On minimum integer representations of weighted games

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Abstract

We study minimum integer representations of weighted games, i.e. representations where the weights are integers and every other integer representation is at least as large in each component. Those minimum integer representations, if they exist at all, are linked with some solution concepts in game theory. Closing existing gaps in the literature, we prove that each weighted game with two types of voters admits a (unique) minimum integer representation, and give new examples for more than two types of voters without a minimum integer representation. We characterize the possible weights in minimum integer representations and give examples for $t \geq 4$ types of voters without a minimum integer representation preserving types, i.e. where we additionally require that the weights are equal within equivalence classes of voters.

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1. Introduction

Simple games, or positive switching functions, can be viewed as models of voting systems in which a single alternative, such as a bill or an amendment, is pitted against the status quo. Weighted games, or positive threshold functions, are possibly the most interesting subclass of simple games. Roughly speaking, in a weighted game a non-negative weight w_i is assigned to each voter $1 \leq i \leq n$ and a quota q is specified. As an abbreviation for a weighted game we use the notation $[q; w_1, \ldots, w_n]$. Winning coalitions are those that can force a victory, i.e. the sum of their weights equals or surpasses the quota. Weighted games naturally appear in several different contexts apart from voting, like reliability analysis of technical systems (see Ramamurthy [27]) or neural networks (see, among others, Elgot [7] or Freixas and Molinero [9]).

The number of simple games on a fixed set $N = \{1, \ldots, n\}$ is finite, of course, but it grows very rapidly with an increasing number of voters n since we are dealing with sets of sets. Indeed, every family of pairwise independent subsets of N can serve as the set of minimal winning coalitions defining a simple game. Two subsets are independent if neither contains the other. Families of independent subsets are sometimes called “Sperner families“, “coherent systems“, or “clutters“, and their enumeration and classification have occupied mathematicians since Dedekind in the 19th century. In his 1897 work he determined the exact number of simple games with four or fewer players. Since that time simple games have been investigated in a variety of different mathematical contexts. An account of some of these

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works can be found in: Sperner [29], Isbell [14], Golomb [13], Muroga et al. [23, 24], Shapley and Shubik [28], Dubey and Shapley [4], Kurz and Tautenhahn [19, 20], Freixas and Molinero [10, 11], Krohn and Sudhölter [18], Keijzer et al. [16, 17]. Although the number of weighted games compared with the number of simple games is small, it grows very rapidly and there do not exist enumeration results for more than nine voters.

Integer representations, i.e. where the weights w_i and the quota q are non-negative integers, are very common in practice and minimum integer representations, if they exist, constitute the most efficient way to represent weighted games. Several algorithms to compute certain power indices require integer weights and benefit from weights of small magnitude. When considering e.g. shareholders of a firm, integer weights, i.e. the number of (equal) shares, arise naturally. Geometrically, the set of equivalent integer representations of a weighted game is an unbounded cone with or without a vertex. Hence, a natural question arises: For which weighted games does a minimum integer representation exist? Or, in other words, for which weighted games does the associated integer cone have a vertex? Symmetric games, i.e. games where all players have an equivalent role in the game and, therefore, are characterized by one single type of equivalent voters, admit a minimum integer representation. But it is known that there does not always exist a minimum integer representation for a weighted game. Muroga et al. [23] in their exhaustive enumeration of threshold functions (or, equivalently, weighted games) uncovered several cases with as few as eight players in which two symmetric players must be given different weights in a minimum sum integer representation; e.g. $[12; 7, 6, 6, 4, 4, 4, 3, 2] = [12; 7, 6, 6, 4, 4, 4, 2, 3]$. Here a minimum sum integer representation is an integer representation such that the sum of weights $\sum_{i=1}^{n} w_i$ is minimal. Moreover, they verified that all weighted games with less than eight players admit a minimum integer representation. We can easily check that this example consists of four types of players (a type here is an element of a partition of N formed by equivalent voters), and each type contains players with the same weights except for the last type, which contains players with weights 3 and 2.

To our knowledge it is not known whether there exist weighted games without a minimum integer representation with either two or three types of players. The main goal of this paper is to ascertain what occurs for these two cases, filling the existing gap in the theory of weighted games. Previous to Muroga et al's example, Isbell [14] had exhibited a remarkable 12-player example in which the affected players are not symmetric. Thus, even if we additionally require that all players of equal type have equal weights, the existence of a minimum integer representation preserving types is not guaranteed. Freixas and Molinero [10, 11] uncovered several cases of weighted games without a minimum integer representation preserving types with as few as 9 players and checked the nonexistence of such examples for less than 9 players; see also [19]. All the examples they listed have at least 5 types of players. So quite naturally, we want to ascertain what occurs for less than 5 types. We would like to remark that homogeneous games[3] admit a minimum integer representation as shown by Ostmann [25].

A natural third issue emerges to be significant, whenever there does not exist a minimum integer representation for a weighted game (either preserving types or not). In that situation at least two integer representations are minimal, but is it possible to generate weighted games with more than two minimal representations? Since integer representations which attain the minimum possible sum of weights are minimal, we ask more generally for constructions of weighted games with an arbitrary number of minimum sum integer representations. As far as we know, all the previously published examples without a minimum integer representation (either preserving types or not) have only two minimum sum integer representations. Additional results, we introduce here, comprise: bounds on the number of non-isomorphic weighted games as a function of the number of voters and the number of types of voters, and the existence of a weighted game in minimum integer representation for any pair of two coprime integer weights.

[3] A weighted game is called *homogeneous* if it admits a representation where all minimal winning coalitions have the same weight.

Minimum integer representations of weighted games are important in game theory: Peleg [26] proved that for *homogeneous* weighted *decisive* games the nucleolus (a well-known solution concept in game theory) coincides with the minimum integer representation preserving types. Also, in the cases where there is no minimum integer representation preserving types, there are connections linking a minimum sum integer representation preserving types with the least core (another solution concept) and the nucleolus of weighted decisive games [18].

The remainder of the paper is organized as follows. In Section 2 we precisely define the classes of complete simple games and weighted games. For complete simple games we state a parameterization theorem by Carreras and Freixas in Subsection 2.1, which completely characterizes these objects up to isomorphism using linear inequalities. The subclass of weighted games can be defined via the non-emptiness of certain polytopes as outlined in Subsection 2.2. The details on minimum integer representations are stated in Subsection 2.3. In Section 3 we present constructions for weighted games without a minimum integer representation for small t (Subsection 3.1) and for those with more than two minimum sum integer representations (Subsection 3.2). In Subsection 3.3 we study the question of which weights may occur in a minimum integer representation. Our main theorem, that each weighted game with two types of voters admits a minimum integer representation, is given in Section 4. Implications for the enumeration or bounds on the number of weighted games, which arise as a byproduct of our previous results, are briefly stated in Section 5. We end with a conclusion in Section 6.

2. Binary voting systems – simple games, complete simple games and weighted games

From a more general point of view, binary voting systems, i.e. those where each voter has the option to vote *yes* or *no*, which then is condensed by a certain voting rule, can be represented by a characteristic function $\chi : 2^N \to \{0, 1\}$, where $N := \{1, \dots, n\}$ is the set of voters and 2^N denotes the set $\{U \mid U \subseteq N\}$ of all subsets of N. A quite natural monotonicity assumption on χ leads to a very prominent class of binary voting systems.

Definition 2.1. *A **simple game** is a function $\chi : 2^N \to \{0, 1\}$, which satisfies $\chi(\emptyset) = 0$, $\chi(N) = 1$, and $\chi(U') \le \chi(U)$ for all $U' \subseteq U \subseteq N$, where N is a finite set.*

So, if we identify 2^N with $\{0, 1\}^n$, each simple game is a monotone Boolean function and except for the all-zero function and the all-one function all monotone Boolean functions are simple games. We will call a subset $U \subseteq N$ a coalition.

Definition 2.2. *A coalition $U \subseteq N$ of a simple game χ is called **winning** if $\chi(U) = 1$ and **losing** otherwise. A coalition U is called a **minimal winning** coalition if $\chi(U) = 1$ and $\chi(U') = 0$ for all proper subsets U' of U. Similarly, a coalition U is called a **maximal losing** coalition if $\chi(U) = 0$ and $\chi(U') = 1$ for all proper supersets U' of U. By $\mathcal{W}$ we denote the set of winning coalitions and by $\mathcal{L}$ the set of losing coalitions for a given simple game. The restrictions to minimal winning coalitions and maximal losing coalitions are denoted by $\mathcal{W}^m$ and $\mathcal{L}^M$, respectively.*

We have $\mathcal{W} \cup \mathcal{L} = 2^N$ and remark that either $\mathcal{W}^m$ or $\mathcal{L}^M$ uniquely characterizes a simple game; see e.g. [30] for the details and additional facts on simple games. A well studied subclass of simple games (and superclass of weighted games) arises from Isbell's desirability relation [15]:

Definition 2.3. *We write $i \sqsupset j$ (or $j \sqsubset i$) for two voters $i, j \in N$ if we have $\chi\Big(\{i\} \cup U \backslash \{j\}\Big) \ge \chi(U)$ for all $\{j\} \subseteq U \subseteq N \backslash \{i\}$ and we abbreviate $i \sqsupset j$, $j \sqsupset i$ by $i \square j$. A simple game χ is called **complete simple game** (also called a "directed game", see [18], or a "linear game", see [30]) if the binary relation $\sqsupset$ is a total preorder, i.e.*

(1) $i \sqsupset i$ *for all* $i \in N$,

(2) $i \sqsupset j$ *or* $j \sqsupset i$ *for all* $i, j \in N$, *and*

(3) $i \sqsupset j$, $j \sqsupset h$ *implies* $i \sqsupset h$ *for all* $i, j, h \in N$.

W.l.o.g. we assume $1 \sqsupset 2 \sqsupset \cdots \sqsupset n$ in the following. Whenever $i \square j$, voter i is as influential in the game as voter j, meaning that it does not matter which one of both takes part in a coalition, i.e. the status of the coalition (winning or losing) does not change after a swap of two equally desirable voters. We can partition the whole set N of voters into equivalence classes $N_1, \ldots, N_t$ and say that the complete simple game consists of t types of voters. By n_i we denote the cardinality of the set N_i for $1 \leq i \leq t$. Coalitions are categorized into different types, which can be described by a vector $(m_1, \ldots, m_t)$ meaning m_i-out-of-n_i voters (from the set N_i) for $1 \leq i \leq t$.

Let us consider an example with $n_1 = n_2 = 2$. Due to the assumed ordering of the players we have $N_1 = \{1, 2\}$ and $N_2 = \{3, 4\}$. With this the vector $(1, 1)$ is the type of the coalitions $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, and $\{2, 4\}$. Since we have $1 \square 2$ and $3 \square 4$ either all those four coalitions are winning or they are all losing and we can therefore speak of a winning or a losing vector.

Definition 2.4. *Let χ be a simple game and N_h, $1 \leq h \leq t$, be the classes of equally desirable voters. We call a vector $\widetilde{\mathbf{m}} := (m_1, \ldots, m_t)$, where $0 \leq m_h \leq |N_h|$ for $1 \leq h \leq t$, a* ***winning vector*** *if $\chi(U) = 1$, where U is an arbitrary subset of N containing exactly m_h elements of N_h for $1 \leq h \leq t$. Analogously, we call such a vector a* ***losing vector*** *if $\chi(U) = 0$, where U is an arbitrary subset of N containing exactly m_h elements of N_h for $1 \leq h \leq t$.*

In the following we will always use a tilde and bold notation to indicate a vector representing a type of a coalition. The concept of inclusion has to be slightly modified for vectors, i.e. types of coalitions:

Definition 2.5. *For two vectors $\widetilde{\mathbf{a}} = (a_1, \ldots, a_t)$ and $\widetilde{\mathbf{b}} = (b_1, \ldots, b_t)$, representing types of coalitions in a complete simple game, we write $\widetilde{\mathbf{a}} \preceq \widetilde{\mathbf{b}}$ if we have $\sum\limits_{i=1}^{k} a_i \leq \sum\limits_{i=1}^{k} b_i$ for all $1 \leq k \leq t$. For $\widetilde{\mathbf{a}} \preceq \widetilde{\mathbf{b}}$ and $\widetilde{\mathbf{a}} \neq \widetilde{\mathbf{b}}$ we use $\widetilde{\mathbf{a}} \prec \widetilde{\mathbf{b}}$ as an abbreviation and say that they are comparable vectors with vector $\widetilde{\mathbf{a}}$ being smaller than vector $\widetilde{\mathbf{b}}$. If neither $\widetilde{\mathbf{a}} \preceq \widetilde{\mathbf{b}}$ nor $\widetilde{\mathbf{b}} \preceq \widetilde{\mathbf{a}}$ holds, we write $\widetilde{\mathbf{a}} \bowtie \widetilde{\mathbf{b}}$ and say that vector $\widetilde{\mathbf{a}}$ and vector $\widetilde{\mathbf{b}}$ are incomparable.*

If $(1, 1)$ is a winning vector in our example, so is $(2, 0)$ while nothing can be deduced for vector $(0, 2)$. With Definition 2.5 at hand, we can define:

Definition 2.6. *A vector $\widetilde{\mathbf{m}} = (m_1, \ldots, m_t)$ in a complete simple game χ is a* ***shift-minimal winning vector*** *if $\widetilde{\mathbf{m}}$ is a winning vector and every vector $\widetilde{\mathbf{m}}'$ with $\widetilde{\mathbf{m}}' \prec \widetilde{\mathbf{m}}$ is losing. Analogously, a vector $\widetilde{\mathbf{m}}$ is a* ***shift-maximal losing vector*** *if $\widetilde{\mathbf{m}}$ is a losing vector and every vector $\widetilde{\mathbf{m}}'$ with $\widetilde{\mathbf{m}}' \succ \widetilde{\mathbf{m}}$ is winning.*

Similarly as for simple games, where the set $\mathcal{W}^m$ or $\mathcal{L}^M$ with the inclusion are enough to generate the entire set of winning coalitions $\mathcal{W}$, for complete simple games the sets $\mathcal{W}^{sm}$ and $\mathcal{L}^{sM}$ of the shift-minimal winning vectors (representing types of coalitions) and the maximal losing vectors uniquely characterize the complete simple game with the operation $\succeq$. Weighted games, which are a subclass of complete simple games, are now formally introduced as follows:

Definition 2.7. *A simple game χ is called a* ***weighted game*** *(or simply* ***weighted****) if there exists a quota $q \in \mathbb{R}_{>0}$ and weights $w_1, \ldots, w_n \in \mathbb{R}_{\geq 0}$ such that $\chi(U) = 1$ if and only if $\sum_{i \in U} w_i \geq q$. As an abbreviation we utilize the notation $\chi = [q; w_1, \ldots, w_n]$ or simply $\chi = [q; w]$ whenever the weight vector $w = (w_1, \ldots, w_n)$ is specified.*

As an example we consider the weighted game $[4;3,2,1,1]$ (which is the same as $[3;2,1,1,1]$), where we have $1 \sqsupset 2 \square 3 \square 4$ for the voters, i.e. $n_1 = 1$ and $n_2 = 3$. The shift-minimal winning vectors are given by $(1,1)$, $(0,3)$ and the smax losing vectors are given by $(1,0)$, $(0,2)$. Since $(1,2) \succ (1,1)$ the coalition type $(1,2)$ is also winning and $(0,2)$ is losing due to $(0,2) \prec (0,3)$. For a more extensive overview on binary voting methods we refer the interested reader to [30].

2.1. A parameterization theorem for complete simple games

Carreras and Freixas have given a full parameterization of complete simple games in [3]. To this end we denote the (decreasing) lexicographic (strict) order by $\gtrdot$, i.e. we have $(a_1,\dots,a_n) \gtrdot (b_1,\dots,b_n)$ iff there is an index $1 \le h \le n$ with $a_i = b_i$ for all $1 \le i < h$ and $a_h > b_h$. An example is given by $(1,2,1) \gtrdot (1,1,3)$.

Theorem 2.8. *(Carreras and Freixas, 1996)*

(a) *Consider a vector*

$$\overline{\mathbf{n}} = (n_1,\dots,n_t) \in \mathbb{N}_{>0}^t$$

and a matrix

$$\mathcal{M} = \begin{pmatrix} m_{1,1} & m_{1,2} & \dots & m_{1,t} \\ m_{2,1} & m_{2,2} & \dots & m_{2,t} \\ \vdots & \vdots & \ddots & \vdots \\ m_{r,1} & m_{r,2} & \dots & m_{r,t} \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbf{m}}_1 \\ \widetilde{\mathbf{m}}_2 \\ \vdots \\ \widetilde{\mathbf{m}}_r \end{pmatrix}.$$

If they satisfy the following properties:

(i) $m_{1,1} > 0$, $0 \le m_{i,j} \le n_j$, $m_{i,j} \in \mathbb{N}_{\ge 0}$ *for* $1 \le i \le r$, $1 \le j \le t$,

(ii) $\widetilde{\mathbf{m}}_i \bowtie \widetilde{\mathbf{m}}_j$ *for all* $1 \le i < j \le r$,

(iii) *for each* $1 \le j < t$ *there is at least one row-index* i *such that* $m_{i,j} > 0$, $m_{i,j+1} < n_{j+1}$, *and*

(iv) $\widetilde{\mathbf{m}}_i \gtrdot \widetilde{\mathbf{m}}_{i+1}$ *for* $1 \le i < t$,

then there exists a complete simple game χ *associated to* $(\overline{\mathbf{n}}, \mathcal{M})$ *with* $\overline{\mathbf{n}}$ *as a vector of the cardinalities of the equivalence classes and matrix* $\mathcal{M}$*, where the rows consist of the shift-minimal winning vectors.*

(b) *Two complete games* $(\overline{\mathbf{n}}_1, \mathcal{M}_1)$ *and* $(\overline{\mathbf{n}}_2, \mathcal{M}_2)$ *are isomorphic* if and only if $\overline{\mathbf{n}}_1 = \overline{\mathbf{n}}_2$ *and* $\mathcal{M}_1 = \mathcal{M}_2$.

In such a vector/matrix representation of a complete simple game the number of voters n is determined by $n = \sum_{i=1}^t n_i$. Although Theorem 2.8 looks technical at first glance, the necessity of the required properties can be explained easily. First we observe that $n_j \ge 1$, $m_{1,1} > 0$, and $0 \le m_{i,j} \le n_j$ must hold for $1 \le i \le r$, $1 \le j \le t$. If $\widetilde{\mathbf{m}}_i \preceq \widetilde{\mathbf{m}}_j$ or $\widetilde{\mathbf{m}}_i \succeq \widetilde{\mathbf{m}}_j$ then we would have $\widetilde{\mathbf{m}}_i = \widetilde{\mathbf{m}}_j$ or either $\widetilde{\mathbf{m}}_i$ or $\widetilde{\mathbf{m}}_j$ cannot be a shift-minimal winning vector. If for a column-index $1 \le j < t$ we have $m_{i,j} = 0$ or $m_{i,j+1} = n_{j+1}$ for all $1 \le i \le r$, then we can check that $g \square h$ for all $g \in N_j$, $h \in N_{j+1}$, which is a contradiction to the definition of the classes N_j and therefore also for the numbers n_j. A complete simple game does not change if two rows of the matrix $\mathcal{M}$ are interchanged. Thus we must require some specific ordering of the rows to avoid duplicities, e.g. $\gtrdot$.

If all voters are equivalent, i.e. $t = 1$, there is a unique shift-minimal winning vector, i.e. $r = 1$. In this case the requirements of Theorem 2.8 are reduced to $1 \le m_{1,1} \le n_1 = n$. Also for $t = 2$ one can easily give a more compact formulation for the requirements in Theorem 2.8. A complete description of the possible values $n_1, n_2, m_{1,1}, m_{1,2}$ corresponding to a complete simple game with parameters n, $t = 2$, and $r = 1$ is given by

$$1 \le n_1 \le n-1, \quad n_1 + n_2 = n, \quad 1 \le m_{1,1} \le n_1, \text{ and } 0 \le m_{1,2} \le n_2 - 1. \tag{1}$$

For $t = 2$ and $r \ge 2$ such a complete and compact description is given by

$$1 \le n_1 \le n-1,\ n_1 + n_2 = n, \text{ and } m_{i,1} \ge m_{i+1,1} + 1,\ m_{i,1} + m_{i,2} + 1 \le m_{i+1,1} + m_{i+1,2} \tag{2}$$

for all $1 \le i \le r-1$.

2.2. Recognizing and representing weighted games

In Definition 2.7 we have introduced the notation $[q; w_1, \dots, w_n]$, consisting of a quota q and weights w_i, for a weighted game. As mentioned in the introduction there are several representations for the same weighted game, e.g. $[3; 2, 1, 1, 1]$, $[4; 3, 2, 1, 1]$, $[11; 9, 5, 5, 4]$, $[q; q-1, x, x, x]$ and $[q; q-2, x, x, x]$ with $q \ge 6$ and $\left\lceil \frac{q}{3} \right\rceil \le x \le \left\lfloor \frac{q-1}{2} \right\rfloor$ all represent the same weighted game because the subsets of N whose weights equal or surpass the quota are invariant for all of them.

So in order to check whether two weighted games are equivalent, it makes sense to have a closer look at the underlying discrete structure as a simple game, i.e. its characteristic function $\chi : 2^N = \{U \mid U \subseteq N\} \to \{0, 1\}$. As weighted games are complete simple games we often find it useful to represent the game using the matrix representation of the previous subsection, especially if we use different weighted representations for the same game or different weights within an equivalence class of voters.

To decide whether a given complete simple game is weighted, we can utilize a linear program; see [30] for an overview on other methods. From Definition 2.7 and the notion of minimal winning and maximal losing coalitions we can conclude that a simple game is weighted *if and only if* the following system of linear inequalities is feasible:

$$\sum_{i \in S} w_i \ge q\ \forall S \in \mathcal{W}^m,\ \sum_{i \in T} w_i < q\ \forall T \in \mathcal{L}^M,\ q \in \mathbb{R}_{>0}, \text{ and } w_i \in \mathbb{R}_{\ge 0}\ \forall 1 \le i \le n. \tag{3}$$

As strict inequalities, i.e., $<$ or $>$, might lead to ill-defined optimization problems like e.g. maximize x subject to $x < 1$, we use an equivalent formulation instead:

$$\sum_{i \in S} w_i \ge q\ \forall S \in \mathcal{W}^m,\ \sum_{i \in T} w_i \le q - 1\ \forall T \in \mathcal{L}^M, \text{ and } w_i \ge 0\ \forall 1 \le i \le n. \tag{4}$$

As $\mathcal{L}^M$ is not empty and the w_i are non-negative, the inequality $q > 0$ is implied by $\sum_{i \in T} w_i \le q - 1$. By rescaling the weights we may achieve that the difference $q - \max_{T \in \mathcal{L}^M} \sum_{i \in T} w_i$ is as large as desired, e.g. at least 1. Of course here we already have integer representations in mind, i.e. where we additionally request $w_i \in \mathbb{N}_{\ge 0}$ (see Definition 2.10). The fact that each weighted game is also a complete simple game can be used to reduce inequality system (4).

Lemma 2.9. *Given a complete simple game χ with t equivalence classes of voters the inequality system (4) has a solution* if and only if

$$\widetilde{\mathbf{x}}^T w \ge q\ \forall \widetilde{\mathbf{x}} \in \mathcal{W}^{sm},\ \widetilde{\mathbf{y}}^T w \le q - 1\ \forall \widetilde{\mathbf{y}} \in \mathcal{L}^{sM}, w_i \ge w_{i+1} + 1\ \forall 1 \le i \le t-1, \textit{ and } w_t \ge 0. \tag{5}$$

has a solution.

PROOF. Let us at first assume that (q, w) is a feasible solution of (5). By setting $q' = q$ and $w'_i = w_h \ge 0$ for all $i \in N_h$ we will obtain a feasible solution (q', w') for (4). Now let $S \in \mathcal{W}^m$ be a minimal winning coalition, $\widetilde{\mathbf{x}}'$ its corresponding type, and let $\widetilde{\mathbf{x}} \in \mathcal{W}^{sm}$ be a vector with $\widetilde{\mathbf{x}} \preceq \widetilde{\mathbf{x}}'$. With this we have

$$\sum_{i \in S} w'_i = \widetilde{\mathbf{x}}'^T w \ge \widetilde{\mathbf{x}}^T w \ge q$$

due to $w_1 > w_2 > \cdots > w_t \geq 0$ for all $1 \leq i \leq t$. Similarly, for a maximal losing coalition $R \in \mathcal{L}^M$ with corresponding type $\widetilde{\mathbf{y}}'$, let $\widetilde{\mathbf{y}} \in \mathcal{L}^{sM}$ be a vector with $\widetilde{\mathbf{y}} \succeq \widetilde{\mathbf{y}}'$, so that

$$\sum_{i \in R} w_i' = \widetilde{\mathbf{y}}'^T w \leq \widetilde{\mathbf{y}}^T w \leq q - 1.$$

For the other direction let (q', w') be a feasible solution of (4). One can easily check that (q', w''), where $w_i'' = \frac{\sum_{j \in N_i} w_j'}{|N_i|}$ for all $1 \leq i \leq t$, is a feasible solution of (5). □

We would like to remark that those complete simple games which are not weighted can be represented as a finite intersection of weighted games, a construction which is also used in practice [8].

2.3. Minimum integer representations

In the previous section we have already seen some different representations of weighted games, e.g. we may assume that the difference between the weight of a winning coalition and the weight of a losing coalition is at least one. A special kind of representation restricts the quota and the weights to integers:

Definition 2.10. *For a given weighted game χ, with minimal winning coalitions $\mathcal{W}^m$, maximal losing coalitions $\mathcal{L}^M$, and t equivalence classes of voters, a vector $(q, w_1, \ldots, w_n) \in \mathbb{N}_{\geq 0}^{n+1}$ is called an* ***integer representation*** *if it is a feasible solution of Inequality system (4). If we have $w_i = w_j$ for all $i, j \in N_h$, where $1 \leq h \leq t$, then we speak of an* ***integer representation preserving types****.*

We remark that each feasible solution $(q, w_1, \ldots, w_n) \in \mathbb{N}_{\geq 0}^{n+1}$ of (3) also satisfies inequality system (4). Given an integer representation we can easily construct a (possibly non-integer) representation, where the weights are equal within equivalence classes of voters by averaging the weights in each equivalence class, as done in the proof of Lemma 2.9. (Every convex combination of solutions of an LP is itself a solution.)

Definition 2.11. *Given an integer representation $(q, w_1, \ldots, w_n)$ for a weighted game χ with equivalence classes $N_1, \ldots, N_t$ the* ***averaged representation*** *$(q, w_1', \ldots, w_t')$ is given by $w_h' = \frac{\sum_{i \in N_h} w_i}{|N_h|}$.*

So indeed each weighted game admits an integer representation preserving types.

Definition 2.12. *Given a weighted game χ we call an integer representation $(q, w_1, \ldots, w_n)$ a* ***minimum sum integer representation****, if we have $\sum_{i=1}^n w_i \leq \sum_{i=1}^n w_i'$ for all integer representations $(q', w_1', \ldots, w_n')$. Similarly we call an integer representation $(q, w_1, \ldots, w_n)$ preserving types a* ***minimum sum integer representation preserving types****, if we have $\sum_{i=1}^n w_i \leq \sum_{i=1}^n w_i'$ for all integer representations $(q', w_1', \ldots, w_n')$ preserving types.*

We remark that each weighted game admits a minimum sum integer representation and a minimum sum integer representation preserving types, but there can exist several such representations. Introducing integer variables changes the linear programs (4) and (5) to integer linear programs (ILP), whose solution is NP-hard in general. So, if we minimize the sum of weights $\sum_{i=1}^n w_i$ subject to the constraints in inequality system (4) restricted to integer variables, each optimal solution corresponds to a minimum sum integer representation. Similarly, if we minimize the sum of weights $\sum_{i=1}^t n_i w_i$ subject to the constraints in inequality system (5) restricted to integer variables, each optimal solution corresponds to a minimum sum integer representation preserving types. To our knowledge there is no known polynomial time algorithm to determine a minimum sum integer representation. For some algebraic techniques, to determine a minimum sum integer representation, we refer the interested reader to [2].

By considering the following LP-relaxation of the ILP for the value of a minimum sum integer representation we can obtain a reasonable lower bound for the sum of weights in an minimum sum integer representation:

$$\min \sum_{i=1}^{t} w_i n_i \tag{6}$$
$$\text{s.t. } \widetilde{\mathbf{x}}^T w \geq q \;\forall \widetilde{\mathbf{x}} \in \mathcal{W}^{sm},\; \widetilde{\mathbf{y}}^T w \leq q-1 \;\forall \widetilde{\mathbf{y}} \in \mathcal{L}^{sM}, w_i \geq w_{i+1}+1 \;\forall 1 \leq i \leq t-1, \text{ and } w_t \geq 0.$$

Lemma 2.13. *For a given weighted game χ with t equivalence classes of voters let φ be the optimal target value of the minimization problem (6), then we have $\sum_{i=1}^{n} w_i' \geq \varphi$ for all integer representations $(q', w_1', \dots, w_n')$ of χ.*

PROOF. For a given integer representation $(q', w_1', \dots, w_n')$ we show that the averaged representation $w_i = \frac{\sum_{j \in N_i} w_j'}{|N_i|}$, $q = q'$ is a feasible solution of inequality system (5) attaining the same sum of its weights as the initial integer representation.

As in the proof of Lemma 2.9 we have $w_{j_1}' > w_{j_2}'$ for all $j_1 \in N_i$, $j_2 \in N_{i+1}$. Since the w_j' are integers we conclude $w_{j_1}' \geq w_{j_2}' + 1$ so that $w_i \geq w_{i+1} + 1$ for all $1 \leq i \leq t-1$. □

A more restrictive integer representation asks for the minimum possible weight for each player simultaneously:

Definition 2.14. *An integer representation $(q, w_1, \dots, w_n)$ for a weighted game χ is called **minimum integer representation** if for all integer representations $(q', w_1', \dots, w_n')$ of χ we have $w_i \leq w_i'$ for all $1 \leq i \leq n$. If we restrict the allowed representations to those where the voters of the same equivalence class N_i have an equal weight, we speak of a **minimum integer representation preserving types**.*

In other words, a minimum integer representation, if it exists, is the least element in the partial order of component-wise comparison of the feasible weight vectors.

In general, both representations need not exist and indeed in this paper we study conditions where they exist and give examples where they do not exist. We would like to note that each minimum integer representation for a weighted game is also a minimum integer representation preserving types, since otherwise the weights could be permuted within equivalence classes of voters. On the other hand, the existence of a minimum integer representation preserving types does not imply the existence of a minimum integer representation. The example $[12; 7, 6, 6, 4, 4, 4, 3, 2] = [12; 7, 6, 6, 4, 4, 4, 2, 3]$ from the introduction has $(14, 8, 7, 7, 5, 5, 5, 3, 3)$ as a minimum integer representation preserving types.

3. Generating conspicuous examples of games without a minimum integer representation

Motivated by the existence of weighted games without a minimum integer representation for more than three equivalence classes of voters; see e.g. Table 3 and Table 4 of [10], we are concerned in this section with this problem in the special case of $t = 3$ types of voters. As we shall see below, we propose a procedure to generate weighted games with three types of voters without a minimum integer representation in Subsection 3.1 based on the famous Coin-Exchange Problem of Frobenius [1]. Similarly, the existence of weighted games without a minimum integer representation preserving types is known for more than four equivalence classes of voters; see e.g. Table 2 in [11]. Thus the case $t = 4$ is under study here and we also propose a procedure to generate weighted games with four types of voters without a minimum integer representation preserving types in Subsection 3.1. Another objective of this section is to generate examples of weighted games with more than two minimum sum integer representations, which is outlined in Subsection 3.2. Finally, Subsection 3.3 concerns weighted games with a minimum integer representation of coprime weights.

The Coin-Exchange Problem of Frobenius considers $n \geq 2$ integers $0 < a_1 < \cdots < a_n$ with $\gcd(a_1, \ldots, a_n) = 1$ as denominations of n different coins. We say that a certain amount of money $A \in \mathbb{N}_{\geq 0}$ can be **represented** by the given coins, if there are n numbers $x_i \in \mathbb{N}_{\geq 0}$ such that $A = \sum\limits_{i=1}^{n} a_i x_i$. As an abbreviation we denote the set of representable integers A by $\langle a_1, \ldots, a_n \rangle$.

If $a_1 > 1$ then some A cannot be represented, e.g. there do no exist representations for all $A \in \{1, \ldots, a_1 - 1\}$. The largest such A for a given problem is called the Frobenius number $g(a_1, \ldots, a_n)$. Well-known results in this context are $g(a_1, a_2) = (a_1 - 1)(a_2 - 1) - 1$ and that exactly $\frac{1}{2}(g(a_1, a_2) + 1) = \frac{1}{2}(a_1 - 1)(a_2 - 1)$ non-negative integers are not representable for $\gcd(a_1, a_2) = 1$. As an example we consider $a_1 = 3$, $a_2 = 5$, where the set of non-negative integers which are not representable is given by $\mathbb{N}_{\geq 0} \backslash \langle a_1, a_2 \rangle = \{1, 2, 4, 7\}$.

Almost all of the following constructions contain the game $\chi_{a,b} = [ab; \overbrace{b, \ldots, b}^{a}, \overbrace{a, \ldots, a}^{b}]$, where $b > a \geq 1$ are coprime integers, as a subgame, i.e. the winning coalitions of $\chi_{a,b}$ are winning coalitions in the larger game and similarly the losing coalitions of $\chi_{a,b}$ are losing coalitions of the larger game. Our first aim is to prove a lower bound on the sum of weights of a minimum sum integer representation of $\chi_{a,b}$. To this end we utilize Bézout's identity stating that there exist integers $u, v \in \mathbb{Z}$ with $ua + bv = \gcd(a, b) = 1$, which can be computed using the extended Euclidean algorithm.

Lemma 3.1. *For coprime integers $b > a \geq 1$ there exist $u, v \in \mathbb{N}_{>0}$ with $ub - va = 1$, $u \leq a$, $v < b$.*

Lemma 3.2. *For coprime integers $b > a \geq 1$ there exist $u, v \in \mathbb{N}_{\geq 0}$ with $ub + va = ab - 1$, $0 \leq u \leq a - 1$, and $1 \leq v \leq b - 1$.*

PROOF. Using Lemma 3.1 and the identity $(a - u)b + va = ab - 1$ yields the stated result. □

In the following we will often use the existence of those integers u, v without explicitly referring to Lemma 3.2. We remark that the (unique) existence of such a pair (u, v) of integers can be concluded from Popoviciu's theorem, which counts the number of representations for a given amount N using two coprime integer coins a and b.

Lemma 3.3. *For every integer representation $(q, w_1, \ldots, w_n)$ of $\chi_{a,b}$ we have $\sum_{i=1}^{n} w_i \geq 2ab$.*

PROOF. Let u, v be integers satisfying the conditions of Lemma 3.2. Due to Lemma 2.13 it suffices to prove that the optimal solution (q', w_1', w_2') of LP (6) has a target value of at least $2ab$. Since (v, u) is a losing vector and $(a, 0)$, $(0, b)$ are winning vectors we have $uw_2' + vw_1' \leq q' - 1$, $aw_1' \geq q'$, and $bw_2' \geq q'$. Multiplying the first inequality by ab yields

$$abuw_2' + abvw_1' \leq abq' - ab.$$

Adding bv times the second inequality and au times the third inequality yields

$$abuw_2' + abvw_1' \geq \underbrace{(au + bv)}_{=ab-1} q'.$$

Thus we conclude $abq' - q' \leq abq' - ab$, which is equivalent to $q' \geq ab$. Next we deduce $w_1' \geq b$ and $w_2' \geq a$ from $aw_1' \geq q' \geq ab$ and $bw_2' \geq q' \geq ab$. Thus we have $aw_1' + bw_2' \geq 2ab$. □

Corollary 3.4. *Let χ be a weighted voting game with equivalence classes $N_1, \ldots, N_t$, $1 \leq i_1 < i_2 \leq t$ be two indices, and $N_1' \subset N_{i_1}$, $N_2' \subset N_{i_2}$ be two subsets. Consider two coprime integers $b > a \geq 1$, such that $|N_1'| = a$ and $|N_2'| = b$. If the restriction of χ to $N_1' \cup N_2'$ is equivalent to $\chi_{a,b}$, then we have $q \geq ab$, $w_{i_1} \geq b$, and $w_{i_2} \geq a$ for the optimal solution $(q, w_1, \ldots, w_t)$ of the linear program (6).*

3.1. Weighted games without a minimum integer representation for small t

In order to construct a weighted game without a minimum integer representation for $t = 3$ equivalence classes of voters we choose two coprime integers $b > a \ge 1$ and an integer c satisfying

(1) $ab - c \notin \langle a, b\rangle$,
(2) $ab - 2c + 1 \in \langle a, b\rangle$,
(3) $ab \ge 2c - 1$, and
(4) $c \ge b + 2$.

With this we consider the weighted game

$$\chi_{a,b,c} = [ab; c - \frac{1}{2}, c - \frac{1}{2}, \underbrace{b, \dots, b}_{a}, \underbrace{a, \dots, a}_{b}],$$

i.e. in the notation of a complete simple game the cardinality vector is given by $\overline{\mathbf{n}} = (2, a, b)$.

These technically looking constraints can be interpreted as follows. Due to $b \ge a+1 \ge 1$ and $c \ge b+2$ the assignment $q = ab$, $w_1 = a$, $w_2 = b$, $w_3 = c - \frac{1}{2}$ is a feasible solution of inequality system (5) for $\chi_{a,b,c}$. We remark $w_1 = c - \frac{1}{2} \notin \mathbb{N}$. Constraint (1) requires that every (shift-maximal) losing vector $\widetilde{\mathbf{l}} = (1, l_2, l_3)$ has a weight of at most $q - \frac{3}{2}$, while constraints (2) and (3) ensure that there exists a winning vector $\widetilde{\mathbf{m}}_t = (2, t_2, t_3)$ whose weight equals exactly the quota q, i.e. $2c - 1 + t_2 b + t_3 a = ab$.

Lemma 3.5. *The sum of weights of a minimum sum integer representation of $\chi_{a,b,c}$ is at least $2c-1+2ab$.*

PROOF. Let (q', w'_1, w'_2, w'_3) be the optimal solution of the linear program minimizing the sum of weights. From Corollary 3.4 we conclude $q' \ge ab$, $w'_2 \ge b$, and $w'_3 \ge a$. Since the, above defined, vector $\widetilde{\mathbf{m}}_t = (2, t_2, t_3)$, is winning we have $2w'_1 + t_2 w'_2 + t_3 w'_3 \ge q' \ge ab$. Using $bt_2 + at_3 = ab - 2c + 1$ we conclude

$$\begin{aligned} & 2w'_1 + aw'_2 + bw'_3 = 2w'_1 + t_2 w'_2 + t_3 w'_3 + (a - t_2)w'_2 + (b - t_3)w'_3 \\ \ge \quad & ab + (a - t_2)b + (b - t_3)a = 2ab + ab - t_2 b - t_3 a = 2c - 1 + 2ab \end{aligned}$$

and finally apply Lemma 2.13. □

Next we show that $\tau_1 = (ab, c, c - 1, b, \dots, b, a, \dots a)$ and $\tau_2 = (ab, c - 1, c, b, \dots, b, a, \dots, a)$ are minimum sum integer representations of $\chi_{a,b,c}$. Due to Lemma 3.5 it remains to show that both vectors are integer representations. Coalitions of type $(0, m_2, m_3)$ or $(2, m_2, m_3)$ have the same weight according to all three different weight vectors (including $\tau_0 = (c-\frac{1}{2}, c-\frac{1}{2}, b, \dots, b, a, \dots, a)$). Now let $(1, m_2, m_3)$ be a winning vector. From the definition of the game $\chi_{a,b,c}$, i.e. τ_0, we conclude $c - \frac{1}{2} + m_2 b + m_3 a \ge ab$, which can be slightly sharpened to $c - 1 + m_2 b + m_3 a \ge ab$. Thus for both weightings τ_1 and τ_2 the lightest coalition, i.e. the one with minimal weight, of type $(1, m_2, m_3)$ has at least a weight of $c - 1 + m_2 b + m_3 a \ge ab$. Now let $(1, l_2, l_3)$ be a losing coalition. Since $ab - c \notin \langle a.b\rangle$ we have $c - \frac{1}{2} + bl_2 + al_3 \le ab - \frac{3}{2}$. Thus for both weightings τ_1 and τ_2 the heaviest coalition of type $(1, l_2, l_3)$ has a weight of at most a weight of $c + m_2 b + m_3 a \le ab - 1$.

The final conclusion is that $\chi_{a,b,c}$ cannot admit a minimum integer representation since it has at least two minimum sum integer representations. An example where the requested technical conditions on a, b, and c are satisfied is given by $c = 12$, $b = 7$, $a = 5$.

Instead of using the general Lemma 3.5 we can consider the example directly. The game $\chi_{5,7,12}$ is a complete simple game with $t = 3$ types of voters whose vector/matrix notation is given by $\overline{\mathbf{n}} = (2, 5, 7)$ and $\mathcal{M} = \begin{pmatrix} 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 5 & 3 & 0 \\ 1 & 3 & 2 & 4 & 5 & 0 & 3 & 7 \end{pmatrix}^T$. The matrix of the shift-maximal losing vectors is given by $\begin{pmatrix} 2 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 1 & 4 & 2 \\ 0 & 2 & 0 & 3 & 1 & 4 \end{pmatrix}^T$. Solving the LP (6) yields the optimal solution $(35, 11.5, 7, 5)$. Thus we conclude from Lemma 2.13 that the sum of the weights in a minimum sum integer representation is at least 93. Now we can easily check that both $(35, 12, 11, 7, 7, 7, 7, 7, 5, 5, 5, 5, 5, 5, 5)$ and $(35, 11, 12, 7, 7, 7, 7, 7, 5, 5, 5, 5, 5, 5, 5)$ are integer representations of $\chi_{5,7,12}$ attaining this lower bound. We remark that the stated representations arise by a swap of weights within the first equivalence class.

For $t = 4$ equivalence classes and for the situation of integer representations preserving types, we apply a similar idea and consider a game χ with cardinality vector $\overline{\mathbf{n}} = (1, 1, a, b)$ containing $\chi_{a,b}$ as a subgame. The rough idea is to choose half-integer weights $w_1, w_2 \in \mathbb{N} + \frac{1}{2}$ such that (ab, w_1, w_2, b, a) is an optimal solution of LP (6) while $\lfloor w_1 \rfloor$ or $\lfloor w_2 \rfloor$ can be attained in (different) integer representations. Similarly as in the example above, sufficient technical conditions can be formulated using the membership or non-membership of certain values in $\langle a, b \rangle$. We refrain from explicitly formulating the details and instead give an example. We choose $b = 11$ and $a = 7$, which satisfy $\{52, 59\} \cap \langle 7, 11 \rangle = \emptyset$ and $52 + 59 - 7 \cdot 11 + 1 \in \langle 7, 11 \rangle$. The game χ now is uniquely chosen by stating its matrix of shift-minimal winning vectors:
$$\mathcal{M} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 5 & 3 & 1 & 0 & 5 & 3 & 1 & 0 & 7 & 6 & 4 & 2 & 0 \\ 2 & 5 & 0 & 3 & 6 & 8 & 1 & 4 & 7 & 9 & 0 & 2 & 5 & 8 & 11 \end{pmatrix}^T.$$
The matrix of the shift-maximal losing vectors is then given by $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 4 & 2 & 0 & 5 & 4 & 2 & 0 & 6 & 5 & 3 & 1 \\ 0 & 3 & 1 & 4 & 7 & 0 & 2 & 5 & 8 & 1 & 3 & 6 & 9 \end{pmatrix}^T$. Solving the LP (6) yields the optimal solution $(77, 24.5, 17.5, 11, 7)$. The sum of weights of a minimum sum integer representation is at least 196. By checking that both $(77, 25, 17, 11, 11, 11, 11, 11, 11, 11, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7)$ and $(77, 24, 18, 11, 11, 11, 11, 11, 11, 11, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7)$ are integer representations of χ we conclude that they are indeed minimum sum integer representations preserving types. Thus χ does not admit a minimum integer representation preserving types.

3.2. *Weighted games with more than two minimum sum integer representations*

It would be nice to have an example of a weighted game with more than two minimum sum representations preserving types. Before we give a construction that works, we briefly remark that not every reasonable system of constraints for the representability of some expressions needs to have a solution, so that not all construction ideas lead to success. Our first idea was to choose $t = 4$, two coprime integers $b > a \geq 1$ and $\overline{\mathbf{n}} = (1, 1, a, b)$. If there existed integers $0 < l_2 < l_1 - 1 < ab$ such that $l_1, l_1+1, l_2, l_2+1 \notin \langle a, b \rangle$ but $ab - l_1 - l_2 + 2 \in \langle a, b \rangle$, then we could check that $(ab, ab - l_2 - 2 + x, ab - l_1 - 2 + (2 - x), b, \ldots, b, a, \ldots, a)$ is a minimum sum integer representation for $x \in \{0, 1, 2\}$. Unfortunately the existence of such integers l_1, l_2 would contradict Popoviciu's theorem counting the number of representations, see e.g. [1]. To be more precisely, $ab - k \in \langle a, b \rangle$ implies $k \notin \langle a, b \rangle$ for all $k \in \mathbb{N}$ with $a, b \not| k$.

For $t = 5$ we have another construction which works:

Proposition 3.6. *Let $b > a \geq 1$ be two coprime positive integers. Suppose we have integers $l_1 < l_2 < l_3$ fulfilling*

(1) $b < \widehat{w_i} = ab - l_i - 1$, $l_i \notin \langle a, b \rangle$ *for* $1 \leq i \leq 3$,

(2) $l_1 + l_2 + l_3 - l_i - ab \notin \langle a, b \rangle$ *for* $1 \leq i \leq 3$, *and*

(3) $0 < l_1 + l_2 + l_3 + 1 - 2ab < ab$, $l_1 + l_2 + l_3 + 1 - 2ab \in \langle a, b \rangle$.

With this the weighted game $\chi = [ab; \widehat{w_1}, \widehat{w_2} + 1, \widehat{w_3} + 1, b, \ldots, b, a, \ldots, a]$, where $\overline{\mathbf{n}} = (1, 1, 1, a, b)$, has the following three minimum sum integer representations preserving types:

- $\tau_1 = (ab, \widehat{w_1}, \widehat{w_2} + 1, \widehat{w_3} + 1, b, \ldots, b, a, \ldots, a)$
- $\tau_2 = (ab, \widehat{w_1} + 1, \widehat{w_2}, \widehat{w_3} + 1, b, \ldots, b, a, \ldots, a)$
- $\tau_3 = (ab, \widehat{w_1} + 1, \widehat{w_2} + 1, \widehat{w_3}, b, \ldots, b, a, \ldots, a)$

PROOF. Let $(q', w_1', w_2', w_3', w_4', w_5')$ be the optimal solution of the linear program minimizing the sum of weights. From Corollary 3.4 we conclude $q' \ge ab$, $w_4' \ge b$, and $w_5' \ge a$. Since $l_1 + l_2 + l_3 + 1 - 2ab \in \langle a, b\rangle$, see constraint (3), there exist integers u, v such that $(1, 1, 1, u, v)$ is a (shift-minimal) winning vector and $ub + va = l_1 + l_2 + l_3 + 1 - 2ab$. Thus we have $w_1' + w_2' + w_3' + uw_4' + vw_5' \ge q'$. Inserting this into the sum of all weights yields

$$\begin{aligned} & w_1' + w_2' + w_3' + aw_4' + bw_5' \ge q' + (a-u)w_4' + (b-v)w_5' \\ \ge \quad & ab + 2ab - (ub + va) = 2ab + \widehat{w}_1 + \widehat{w}_2 + \widehat{w}_3 + 2, \end{aligned}$$

i.e. each minimum sum integer representation has a weight of at least $2ab + \widehat{w}_1 + \widehat{w}_2 + \widehat{w}_3 + 2$ due Lemma 2.13.

The next step is to prove that the three stated weightings represent the game χ. So for each vector $(m_1, m_2, m_3, m_4, m_5)$ we have to prove that its weight is less then ab or at least ab in all three different weightings simultaneously. This can be easily verified for the cases where $m_1 + m_2 + m_3 \in \{0, 3\}$.

For $m_1 + m_2 + m_3 = 1$ let $(1, 0, 0, m_4, m_5)$ be a losing vector, i.e. we have $\widehat{w}_1 + m_4 b + m_5 a = ab - l_1 - 1 + m_4 b + m_5 a \le ab - 1$. In τ_2 and τ_3 the weight of the first player is increased by one, so we need $ab - l_1 - 1 + m_4 b + m_5 a \le ab - 2$, which is indeed true since $ab - l_1 - 1 + m_4 b + m_5 a = ab - 1$ is equivalent to $l_1 \in \langle a, b\rangle$, i.e. it contradicts constraint (1). Increasing the weight of player one in a winning coalition does not affect its status. Now we consider a winning vector $(0, 1, 0, m_4, m_5)$, i.e. we have $\widehat{w}_2 + 1 + m_4 b + m_5 a = ab - l_2 + m_4 b + m_5 a \ge ab$. In τ_2 the weight of the second player is decreased by one, so that we need $ab - l_2 + m_4 b + m_5 a \ge ab + 1$, which is true since $ab - l_2 + m_4 b + m_5 a = ab$ is equivalent to $l_2 \in \langle a, b, \rangle$. Due to symmetry we conclude that there are no contradictions for $m_3 = 1$.

For $m_1 + m_2 + m_3 = 2$ let $(1, 1, 0, m_4, m_5)$ be a losing vector, i.e. we have $\widehat{w}_1 + \widehat{w}_2 + 1 + m_4 b + m_5 a = 2ab - l_1 - l_2 - 1 + m_4 b + m_5 a \le ab - 1$. Due to $l_1 + l_2 \notin \langle a, b\rangle$ the vector has a weight of at most $ab - 2$ using the weights from τ_1. Thus the vector remains losing in τ_2 and τ_3. Increasing the weight of player one in a winning coalition does not affect its status. A symmetric argument applies for vectors of type $(1, 0, 1, m_4, m_5)$. Now let $(0, 1, 1, m_4, m_5)$ be a winning vector, i.e. we have $\widehat{w}_2 + \widehat{w}_3 + 2 + m_4 b + m_5 a = 2ab - l_1 - l_2 + m_4 b + m_5 a \ge ab$. Due to $l_1 + l_2 \notin \langle a, b\rangle$ the vector has a weight of at least $ab + 1$ using the weights from τ_1. Thus the vector remains winning in τ_2 and τ_3. Decreasing the weight of either player two or player three by one does not cause any problems for a losing coalition.

Thus we have exhaustively checked that τ_1, τ_2, and τ_3 are minimum sum integer representations preserving types. $\square$

An example where the requirements of the previous proposition are fulfilled is given by $a = 17$, $b = 13$, $l_1 = 157$, $l_2 = 161$, $l_3 = 174$, $\widehat{w_1} = 63$, $\widehat{w_2} = 59$, and $\widehat{w_3} = 46$. A smaller example is given by $a = 13$, $b = 11$, $l_1 = 93$, $l_2 = 97$, $l_3 = 106$, $\widehat{w_1} = 49$, $\widehat{w_2} = 45$, and $\widehat{w_3} = 36$. Furthermore we have the following straightforward generalization:

Proposition 3.7. *Let $b > a \ge 1$ be two coprime integers with $a > b$ and t be an integer with $t \ge 2$. Suppose we have integers $l_1 < l_2 < \cdots < l_t$ fulfilling*

(1) $b < \widehat{w_i} = ab - l_i - 1$, $l_i \notin \langle a, b\rangle$ for $1 \le i \le 3$,

(2) $\sum_{j=1}^{z} l_{i_j} - (z-1)ab \notin \langle a,b\rangle$ *for all* $2 \le z < t$ *and all subsets* $\{i_1,\dots,i_z\} \subseteq \{1,\dots,t\}$ *of cardinality* z*, and*

(3) $0 < \sum_{j=1}^{t} l_j + 1 - (t-1)ab < ab$, $\sum_{j=1}^{t} l_j + 1 - (t-1)ab \in \langle a,b\rangle$.

With this the weighted game $\chi = [ab; \widehat{w_1}, \widehat{w_2}{+}1, \dots, \widehat{w_t}{+}1, b, \dots, b, a, \dots, a]$*, where* $\overline{\mathbf{n}} = (\underbrace{1,\dots,1}_{t}, a, b)$*, has the following* t *minimum sum integer representations preserving types:*

- $(ab, \widehat{w_1}, \widehat{w_2}+1, \dots, \widehat{w_t}+1, b, \dots, b, a, \dots, a)$
- $(ab, \widehat{w_1}+1, \widehat{w_2}, \widehat{w_3}+1 \dots, \widehat{w_t}+1, b, \dots, b, a, \dots, a)$

  $\vdots$
- $(ab, \widehat{w_1}+1, \dots, \widehat{w_{t-1}}+1, \widehat{w_t}, b, \dots, b, a, \dots, a)$

For $t \ge 4$ we have the following examples:

t	a	b	$\mathbf{l_1}, \dots, \mathbf{l_t}$
4	19	11	141, 157, 160, 179
5	19	17	249, 251, 253, 268, 287
6	29	17	389, 396, 401, 418, 430, 447
7	31	29	746, 750, 752, 777, 779, 808, 810
8	37	29	883, 891, 920, 941, 949, 970, 978, 1007
9	41	31	1086, 1100, 1106, 1117, 1127, 1137, 1158, 1168, 1199
10	43	41	1513, 1550, 1552, 1554, 1593, 1595, 1597, 1636, 1638, 1679

Conjecture 3.8. *Each weighted game with* t *equivalence classes of voters can have at most* $t-2$ *different minimum sum representations preserving types.*

3.3. Possible weights of minimum integer representations

Instead of asking which classes of weighted games admit a minimum integer representation or a minimum integer representation preserving types one can ask which weights are possible in a minimum integer representation. The following theorem and remarks resolve this question for two different weights almost completely. The stated lower bounds on the number of necessary voters n might be improved.

Theorem 3.9. *For two coprime integers* $b > a \ge 1$ *the weighted game* $\chi = [q = ab; \underbrace{b,\dots,b}_{n_1}, \underbrace{a,\dots,a}_{n_2}]$*, where* $n_1 \ge a$ *and* $n_2 \ge b$*, is in minimum integer representation.*

PROOF. Let $(q', b'_1, \dots, b'_{n_1}, a'_1, \dots, a'_{n_2})$ be an arbitrary integer representation of χ, where we assume $b'_1 \ge \dots \ge b'_{n_1}$ and $a'_1 \ge \dots \ge a'_{n_2} \ge 0$ w.l.o.g. From Isbell's desirability relation we conclude $b'_{n_1} > a'_1$. By Corollary 3.4 every integer representation of χ has a sum of weights of at least $n_1 b + n_2 a$ and $q' \ge ab$, $\sum_{i=1}^{n_1} b'_i \ge n_1 b$, $\sum_{i=1}^{n_2} a'_i \ge n_2 a$. It suffices to show $b'_{n_1} \ge b$ and $a'_{n_2} \ge a$.

If $a'_{n_2} < a$ we can assume $a'_{n_2} = a-1$, since convex combinations of feasible weightings are feasible. By averaging the weights $a'_1, \dots a'_{n_2-1}$ and $b'_1, \dots, b'_{n_1}$ we obtain the (feasible, possibly non-integer) weighting $(q', b+t, \dots, b+t, a+s, \dots, a+s, a-1)$, where $s \in \mathbb{Q}_{>0}$, $t \in \mathbb{Q}_{\ge 0}$.

Let $0 \le u \le a-1$ and $0 \le v \le b-1$ be two integers with $ub + va = q-1 = ab-1$. Rearranging yields $u = a - \frac{1+av}{b}$ so that b divides $1+av$ and we have $b < av$. Since (u,v) is a losing vector and $(a,0)$, $(0,b)$ are winning vectors we have

$$ub + ut + av + vs \le q' - 1, \quad ab + at \ge q', \quad \text{and} \quad ab + bs - s - 1 \ge q'. \tag{7}$$

Multiplying the first inequality by ab yields

$$ab^2u + abut + a^2bv + abvs \leq abq' - ab$$

and bu times the second inequality plus av times the third inequality yields

$$ab^2u + abtu + a^2bv + absv - avs - av \geq abq' - q'$$

Combining the last two inequalities yields

$$q' \geq ab + (s+1)av. \tag{8}$$

We already know $ab + bs - s - 1 \geq q'$ and conclude $(b-1)s - 1 \geq (s+1)av$. Inserting $b < av$ yields the contradiction $-s - 1 > b$. Thus $a'_{n_2} \geq a$.

If $b'_{n_1} < b$ we can assume $b'_{n_1} = b - 1$ and consider the weighting $(q', b+t, \dots, b+t, b-1, a+s, \dots, a+s)$, with $t \in \mathbb{Q}_{>0}$, $s \in \mathbb{Q}_{\geq 0}$. Let again $0 \leq u \leq a-1$ and $0 \leq v \leq b-1$ be two integers with $ub + va = q - 1 = ab - 1$. Here we have $a < ub$. An analogous calculation as before yields

$$q' \geq ab + (t+1)ub$$

and

$$q' \leq ab + (a-1)t - 1.$$

Combining these two inequalities yields $(a-1)t - 1 \geq (t+1)ub$. Inserting $a < ub$ ends up in the contradiction $-t - 1 > a$. $\square$

In the following remark we want to emphasize that most of the requirements of Theorem 3.9 are necessary:

Remark 3.10.

(1) If $r = \gcd(a,b) > 1$, then $\left[\frac{q}{r}, \frac{b}{r}, \dots, \frac{b}{r}, \frac{a}{r}, \dots, \frac{a}{r}\right]$ is a smaller representation for the same game.

(2) If $b = a$ then there is only one type of voters with minimum representation $[q'; 1, \dots, 1]$ for a suitable quota q'. If $b < a$ then the voters of type 2 would be more powerful than the voters of type 1, which is not possible by definition.

(3) If $a = 0$ and $b > 1$ then $\left[\left\lceil \frac{q}{b} \right\rceil, 1, \dots, 1, 0, \dots, 0\right]$ is a smaller representation for the same game, which is indeed the minimum integer representation.

(4) The lower bounds on n_1 and n_2 can be improved, e.g. based on the knowledge of u and v.

There is a generalization to weighted games with more than two types of voters:

Theorem 3.11. *Let $a_1, \dots, a_t$ be integers such that $a_1 > a_2 > \cdots > a_t > 0$ and for each $1 \leq i \leq t$ there is an index $1 \leq j \leq t$ with $\gcd(a_i, a_j) = 1$. The weighted game*

$$\chi = [q = lcm(a_1, \dots, a_t); \underbrace{a_1, \dots, a_1}_{n_1}, \dots, \underbrace{a_t, \dots, a_t}_{n_t}],$$

where $n_i \geq lcm(a_1, \dots, a_t)/a_i$ for all $1 \leq i \leq t$, is in minimum integer representation.

PROOF. For an arbitrary integer representation of χ let $(q', a'_1, \ldots, a'_t)$ be the averaged representation with equal (possibly non-integer) weights within each equivalence class of voters. For each index $1 \le i \le t$ choose a suitable index $j \neq i$ such that $\gcd(a_i, a_j) = 1$. Let $z = \frac{q}{a_i a_j} \in \mathbb{N}$ and $0 \le u \le a_j - 1$, $0 \le v \le a_i - 1$ be two integers such that $ua_i + va_j = a_i a_j - 1$. With this we have

$$\begin{aligned} -\Big((z-1)\, a_j + u \Big) a'_i - va'_j + q' &\ge 1 \\ (z-1) a_j a'_i + a_i a'_j - q' &\ge 0 \\ z a_j a'_i - q' &\ge 0. \end{aligned}$$

Combining these inequalities with the vectors $(a_i, v, a_i - v)$, $(a_j, a_j - u, u)$, and $(za_i a_j, za_j v, (z-1)a_j(a_i - v) + a_i u)$ as multipliers yields $a'_i \ge a_i$, $a'_j \ge a_j$, and $q' \ge q$.

We can treat the case of different weights within equivalence classes of voters analogously to the proof of the previous theorem. $\square$

In the next theorem we pay for less restrictive conditions on the weights a_i by a rather large bound on the number of voters n. To this end we generalize Lemma 3.2 for more than two integers:

Lemma 3.12. *Let $a_1, \ldots, a_t$ be positive integers with $t \ge 2$ and $\gcd(a_1, \ldots, a_t) = g$. There exist t integers u_i with $\sum\limits_{i=1}^{t} u_i a_i = \prod_{i=1}^{t} a_i - g$ and $0 \le u_i \le \prod\limits_{j=1, j\neq i}^{t} a_j - 1$ for all $1 \le i \le t$.*

PROOF. We prove by induction on t. For $t = 2$ we apply Lemma 3.2 for the two integers $\frac{a_1}{g}$, $\frac{a_2}{g}$. For $t > 2$ let u'_i be integers with $\sum\limits_{i=1}^{t-1} u'_i a_i = \prod_{i=1}^{t-1} a_i - g' =: k$ and $0 \le u'_i \le \prod\limits_{j=1, j\neq i}^{t-1} a_j - 1$ for all $1 \le i \le t-1$, where $g' = \gcd(a_1, \ldots, a'_{t-1})$. We remark $\gcd(k, a_t) = g$ and apply Lemma 3.12 for $t = 2$. $\square$

Theorem 3.13. *For integers $a_1 > a_2 > \cdots > a_t > 0$ with $\gcd(a_1, \ldots, a_t) = 1$ the weighted game*

$$\chi = [q = \prod_{j=1}^{t} a_j; \underbrace{a_1, \ldots, a_1}_{n_1}, \ldots, \underbrace{a_t, \ldots, a_t}_{n_t}],$$

where $n_i \ge 2 \prod\limits_{j=1, j\neq i}^{t} a_j$ for all $1 \le i \le t$, is in minimum integer representation.

PROOF. Due to Lemma 3.12 there are t integers u_i with $\sum\limits_{i=1}^{t} u_i a_i = q - 1$ and $0 \le u_i \le \prod\limits_{j=1, j\neq i}^{t} a_j - 1$ for all $1 \le i \le t$. For an arbitrary integer representation of χ let $(q', a'_1, \ldots, a'_t)$ be the averaged representation with equal (possibly non-integer) weights within each equivalence class of voters. With this the following inequalities have to be valid:

$$\frac{q}{a_i} \cdot a'_i - q' \ge 0 \text{ for all } 1 \le i \le t \quad \text{and} \quad -\sum_{i=1}^{t} u_i a'_i + q' \ge 1.$$

Summing up $a_i u_i$ times the ith inequality plus q times the last inequality yields $q' \ge q$. Inserting this into the ith inequality gives $a'_i \ge a_i$ for all $1 \le i \le t$.

We can treat the case of different weights within equivalence classes of voters analogously to the proofs of the previous theorems. $\square$

The condition $\gcd(a_1,\dots,a_t)=1$ is necessary. If we also want to use zero weights we can utilize the next lemma:

Lemma 3.14. *The weighted game* $[q;\underbrace{a_1,\dots,a_1}_{n_1},\dots,\underbrace{a_t,\dots,a_t}_{n_t}]$, *where* $a_i>0$ *for all* $1\le i\le t$, *is in minimum integer representation* if and only if *the weighted game* $[q;\underbrace{a_1,\dots,a_1}_{n_1},\dots,\underbrace{a_t,\dots,a_t}_{n_t},\underbrace{0,\dots,0}_{n_{t+1}}]$ *is in minimum integer representation.*

4. Weighted games with minimum integer representations for two types of voters

As we have remarked in the introduction, all complete simple games with just one type of voters, $t=1$, are weighted and admit a minimum integer representation with all weights being equal to 1. These games are called "symmetric", "anonymous" or "k-out-of-n-games" in the literature. In the previous section we have constructed weighted games with $t=3$ equivalence classes without a minimum integer representation. So the central question of this section (and the paper) is: what happens for two types of voters? We first state the main result.

Theorem 4.1. *Each weighted game with two types of voters admits a minimum integer representation* $(q,\overbrace{w_1,\dots,w_1}^{n_1},\overbrace{w_2,\dots,w_2}^{n_2})$, *with* $1\le w_1\le\max(n_1+1,n_2)$, $0\le w_2\le\max(n_1,n_2-1)$, *and* $1\le q\le(n_1+n_2-1)\cdot\max(n_1+1,n_2)$. *For* $r\ge 2$ *shift-minimal winning vectors the bounds of minimum weights can be sharpened to* $1\le w_1\le n_2$, $1\le w_2\le n_1$, *and* $w_2+1\le q\le 2n_1n_2$.

In Section 5 we will use the bounds on q, w_1, and w_2 to give an upper bound on the number of weighted voting games with two types of voters. As preliminary work we prove Theorem 4.1 for some special cases of weighted games with two types of voters by a direct argumentation on the possible integer representations in Subsection 4.1. The proof strategy for the remaining part is more involved. We study linear minimization problems subject to the constraints in (5). It turns out that each target function without negative coefficients admits an optimal integer solution. By additionally using some structure result on the set of inequalities, which attain equality, called *tight* later on, we deduce the existence of a minimum integer representation.

4.1. Proof of the main theorem for $r=1$ *shift-minimal winning vectors*

Theorem 4.2. *For a weighted game* χ *with two types of voters, a cardinality vector* $\overline{\mathbf{n}}=(n_1,n_2)$ *and a unique minimal winning coalition* $\widetilde{m}=(m_1,m_2)$, *i.e.* $t=2$ *and* $r=1$, *there exists a minimum integer representation.*

PROOF. Since the game is weighted there are some restrictions on the parameters m_1,m_2 beyond those from (1), i.e. $1\le n_1\le n-1$, $n_1+n_2=n$, $1\le m_1\le n_1$, and $0\le m_2\le n_2-1$. First we exclude the cases where $1\le m_1\le n_1-1$ and $2\le m_2\le n_2-2$. Assume that (q,w_1,w_2) is a feasible solution of (5). Since (m_1-1,m_2+2), (m_1+1,m_2-2) are losing vectors and (m_1,m_2) is a winning vector we have

$$(m_1-1)w_1+(m_2+2)w_2\le q-1\le m_1w_1+m_2w_2-1,$$
$$(m_1+1)w_1+(m_2-2)w_2\le q-1\le m_1w_1+m_2w_2-1,$$

from which we conclude $2w_2\le w_1-1\le 2w_2-2$; a contradiction. It will turn out that χ is weighted in the remaining cases, i.e. for $m_1=n_1$ or $m_2\in\{0,1,n_2-1\}$.

Let $(q,a_1,\dots,a_{n_1},b_1,\dots,b_{n_2})$ be an arbitrary integer representation with $a_1\ge\dots\ge a_{n_1}$ and $b_1\ge\dots\ge b_{n_2}$ of the game χ. Due to Isbell's desirability relation we have $a_{n_1}\ge b_1+1$.

- $1 \leq m_1 \leq n_1 - 1$, $m_2 = 0$:
  We can easily check that $a_1 = \cdots = a_{n_1} = 1$, $b_1 = \cdots = b_{n_2} = 0$, $q = m_1$ is an integer representation of χ. Since we have $a_i \geq 1$ and $b_j \geq 0$ it is also a minimum integer representation.

- $1 \leq m_1 \leq n_1 - 1$, $m_2 = 1$:
  Since $(m_1, 0)$ and $(m_1 - 1, n_2)$ are losing vectors, we can conclude $b_i \geq 1$ for all $1 \leq i \leq n_2$ and $a_i \geq n_2$ for all $1 \leq i \leq n_1$. We can easily check that $a_1 = \cdots = a_{n_1} = n_2$, $b_1 = \cdots = b_{n_2} = 1$, $q = m_1 n_2 + 1$ is an integer representation of χ and thus is indeed a minimum integer representation.

- $1 \leq m_1 \leq n_1 - 1$, $m_2 = n_2 - 1$:
  Since the cases $m_2 \in \{0, 1\}$ were dealt previously, we assume $m_2 \geq 2$.

  For $m_1 + n_2 - 1 \leq n_1$ the vector $(m_1 + n_2 - 2, 0)$ is losing. Comparing the weights of its corresponding coalitions with those from the shift-minimal winning vector and inserting $a_i \geq b_j + 1$ yields $b_j \geq n_2 - 1$ and $a_i \geq n_2$. We can easily check that $a_1 = \cdots = a_{n_1} = n_2$, $b_1 = \cdots = b_{n_2} = n_2 - 1$, $q = m_1 n_2 + (n_2 - 1)^2$ is an integer representation of χ.

  For $m_1 + n_2 - 1 > n_1$ we compare the losing vector $(n_1, m_1 + n_2 - 2 - n_1)$ with the shift-minimal winning vector and insert $a_i \geq b_j + 1$ to deduce $b_j \geq n_1 - m_1 + 1$ and $a_i \geq n_1 - m_1 + 2$. We can easily check that $a_1 = \cdots = a_{n_1} = n_1 + 2 - m_1$, $b_1 = \cdots = b_{n_2} = n_1 + 1 - m_1$, $q = (m_1 + n_2)(n_1 + 1 - m_1) + 2m_1 - n_1 - 1$ is an integer representation of χ.

- $m_1 = n_1$, $0 \leq m_2 \leq n_2 - 1$:
  If $m_2 = 0$ then $a_1 = \cdots = a_{n_1} = 1$, $b_1 = \cdots = b_{n_2} = 0$, $q = n_1$ is a minimum integer representation. Otherwise we have the losing vectors $(n_1, m_2 - 1)$ and $(n_1 - 1, n_2)$ from which we conclude $b_i \geq 1$ for all $1 \leq i \leq n_2$ and $a_i \geq n_2 - m_2 + 1$, respectively. Thus we have $q \geq n_1(n_2 - m_2 + 1) + m_2$. We can easily check that equality is possible, so that $a_i = n_2 - m_2 + 1 \geq 2$, $b_j = 1$, $q = n_1(n_2 - m_2 + 1) + m_2$ is a minimum integer representation.

□

Going over the cases of the proof of Theorem 4.2 we can check that all stated minimum integer representations satisfy $1 \leq w_1 \leq \max(n_1 + 1, n_2)$ and $0 \leq w_2 \leq \max(n_1, n_2 - 1)$ so that $1 \leq q \leq (n_1 + n_2 - 1) \cdot \max(n_1 + 1, n_2)$, as stated in Theorem 4.1.

To reduce the need for case differentiations in the remaining part we now completely handle the cases where null voters or dummies, i.e. voters i such that $\chi(U) = \chi(U \cup \{i\})$ for all subsets $U \subseteq N \backslash \{i\}$, occur.

Lemma 4.3. *Weighted games with two types of voters, where one class consists of null voters, admit a minimum integer representation.*

PROOF. If, as usual, the equivalence classes of the game χ are given by N_1, N_2, then N_2 has to be the set of null voters. By the definition of a null voter each shift-minimal winning vector (m_1, m_2) has to satisfy $m_2 = 0$. Since shift-minimal winning vectors are incomparable, we have $r = 1$ and can apply Theorem 4.2. □

Also in general we can drop null voters from given games when determining minimum integer representations or minimum sum integer representations (preserving types or not).

Lemma 4.4. *Let χ be a weighted game with k null voters and χ' be the (weighted) game arising from χ by deleting the k null voters. If $(q, w_1, \ldots, w_{n-k})$ is an integer representation of χ', then $(q, w_1, \ldots, w_{n-k}, 0, \ldots, 0)$ is an integer representation of χ.*

4.2. Proof of the main theorem for $r > 1$ shift-minimal winning vectors

In the following we restrict our considerations to games without null voters and $r > 1$ shift-minimal winning vectors. In this case we can drop two constraints from inequality system (5).

Lemma 4.5. *For a weighted game χ without null voters and with $t = 2$, $r > 1$ every vector (q, w_1, w_2) is feasible for inequality system (5) if and only if it satisfies*

$$\widetilde{\mathbf{x}}^T w \geq q \;\forall \widetilde{\mathbf{x}} \in \mathcal{W}^{sm},\; \widetilde{\mathbf{y}}^T w \leq q - 1 \;\forall \widetilde{\mathbf{y}} \in \mathcal{L}^{sM}. \tag{9}$$

PROOF. It remains to prove that the constraints from (9) imply $w_1 \geq w_2 + 1$ and $w_2 \geq 0$.

Let $(a, b) \in \mathcal{W}^{sm}$ with minimal a, i.e. all $(m_1, m_2) \in \mathcal{W}^{sm}$ satisfy $m_1 \geq a$. If $a \geq 1$ and $b < n_2$, then $(a - 1, n_2) \in \mathcal{L}^{sM}$. With this we conclude

$$aw_1 + bw_2 \geq q \geq (a - 1)w_1 + n_2 w_2 + 1,$$

which is equivalent to $w_1 \geq (n_2 - b)w_2 + 1 \geq w_2 + 1$. If $a = 0$ or $b = n_2$ then let $(c, d) \in \mathcal{W}^{sm}$ with minimal $c > a$, i.e. for all $(m_1, m_2) \in \mathcal{W}^{sm}$ we either have $m_1 = a$ or $m_1 \geq c$. With this we have $(c - 1, a + b - c) \in \mathcal{L}^{sM}$ and conclude

$$cw_1 + dw_2 \geq q \geq (c - 1)w_1 + (a + b - c)w_2 + 1,$$

which is equivalent to $w_1 \geq (a + b - c - d)w_2 + 1 \geq w_2 + 1$. Thus in both cases the constraints from (9) imply $w_1 \geq w_2 + 1$.

In order to deduce $w_2 \geq 0$ we consider a winning vector (m_1, m_2) with $m_2 > 0$, which must exist since χ does not contain null voters. Thus $(m_1, m_2 - 1)$ is a losing vector. Now let (l_1, l_2) be a shift-maximal losing vector with $(l_1, l_2) \succeq (m_1, m_2 - 1)$, i.e. we have $l_1 \geq m_1$ and $l_1 + l_2 \geq m_1 + m_2 - 1$. From $(l_1, l_2) \not\succeq (m_1, m_2)$ we conclude $l_1 + l_2 < m_1 + m_2$, so that only $l_1 + l_2 = m_1 + m_2 - 1$ is possible. With this we have $m_1 w_1 + m_2 w_2 \geq q \geq l_1 w_1 + l_2 w_2 + 1$, which is equivalent to $(l_1 - m_1)w_1 + 1 \leq (m_2 - l_2)$. Inserting $w_1 \geq w_2 + 1$ yields $(l_1 - m_1)w_2 + 1 + (l_1 - m_1) \leq (m_2 - l_2)w_2$, so that we have $w_2 \geq 1 + l_1 - m_1 \geq 1$. $\square$

Let us consider an example of inequality system (9) for the complete simple game χ uniquely characterized by $\overline{\mathbf{n}} = (3, 3)$ and $\mathcal{M} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 5 \end{pmatrix}^T$. The shift-maximal losing vectors are given by $(1, 2)$ and $(0, 4)$, so that Inequality system (9) reads as follows.

$$2w_1 \geq q, \quad w_1 + 3w_2 \geq q, \quad 5w_2 \geq q, \quad w_1 + 2w_2 \leq q - 1, \;\text{ and }\; 4w_2 \leq q - 1.$$

If we minimize one of the objective functions $f_1(q, w_1, w_2) = q$, $f_2(q, w_1, w_2) = w_1$, or $f_3(q, w_1, w_2) = w_2\ w_1$ subject to those constraints, we obtain the optimal solution $q = 10$, $w_1 = 5$, $w_2 = 2$ in all three cases. It is quite remarkable that those values are integers while we have only requested that they are real-valued. It will turn out that this is a general phenomenon in our context.

Optimal solutions of linear programs are strongly connected with solutions of linear equation systems, since it is well known that, if a linear program admits an optimal solution, then there is an optimal solution attained at a corner of the set of feasible points. To this end we say that an inequality of a linear program is **tight** for a given feasible point if equality is attained. In our example the inequalities $2w_1 \geq q$, $5w_2 \geq q$, and $w_1 + 2w_2 \leq q - 1$ are tight for the point $(10, 5, 2)$, while the inequalities $w_1 + 3w_2 \geq q$ and $4w_2 \leq q - 1$ are not. In our context each corner is the solution of an equation system of three tight inequalities, as we have three (linearly independent) variables.

In Lemma 4.6, Lemma 4.7, and Lemma 4.8 we check all possible 3-element subsets of the inequalities of (9). It turns out that whenever the corresponding 3×3-equation system has a unique solution, all variables attain integer values.

So each optimal vertex of the linear program in Lemma 4.5 is determined by three tight inequalities of one of the types $\widetilde{m}^T w \geq q$ or $\widetilde{l}^T w \leq q-1$, since $w_1, w_2, q \geq 0$ cannot be attained with equality. In the following three lemmas we consider the possible cases.

Lemma 4.6. *For Inequality system (9), three tight inequalities of type $\widetilde{m}^T w \geq q$ or three tight inequalities of type $\widetilde{l}^T w \leq q-1$ have to be either linearly dependent or do not determine a solution at all.*

PROOF. Consider the equation system $aw_1+bw_2 = cw_1+dw_2 = ew_1+fw_2 = z$, where $z \in \{q, q-1\}$. Eliminating z leaves $(a-c)w_1+(b-d)w_2 = (c-e)w_1+(d-f)w_2 = 0$, which has either the unique solution $w_1 = w_2 = 0$, which is infeasible for the whole inequality system, or an infinite number of solutions due to scaling. (In the latter case the equations are linearly dependent.) $\square$

Lemma 4.7. *For Inequality system (9), two tight inequalities of type $\widetilde{m}^T w \geq q$ and one tight inequality of type $\widetilde{l}^T w \leq q-1$ lead to an integer solution $(\widehat{q}, \widehat{w_1}, \widehat{w_2})$ such that $w_1 \geq \widehat{w_1}$, $w_2 \geq \widehat{w_2}$, and $q \geq \widehat{q}$ for all feasible (q, w_1, w_2) or do not determine a solution at all.*

PROOF. Let $(a,b),(c,d) \in \mathcal{W}^{sm}$ and $(e,f) \in \mathcal{L}^{sM}$ be the vectors corresponding to the tight inequalities, where we assume $a > c$. From $(a,b) \bowtie (c,d)$ and $a > c$ we conclude $d > b+1$. Solving the corresponding equation system yields $\widehat{w_1} = \frac{d-b}{Q}$, $\widehat{w_2} = \frac{a-c}{Q}$, and $\widehat{q} = \frac{ad-bc}{Q}$, where $Q := fc - fa + ad - bc - ed + eb \in \mathbb{Z}$. The case $Q = 0$ corresponds to an equation system which does not have a unique solution. Since we know that each feasible solution of (9) satisfies $w_1, w_2 > 0$ we can assume $Q > 0$ in the following.

Let $g := \gcd(a-c, d-b) \geq 1$. For the weights $\widehat{w}_1, \widehat{w}_2$ we can easily check that coalition type (a,b) has the same weight as coalition type $(a',b') = \left(a - \frac{a-c}{g}, b + \frac{d-b}{g}\right)$. If (a',b') is not a winning vector, then $(\widehat{q}, \widehat{w}_1, \widehat{w}_2)$ cannot be a feasible solution. Thus (a',b') is a shift-minimal winning vector too. If $g > 1$ then we have $a > a' > c$. We can check $Q' := fc - fa' + a'd - b'c - ed + eb' = \left(1 - \frac{1}{g}\right) \cdot Q > 0$. Thus we can assume w.l.o.g. that $a > c$ is minimal within the set of shift-minimal winning vectors corresponding to tight inequalities, i.e. we can assume $g = 1$.

Now we apply Lemma 3.1 and choose unique integers u, v fulfilling $u(d-b) - v(a-c) = 1$, where $0 < u \leq a-c$ and $0 \leq v < d-b$. The coalition type $(e',f') = (a-u, b+v)$ has weight $\widehat{q} - \frac{1}{Q}$ and thus is losing. Since all losing coalitions have weight at most $q-1$ we conclude $Q = 1$. Thus (e',f') is indeed a shift-maximal losing vector corresponding to a tight inequality. We can easily check $Q' = f'c - f'a + ad - bc - e'd + e'b = 1$ so that we can assume $(e,f) = (e',f')$ since this characterizes the same solution.

Let us have a closer look at the corresponding inequality system again:

$$aw_1 + bw_2 - q \geq 0, \quad cw_1 + dw_2 - q \geq 0, \quad \text{and} \quad -ew_1 - fw_2 + q \geq 1.$$

For the basis (w_1, w_2, q) the inverse matrix is given by

$$M^{-1} = \frac{1}{Q} \cdot \begin{pmatrix} d-f & f-b & d-b \\ e-c & a-e & a-c \\ ed-cf & af-eb & ad-bc \end{pmatrix}.$$

If we can show that all entries of M^{-1} are non-negative, then we have $w_1 \geq \widehat{w_1}$, $w_2 \geq \widehat{w_2}$, and $q \geq \widehat{q}$ for all feasible (w_1, w_2, q).

From $a > c$ and $(a,b) \bowtie (c,d)$ we conclude $a + b < c + d$, so that we have $a - c \geq 1$ and $d - b \geq 2$. Since $e = a-u$, $f = b+v$ with $0 < u \leq a-c$, $0 \leq v < d-b$ we have $a-e \geq 1$, $f-b \geq 0$, $e-c \geq 0$, and $d-f \geq 1$. Thus, the entries of the first two rows of M^{-1} are non-negative integers. For $Q = 1$ we have $ad - bc = \widehat{q} \geq 1$. From $f = b + v$, $e = a - u$ we conclude $af - eb = av + bu \geq 0$. The last inequality arises from

$$ed - cf \underbrace{=}_{Q=1} ad - bc - (af - eb) - 1 = a\underbrace{(d - f)}_{\geq 1} + b\underbrace{(e - c)}_{\geq 0} - 1 \geq 0.$$

□

Let us illustrate how Lemma 4.7 works by an example. For this purpose let the weighted game χ be uniquely characterized by its cardinality vector $\overline{\mathbf{n}} = (4, 8)$ and its matrix of shift-minimal winning vectors $\mathcal{M} = \begin{pmatrix} 4 & 3 & 2 & 1 & 0 \\ 0 & 1 & 4 & 6 & 8 \end{pmatrix}^T$. An integer representation preserving types is given by the weights $w_1 = 7$, $w_2 = 3$, and quota $q = 24$. Let us assume the that winning vectors $(3, 1)$, $(0, 8)$ and the losing vector $(1, 5)$ would correspond to tight inequalities. The solution of the corresponding equation system is given by $w_1 = \frac{7}{2}$, $w_2 = \frac{3}{2}$, $q = 12$. Here the weights w_1 and w_2 are non-integer. So Lemma 4.7 says that (q, w_1, w_2) cannot be a feasible solution of inequality system (9). Thus there must be a constraint which is violated. The construction of (e', f') in the proof precisely gives such a violation. Since $1 \cdot (d - b) - 2 \cdot (a - c) = 1$ the coalition $(2, 3)$ is a losing vector with weight 11.5. We can easily check that it is indeed a shift-maximal losing vector having a weight strictly larger than $q - 1 = 11$.

Starting from the infeasible vector $(12, 3.5, 1.5)$ the proof provides us even another candidate for a 3-element subset of tight inequalities. If we replace the losing vector $(1, 5)$ by $(e', f') = (2, 3)$, then we obtain the solution $w_1 = 7$, $w_2 = 3$, $q = 24$, which now consists of integers. Here we have

$$M = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 8 & -1 \\ -2 & -3 & 1 \end{pmatrix} \quad \text{and} \quad M^{-1} = \begin{pmatrix} 5 & 2 & 7 \\ 2 & 1 & 3 \\ 16 & 7 & 24 \end{pmatrix}.$$

Since the inverse matrix M^{-1} consists of non-negative entries, as generally shown in the proof, we have $w'_1 \geq 7$, $w'_2 \geq 3$, and $q' \geq 24$ for every averaged integer representation (q', w'_1, w'_2). To be more precise: if we combine the inequalities $3w_1 + w_2 - q \geq 0$, $8w_2 - q \geq 0$, and $-2w_1 - 3w_2 + q \geq 1$ with non-negative multipliers given by the first row on M^{-1}, we conclude $w_1 \geq 7$. For the second and third row we similarly obtain $w_2 \geq 3$ and $q \geq 24$, respectively. Thus we have found a minimum sum integer representation preserving types.

Lemma 4.8. *For Inequality system (9), one tight inequality of type $\widetilde{m}^T w \geq q$ and two tight inequalities of type $\widetilde{l}^T w \leq q - 1$ lead to an integer solution $(\widehat{w_1}, \widehat{w_2}, \widehat{q})$ such that $w_1 \geq \widehat{w_1}$, $w_2 \geq \widehat{w_2}$, and $q \geq \widehat{q}$ for all feasible (w_1, w_2, q) or do not determine a solution at all.*

PROOF. Let $(a, b) \in \mathcal{W}^{sm}$ and $(c, d), (e, f) \in \mathcal{L}^{sM}$ be the vectors corresponding to the tight inequalities, where we assume $e > c$. Solving the corresponding equation system yields $\widehat{w_1} = \frac{d-f}{Q}$, $\widehat{w_2} = \frac{e-c}{Q}$, and $\widehat{q} = \frac{ad-fa+eb-bc}{Q}$, where $Q := fc - fa + ad - bc - ed + eb \in \mathbb{Z}$. The case $Q = 0$ corresponds to an equation system which does not have a unique solution. Since we know that each feasible solution of (9) satisfies $w_1, w_2 > 0$ we can assume $Q > 0$ in the following.

Let $g := \gcd(e - c, d - f) \geq 1$. The vector $(a', b') = \left(e - \frac{e-c}{g}, c + \frac{d-f}{g}\right)$ has the same weight as (a, b). So similarly to the proof of Lemma 4.7 we conclude that (a', b') is a shift-minimal winning vector, which corresponds to a tight inequality. We again check that replacing (a, b) by (a', b') is compatible with $Q' > 0$ so that we can finally assume $g = 1$ w.l.o.g.

Now we apply Lemma 3.1 and choose unique integers u, v fulfilling $u(d - f) - v(e - c) = 1$, where $0 < u \leq e - c$ and $0 \leq v < d - f$. The coalition type $(a', b') = (c + u, d - v)$ has weight $q - 1 + \frac{1}{Q}$. Since losing vectors have a weight of at most $q - 1$ the vector is winning and we have $Q = 1$. Using a similar argument as in the proof of Lemma 4.7 we conclude that (a', b') is indeed a shift-minimal winning vector corresponding to a tight inequality. We can easily check $Q' = fc - fa' + a'd - b'c - ed + eb' = 1$ so that we can assume $(a, b) = (a', b')$ since this characterizes the same solution.

Let us have a closer look at the corresponding inequality system again:

$$aw_1 + bw_2 - q \geq 0, \quad -cw_1 - dw_2 + q \geq 1, \quad \text{and} \quad -ew_1 - fw_2 + q \geq 1.$$

For the basis (w_1, w_2, q) the inverse matrix is given by

$$M^{-1} = \frac{1}{Q} \cdot \begin{pmatrix} d-f & b-f & d-b \\ e-c & e-a & a-c \\ ed-cf & eb-af & ad-bc \end{pmatrix}.$$

If we can show that all entries of M^{-1} are non-negative, then we have $w_1 \geq \widehat{w_1}$, $w_2 \geq \widehat{w_2}$, and $q \geq \widehat{q}$ for all feasible (w_1, w_2, q).

From $e > c$ and $(c, d) \bowtie (e, f)$ we conclude $e + f < c + d$, so that we have $e - c \geq 1$ and $d - f \geq 2$. Since $a = c + u$, $b = d - v$ with $0 < u \leq e - c$, $0 \leq v < d - f$ we have $a - c \geq 1$, $d - b \geq 0$, $e - a \geq 0$, and $b - f \geq 1$. Thus, the entries of the first two rows of M^{-1} are non-negative integers. From $e > c$ we conclude $ed - cf \geq c(d - f) \geq 0$ and from $a = c + u$, $b = d - v$ we conclude $ad - bc = ud + vc \geq 1$. The last inequality arises from

$$eb - af \underbrace{=}_{Q=1} 1 + (ed - cf) - (ad - bc) = 1 + d\underbrace{(e - a)}_{\geq 0} + c\underbrace{(b - f)}_{\geq 1} \geq 1.$$

□

Theorem 4.9. *Let χ be a weighted game without null voters and with $t = 2$, $r > 1$. Minimizing the target function $c_1w_1 + c_2w_2 + c_3q$, where $c_1, c_2, c_3 \geq 0$ and $c_1 + c_2 + c_3 > 0$, subject to the constraints in (9) results in a unique optimal integer solution $(q, w_1, w_2) \in \mathbb{N}^3_{>0}$ satisfying $1 \leq w_1 \leq n_2$, $1 \leq w_2 \leq n_1$, and $w_2 + 1 \leq q \leq 2n_1n_2$.*

PROOF. Let (q, w_1, q_2) be the minimum value of $n_1w_1 + n_2w_2$ subject to the constraints in (9). We already know that the optimum exists. This minimum is attained at a corner of the corresponding feasible set and thus arises as the unique solution of a 3×3-equation system, consisting of three tight inequalities. Due to Lemma 4.6 we can apply either Lemma 4.7 or Lemma 4.8. Thus, each feasible solution (q', w_1', w_2') of inequality system (9) has to satisfy $q' \geq q$, $w_1' \geq w_1$, and $w_2' \geq w_2$. So we have $c_1w_1' + c_2w_2' + c_3q' \geq c_1w_1 + c_2w_2 + c_3q$, where equality is attained if and only if $(q', w_1', w_2') = (q, w_1, w_2)$. The formulas for w_1, w_2 and q in Lemma 4.7 and Lemma 4.8 give the upper bounds $w_1 \leq n_2$, $w_2 \leq n_1$, and $q \leq 2n_1n_2$. Since χ does not contain null voters we also have $w_1, w_2 \geq 1$. If $q \leq w_2$, then every single voter would form a winning coalition, so that we only have one equivalence class, which contradicts $t = 2$. □

To prove Theorem 4.1, we show that the unique optimal integer solution (q, w_1, w_2) from Theorem 4.9 is indeed a minimum integer representation. To this end we state that for two feasible solutions (q, w) and (q', w') of Inequality system (4) the vector $\lambda \cdot (q, w) + (1 - \lambda) \cdot (q', w')$ is also a feasible solution for all $\lambda \in [0, 1]$.

Lemma 4.10. *Given a weighted game χ without null voters and with $t = 2$, $r > 1$, let $(\widehat{q}, \widehat{w}_1, \widehat{w}_2) \in \mathbb{N}^3_{>0}$ be a feasible solution of (9), which minimizes the sum of weights $n_1\widehat{w}_1 + n_2\widehat{w}_2$. For each integer representation $(q, a_1, \ldots, a_{n_1}, b_1, \ldots b_{n_2})$ of χ we have $a_i \geq \widehat{w}_1$ for all $1 \leq i \leq n_1$ and $b_j \geq \widehat{w}_2$ for all $1 \leq j \leq n_2$.*

PROOF. It suffices to conclude a contradiction both from $a_{n_1} \leq \widehat{w}_1 - 1$ and $b_{n_2} \leq \widehat{w}_2 - 1$. To shorten the presentation we deal with the first case only. Since $\frac{1}{n_1} \cdot \sum_{i=1}^{n_1} a_i \geq \widehat{w}_1$ we can assume $n_1 \geq 2$ and since every convex combination of a feasible weighting is feasible we can assume $a_{n_1} = \widehat{w}_1 - 1$ w.l.o.g. Next we set $a := \frac{\sum_{i=1}^{n_1-1} a_i}{n_1-1}$ and $b := \frac{\sum_{i=1}^{n_2} b_i}{n_2}$. With this the vector

$$(q, \underbrace{a, \dots, a}_{n_1-1}, \widehat{w}_1 - 1, \underbrace{b, \dots, b}_{n_2})$$

is also a feasible solution of (4), where we have $a \ge \widehat{w}_1 + \frac{1}{n_1-1} > \widehat{w}_1$.

Next we want to utilize the concept of tight inequalities to use a formula between the parameters of the tight inequalities and $\widehat{w}_1$. Due to Lemma 4.6 we have to distinguish the cases of Lemma 4.7 and Lemma 4.8 only.

If there are two tight inequalities of type $\widetilde{m}^T w \ge q$ for $(\widehat{q}, \widehat{w}_1, \widehat{w}_2)$, see Lemma 4.7, then let (c_1, d_1), (c_2, d_2) be the two corresponding winning vectors satisfying $c_1 > c_2$ and $d_1 < d_2$. Due to Lemma 4.7 we have $\widehat{w}_1 = d_2 - d_1$. Next we choose two non-negative integers $u \le c_1 - c_2$ and $v \le d_2 - d_1$ such that $u \cdot (d_2 - d_1) - v(c_1 - c_2) = 1$. We remark $u \ge 1$. With this $(c_3, d_3) := (c_1 - u, d_1 + v)$ is a losing vector corresponding to a tight inequality. Since (c_1, d_1), (c_2, d_2) are winning, (c_3, d_3) is losing, $c_1 \ge 1$, $c_2 < n_1$, and $c_3 < n_1$ we have

$$(c_1 - 1) \cdot a + 1 \cdot (\widehat{w}_1 - 1) + d_1 \cdot b - q \ge 0, \quad c_2 \cdot a + d_2 \cdot b - q \ge 0, \quad \text{and} \quad -c_3 \cdot a - d_3 \cdot b + q \ge 1.$$

Summing up $d_2 - d_3$ times the first, $d_3 - d_1$ times the second, and $d_2 - d_1$ times the third inequality yields

$$\underbrace{(1 - d_2 + d_3)}_{\le 0} \underbrace{a}_{> \widehat{w}_1} + \underbrace{(d_2 - d_3)}_{> 0} \widehat{w}_1 - \underbrace{(d_2 - d_3)}_{\ge 0} \ge d_2 - d_1,$$

from which we conclude the contradiction

$$\widehat{w}_1 > d_2 - d_1 = \widehat{w}_1.$$

If there are two tight inequalities of type $\widetilde{m}^T w \le q - 1$ for $(\widehat{q}, \widehat{w}_1, \widehat{w}_2)$, see Lemma 4.8, then let (c_1, d_1), (c_2, d_2) be the two corresponding losing vectors satisfying $c_1 < c_2$ and $d_1 > d_2$. Due to Lemma 4.8 we have $\widehat{w}_1 = d_2 - d_1$. Next we choose two non-negative integers $u \le c_2 - c_1$ and $v \le d_1 - d_2$ such that $u \cdot (d_1 - d_2) - v(c_2 - c_1) = 1$. We remark $u \ge 1$. With this $(c_3, d_3) := (c_1 + u, d_1 - v)$ is a winning vector corresponding to a tight inequality. Thus we have

$$(c_3 - 1) \cdot a + 1 \cdot (\widehat{w}_1 - 1) + d_3 \cdot b - q \ge 0, \quad -c_1 \cdot a - d_1 \cdot b + q \ge 1, \quad \text{and} \quad -(c_2 - 1) \cdot a - 1 \cdot (\widehat{w}_1 - 1) - d_2 \cdot b + q \ge 1.$$

Summing up $d_1 - d_2$ times the first, $d_3 - d_2$ times the second, and $d_1 - d_3$ times the third inequality yields the contradiction $\widehat{w}_1 > \widehat{w}_1$.

Thus the assumption $a_{n_1} \le \widehat{w}_1 - 1$ cannot be true and we have $a_i \ge \widehat{w}_1$ for all $1 \le i \le n_1$. Similar arguments can be outlined for $b_j \ge \widetilde{w}_2$ for all $1 \le j \le n_2$. □

Remark 4.11. *Due to the above lemmas we can algorithmically determine a minimum integer representation in* $O\Big(|\mathcal{W}^{sm}|^3 \log(n) \log\log(n) + |\mathcal{W}^{sm}|^2 \log^2(n) \log\log(n)\Big)$ *time. The case* $|\mathcal{W}^{sm}| = r = 1$ *can be dealt directly using Lemma 4.2. For* $r \ge 2$ *we consider all pairs of shift-minimal winning vectors and all pairs of shift-maximal losing vectors. Here we have* $\left|\mathcal{L}^{sM}\right| \le |\mathcal{W}^{sm}| + 1$ *and* $|\mathcal{W}^{sm}| \le \min\left(n_1 + 1, \left\lfloor \frac{n_2+2}{2} \right\rfloor\right) \le \left\lfloor \frac{n+3}{3} \right\rfloor$ *due to Inequality system (2). For each, in* $\mathcal{W}^{sm} \times \mathcal{W}^{sm}$ *or* $\mathcal{L}^{sM} \times \mathcal{L}^{sM}$ *we calculate the parameters* u *and* v *via the Euclidean algorithm to determine the third tight vector, see Lemma 4.7 and Lemma 4.8, respectively. So we have to consider at most* $|\mathcal{W}^{sm}|^2 + \left|\mathcal{L}^{sM}\right|^2$ *cases. In each case the Euclidean algorithm performs at most* $\log(n)$ *steps where numbers between* $-n$ *and* n *are added and divided. After solving the* 3×3*-equation system, which can be done in time* $O(\log(n) \log\log(n))$, *we only have to check if the solution is feasible. Checking the feasibility means determining the minimal weight of a winning vector and the maximal weight of a losing vector, which can be done using $O(|\mathcal{W}^{sm}|)$ multiplications and additions.*

Since the minimal possible values of w_1, w_2, and q can be bounded via $w_1 \le \max(n_1+1, n_2)$, $w_2 \le \max(n_1, n_2-1)$, and $q \le (n_1+n_2) \cdot \max(n_1+1, n_2)$ we may also determine a minimum integer representation by trying out all possibilities, which results in a pseudo-polynomial algorithm.

Due to the famous LLL-algorithm [21, 22] integer linear programs with a fixed number of dimensions, i.e. the number of variables, and a fixed number of constraints can be solved in polynomial time. For a two variables integer program defined by m constraints involving coefficients with at most s bits there is a $O(m + \log m \log s)M(s)$ algorithm [6], where $M(s)$ is the time needed for s-bit integer multiplication (we assume $M(s) = s \log s \log\log s$). For $t = 2$ types of voters we have $\left|\mathcal{L}^{sM}\right|, |\mathcal{W}^{sm}| \le \left\lfloor \frac{n+6}{3} \right\rfloor$, so that $m = |\mathcal{W}^{sm}| \cdot \left|\mathcal{L}^{sM}\right| + n \in O(n^2)$, and $s \in O(\log n)$ using the ILP formulation without the quota q. For a general but fixed number of variables Clarkson's sampling algorithm needs an expected number of $O(m + s \log m)$ arithmetic operations [5]. Using the ILP formulation with an extra variable for the quota q we have $m = |\mathcal{W}^{sm}| + \left|\mathcal{L}^{sM}\right| + n \in O\big(n^{t-1}\big)$ and $s \in O(\log n)$ for t types of voters. We would like to remark that the number of minimal winning vectors can be exponential in n whenever the number t of types of voters is not restricted; see e.g. [18].

5. Enumerations and bounds for the number of weighted games

Besides studying properties of complete simple games and weighted games one can also enumerate these special classes of cooperative games for small numbers of players n. In some cases enumeration results provide a deeper understanding. So far the number of complete simple games of weighted games is only known up to $n = 9$; see e.g. [10, 20]. Additionally restricting the parameters t (the number of types of voters) and/or r (the number of shift-minimal winning vectors) opens the possibility to determine enumeration formulas in some cases. A widely known result in this context is $csg(n,1) = wvg(n,1) = n$, where $csg(n,t)$ denotes the number of complete simple games with n voters partitioned into t equivalence classes. Similarly $wvg(n,t)$ denotes the number of weighted games with n voters occurring in t different types. In [12] the authors have determined the formula $cs(n,2) = Fib(n+6) - (n^2+4n+8)$, where $Fib(n)$ denotes the n-th Fibonacci number; see also [20] for an alternative proof. So we know that $cs(n,t)$ is at least exponential in n for $t \ge 2$. In this section we want to show that the situation changes for weighted games by proving a polynomial upper bound on $wm(n,t)$ in Theorem 5.2 and Theorem 5.3. It remains to come up with an exact formula for $wm(n,2)$.

If we refine our counts to the numbers $csg(n,t,r)$ and $wvg(n,t,r)$ by additionally considering the number r of shift-minimal winning vectors, more results can be obtained. In [20] an algorithm is given to principally determine an exact formula for $csg(n,t,r)$ whenever t and r are fixed. So far it is not known whether this can also be done for the number $wvg(n,t,r)$ of weighted games with t types of voters and r shift-minimal winning vectors. For $r = 1$ it is not too difficult to come up with such enumeration formulas as we will demonstrate for $t = 2$. Having an exact characterization of the weighted games with $t = 2$ and $r = 1$ at hand, see the proof of Theorem 4.2, we can easily determine a formula for their number:

Corollary 5.1. *The number $wm(n,2,1)$ of weighted games with $t = 2$ and $r = 1$ is given by $n-1$ for $n \le 2$ and $2(n-2)^2 + 2$ for $n \ge 3$.*

If we skip the parameter r then we can only state an upper bound:

Theorem 5.2. $wm(n,2) \le \frac{n^5}{15} + 4n^4$.

PROOF. Due to the bounds in the minimum integer representation for $r \geq 2$ in Theorem 4.1 and Corollary 5.1 the number $wm(n, 2)$ of weighted games with n voters and two types of voters is upper bounded by

$$2(n-2)^2+2+\sum_{n_1=1}^{n-1}\sum_{w_1=1}^{n-n_1}\sum_{w_2=0}^{n_1}\sum_{q=1}^{2n_1(n-n_1)} 1 = 2(n-2)^2+2+2\sum_{n_1=1}^{n-1}(n-n_1)^2(n_1+1)n_1 \leq \frac{n^5}{15}+4n^4.$$

□

For an arbitrary number t of types of voters we can determine the following polynomial upper bound:

Theorem 5.3.

$$wm(n,t) < (tn)^{t^3+2t^2}.$$

PROOF. Let us denote the weight vector by w, the shift-minimal winning vectors by $\widetilde{m}_i$, and the shift-maximal losing vectors by $\widetilde{l}_j$. A complete simple game described by the $\widetilde{m}_i$ or the $\widetilde{l}_j$ is weighted if and only if the system of inequalities

$$\left(\widetilde{m}_i - \widetilde{l}_j\right) w^T > 0 \tag{10}$$

has a non-negative solution w (for all i, j).

Since λw is also a solution for all $\lambda > 0$ whenever w is a solution, we consider the equivalent system

$$\left(\widetilde{m}_i - \widetilde{l}_j\right) w^T \geq 1. \tag{11}$$

Such a system of linear inequalities corresponds to a polytope whose vertices correspond to n-element subsets of the constraints which are attained with equality. Using the fact that the coefficients of this system of linear inequalities are integers between $-(n-1)$ and $n-1$ we can apply Cramers rule to conclude that vertices of this polytope can be written as $v_i = \begin{pmatrix} w_1 & \dots & w_t \end{pmatrix} = \begin{pmatrix} \frac{a_{2,i}}{b_{2,i}} & \cdots & \frac{a_{t,i}}{b_{t,i}} \end{pmatrix}$, where $0 \leq a_{j,i} \leq (t-1)!(n-1)^t$ and $1 \leq b_{j,i} \leq (t-1)!(n-1)^t$. Here the common denominator g is bounded from above by $\left((t-1)!(n-1)^t\right)^t$.

Thus multiplying vertex v_i with g yields integer weights $\widetilde{w}_i$ between 0 and $\left((t-1)!(n-1)^t\right)^{t+1}$. There are at most $(tn)^{t^3+t^2}$ possible tuples of integer weights to be considered. The quota can be chosen as the minimum weight of a winning coalition. Since there are less than n^t possibilities for the numbers n_i of voters in the t equivalence classes, the proposed upper bound on $wm(n,t)$ follows. □

6. Concluding remarks

The main result of this paper is that weighted games with two types of players admit a minimum integer representation. For three types of players this need not to be the case. We have shown that by providing examples of games without a minimum integer representation.

We found examples of weighted games with four types of voters without a minimum integer representation preserving types. It is still an open problem to clarify whether all weighted games with three types of voters admit a minimum integer representation preserving types. To adres this lacuna we have tried to generalize our technique from Subsection 4.2. One may consider the linear program minimizing the sum of the weights and have a closer look at the corners of the corresponding polytope, which are characterized by four equations corresponding to four *tight types of coalitions* (shift-maximal losing or shift-minimal winning vectors).

As demonstrated in Subsection 4.2 for three tight types of coalitions, the resulting weights and the quota could be fractional. But using the extended Euclidean algorithm we were able to construct another type of a coalition which contradicts the tightness of the starting three vectors in these cases. For four tight types of coalitions (and the variables q, w_1, w_2, and w_3) we may go along the same lines and use the extended Euclidean algorithm for three integers in order to deduce some restrictions on quadruples of tight types of coalitions. This indeed works, but there still remain cases where the optimal LP solution is fractional. By generating random weighted games with three types of voters we have discovered several such examples, some of them are given below. For each example we state the sizes of the equivalence classes $\overline{\mathbf{n}} = (n_1, n_2, n_3)$, the non-integer minimum sum representation preserving types $\tau_r = (q, w_1, w_2, w_3)$, and the minimum sum integer representation preserving types $\tau_i = (q, w_1, w_2, w_3)$:

(1) $\overline{\mathbf{n}} = (9, 62, 71)$, $\tau_r = (154.\overline{3}, 38.\overline{3}, 22.\overline{6}, 6.\overline{6})$, $\tau_i = (185, 46, 27, 8)$

(2) $\overline{\mathbf{n}} = (19, 52, 65)$, $\tau_r = (3984.2, 200, 110, 76.6)$, $\tau_i = (5617, 282, 155, 108)$

(3) $\overline{\mathbf{n}} = (30, 93, 30)$, $\tau_r = (122.\overline{3}, 22.\overline{3}, 16, 9.\overline{3})$, $\tau_i = (131, 24, 17, 10)$

(4) $\overline{\mathbf{n}} = (8, 99, 10)$, $\tau_r = (51, 17, 10.5, 4.5)$, $\tau_i = (57, 19, 12, 5)$

(5) $\overline{\mathbf{n}} = (3, 71, 37)$, $\tau_r = (347.5, 100, 31.5, 15)$, $\tau_i = (441, 127, 40, 19)$

Originally we have obtained the values of τ_i by minimizing $n_1w_1 + n_2w_2 + n_3w_3$ but it turned out that in all of these (and the other found) cases we have a minimum integer representation preserving types, so that minimizing w_1, w_2, w_3, or q would yield the same result. We would like to remark that we have also found some example where only one value is non-integer. Although in our experiments the only occurring denominators were 2, 3, and 5, we do not think that the denominators are bounded by a constant. So far we have a very poor probabilistic model which generates those examples with a very low probability. Nevertheless we have a strong feeling that each weighted game with three types of voters admits a minimum integer representation preserving types. As a small justification we would like to remark that we have tried some specific parametric constructions which provably do not contain counter examples.

We leave the challenging question of whether each weighted games with three types of voters admits an minimum integer representation preserving types open for the interested reader and hope that our specific examples might help to get some useful insights. One can get a first glimpse of the difficulty of this problem by comparing the values of τ_r and τ_i in our examples.

Weighted games with an arbitrary number of minimum sum integer representations have been generated in Subsection 3.2. Moreover, some bounds have been obtained for the number of non-isomorphic weighted games depending on the number of voters and on the number of types of voters, and the existence of a weighted game, in minimum integer representation for any pair of two coprime integer weights, has been determined.

Other interesting open problems in the context of this paper are the question for a weighted game with a unique minimum sum integer representation, but without a minimum integer representation, and the question for a polynomial time algorithm to determine minimum sum integer representations for weighted games or a proof that this problem is NP-hard.

Another important line of research would be to deepen our understanding of the link between minimum integer representations of weighted games and one-point solution concepts, like the nucleolus, least core, etc.; see e.g. [18, 26].

Of course the techniques presented in this paper may be applied to study similar questions for roughly weighted games.

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