + xN(z))$. Similarly the weighted generating function $P(z)$ of parallel plane networks satisfies $P(z) = (N(z) - P(z))N(z)$, so that $P(z) = N(z)^2/(1 + N(z))$. Hence

$$N(z) = S(z) + P(z) + \widehat{N}(z), \tag{13}$$

where

$$S(z) = \frac{xN(z)^2}{1 + xN(z)}, \quad P(z) = \frac{N(z)^2}{1 + N(z)}, \quad \widehat{N}(z) = T(N(z)).$$

An important remark is that a random plane network C with n edges and weight x at vertices can be seen as a random 2-connected map with $n + 1$ edges, weight x at vertices, and where the root-edge has been deleted. Similarly as in Section 3.3, for a random plane network N with n edges and weight x at vertices, and conditioned to having a 3-connected core T of size k, T is a random rooted 3-connected map with k edges and weight x at vertices; and each piece C_e conditioned to have a given size n_e is a random plane network with n_e edges and weight x at vertices.

For proving the diameter estimate for 3-connected maps, we need the following lemma, ensuring that the root-face degree of a random 2-connected map is small.

Lemma 3.8. *Let $C(z, u)$ be the generating function of rooted 2-connected maps, where z marks the number of edges, u marks the root-face degree, and with weight x at each non-root vertex. Let R be the radius of convergence of $C(z, 1)$. Then there is $v_0 > 1$ such that $C(\rho, v_0)$ converges. In addition for $0 < a < b$, the value of v_0 can be chosen uniformly over $x \in [a, b]$, and $C(z, v_0)$ is uniformly bounded over $x \in [a, b]$.*

Proof. The result has been established for arbitrary rooted maps in Lemma 3.4. To prove the result for 2-connected maps, we rewrite Equation 11 taking account of the root-face degree, which is (with $M(z) = M(z, 1)$)

$$M(z, u) = C\left(z(1 + M(z))^2, \frac{M(z, u)}{M(z)}\right).$$

Since the composition scheme is "critical" [2], it is known that, if ρ denotes the radius of convergence of $M(z, 1)$, then $R = \rho \cdot (1 + M(\rho))^2$ is the radius of convergence of $C(z, 1)$. Hence, since $M(\rho, u_0)$ converges, $C(R, v_0)$ converges for $v_0 = M(\rho, u_0)/M(\rho) > 1$. The uniformity statement for $C(z, u)$ (for $x \in [a, b]$) follows from the uniformity statement for $M(z, u)$, established in Lemma 3.4, and the fact that v_0 is uniformly bounded away from 1 when x lies in a compact interval. □

Theorem 3.9. *Let $0 < a < b$. The diameter of a random 3-connected map with n edges with weight x at vertices is, a.a.s. with exponential rate, in the interval $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$, uniformly over $x \in [a, b]$.*

Proof. Let ρ be the radius of convergence (depending on the weight x at vertices) of $N(z)$, which is the same as the radius of convergence of $C(z) = z + xz + xzN(z)$. And let

$$\alpha = \frac{N(\rho)}{\rho N'(\rho)}.$$

Again the results in [2] ensure that, for a random plane network C with n edges and weight x at vertices, the probability of having a 3-connected core T of size $\lfloor \alpha n \rfloor$ is $\Theta(n^{-2/3})$, hence polynomially small, whereas the probability that $\mathrm{diam}(C) > n^{1/4+\epsilon}$ is exponentially small. Since $\mathrm{diam}(C) \geq \mathrm{diam}(T)$, and since T is a random rooted 3-connected map with $k = \lfloor \alpha n \rfloor$ edges and weight x at vertices, we conclude that $\mathrm{diam}(T) \leq n^{1/4+\epsilon}$ a.a.s. with exponential rate. For the lower bound we look at the second inequality in (6):

$$\mathrm{diam}(C) \leq \mathrm{diam}(T) \cdot \max_{e\in T}(d_e) + 2\max_{e\in T}(\mathrm{diam}(C_e)),$$

where for each edge e of T, C_e denotes the piece substituted at e and d_e denotes the root-face degree of C_e.

Lemma 2.1 and Lemma 3.8 ensure that the distribution of the root-face degree of a random rooted 2-connected map has exponentially fast decaying tail. Hence $\max_{e\in T}(d_e) \leq n^{\epsilon}$ a.a.s. with exponential rate. Moreover, in the same way as in Lemma 3.6, one can show that the probability of the event $\mathcal{E} = \{|T| = \lfloor \alpha n \rfloor,\ \max(|C_e|) \leq n^{3/4}\}$ is $\Theta(n^{-2/3})$. Since $\max_{e\in T}(d_e) \leq n^{\epsilon}$ and $\mathrm{diam}(C) \geq n^{1/4-\epsilon}$ a.a.s. with exponential rate, Equation (6) easily implies that, conditioned on $\mathcal{E}$, $\mathrm{diam}(T) \geq n^{1/4-\epsilon}$ a.a.s. with exponential rate. Since $\mathcal{E}$ occurs with polynomially small probability, we conclude that $\mathrm{diam}(T) \geq n^{1/4-\epsilon}$ a.a.s. with exponential rate. Finally the uniformity of the estimate over $x \in [a,b]$ follows from the uniformity over $x \in [a,b]$ in Theorem 3.7 and in Lemma 3.8. □

4. Diameter estimates for families of graphs

4.1. **3-connected planar graphs.** For the time being we need 3-connected graphs labelled at the edges (this is enough to avoid symmetries). The number of edges is denoted m, and n is reserved for the number of vertices. By Whitney's theorem, 3-connected planar graphs with at least 4 vertices a unique embedding on the oriented sphere. Hence Theorem 3.9 gives:

Theorem 4.1. *Let $0 < a < b$. The diameter of a random 3-connected planar graph with m edges and weight x at vertices is, a.a.s. with exponential rate, in the interval $(m^{1/4-\epsilon}, m^{1/4+\epsilon})$, uniformly over $x \in [a,b]$.*

4.2. **Networks.** Before handling 2-connected planar graphs we treat the closely related family of planar *networks*. A *network* is a connected simple planar graph with two marked vertices called the poles, such that adding an edge between the poles, called the root-edge, makes the graph 2-connected. First it is convenient to consider networks as labelled at the edges.

Theorem 4.2. *Let $0 < a < b$. The diameter of a random network with m labelled edges and weight x at vertices is, a.a.s. with exponential rate, in the interval*

$$(m^{1/4-\epsilon}, m^{1/4+\epsilon}),$$

uniformly over $x \in [a,b]$.

The proof, which is quite technical, is delayed to Section 5; it relies on the decomposition into 3-*connected components* described in Section 2.3.2 and the inequalities (8). The proof of Theorem 4.9 in the next section, which relies on the decomposition into *2-connected components* gives a good idea (with less technical details), of the different steps needed to prove Theorem 4.2.

Lemma 4.3. *Let $1 < c < d < 3$. Let $N_{n,m}$ be a network with n vertices and m labelled edges, taken uniformly at random. Then $\mathrm{diam}(N_{n,m}) \in (n^{1/4-\epsilon}, n^{1/4+\epsilon})$ a.a.s. with exponential rate, uniformly over $m/n \in [c,d]$.*

Proof. Let $\mu \in [c,d]$, let $m \geq 0$, and define $n = \lfloor m/\mu \rfloor$. For $x > 0$, let X_m be the number of vertices of a random network N with m edges and weight x at vertices. The results in [3] ensure that there exists $x_\mu > 0$ such that, for $x = x_\mu$, $P(X_m = n) = \Theta(m^{-1/2})$, uniformly over $\mu \in [c,d]$. In addition x_μ is a continuous function of μ, so it maps $[c,d]$ into a compact interval. Therefore, Theorem 4.2 implies that, for $x = x(\mu)$, $\mathrm{diam}(N) \in [m^{1/4-\epsilon}, m^{1/4+\epsilon}]$ a.a.s. with exponential rate uniformly over $\mu \in [c,d]$. Since $P(X_m = n) = \Theta(m^{-1/2})$, uniformly over $\mu \in [c,d]$, we conclude that the event $\mathrm{diam}(N) \in [m^{1/4-\epsilon}, m^{1/4+\epsilon}]$, conditioned on $X_m = n$, holds a.a.s. with exponential rate uniformly over $\mu \in [c,d]$, which concludes the proof (note that the distribution of N conditioned on $X_m = n$ is the uniform distribution on networks with m edges and n vertices). □

The proof of Lemma 4.3 is the only place where uniformity of the estimates according to x (for x in an arbitrary compact interval) is needed. In the following, the weight x will be at edges, and we will not need anymore to check that the statements hold uniformly in x (even though they clearly do). Another important remark is that networks with n vertices and m edges can be labelled either at vertices or at edges, and the uniform distribution in one case corresponds to the uniform distribution in the second case. Hence the result of Lemma 4.3 holds for random networks with n labelled vertices and m unlabelled edges.

Lemma 4.4. *Let $x > 0$. Let N be a random network with n labelled vertices and weight x at edges (which are unlabelled). Then* $\mathrm{diam}(N) \in (n^{1/4-\epsilon}, n^{1/4+\epsilon})$ *a.a.s. with exponential rate.*

Proof. As shown in [3], the ratio $r = \#(edges)/\#(vertices)$ of N is concentrated around some value $\mu = \mu(x) \in (1, 3)$. Precisely, for each $\delta > 0$, there is $c = c(\delta) > 0$ such that

$$\mathbb{P}\{r \notin (\mu - \delta, \mu + \delta)\} \leq \exp(-cn).$$

Take δ small enough so that $r - \delta > 1$ and $r + \delta < 3$. Then Lemma 4.3 ensures that $\mathrm{diam}(N) \in [n^{1/4-\epsilon}, n^{1/4+\epsilon}]$ a.a.s. with exponential rate. □

4.3. **2-connected planar graphs.** Networks are very closely related to edge-rooted 2-connected planar graphs. In fact, an edge-rooted (i.e., with a marked oriented edge) 2-connected planar graph yields two networks: one where the marked edge is kept and one where the marked edge is deleted. Consequently the statement of Lemma 4.4 also holds for N a random edge-rooted 2-connected planar graph with n (labelled) vertices and weight x at edges. And the statement still holds for a random 2-connected planar graph (unrooted) with n vertices, since the number of edges can vary only from n to $3n$ (hence the effect of unmarking a root-edge biases the distribution by a factor of at most 3). We obtain:

Theorem 4.5. *Let $x > 0$. The diameter of a random 2-connected planar graph with n vertices and weight x at edges is, a.a.s. with exponential rate, in the interval $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$.*

4.4. **Connected planar graphs.** Here we deduce from Theorem 4.5 that a random connected planar graph with n vertices has diameter in $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$ a.a.s. with exponential rate. We use the block decomposition presented in Section 2.3.1, and the inequality (7). Again an important point is that if C is a random connected planar graph with n vertices and weight x at edges, then each block B of size k in C is a random 2-connected planar graph with k vertices and weight x at edges. Note that, formulated on pointed graphs, the block-decomposition ensures that a pointed connected planar graph is obtained as follows: take a collection of 2-connected pointed planar graphs, and merge their pointed vertices into a single vertex; then attach at each non-marked vertex v in these blocks a pointed connected planar graph C_v. Fix $x > 0$. Call $C(z)$ and $B(z)$ the weighted generating functions, respectively, of pointed connected and 2-connected planar graphs with weight x at edges. Then the decomposition above yields

$$C'(z) = \exp(B'(zC'(z))). \tag{14}$$

Lemma 4.6. *For $x > 0$, a random connected planar graph with n vertices and weight x at edges has a block of size at least $n^{1-\epsilon}$ a.a.s. with exponential rate.*

Proof. Denote by $E(z) = zC'(z)$ the series counting pointed connected planar graphs with weight x at edges. Note that the functional inverse of $E(z)$ is $\phi(u) = u \exp(-g(u))$, where $g(u) = B'(u)$. Call ρ the radius of convergence of $C(z)$ and R the radius of convergence of $B(u)$. Call $b_i = [u^i]g(u)$, $g_k(u) = \sum_{i \leq k} b_i u^i$, and call $E_k(z)$ the series of pointed connected planar graphs where all blocks have size at most k. Note that the probability of a random connected planar graphs with n vertices to have all its blocks of size at most k is $[z^n]E_k(z)/[z^n]E(z)$. Clearly

$$E_k(z) = z \exp(g_k(E_k(z)),$$

hence the functional inverse of $E_k(z)$ is $\phi_k(u) = u \exp(-g_k(u))$. Call ρ_k the singularity of $E_k(z)$. Since $\phi_k(u)$ is analytic everywhere, the singularity at ρ_k is caused by a branch point, i.e., $\rho_k = \phi_k(R_k)$, where R_k is the unique $u > 0$ such that $\phi'_k(u) = 0$: $\phi'_k(u) > 0$ for $0 < u < R_k$ and

$\phi'(u) < 0$ for $u > R_k$. According to (2), $[z^n]E_k(z) \leq E_k(x)x^{-n}$ for $x < \rho_k$, or equivalently, writing $u = E_k(x)$,

$$[z^n]E_k(z) \leq u\phi_k(u)^{-n} \quad \text{for all } u \text{ such that } \phi_k'(u) > 0. \tag{15}$$

Define $u_k = R \cdot (1 + 1/(k \log k))$. Note that

$$g_k(R) \leq g_k(u_k) \leq \left(\frac{u_k}{R}\right)^k g_k(R).$$

Since $(u_k/R)^k \to 1$ we have $g_k(u_k) \to g(R)$. Similarly $g_k'(u_k) \to g'(R)$, hence $\phi_k'(u_k) \to \phi'(R)$. It is shown in [16] that $a = \phi'(R)$ is strictly positive (i.e., the singularity of $F(z)$ is not due to a branch point), so for k large enough, $\phi_k'(u_k) \geq a/2 > 0$, i.e., the bound (15) can be used, giving

$$[z^n]E_k(z) \leq 2\,R\,\phi_k(u_k)^{-n} \quad \text{for } k \text{ large enough and any } n \geq 0.$$

Moreover

$$\phi_k(u_k) - \rho = (\phi_k(u_k) - \phi_k(R)) + (\phi_k(R) - \phi(R)) \sim a \cdot (u_k - R) + O(k^{-3/2}) \sim \frac{a\,R}{k \log k},$$

where $\phi_k(R) - \phi(R) = O(k^{-3/2})$ is due to $g(R) - g_k(R) = O(k^{-3/2})$, which itself follows from the estimate $b_i = \Theta(R^{-i} i^{-5/2})$ shown in [16]. Hence for k large enough and any $n \geq 0$:

$$[z^n]E_k(z) \leq 2\left(\rho + \frac{a\,R}{2k \log k}\right)^{-n}.$$

Hence, for $k = n^{1-\epsilon}$, $[z^n]E_k(z) = \Theta(\rho^{-n} \exp(-n^{\epsilon/2}))$. Finally, according to [16], $[z^n]E(z) = \Theta(\rho^{-n} n^{-5/2})$, so $[z^n]E_k(z)/[z^n]E(z) = O(\exp(-n^{\epsilon/3}))$. □

Remark. It is shown in [18] and [26] that a random connected graph has a.a.s. a block of linear size, but no with exponential rate. This is reason for the previous lemma.

Lemma 4.6 directly implies that a random connected planar graph with n vertices has diameter at least $n^{1/4-\epsilon}$. Indeed it has a block of size $k \geq n^{1-\epsilon}$ a.a.s. with exponential rate and since the block is uniformly distributed in size k, it has diameter at least $k^{1/4-\epsilon}$ a.a.s. with exponential rate.

Let us now prove the upper bound. For this purpose we use the inequality given in Section 2.3.1:

$$\text{diam}(C) \leq \text{diam}(\tau) \cdot \max_i(\text{diam}(B_i)),$$

where C denotes a connected planar graph, τ is the Bv-tree, and the B_i's are the blocks of C. We show that $\text{diam}(\tau) \leq n^\epsilon$ a.a.s. and that $\max_i(\text{diam}(B_i)) \leq n^{1/4+\epsilon}$ a.a.s., both with exponential rate.

To show that $\text{diam}(\tau) \leq n^\epsilon$ we need a counterpart of Lemma 3.1 for so-called *critical* equations of the form (9). (Indeed, note that $y \equiv y(z) = C'(z)$ is solution of $y = F(z, y)$, where $F(z, y) = \exp(B'(zy)$; in addition the height of the Bv-tree, rooted at the pointed vertex, is a height-parameter of that system.) Let $y \equiv y(z) = \sum_{\tau \in \mathcal{T}} z^{|\tau|} w(\tau)$ be the weighted generating function of some combinatorial class $\mathcal{T}$, with strictly positive radius of convergence ρ. Assume $y \equiv y(z)$ satisfies an equation of the form $y = F(z, y)$, with $F(z, y)$ a bivariate function with nonnegative coefficients, nonlinear in y, analytic around $(0, 0)$, and $F(0, y) = 0$. By the non-linearity of (9) with respect to y, $y(\rho)$ is finite; let $\tau = y(\rho)$. An equation of the form $y = F(z, y)$ is called *critical* if $F_y(\rho, \tau) < 1$, which is also equivalent to the fact that $y'(z)$ converges at ρ.

Lemma 4.7. *Let $\mathcal{T}$ be a combinatorial class endowed with a weight-function $w(\cdot)$ so that the corresponding (weighted) generating function $y(z)$ satisfies an equation of the form $y = F(z, y)$, which is critical.*

Let ξ be a height-parameter for (9) and let T_n be taken at random in $\mathcal{T}_n$ under the weighted distribution in size n. Assume that $[z^n]y(z) = \Omega(n^{-\alpha}\rho^{-n})$ for some α. Then $\xi(T_n) \leq n^\epsilon$ a.a.s. with exponential rate.

Proof. For $h \geq 0$ we define the generating functions $y_h(z) = \sum_{\tau \in \mathcal{T},\ \xi(\tau) \leq h} z^{|\tau|} w(\tau)$, and $\overline{y}_h(z) = \sum_{\tau \in \mathcal{T},\ \xi(\tau) = h} z^{|\tau|} w(\tau)$ (i.e., $\overline{y}_h(z) = y_h(z) - y_{h-1}(z)$). Let $\tau_h = y_h(\rho)$ and $\overline{\tau}_h = \overline{y}_h(\rho)$. Note that

$y(z,u) = \sum_h \overline{y}_h(z)u^h$ is the bivariate generating function of $\mathcal{T}$ where z marks the size and u marks the height. For $h > 0$ we have

$$\tau_{h+1} - \tau_h = F(\rho, \tau_h) - F(\rho, \tau_h) = F_y(\rho, u_h) \cdot (\tau_h - \tau_{h-1}), \quad \text{for some } u_h \in [\tau_{h-1}, \tau_h].$$

Since τ_h converges to τ as $h \to \infty$, u_h also converges to τ, hence $F_y(\rho, u_h)$ converges to $F_y(\rho, \tau) < 1$. Consequently $\overline{\tau}_h = \tau_h - \tau_{h-1}$ is $O(\exp(-ch))$ for some $c > 0$, so that $y(\rho, u)$ converges for $u < \exp(c)$. Hence, by Lemma 2.1, we conclude that $\xi(T_n) \leq n^\epsilon$ a.a.s. with exponential rate. □

Lemma 4.8. *For $x > 0$, the block-decomposition tree τ of a random connected planar graph with n vertices and weight x at edges has diameter at most n^ϵ a.a.s. with exponential rate.*

Proof. Let C be a pointed connected planar graph, and τ the associated Bv-tree, rooted at the marked vertex of C. Define the block-height $h(\tau)$ of τ as the maximal number of blocks (B-nodes) over all paths starting from the root. Clearly $\text{diam}(\tau) \leq 4h(\tau) + 4$. In addition the block-height is clearly a height-parameter for the equation

$$y = F(z, y), \quad \text{where } F(z, y) = \exp(B'(zy))$$

satisfied by the (weighted) generating function $y(z) = C'(z)$ of pointed connected planar graphs. It is shown in [16] that $y'(z)$ converges at its radius of convergence ρ. Hence the equation is critical; by Lemma 4.7, $h(\tau) \leq n^\epsilon$ a.a.s. with exponential rate, hence $\text{diam}(\tau) \leq n^\epsilon$ a.a.s. with exponential rate. □

Lemma 4.8 easily implies that the diameter of a random connected planar graph C with n vertices is at most $n^{1/4+\epsilon}$ a.a.s. with exponential rate. Indeed, calling τ the block-decomposition tree of C and B_i the blocks of C, one has

$$\text{diam}(C) \leq \text{diam}(\tau) \cdot \max_i(\text{diam}(B_i)).$$

Lemma 4.8 ensures that $\text{diam}(\tau) \leq n^\epsilon$ a.a.s. with exponential rate. Moreover Theorem 4.5 easily implies that a random 2-connected planar graph of size $k \leq n$ has diameter at most $n^{1/4+\epsilon}$ a.a.s. with exponential rate, whatever $k \leq n$ is (proof by splitting in two cases: $k \leq n^{1/4}$ and $n^{1/4} \leq k \leq n$, similarly as in the proof of Theorem 3.7). Hence, since each of the blocks has size at most n, $\max_i(\text{diam}(B_i)) \leq n^{1/4+\epsilon}$ a.a.s. with exponential rate. Therefore we have

Theorem 4.9. *For $x > 0$, the diameter of a random connected planar graph with n vertices and weight x at edges is, a.a.s. with exponential rate, in the interval $(n^{1/4-\epsilon}, n^{1/4+\epsilon})$.*

We can now complete the proof of Theorems 1.1 and 1.2. Theorem 1.1 is just Theorem 4.9 for $x = 1$. To show Theorem 1.2, one uses the fact (proved in [16]) that for each $\mu \in (1, 3)$ there exists $x > 0$ such that a random connected planar graph with n edges and weight x at edges has probability $\Theta(n^{-1/2})$ to have $\lfloor \mu n \rfloor$ edges.

5. Proof of Theorem 4.2

The proof of Theorem 4.2 follows the same lines as the proof of Theorem 4.9, with the RMT-tree playing the role that the Bv-tree had in Theorem 4.9. The lower bound is obtained from the fact, established in Lemma 5.2, that a random planar network has a.a.s. a "big" 3-connected component. The upper bound is obtained from the inequality given in Section 2.3.2,

$$\text{diam}(G) \leq \max_i(\text{diam}(B_i)) \cdot \text{diam}(\tau) \cdot \max_{(u,v)\in\mathcal{E}_{\text{virt}}} \text{Dist}_G(u, v) \tag{16}$$

where G is the 2-connected planar graph obtained by connecting the two poles of the considered planar network, τ is the RMT-tree of G, the B_i are the bricks of G, and $\mathcal{E}_{\text{virt}}$ is the set of virtual edges of G. To get the upper bound we will successively prove that a.a.s. with exponential rate we have $\text{diam}(\tau) \leq m^\epsilon$ (in Lemma 5.4), $\max_i(\text{diam}(B_i)) \leq m^{1/4+\epsilon}$ (in Lemma 5.5), and $\max_{(u,v)\in\mathcal{E}_{\text{virt}}} \text{Dist}_G(u, v) \leq m^\epsilon$ (in Lemma 16).

First we need the following lemma, which is a counterpart of Lemmas 3.4 and 3.8 for 3-connected maps.

Lemma 5.1. *Let $T(z,u)$ be the generating function of rooted 3-connected maps where z marks the number of non-root edges, u marks the root-face degree, and with weight x at each vertex not incident to the root-edge. Let ρ be the radius of convergence of $T(z,1)$. Then there is $u_0 > 1$ such that $T(\rho, u_0)$ converges. In addition for $0 < a < b$, the value of u_0 can be chosen uniformly over $x \in [a,b]$, and $M_i(z,u_0)$ is uniformly bounded over $x \in [a,b]$.*

Proof. The result is derived from Lemma 3.8 using a bivariate version of Equation (13), in the very same way that Lemma 3.8 is derived from Lemma 3.4 using a bivariate version of Equation (11). □

To carry out the proof it is useful to rely on a well-known recursive decomposition of planar networks (shortly called networks thereafter) that derives from the RMT-tree. Call a network D polyhedral if the poles are not adjacent and the addition of an edge between the poles gives a 3-connected planar graph with at least 4 vertices. Similarly as for a plane network (see Section 2.2.4), a network is either obtained as several networks in series (S-network), or as several networks in parallel (P-network), or as a polyhedral network where each edge is substituted by an arbitrary planar network (H-network). This can also be seen using the RMT-tree. Indeed let $B = D + e$ be the 2-connected planar graph obtained from D by adding an edge e between the two poles, and let τ be the RMT-tree of B. Then e corresponds to a leaf ℓ of τ, and the type of the inner node ν of τ adjacent to ℓ gives the type of the network (S-network if ν is an R-node, P-network if ν is an M-node, H-network if ν is a T-node). Let $D \equiv D(z)$, $S \equiv S(z)$, $P \equiv P(z)$, $H \equiv H(z)$ be respectively the generating functions of networks, series-networks, parallel networks, and polyhedral networks, where z marks the number of edges and with weight x at each non-pole vertex. And let $T(z)$ be the series of edge-rooted 3-connected planar graphs where z marks the number of non-root edges. One finds (see [30]):

$$\left\{ \begin{array}{lcl} D & = & z + S + P + H, \\ S & = & (z + P + H)xD, \\ P & = & (1+z)\exp(S+H) - 1 - z - S - H, \\ H & = & T(D). \end{array} \right. \tag{17}$$

The equation system above is similar to the one for plane networks; the difference is that for networks assembled in parallel, the order does not matter (since the graph is not equipped with a plane embedding). Note that the 2nd equation gives $S = (D-S)xD$, i.e., $S = xD^2/(1+xD^2)$, and the 3rd equation gives $z + S + P + H = (1+z)\exp(S+H) - 1$. Since $D = z + S + P + H$, we finally obtain

$$D = (1+z)\exp\left(\frac{xD^2}{1+xD} + T(D)\right) - 1. \tag{18}$$

Lemma 5.2. *For $x > 0$, let N be a random network with m (labelled) edges and weight x at (unlabelled) vertices. Then N has a 3-connected component (a T-brick in the tree-decomposition) of size at least $m^{1/4-\epsilon}$ a.a.s. with exponential rate.*

Proof. The proof is very similar to the one of Lemma 4.6. For $k \geq 1$ define $T_k(z)$ as the weighted generating function of rooted 3-connected planar graphs with at least 4 vertices and at most k edges, where z marks the number of non-root edges, with weight x at non-pole vertices (hence $T(z) = \lim_{k\to\infty} T_k(z)$). And define $D_k \equiv D_k(z)$ as the weighted generating function of networks with weight x at vertices, and where all 3-connected components (T-bricks) have at most k edges. Then clearly

$$D_k = (1+z)\exp\left(\frac{xD_k^2}{1+xD_k} + T_k(D_k)\right) - 1,$$

so T_k and D_k are related by the same equation as T with D. Note that the functional inverse of D is the function $\phi(u) = (u+1)\exp(-xu^2/(1+u) - T(u)) - 1$ and the functional inverse of D_k is the function $\phi_k(u) = (u+1)\exp(-xu^2/(1+u) - T_k(u)) - 1$. The arguments are then the same as in the proof of Lemma 4.6: one defines $u_k = R(1 + 1/(k\log(k)))$, where R is the radius of convergence of $\phi(u)$ (it is proved in [3] that R is also the radius of convergence of $T(u)$ and that $a = \phi'(R)$ is strictly positive), and one proves that for k large enough and $n \geq 0$,

$$[z^n]D_k \leq 2\left(\rho + \frac{a}{2k\log(k)}\right)^{-n},$$

where $\rho = \phi(R)$ is the radius of convergence of $D(z)$. One concludes the proof using the fact, proved in [3], that $[z^n]D(z) = \Theta(\rho^{-n}n^{-5/2})$. □

Note that Lemma 5.2 directly gives the lower bound in Theorem 4.2, using the fact (proved in Theorem 4.1) that the diameter of a random 3-connected planar graph of size k is at least $k^{1/4-\epsilon}$ a.a.s. with exponential rate.

The rest of the section is now devoted to the proof of the upper bound in Theorem 4.2. Let D be a random network with m labelled edges and weight $x > 0$ at vertices, let G be the 2-connected planar graph obtained by connecting the poles of D, and let τ be the RMT-tree of G. To show that $\mathrm{diam}(\tau) \leq n^\epsilon$ we need to extend Lemma 3.1 to vectorial equation systems. Assume $\mathbf{y} \equiv (y_1(z), \ldots, y_r(z))$ satisfies an equation of the form

$$\mathbf{y} = \mathbf{F}(z, \mathbf{y}), \tag{19}$$

with $\mathbf{F}(z, y)$ an r-vector of bivariate functions $F_i(z, \mathbf{y})$ each with nonnegative coefficients, analytic around $(0, 0)$, with $F_i(0, y) = 0$. Assume also that at least one of the F_i is nonaffine in one of the y_js, and that the dependency graph for $\mathbf{F}$ (i.e., there is an edge from i to j if $\partial_i F_j \neq 0$) is strongly connected. The two latter conditions imply that $\mathbf{y}(\rho)$ is finite; let $\tau = \mathbf{y}(\rho)$. Define $\mathrm{Jac}\mathbf{F}(z, \mathbf{y})$ as the $r \times r$ matrix $M = (M_{i,j})$ of formal power series in $(z, \mathbf{y})$ where $M_{i,j} = \partial_i F_j$. Equation 19 is called *critical* if the largest eigenvalue of $\mathrm{Jac}(\rho, \tau)$ (which is a real number by the Perron Frobenius theory) is strictly smaller than 1, which is also equivalent to the fact that $\mathbf{y}'(z)$ converges at ρ.

Assume that, for $i \in [1..r]$, $y_i(z)$ is the weighted generating function of a combinatorial class $\mathcal{G}_i$. A *height-parameter* for (19) is a parameter ξ for the classes $\mathcal{G}_i$ such that, if we define

$$y_{i,h}(z) = \sum_{\alpha \in \mathcal{G}_i,\ \xi(\alpha) \leq h} w(\alpha) z^{|\tau|}, \quad \mathbf{y}_h = (y_{1,h}, \ldots, y_{r,h}),$$

then we have

$$\mathbf{y}_{h+1} = \mathbf{F}(z, \mathbf{y}_h) \ \text{ for } h \geq 0, \quad \mathbf{y}_0 = 0.$$

As an easy extension of Lemma 4.7 relying on standard arguments of the the Perron-Frobenius theory, one has the following extension of Lemma 4.7:

Lemma 5.3. *Let $\mathcal{T}$ be a combinatorial class endowed with a weight-function $w(\cdot)$ so that the corresponding (weighted) generating function $y(z)$ is the first component of a vector $\mathbf{y} = (y_1(z), \ldots, y_r(z))$ of generating functions satisfying an equation* (19) *that is critical.*

Let ξ be a height-parameter for (19) *and let T_n be taken at random in $\mathcal{T}_n$ under the weighted distribution in size n. Assume that $[z^n]y(z) = \Omega(n^{-\alpha}\rho^{-n})$ for some α. Then $\xi(T_n) \leq n^\epsilon$ a.a.s. with exponential rate.*

Lemma 5.4. *For $0 < a < b$, the RMT-tree τ of a random network with m (labelled) edges and weight x at vertices has diameter at most m^ϵ a.a.s. with exponential rate, uniformly over $x \in [a, b]$.*

Proof. Let B be an edge-rooted 2-connected planar graph, and τ the associated RMT-tree, rooted at the leaf corresponding to the root-edge of B. Define the *brick-height* $h(\tau)$ of τ as the maximal number of bricks (nodes of type R, M, or T) over all paths starting from the root. Clearly $\mathrm{diam}(\tau) \leq 2h(\tau) + 4$. In addition the brick-height is clearly a height-parameter for the equation-system

$$\left\{\begin{array}{lcl} S & = & \frac{x(z+P+H)^2}{1-x(z+P+H)}, \\ P & = & (1+z)\exp(S+H) - 1 - z - S - H, \\ H & = & T(z+S+P+H). \end{array}\right. \tag{20}$$

which is equivalent to (17). Moreover it follows from the results in [3] that (20) is critical (e.g. because the derivative of the generating function of networks converges at the dominant singularity). Hence the brick-height of a random network with m labelled edges and weight x at vertices has diameter at most m^ϵ a.a.s. with exponential rate, and the calculations are readily checked to hold uniformly over $x \in [a, b]$. □

Lemma 5.5. *Let $0 < a < b$, and let $x \in [a, b]$. Let D be a random 2-connected planar graph with m labelled edges and weight x at vertices. Let G be the 2-connected planar graph obtained by connecting the two poles of D, and let $B_1, \ldots, B_k$ be the bricks of G. Then* $\max(\mathrm{diam}(B_i)) \leq n^{1/4+\epsilon}$ *a.a.s. with exponential rate, uniformly over $x \in [a, b]$.*

Proof. Consider a brick B_i of G. If B_i is 3-connected and conditioned to have m_i edges, B_i is a random 3-connected planar graph with m_i edges and weight x at vertices. Hence, according to Theorem 4.1, the diameter of B_i is at most $m^{1/4+\epsilon}$ a.a.s. with exponential rate (uniformly over $x \in [a, b]$). Now a brick B_i can also be a multiedge-graph, in which case $\mathrm{diam}(B_i) = 1$, or can be a ring-graph (polygon) with diameter $\lfloor m_i/2 \rfloor$ (with m_i the number of edges of B_i). So it remains to show that the largest R-brick of G is of size at most m^ϵ a.a.s. with exponential rate (uniformly over $x \in [a, b]$). Let $A(z, u)$ be the generating function of 2-connected planar graphs with a marked oriented R-brick, where z marks the number of edges, u marks the size of the marked R-brick, and with weight x at vertices. Clearly $A(z, u)$ is given by

$$A(z, u) = \log\left(\frac{1}{1 - ux(D(z) - S(z))}\right).$$

Let ρ be the radius of convergence of $D(z)$. Note that $S(z) = x(D(z)-S(z))^2/(1-x(D(z)-S(z))$. Since $S(z)$ converges at $z = \rho$ (as proved in [3]), we have $x(D(\rho) - S(\rho)) < 1$, so that $G(z, u)$ is finite for $z = \rho$ and u in a neighborhood of 1. Hence by Lemma 2.1, the distribution of the size of the marked R-brick has exponentially fast decaying tail. This ensures in turn that the largest R-brick is of size at most m^ϵ a.a.s. with exponential rate. And the estimates are readily checked to hold uniformly for $x \in [a, b]$. □

Consider the following parameter χ defined recursively for each network N:

- If N is reduced to a single edge, then $\chi(N) = 1$.
- If N is made of several networks $N_1, \ldots, N_k$ in parallel or in series, then $\chi(N) = \chi(N_1) + \cdots + \chi(N_k)$.
- If N has a 3-connected core T, and if $N_1, \ldots, N_k$ are the networks substituted at the edges of the outer face of T, then $\chi(N) = \chi(N_1) + \cdots + \chi(N_k)$.

It is easy to check recursively that $\chi(N)$ is at least the distance between the two poles of N. For each $x > 0$, denote by $D(z, u)$ (resp. $S(z, u)$, $P(z, u)$, $H(z, u)$) the bivariate generating function of networks (resp. series-networks, parallel networks, polyhedral networks) where z marks the number of edges, u marks the parameter χ, and with weight x at each non-pole vertex. Let $T(z, u)$ be the series of edge-rooted 3-connected planar graphs where z marks the number of non-root edges and u marks the number of non-root edges incident to the outer face, and with weight x at each vertex not incident to the root-edge. Then (with $D(z) = D(z, 1)$):

$$\text{(21)} \qquad \begin{cases} D(z,u) &=& zu + S(z,u) + P(z,u) + H(z,u), \\ S(z,u) &=& (zu + P(z,u) + H(z,u))xD(z,u), \\ P(z,u) &=& (1+zu)\exp(S(z,u) + H(z,u)) - 1 - zu - S(z,u) - H(z,u), \\ H(z,u) &=& T(D(z), D(z,u)/D(z)). \end{cases}$$

which coincides with (17) for $u = 1$.

Lemma 5.6. *For each $x > 0$, let ρ be the radius of convergence of $D(z, 1)$. Then there exists $u_0 > 1$ such that the generating function $D(\rho, u_0)$ converges. In addition, for $0 < a < b$ there exists some value $u_0 > 1$ that works uniformly over $x \in [a, b]$, and such that $D(\rho, u_0) = O(1)$ for $x \in [a, b]$.*

Proof. Let $R = T(D(\rho), 1)$. As shown in [3], R is the radius of convergence of $w \to T(w, 1)$. In addition, Lemma 5.1 ensures that there is some $v_0 > 1$ such that $T(R, v_0)$ converges. It follows from the results in [3] that, at $z = \rho$ the largest eigenvalue of the Jacobian matrix of (20) is strictly smaller than 1. Hence by continuity, at $z = \rho$ the largest eigenvalue of the Jacobian matrix of (21) is strictly smaller than 1 in a neighborhood of $u = 1$. Hence $D(\rho, u)$ converges for u close to 1. Finally, the uniformity of the statement for $x \in [a, b]$ follows from the uniformity over $x \in [a, b]$ in Lemma 5.1 and from the fact that (21) is continuous according to x. □

Let G be a 2-connected planar graph with a marked virtual edge $e = \{v, v'\}$. The edge e corresponds to an edge e^* in the RMT-tree connecting two nodes ν_1 and ν_2. The subtree of the RMT-tree hanging from ν_1 (resp. ν_2) corresponds to a network N_1 (resp. N_2). Define $\widetilde{\chi}(G) = \chi(N_1) + \chi(N_2)$. Clearly $\widetilde{\chi}(G)$ is an upper bound on the distance (in G) between v and v'. We denote by $G(z, u)$ the generating function of 2-connected planar graphs with a marked virtual edge, where z marks the number of edges and u marks the parameter $\widetilde{\chi}$. Looking at the possible types for the nodes ν_1 and ν_2, we obtain (the terms $S(z, u)^2$ and $P(z, u)^2$ do not appear since there are no adjacent R-nodes nor adjacent M-nodes in the RMT-tree):

$$G(z, u) = 2S(z, u)P(z, u) + 2S(z, u)H(z, u) + 2P(z, u)H(z, u) + H(z, u)^2.$$

Lemma 5.7. *For each $x > 0$, let ρ be the radius of convergence of $G(z, 1)$. Then there exists $u_0 > 1$ such that the generating function $G(\rho, u_0)$ converges. In addition, for $0 < a < b$ there some value $u_0 > 1$ that works uniformly over $x \in [a, b]$, and such that $G(\rho, u_0) = O(1)$ for $x \in [a, b]$.*

Proof. First the expression of $G(z, u)$ in terms of the generating functions of networks ensures that ρ is the radius of convergence of $D(z, 1)$, and that the property for $G(z, u)$ is just inherited from the same property satisfied by $D(z, u)$ (and the other network generating functions $S(z, u)$, $P(z, u)$, $H(z, u)$) that has been proved in Lemma 5.6. □

Lemma 5.8. *For $0 < a < b$ and $x \in [a, b]$, let D be a random network with m (labelled) edges and weight x at vertices. Let G be the 2-connected planar graph obtained by connecting the pole of D. For each virtual edge $e = \{u, v\}$ of G, let d_e be the distance in G between u and v, and let $d_{\max}$ be the maximum of d_e over all virtual edges of G. Then $d_{\max} \leq m^{\epsilon}$ a.a.s. with exponential rate, uniformly over $x \in [a, b]$.*

Proof. A network N with a marked virtual edge e can be seen as a 2-connected planar graph G rooted at a virtual edge $e = \{u, v\}$ and with a secondary marked edge whose ends play the role of poles of the network. Let G be a random 2-connected planar graph rooted at a virtual edge, with m edges and weight x at vertices. By Lemma 5.7, the distribution of the distance between u and v in G has exponentially fast decaying tail. Hence, for N a random network with m edges, weight x at vertices, and with a marked virtual edge $e = \{u, v\}$, the distribution of the distance d_e between u and v in G has exponentially fast decaying tail as well. In addition it is easy to prove inductively (on the number of nodes in the RMT-tree) that a network with m edges has $O(m)$ virtual edges. Hence $d_{\max} \leq m^{\epsilon}$ a.a.s. with exponential rate, and the uniformity over $x \in [a, b]$ follows from the uniformity over $x \in [a, b]$ in Lemma 5.7. □

To conclude, Lemmas 5.4, 5.5, and 5.8 together with the inequality (16) yield the upper bound in Theorem 5.

6. Diameter estimates for subcritical graph families

We conclude with a remark on so-called "subcritical" graph families, these are the families where the system

$$y = z \exp(B'(y)) \tag{22}$$

to specify pointed connected from pointed 2-connected graphs in the family is admissible, i.e., $F(z, y) = z \exp(B'(y))$ is analytic at (ρ, τ) where ρ is the radius of convergence of $y = y(z)$ and $\tau = y(\rho)$. Examples of such families are cacti graphs, outerplanar graphs, and series-parallel graphs.

Define the *block-distance* of a vertex v in a vertex-pointed connected graph G as the minimal number of blocks one can use to travel from the pointed vertex to v; and define the *block-height* of G as the maximum of the block-distance over all vertices of G. With the terminology of Lemma 3.1, one easily checks that the block-height is a height-parameter for the system (22). Hence by Lemma 3.1, the block-height h of a random pointed connected graph G with n vertices from a subcritical family is in $(n^{1/2-\epsilon}, n^{1/2+\epsilon})$ a.a.s. with exponential rate. Clearly $\mathrm{diam}(G) \geq h-1$ since the distance between two vertices is at least the block-distance minus 1. Hence $\mathrm{diam}(G) \geq n^{1/2-\epsilon}$ a.a.s. with exponential rate. For the upper bound, note that $\mathrm{diam}(G) \leq h \cdot \max_i(|B_i|)]$, where the B_i's are the blocks of G. Using Lemma 2.1 and the subcritical condition one easily shows that $\max_i(|B_i|) \leq n^{\epsilon}$ a.a.s. with exponential rate. This implies that $\mathrm{diam}(G) \leq n^{1/2+\epsilon}$ a.a.s. with exponential rate. It would be interesting to obtain explicit limit laws (in the scale $n^{1/2}$) for the diameter of random graphs in subcritical families such as outerplanar graphs and series-parallel graphs. Such a result has for instance recently been obtained for stacked triangulations [1].

References

[1] M. Albenque, and J. F. Marckert. Some families of increasing planar maps. *Electronic journal of probability*, 13(56):1624–1671, 2008.

[2] C. Banderier, P. Flajolet, G. Schaeffer, and M. Soria. Random maps, coalescing saddles, singularity analysis, and Airy phenomena. *Random Structures Algorithms*, 19(3/4):194–246, 2001.

[3] E. Bender, Z. Gao, and N. Wormald. The number of labeled 2-connected planar graphs. *Electron. J. Combin.*, 9:1–13, 2002.

[4] E. A. Bender, E. R. Canfield, and L. B. Richmond. The asymptotic number of rooted maps on surfaces. ii. enumeration by vertices and faces. *J. Combin. Theory Ser. A*, 63(2):318–329, 1993.

[5] J. Bouttier, P. Di Francesco, and E. Guitter. Geodesic distance in planar graphs. *Nucl. Phys.*, B663:535–567, 2003.

[6] G. Chapuy, M. Marcus, and G. Schaeffer. A bijection for rooted maps on orientable surfaces. *SIAM J. Discrete Math.*, 23(3):1587–1611, 2009.

[7] P. Chassaing and G. Schaeffer. Random planar lattices and integrated superBrownian excursion. *Probab. Theory Related Fields*, 128(2):161–212, 2004.

[8] R. Cori and B. Vauquelin. Planar maps are well labeled trees. *Canad. J. Math.*, 33(5):1023–1042, 1981.

[9] M. Drmota, O. Giménez, M. Noy. The maximum degree of series-parallel graphs. *Combin. Probab. Comput.* 20 (2011), 529–570.

[10] M. Drmota, O. Giménez, M. Noy, K. Panagiotou, and A. Steger. The maximum degree of planar graphs (in preparation).

[11] P. Flajolet, Z. Gao, A. M. Odlyzko, and L. B. Richmond. The distribution of heights of binary trees and other simple trees. *Combin. Probab. Comput.*, 2(145-156), 1993.

[12] J. F. Le Gall. The topological structure of scaling limits of large planar maps. *Invent. Math.*, 169(3):621–670, 2007.

[13] J. F. Le Gall and F. Paulin. Scaling limits of bipartite planar maps are homeomorphic to the 2-sphere. *Geom. Funct. Anal.*, 18:893–918, 2008.

[14] J. F. Le Gall. Uniqueness and universality of the Brownian map. arxiv.org/abs/1105.4842.

[15] J. Gao and N. Wormald. The size of the largest components in random planar maps. *SIAM J. Discrete Math.*, 12(2):217–228, 1999.

[16] O. Giménez and M. Noy. Asymptotic enumeration and limit laws of planar graphs. *J. Amer. Math. Soc.*, 22:309–329, 2009.

[17] O. Giménez, M. Noy. Counting planar graphs and related families of graphs. In *Surveys in combinatorics 2009*, 169–210, Cambridge Univ. Press, Cambridge, 2009.

[18] O. Giménez, M. Noy and J. Rué. Graph classes with given 3-connected components: asymptotic enumeration and random graphs. *Random Structures Algorithms* (to appear).

[19] J. Hopcroft and R. E. Tarjan Dividing a graph into triconnected components. *SIAM J. Comput.*, 2:135–158, 1973.

[20] J. F. Marckert and G. Miermont. Invariance principles for random bipartite planar maps. *Ann. Probab.*, 35(5):1642–1705, 2007.

[21] J. F. Marckert and A. Mokkadem. Limit of normalized random quadrangulations: The brownian map. *Ann. Probab.*, 34(6):2144–2202, 2006.

[22] G. Miermont. An invariance principle for random planar maps. In *Fourth Colloquium in Mathematics and Computer Sciences CMCS'06, DMTCS Proceedings AG*, pages 39–58, 2006.

[23] G. Miermont. On the sphericity of scaling limits of random planar quadrangulations. *Elect. Comm. Probab.*, 13:248–257, 2008.

[24] G. Miermont. The Brownian map is the scaling limit of uniform random plane quadrangulations. arxiv.org/abs/1104.1606.

[25] B. Mohar and C. Thomassen *Graphs on surfaces*. John Hopkins University Press, 2001.
[26] K. Panagiotou, A. Steger. Maximal biconnected subgraphs of random planar graphs. *ACM Trans. Algorithms* 6 (2010), Art. 31, 21 pp.
[27] G. Schaeffer. *Conjugaison d'arbres et cartes combinatoires aléatoires*. PhD thesis, Université Bordeaux I, 1998.
[28] W. T. Tutte. A census of planar maps. *Canad. J. Math.*, 15:249–271, 1963.
[29] W.T. Tutte. *Connectivity in graphs*. Oxford U.P, 1966.
[30] T.R.S. Walsh. *Counting labeled three-connected and homeomorphically irreducible two-connected graphs. J. Comb. Theory, Ser. B*, 32(1): 1–32, 1982.