The graph distance game and some graph operations *

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Abstract. In the graph distance game, two players alternate in constructing a maximal path. The objective function is the distance between the two endpoints of the path, which one player tries to maximize and the other tries to minimize. In this paper we examine the distance game for various graph operations: the join, the corona and the lexicographic product of graphs. We provide general bounds and exact results for special graphs.

Key words: Distance game, graph operations.

1 Introduction

Combinatorial games have been widely studied and are constantly developed. Many such games are modifications of some previous ones [17], other are entirely new. Combinatorial games remain an active field of new and interesting research. For an extensive bibliography on combinatorial games and related topics see [12].

Notice that, as shown in [14], in many combinatorial games, the winner is determined by who moves last, as studied for example in [3]. In others, the players compete to construct a desired goal, by taking one element at a time from the universe, as studied for example in [2]. Another class of games consists of those where the players compete to maximize or minimize some quantity, such as the game chromatic number introduced in [11], the competition chromatic number introduced in [21] , graph competition independence introduced in [18] or the domination game introduced in [4]. We consider here a game that falls in the latter category. (See [19] for more on such competitive games.)

Some of this games remained more or less unnoticed for many years, as for example, the game chromatic number. However, in the last several years

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various authors focus their attention in this topics. Many of this combinatorial games are specially studied in some families of graphs, such as trees, forests, outerplanar graphs, cactuses or wheels [5,6,15,20,16]. Others authors infer interrelationship between this games and some operations in graphs [9,1,22,10].

In [14], the graph distance game is introduced as follows. Given a graph G, two players alternate in constructing a path. The first player picks a vertex, the second player picks a neighbor of the first vertex, the first player picks a neighbor of the second vertex that has not yet been picked, and so on. This is continued until the path cannot be extended. One player tries to maximize the final distance from the start, and the other player tries to minimize this distance. That is, the value at the end of the game is the distance between the start and the finish, regardless of the path taken. We call this the distance game.

2 Basics and known results of graph distance game

In [14], the authors explore the distance game for various graphs and provide general bounds and exact results for simple graphs. They also show that the parameter can be calculated in a tree in linear time. Further, the values for small grids are determined.

There are two versions, depending on which of the Minimizer or Maximizer moves first. We let $S_m(G)$ denote the value of the game on graph G when the minimizer chooses the first vertex, and $S_M(G)$ the value when the maximizer chooses the first vertex. We call the first vertex of the path the *source*. Clearly, if the graph is vertex transitive, all sources are equivalent. If it does not matter who goes first, then we drop the subscript and write $S(G)$.

For a trivial example, the value of this parameters is always 1 in the complete graph. Table 1 shows the values of these parameters for paths, cycles, complete bipartite graphs and wheels.

Table 1: Distance game parameters of some basic graphs.

G	P_n	C_n	K_n	$K_{p,q}$	W_n
$S_m(G)$	$\lfloor \frac{n}{2} \rfloor$	1	1	1 (if $p = q$)	1
$S_M(G)$	$n-1$	1	1	$1, 2$ (if $p \neq q$)	1

The radius and the diameter of the graph give bounds of these parameters.

Proposition 1. *Let G be a graph of order n, radius $rad(G) = r$ and diameter $diam(G) = d$. Then, $1 \leq S_m(G) \leq r$ and $1 \leq S_M(G) \leq d$.*

In Figure 1, we show the distance game pairs (x, y) of alls vertices of P_6. In these terms, S_m is the minimum of the x-numbers and S_M is the maximum of the y-numbers. In this example, the Maximizer choose as first vertex one of the leaves and Minimizer select one central vertex as source.

Fig. 1: Distance game pairs of all vertices of P_6. Notice that $S_m(P_6) = 3$ and $S_M(P_6) = 5$.

In other graphs, the same vertex could be a good source for the two players, the Maximizer and the Minimizer.

3 Operations

In this paper we examine the distance game for various graph operations. We follow the notation of [7].

In [14], the authors explore the distance game for small grids. Specifically, the following results were obtained.

Theorem 1. *i)* $S_M(P_2 \Box P_m) = 1$ *and* $S_m(P_2 \Box P_m) = 2\lfloor m/4 \rfloor + 1$,

$$ii) S_M(P_3 \Box P_m) = \begin{cases} m+1, & \textit{for odd } m, \\ 1, & \textit{for even } m. \end{cases}$$

$$S_m(P_3 \Box P_m) = \begin{cases} 2, \textit{ for odd } m \geq 3, \\ 1, \textit{ for even } m. \end{cases}$$

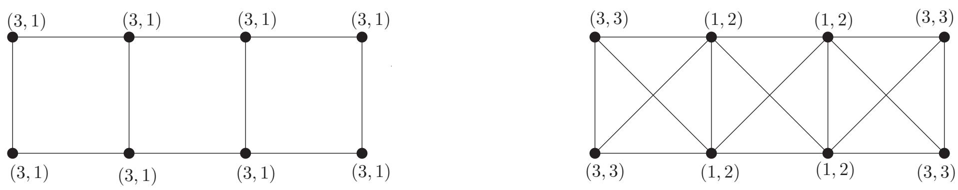

Fig. 2: Distance game pairs of vertices of the cartesian and the strong product of P_2 and P_4.

We focus our attention in the study of the distance game for join, corona and lexicographic product of graphs.

3.1 Join

The *join* $G = G_1 \vee G_2$ is the graph such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.

Remark 1. Notice that $1 \leq S_m(G \vee H) \leq 2$ and $1 \leq S_M(G \vee H) \leq 2$, since $1 \leq diam(G \vee H) \leq 2$.

Theorem 2. *Let G, H be graphs. Then, $S_m(G \vee H) = 1$.*

Remark 2. Notice that $S_M(P_2 \vee P_3) = 1$ and $S_M(P_2 \vee P_4) = 2$.

3.2 Corona product

Let G and H be two graphs and let n be the order of G. The *corona product* of graphs G and H is the graph $G \odot H$ obtained by taking one copy of G and n copies of H, and then joining by an edge the i^{th} vertex of G to every vertex in the i^{th} copy of H. The corona product is neither associative nor commutative.

Theorem 3. *Let G, H be graphs. Then,*
$1 \leq S_m(G \odot H) \leq 2$ *and* $1 \leq S_M(G \odot H) \leq 3$.

Remark 3. For both parameters all values are possible:

- If $H \cong (K_2 + K_2) \vee K_1$, then $S_m(G \odot H) = 1$.
- If $H \cong P_3$ (or $H \cong K_q$) and $G \ncong K_1$, then $S_m(G \odot H) = 2$.

And

- If $H \cong K_p$ and $G \cong K_1$, then $S_M(G \odot H) = 1$.
- If $H \cong P_3 \odot K_1$, then $S_M(G \odot H) = 2$.
- If $H \cong K_q$ and $G \ncong K_1$, then $S_M(G \odot H) = 3$.

Remark 4. Notice that

- If $H \cong P_n$ and $n \geq 5$, then $S_M(G \odot H) \leq 3 < n - 1 = S_M(H)$.
- If $H \cong K_r$ and $G \ncong K_1$, then $S_M(G \odot H) = 3 > 1 = S_M(H)$.
- If $H \cong P_3 \odot K_1$ and $G \cong K_r$, then $S_M(G \odot H) = 2 > 1 = S_M(G)$.
- If $G \cong P_n$ with $n \geq 5$ and $H \cong K_r$, then $S_M(G \odot H) = 3 < n - 1 = S_M(G)$.

3.3 Lexicographic product

The *lexicographic product* of graphs G and H is the graph $G \circ H$ on vertex set $V(G) \times V(H)$ in which vertices (g_1, h_2) and (g_2, h_2) are adjacent if and only if either $g_1 g_2 \in E(G)$ or $g_1 = g_2$ and $h_1 h_2 \in E(H)$. This graph operation is also known as the graph *composition* and denoted by $G[H]$. The graph $G \circ H$ is called nontrivial if both factors are graphs on at least two vertices. Next, we show a basic list of properties of this graph operation, whose proofs are direct consequences of the definition.

Proposition 2. *Let G, H be graphs.*

1. *The graph $G \circ H$ is connected if and only if G is connected.*
2. *The lexicographic product is associative but not commutative.*
3. *If G is connected, then*
   - $d_{G\circ H}((g,h),(g',h')) = d_G(g,g')$ *if* $g \neq g'$,
   - $d_{G\circ H}((g,h),(g,h')) = 2$ *if* $hh' \in E(H)$,
   - $d_{G\circ H}((g,h),(g,h')) = 1$ *if* $hh' \in E(H)$.

We have obtained the following result.

Theorem 4. *Let G, H be graphs. Then,*

$$1 \leq S_m(G \circ H) \leq S_m(G) \quad \text{and} \quad S_M(G) \leq S_M(G \circ H).$$

Conjecture 1. Let G be a graph. Then,

$$1 \leq S_m(G \circ K_q) = S_m(G) \quad \text{and} \quad S_M(G) = S_M(G \circ K_q).$$

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