4-Holes in Point Sets

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Abstract

We consider a variant of a question of Erdős on the number of empty k-gons (k-holes) in a set of n points in the plane, where we allow the k-gons to be non-convex. We show bounds and structural results on maximizing and minimizing the number of general 4-holes, and maximizing the number of non-convex 4-holes. In particular, we show that for $n \geq 9$, the maximum number of general 4-holes is $\binom{n}{4}$; the minimum number of general 4-holes is at least $\frac{5}{2}n^2 - \Theta(n)$; and the maximum number of non-convex 4-holes is at least $\frac{1}{2}n^3 - \Theta(n^2 \log n)$ and at most $\frac{1}{2}n^3 - \Theta(n^2)$.

Keywords: Erdős-Szekeres type problems, k-holes, empty k-gons

1. Introduction

Let S be a set of n points in general position in the plane (i.e., no three points of S are collinear). A k-gon is a simple polygon spanned by k points of S. A k-hole is an empty k-gon; that is, a k-gon which does not contain any points of S in its interior.

In 1978 Erdős [9] raised the following question for convex k-holes: "What is the smallest integer $h(k)$ such that any set of $h(k)$ points in the plane contains at least one convex k-hole?" As already observed by Esther Klein, every set of 5 points determines a convex 4-hole, and 10 points always contain a convex 5-hole, a fact proven by Harborth [15]. However, in 1983 Horton showed that there exist arbitrarily large sets of points containing no convex 7-hole [16]. It again took almost a quarter of a century after Horton's construction to answer the existence question for 6-holes. In 2007–08 Nicolás [18] and independently Gerken [13] proved that every sufficiently large point set contains a convex 6-hole.

Erdős also proposed the following variation of the problem [10]. "What is the least number $h_k(n)$ of convex k-holes determined by any set of n points in the plane?" We know by Horton's construction that $h_k(n) = 0$ for $k \geq 7$. For $k \leq 6$, upper and lower bounds on $h_k(n)$ exist; see [1] for a survey.

In this paper we generalize the latter problem by allowing a k-hole to be non-convex. Thus, whenever we refer to a k-hole, it might be convex or non-convex, and we will explicitly state it when we restrict investigations to one of these two classes.

Note that a set of four points in non-convex position might span up to three 4-holes; that is, the number of k-holes can be larger than $\binom{n}{k}$, the maximum number of convex k-holes. Further, note that the number of k-holes (of whatever type) only depends on the combinatorics of the point set, and is thus determined by its order type; see [14, 6, 2].

We first investigate point sets of small cardinality (Section 2), and then consider the following tasks: maximizing the number of 4-holes (Section 3), maximizing the number of non-convex 4-holes (Section 4), and minimizing the number of 4-holes (Section 5). In addition to lower and upper bounds on their number, we also show which families of point sets achieve these bounds. Table 1 gives an overview on the bounds on the minimum and maximum numbers of general and non-convex holes. It also contains information about the classes of point sets achieving these bounds.

4-hole type	minimum number	maximum number
general	$\geq\ \frac{5}{2}n^2-\Theta(n)$ [Thm. 6] *(all sets)* $\leq\ O(n^{\frac{5}{2}}\log n)$ [3] *("squared Horton sets" [20])*	$=\ \binom{n}{4}$ for $n \geq 9$ [Thm. 2] *(convex sets)*
non-convex	$=\ 0$ *(convex sets)*	$\leq\ \frac{n^3}{2}-\Theta(n^2)$ [Lem. 3] *(all sets)* $\geq\ \frac{n^3}{2}-\Theta(n^2\log n)$ [Thm. 4] *("mostly convex" sets [17])*

Table 1: Bounds on the minimum and maximum numbers of general and non-convex 4-holes, together with the classes of point sets achieving them.

A natural generalization of this work is to consider similar questions for larger k-holes, i.e., $k > 4$, as well as k-gons (for 4-gons, these questions are directly related to the rectilinear crossing number). During the preparation of the full version of this paper, we have been able to obtain some results in this direction. In [5], we show that also the number of non-convex 5-gons can be expressed in terms of the rectilinear crossing number. Further, we provide an improved lower bound for the minimum number of convex 5-holes and show that the principles of Theorems 2 and 6 can be used to obtain similar results for $k=5$. In a forthcoming paper, see [3] for an extended abstract, we investigate the asymptotic behavior of the numbers of k-gons and k-holes in large point sets. Most noteworthy, we provide an upper bound of $O(n^{\frac{k+1}{2}}(\log n)^{k-3})$ for the minimum number of general k-holes (for any constant $k \geq 4$) and show that for large n, this number is still maximized by n-point sets in convex position. We also consider the case where k is not constant with respect to n. For example, we show that if k is a sufficiently large linear fraction of n, then the convex set contains strictly less k-holes than the double chain [12] of same cardinality. The PhD thesis [21] summarizes most results obtained for $k \geq 4$.

Note that while the proof for the variation of Theorem 2 in [3] is simpler than the according proofs in [5] and in the work at hand, it provides weaker results with respect to the obtained bounds on the cardinality of the point sets. In particular, it leaves a gap for $k=4$, while Theorem 2 in this paper is best possible in that sense.

2. Small Sets

Even to determine the number of 4-holes in point sets of small cardinality is surprisingly intriguing. For $n \leq 11$, Table 2, which is partly taken from [1], shows the minimum number of convex 4-holes, the maximum number of non-convex 4-holes, the minimum and maximum number of (general) 4-holes, and, for easy comparison, the number of (unordered) 4-tuples. The numbers are calculated using a straight-forward program and the order type database [6, 2].

n	convex min	non-convex max	general min	general max	$\binom{n}{4}$
4	0	3	1	3	1
5	1	8	5	9	5
6	3	18	15	22	15
7	6	36	35	43	35
8	10	64	66	77	70
9	15	100	102	126	126
10	23	150	147	210	210
11	32	216	203	330	330

Table 2: Number of 4-holes for $n = 4, \ldots, 11$ points; see [1] for details.

Obviously, the maximum number of convex 4-holes is $\binom{n}{4}$, obtained by sets in convex position. For minimizing the number of convex 4-holes, the currently best known bounds are $\frac{n^2}{2} - \frac{9}{4}n - o(n) \leq h_4(n) \leq 1.9397n^2 + o(n^2)$, where the upper bound is by Bárány and Valtr [7], and the lower bound is by García [11] and Aichholzer et al. [4]. A result of independent interest is by Pinchasi et al. [19], who showed $h_4(n) \geq h_3(n) - \frac{n^2}{2} - O(n)$. Currently, the best known bound on $h_3(n)$ is $h_3(n) \geq n^2 - \frac{32}{7}n + \frac{22}{7}$ [11, 4]. Thus, improving the dominant factor in the lower bound for $h_3(n)$ would also imply a better lower bound for $h_4(n)$.

For $n = 4, \ldots, 7$ it can be seen from Table 2 that the minimum number of 4-holes is $\binom{n}{4}$. In contrast, $\binom{n}{4}$ is the maximum number of 4-holes for $n = 9, 10, 11$. This implies that convex sets are minimizing examples for $n \leq 7$ and maximizing examples for $n = 9, 10, 11$. So the structure of extremal sets seems to switch.

Figure 1 shows point sets maximizing the number of 4-holes for $n = 4, \ldots, 8$. The results for $n \geq 9$ suggest that sets in convex position might

maximize the number of 4-holes for $n \geq 9$. Indeed, this will be the first result we prove for general 4-holes (Section 3).
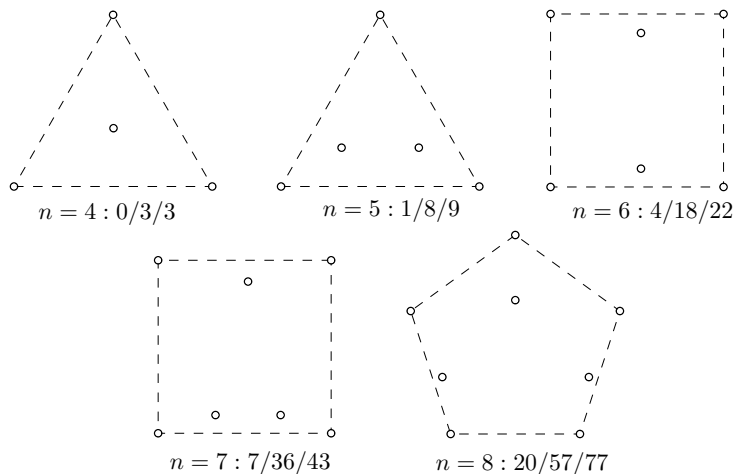

$$n = 4 : 0/3/3 \qquad n = 5 : 1/8/9 \qquad n = 6 : 4/18/22$$

$$n = 7 : 7/36/43 \qquad n = 8 : 20/57/77$$

Figure 1: Point sets maximizing the number of 4-holes for $n = 4, \ldots, 8$. Shown are the number of convex, non-convex, and general 4-holes. With the exception of $n = 7$, each point set represents the unique maximizing order type.

Figure 2 shows two extremal sets for $n = 11$ points. Each point set represents the unique order type which reaches the extreme value. The left set maximizes the number of non-convex 4-holes, namely 216, and consists of a convex 5-hole inside a convex 6-gon. The total number of 4-holes in this set is 267; i.e., it contains in addition 51 convex 4-holes. The set on the right side minimizes the number of general 4-holes. It contains 51 convex and 152 non-convex 4-holes, thus in total the minimum of 203 4-holes.

3. Maximizing the Number of (General) 4-Holes

In this section we show that for large enough n, the number of 4-holes is maximized by convex sets, as suggested by the enumerative results.

Lemma 1. *Let Δ be a non-empty triangle in S. There are at most three non-convex 4-holes spanned by the three vertices of Δ plus a point of S in the interior of Δ.*

Proof. Let p_1, p_2, and p_3 be the vertices of Δ. Observe that any non-convex 4-hole has to use two edges of Δ. Thus there are three choices for the unused
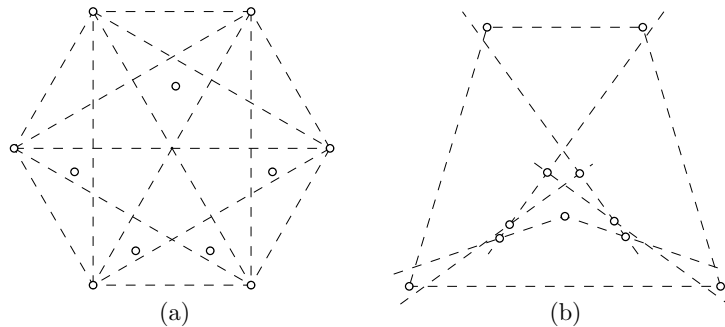
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Figure 2: Two unique extremal sets for $n = 11$ points: (a) maximizes the number of non-convex 4-holes, and (b) minimizes the number of general 4-holes.

edge of Δ, and for each choice there is at most one way to complete the two used edges of Δ to a 4-hole. Assume to the contrary that two different 4-holes avoid the edge p_2p_3 and use points q_1 and q_2, respectively, in the interior. Then q_2 lies outside the two triangles formed by $p_1q_1p_2$ and $p_1q_1p_3$. Thus q_2 lies in the triangle formed by $p_2q_1p_3$. But then q_1 must lie in one of the triangles spanned by $p_1q_2p_2$ and $p_1q_2p_3$, a contradiction. $\qquad\square$

Theorem 2. *For $n \geq 9$, the number of 4-holes is maximized by a set of n points in convex position.*

Proof. In the following we assign every non-convex 4-tuple of points to the three vertices of its convex hull and call this the *representing* triangle of the potential non-convex 4-holes. By Lemma 1, any non-empty triangle represents at most three 4-holes, and any convex 4-tuple gives at most one 4-hole.

Let $T = T(S)$ be the number of non-empty triangles in a point set S. As any non-empty triangle induces at least one 4-tuple in non-convex position, we get

$$\binom{n}{4} - T + 3T = \binom{n}{4} + 2T \qquad (1)$$

as a first upper bound on the number of 4-holes in S.

Note that a triangle Δ with $k \geq 1$ interior points is counted $k+2$ times in (1), namely k times in the $\binom{n}{4}$ 4-tuples and twice in the term $2T$, as Δ is non-empty. Thus for $k > 1$ we have over-counted the number of non-convex 4-holes assigned to Δ; cf. Lemma 1. Moreover, many of the convex 4-gons might not be empty and thus not 4-holes. Therefore we now analyze

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how many 4-holes can be deduced from (1) as overcounted. We will do this by assigning (possibly multiple) markers for over-counted 4-holes to convex 4-tuples and non-empty triangles.

As above, let Δ be a triangle with $k \geq 1$ interior points, and consider all 4-tuples consisting of the three vertices of Δ plus an extra point p. We distinguish two cases.

Case 1. Let p be one of the $n-k-3$ points outside Δ. If the resulting 4-tuple is convex, we mark this 4-tuple, as it is not empty and thus not a 4-hole. If the 4-tuple is non-convex, we mark the triangle which represents the potential non-convex 4-hole, as at least one of the three possible 4-holes of this 4-tuple is non-empty.

Case 2. Consider the k points inside Δ. As argued above, Δ was counted $k+2$ times. But by Lemma 1, there are at most three 4-holes using one interior point of Δ and thus represented by Δ. Therefore we assign $k-1$ markers to Δ.

Adding up both cases, we have distributed $n-k-3+(k-1) = n-4$ markers while considering Δ. Repeating this for all non-empty triangles, we obtain a total of $(n-4) \cdot T$ markers.

A non-empty convex 4-tuple might have received up to 4 markers in this way, one from each of its sub-triangles. That is, we have at most 4 times as many markers as convex 4-tuples which we can reduce from the upper bound (1).

A non-empty triangle Δ with $k \geq 1$ interior points might have received $4 \cdot (k-1)$ markers: For its interior points, Δ received $k-1$ markers from the second case, and for each non-empty triangle formed by two vertices of Δ and one point inside Δ, it received one marker from the first case. Observe that at least three of the considered inner triangles are empty, namely the ones spanned by an edge e of Δ and the interior point closest to e. Thus, the first case gives at most $3 \cdot (k-1)$ additional markers, resulting in a total of at most $4 \cdot (k-1)$ markers for Δ. As Δ was counted $k+2$ times but represents at most three 4-holes (Lemma 1), we have at most $4 \cdot (k-1)$ markers for at least $(k+2)-3 = k-1$ over-counted objects. Thus, in both cases we get at most four times as many markers as over-counted 4-holes which we may deduct from (1). We therefore can reduce the number of potential 4-holes by one quarter of the distributed markers, namely by $\frac{n-4}{4} \cdot T$. This leads to the improved upper bound

$$\binom{n}{4} + 2T - \frac{n-4}{4} \cdot T \;=\; \binom{n}{4} - \frac{n-12}{4} \cdot T$$

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for the number of 4-holes. For $n \geq 12$ this is at most $\binom{n}{4}$, the number of 4-holes for a set of points in convex position. Together with the results from Table 2 for $n = 9, \ldots, 11$, this proves the theorem. $\qquad\square$

Remark. From the proof of the previous theorem it follows that for $n \geq 13$ any non-convex set of n points has strictly less general 4-holes than the convex set of the same cardinality. Moreover, Theorem 2 is tight in the sense that for $n < 9$, the statement is not true.

4. Maximizing the Number of Non-Convex 4-Holes

In the previous section we have shown that for $n \geq 9$, the number of general 4-holes is maximized for sets in convex position. This obviously also holds for the number of convex 4-holes.

We now consider maximizing the number of non-convex 4-holes. From Lemma 1 we obtain the following.

Lemma 3. *The number of non-convex 4-holes of any set of n points is at most $\frac{n(n-1)(n-2)}{2} = \frac{n^3}{2} - \Theta(n^2)$.*

Proof. By Lemma 1, any non-empty triangle generates at most three non-convex 4-holes, and there are at most $\binom{n}{3}$ such triangles in a set of n points. $\qquad\square$

Theorem 4. *For any $m \geq 1$ there exist point sets of cardinality $n = 2^{m+1} - 2$ that contain $\frac{n^3}{2} - \Theta(n^2 \log n)$ non-convex 4-holes.*

Proof. We consider special point sets $\mathcal{X}_m$, $m \geq 1$, with $|\mathcal{X}_m| = n = 2^{m+1} - 2$ points, that have been introduced in [17]. The point sets are defined recursively in layers, starting with two points $\mathcal{X}_1 := \mathcal{R}_1$ in the first layer. An additional layer $\mathcal{R}_i$ is added to $\mathcal{X}_{i-1} := \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_{i-1}$ by placing two new points close to each point in $\mathcal{R}_{i-1}$ outside the convex hull of $\mathcal{X}_{i-1}$, such that the following conditions hold:

1. $\mathcal{X}_i = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_i$ is in general position,
2. the points in $\mathcal{R}_i$ are precisely the extremal points (i.e., vertices on the convex hull) of $\mathcal{X}_i$, and
3. any triangle determined by $\mathcal{R}_i$ contains precisely one point of $\mathcal{X}_i$ in its interior.

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See Figure 3 for an example and [17] for a detailed description of the construction. Furthermore, in [17] it is shown that every triangle spanned by $\mathcal{X}_m$ contains at most one interior point of $\mathcal{X}_m$; i.e., every non-empty triangle of $\mathcal{X}_m$ contains exactly one point. Thus, using Lemma 1, the number of non-convex 4-holes of $\mathcal{X}_m$ is three times the number of non-empty triangles.
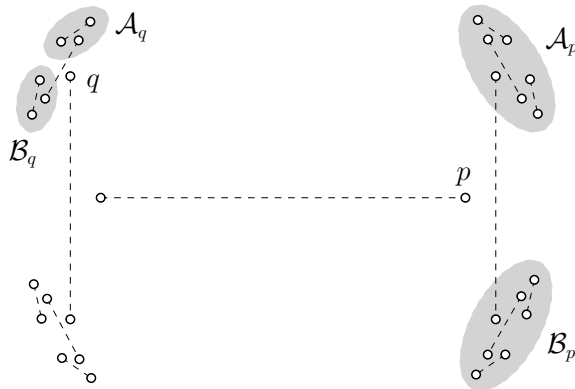

Figure 3: Sketch of the special point set defined in [17], for $m = 4$.

For each point $x \in \mathcal{R}_{i-1}$, $2 \leq i \leq m$, we will count the number of triangles that contain x in its interior. Note that points in $\mathcal{R}_m$ are extremal points of $\mathcal{X}_m$ and thus not in the interior of any triangle spanned by points of $\mathcal{X}_m$. To this end, we consider the two points in the next layer $\mathcal{R}_i$ which are closest to x. We denote by $\mathcal{A}_x$ and $\mathcal{B}_x$ the subsets of $\mathcal{X}_m$ which consist of one of these points of $\mathcal{R}_i$, respectively, plus all points in outer layers $\mathcal{R}_j$ $(j > i)$ which were subsequently added to this point of $\mathcal{R}_i$ by the construction. We say that $\mathcal{A}_x$ and $\mathcal{B}_x$ are the *induced subsets* of x, and that $\mathcal{C}_x := \mathcal{X}_m \backslash \{\mathcal{A}_x \cup \mathcal{B}_x \cup \{x\}\}$ is the *remainder* (of $\mathcal{X}_m$) for x.

First, fix a point in the first layer $\mathcal{R}_1$, say p in Figure 3. Any triangle containing p in its interior is formed by one point of $\mathcal{A}_p$, one point of $\mathcal{B}_p$, and one point of the remainder $\mathcal{C}_p$. As $a_1 := |\mathcal{A}_p| = |\mathcal{B}_p| = \frac{n-2}{4}$ and $c_1 := |\mathcal{C}_p| = n - 2 \cdot a_1 - 1 = \frac{n}{2}$, this gives $a_1^2 \cdot c_1$ triangles containing p in the interior, and thus the number of triangles containing a point of $\mathcal{R}_1$ in the interior is $2 \cdot a_1^2 \cdot c_1 = 2 \cdot \left(\frac{n-2}{4}\right)^2 \cdot \frac{n}{2}$.

Now consider a point q in the second layer $\mathcal{R}_2$. Its induced subsets $\mathcal{A}_q$ and $\mathcal{B}_q$ have size $a_2 = \frac{n-6}{8}$, and the remainder $\mathcal{C}_q$ has $c_2 = n - 2 \cdot a_2 - 1 = \frac{3n+2}{4}$ points. In combination with $r_2 := |\mathcal{R}_2| = 4$ this gives a total of $4 \cdot \left(\frac{n-6}{8}\right)^2 \cdot \frac{3n+2}{4}$ triangles containing a point of $\mathcal{R}_2$ in the interior.

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In general, $|\mathcal{R}_i| = r_i = 2^i$, and the size of the two induced subsets of a point p_i in $\mathcal{R}_i$ is

$$a_i = \frac{1}{r_{i+1}}(n - |\mathcal{X}_i|) = \frac{n - (2^{i+1} - 2)}{2^{i+1}}.$$

Thus with the size of the corresponding remainder $\mathcal{C}_{p_i}$ of

$$c_i = n - 2 \cdot a_i - 1 = \frac{(2^i - 1)n + 2^i - 2}{2^i},$$

we get $r_i \cdot a_i^2 \cdot c_i$ triangles containing one point of $\mathcal{R}_i$ in the interior.

Using that every non-empty triangle of $\mathcal{X}_m$ gives three non-convex 4-holes, and summing up over all layers $\mathcal{R}_i$, we obtain

$$
\begin{aligned}
3 \cdot \sum_{i=1}^{m} r_i \cdot a_i^2 \cdot c_i &= 3 \cdot \sum_{i=1}^{m} 2^i \left(\frac{n - (2^{i+1} - 2)}{2^{i+1}} \right)^2 \frac{(2^i - 1)n + 2^i - 2}{2^i} \\
&= \frac{1}{2}n^3 - 3n^2 \log_2(n+2) + \frac{39}{4}n^2 \\
&\quad - 12n \log_2(n+2) + \frac{41}{2}n - 12 \log_2(n+2) + 12
\end{aligned}
$$

for the total number of non-convex 4-holes of $\mathcal{X}_m$. $\qquad\square$

5. Minimizing the Number of (General) 4-Holes

As already mentioned, we have $\frac{n^2}{2} - \frac{9}{4}n - o(n) \leq h_4(n) \leq 1.9397n^2 + o(n^2)$ for the minimal number $h_4(n)$ of convex 4-holes. For non-convex 4-holes, the lower bound trivially is zero. In the following we will show a lower bound for the number of (general) 4-holes.

By checking all order types of cardinality eight from the order type data base [6, 2], we obtained the following observation for general 4-holes.

Observation 5. *Let S be a set of $n = 8$ points in the plane in general position, and $p_1, p_2 \in S$ two arbitrary points of S. Then S contains at least five 4-holes having p_1 and p_2 among their vertices.*

Let us make a few remarks on this observation. On the one hand, consider any set S with $n \geq 8$ points, and any two points $p_1, p_2 \in S$. Then p_1 and p_2 together with the six points of $S \setminus \{p_1, p_2\}$ that are closest to the segment
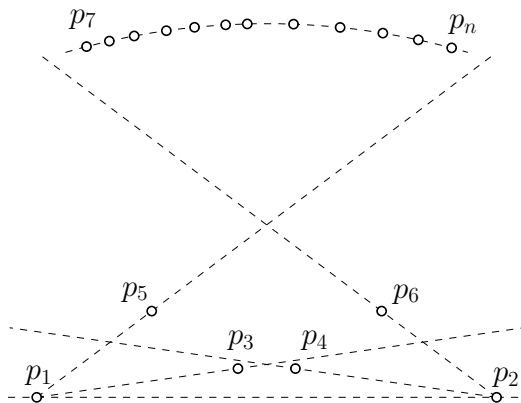
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Figure 4: A point set containing only five 4-holes (one convex and four non-convex) that have both p_1 and p_2 as a vertex.

$p_1 p_2$ form a set S' of eight points. Obviously, the convex hull of S' does not contain any points from $S \setminus S'$. By Observation 5, S' contains at least five 4-holes having p_1 and p_2 among its vertices and thus S does as well.

On the other hand, there exist arbitrarily large point sets S such that there exist points $p_1, p_2 \in S$ which are contained in at most five 4-holes. For example, given the point set shown in Figure 4, we consider 4-holes having both p_1 and p_2 as a vertex. Note that such a 4-hole cannot contain any of the points $p_7, \ldots, p_n$. The reason is that every triangle $p_1 p_2 p_k$, $7 \leq k \leq n$, contains $p_3, \ldots, p_6$ and thus cannot be completed to a 4-hole; cf. the proof of Lemma 1. Thus, the vertex set of any 4-hole in S having both p_1 and p_2 as a vertex is a subset of $\{p_1, \ldots, p_6\}$. As $\{p_1, \ldots, p_6\}$ contains only five such 4-holes, Observation 5 is tight in that respect also for $n > 8$.

Moreover, for $n \leq 7$, there are point sets S_n and points $p_1, p_2 \in S_n$ for which only $n{-}3 {<} 5$ 4-holes are incident to both, p_1 and p_2. Figure 5 shows an example with 7 points. According examples with less points can be derived by disregarding the points with higher indices. Also, adding an 8th point on the chain yields the second 8-point order type having only five 4-holes incident to both p_1 and p_2.

Based on Observation 5, we can derive a lower bound for the number of general 4-holes. Note that there exist sets which contain fewer than $1.94 n^2$ convex 4-holes, while by the following result any set contains at least $2.5 n^2 - \Theta(n)$ general 4-holes.
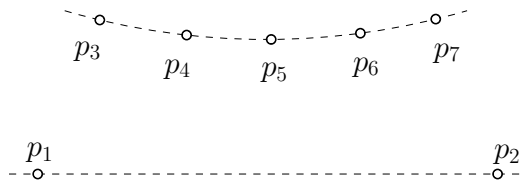
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Figure 5: A 7-point set containing only four 4-holes (all convex) that have both p_1 and p_2 as a vertex.

Theorem 6. *Let S be a set of $n \geq 8$ points in the plane in general position. Then S contains at least $\frac{5}{2}n^2 - \Theta(n)$ 4-holes.*

Proof. Assume w.l.o.g. that no two points of the set S have identical x-coordinates. We consider S in x-sorted order, $S = \{p_1, \ldots, p_n\}$, and define subsets $S_{i,j} = \{p_i, \ldots, p_j\} \subseteq S$. The number of sets $S_{i,j}$ having at least 8 points is

$$\sum_{i=1}^{n-7} \sum_{j=i+7}^{n} 1 = \sum_{i=1}^{n-7} (n - i - 6) = \frac{n^2}{2} - \frac{13}{2}n + 21.$$

By Observation 5, each set $S_{i,j}$ contains at least five 4-holes having p_i and p_j among their vertices. Moreover, as p_i and p_j are the left- and rightmost points of $S_{i,j}$, they are also the left- and rightmost points for each of these 4-holes. This implies that any 4-hole of S counts for at most one set $S_{i,j}$, which gives a lower bound of $\frac{5}{2}n^2 - \Theta(n)$ for the number of 4-holes in S. $\square$

The principle of the proof of the above theorem can also be used to obtain an asymptotic lower bound of $\Omega(n^2)$ on the number of k-holes (for any $c < 1$ and $3 \leq k \leq c \cdot n$) [3]. Expectedly, such a generalization cannot provide specific constants, like the dominant factor of $\frac{5}{2}$ in Theorem 6, which is obtained by using Observation 5.

6. Conclusion

We have shown lower and upper bounds on the numbers of general and non-convex 4-holes in point sets; see again Table 1 on page 3.

Several questions remain open, of which the following is perhaps the most intriguing: What is the minimum number of general 4-holes that any n-element point set in general position must have? In Section 5 we have shown a quadratic lower bound for this number. By Proposition 7 below,

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a super-quadratic lower bound would solve a conjecture of Bárány in the affirmative, showing that every point set in general position contains a pair of points that is incident to a super-constant number of 3-holes; see e.g. [8], Chapter 8.4, Problem 4.

Proposition 7. *Assume that a point set S in general position contains a super-quadratic number of general 4-holes. Then S also contains a pair of points that is incident to a super-constant number of 3-holes.*

Proof. Consider two arbitrary points $p, q \in S$ and let $\Diamond(p, q)$ be the number of 4-holes in S for which pq is an interior diagonal. A 4-hole counted by $\Diamond(p, q)$ consists of two empty triangles which lie on opposite sides of pq and both have pq as an edge. Let $\Delta_1(p, q)$ and $\Delta_2(p, q)$ be the numbers of empty triangles incident to pq on each of the two sides of pq, respectively. Then $\Diamond(p, q) = \Delta_1(p, q) \cdot \Delta_2(p, q)$.

A convex 4-hole has two interior diagonals, while a non-convex 4-hole has only one. Hence, summing up $\Diamond(p, q)$ over all pairs of points $p, q \in S$ gives twice the number of convex plus once the number of non-convex 4-holes in S, which is at least the number of general 4-holes in S and thus super-quadratic. This implies that $\Diamond(p^*, q^*) = \max\{\Diamond(p, q) : p, q \in S, p \neq q\}$ is super-constant. Thus, the number of empty triangles incident to p^*q^*, which is $\Delta_1(p^*, q^*) + \Delta_2(p^*, q^*) \geq 2 \cdot \sqrt{\Diamond(p^*, q^*)}$, is super-constant as well. $\square$

Acknowledgments

References

[1] O. Aichholzer, [Empty] [colored] k-gons - Recent results on some Erdős-Szekeres type problems, in: Proc. XIII Encuentros de Geometría Computacional ECG2009, Zaragoza, Spain, pp. 43–52.

[2] O. Aichholzer, Enumerating order types for small point sets with applications, http://www.ist.tugraz.at/aichholzer/research/rp/triangulations/ordertypes/, 2012.

[3] O. Aichholzer, R. Fabila-Monroy, H. González-Aguilar, T. Hackl, M.A. Heredia, C. Huemer, J. Urrutia, P. Valtr, B. Vogtenhuber, On k-gons and k-holes in point sets, in: Proc. 23th Canadian Conference on Computational Geometry CCCG'11, Toronto, Canada, pp. 21–26.

[4] O. Aichholzer, R. Fabila-Monroy, T. Hackl, C. Huemer, A. Pilz, B. Vogtenhuber, Lower bounds for the number of small convex k-holes, in: Proc. 24th Canadian Conference on Computational Geometry CCCG'12, Charlottetown, Canada. To appear.

[5] O. Aichholzer, T. Hackl, B. Vogtenhuber, On 5-holes and 5-gons, in: A. Marquez, P. Ramos, J. Urrutia (Eds.), Special issue: XIV Encuentros de Geometría Computacional ECG2011, Lecture Notes in Computer Science (LNCS), Springer, 2012. To appear.

[6] O. Aichholzer, H. Krasser, The point set order type data base: A collection of applications and results, in: Proc. 13th Canadian Conference on Computational Geometry CCCG'01, volume 13, pp. 17–20.

[7] I. Bárány, P. Valtr, Planar point sets with a small number of empty convex polygons, Studia Scientiarum Mathematicarum Hungarica 41 (2004) 243–266.

[8] P. Brass, W. Moser, J. Pach, Research Problems in Discrete Geometry, Springer, 2005.

[9] P. Erdős, Some more problems on elementary geometry, Australian Mathematical Society Gazette 5 (1978) 52–54.

[10] P. Erdős, Some old and new problems in combinatorial geometry, in: M. Rosenfeld, J. Zaks (Eds.), Annals of Discrete Mathematics, Convexity and Graph Theory, volume 20, 1984, pp. 129–136.

[11] A. García, A note on the number of empty triangles, in: A. Marquez, P. Ramos, J. Urrutia (Eds.), Special issue: XIV Encuentros de Geometría Computacional ECG2011, Lecture Notes in Computer Science (LNCS), Springer, 2012. To appear.

[12] A. García, M. Noy, J. Tejel, Lower bounds on the number of crossing-free subgraphs of K_n, Computational Geometry: Theory and Applications 16 (2000) 211–221.

[13] T. Gerken, Empty convex hexagons in planar point sets, Discrete and Computational Geometry 39(1–3) (2008) 239–272.

[14] J.E. Goodman, R. Pollack, Multidimensional sorting., SIAM Journal on Computing 12 (1983) 484–507.

[15] H. Harborth, Konvexe Fünfecke in ebenen Punktmengen, Elemente der Mathematik 33 (1978) 116–118.

[16] J. Horton, Sets with no empty convex 7-gons, Canadian Mathematical Bulletin 26(4) (1983) 482–484.

[17] G. Károlyi, J. Pach, G. Tóth, A modular version of the Erdős-Szekeres theorem, Studia Scientiarum Mathematicarum Hungarica 38 (2001) 245–260.

[18] C. Nicolás, The empty hexagon theorem, Discrete and Computational Geometry 38(2) (2007) 389–397.

[19] R. Pinchasi, R. Radoičić, M. Sharir, On empty convex polygons in a planar point set, J. Comb. Theory Ser. A 113 (2006) 385–419.

[20] P. Valtr, Convex independent sets and 7-holes in restricted planar point sets, Disc. Comp. Geom. 7 (1992) 135–152.

[21] B. Vogtenhuber, Combinatorial Aspects of [Colored] Point Sets in the Plane, Ph.D. thesis, Institute for Software Technology, Graz University of Technology, Graz, Austria, 2011.