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The Picard–Fuchs equations for complete hyperelliptic integrals of even order curves, and the actions of the generalized Neumann system

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We consider a family of genus 2 hyperelliptic curves of even order and obtain explicitly the systems of 5 linear ordinary differential equations for periods of the corresponding Abelian integrals of first, second, and third kind, as functions of some parameters of the curves. The systems can be regarded as extensions of the well-studied Picard–Fuchs equations for periods of complete integrals of first and second kind on odd hyperelliptic curves. The periods we consider are linear combinations of the action variables of several integrable systems, in particular the generalized Neumann system with polynomial separable potentials. Thus the solutions of the extended Picard–Fuchs equations can be used to study various properties of the actions. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4868965]

I. INTRODUCTION

Given a family of elliptic curves $\mathcal{E} \subset \mathbb{P}^2$ in the Legendre form

$$w^2 = (1-z^2)(1-k^2z^2),$$

it is known that the complete elliptic integrals of first kind

$$K(k) = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}, \quad K'(k) = \int_1^{1/k} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

as functions of the modulus $k \in \mathbb{C}$, give 2 independent solutions of the hypergeometric equation of the Legendre type

$$k(1-k^2)\frac{d^2y}{dk^2} - (1+k^2)\frac{dy}{dk} + ky = 0, \tag{1}$$

that is, $K(k) = \frac{\pi}{2}F(\frac{1}{2}, \frac{1}{2}, 1; k^2)$. The equation has singular points $z_{1,2,3} = -1, 0, 1$, which means that the solutions $K(k)$, $K'(k)$ are not single-valued: when k goes around z_i, these functions transform to a linear combination of $K(k)$, $K'(k)$. That is, the solutions $y(k)$ undergo a monodromy.

Equivalently, (1) can be rewritten as a system of first order equations for $K(k)$ and the complete integral of the *second kind*[17]

$$\bar{E}(k) = \int_0^1 \frac{z^2dz}{\sqrt{(1-z^2)(1-k^2z^2)}},$$

namely,

$$\frac{dK}{dk} = \frac{1}{k(1-k^2)}(k^2K - \bar{E}), \quad \frac{d\bar{E}}{dk} = \frac{k}{1-k^2}(K - \bar{E}) \tag{2}$$

(see, e.g., Ref. 8).

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Starting from the pioneer work,[16] the above description has been generalized to the case of curves of higher genus, hyperelliptic, and non-hyperelliptic, in many publications. Apparently, one of the most generic results was obtained in Ref. 14, which, amongst others, considered the family of hyperelliptic curves

$$G_t = \{y^2/2 + x^{n+1} + c_{n-1}x^{n-1} + \cdots + c_0 = t \mid t \in \mathbb{C}\}, \qquad c_0, \ldots, c_{n-1} = \text{const.}$$

For generic values of t, the curves have genus $[n/2]$. Choose the following set of meromorphic differentials:

$$\bar{\omega}_i = x^{i-1} y\, dx, \qquad i = 1, \ldots, n,$$

and let $\gamma \in H_1(\Gamma, G_t)$ be a cycle on G_t. Then the periods of the above differentials,

$$I_1 = \oint_\gamma \bar{\omega}_1, \quad \ldots, \quad I_n = \oint_\gamma \bar{\omega}_n, \tag{3}$$

become functions of the parameter t. As was shown in Ref. 14, the vector $X = (I_1, \ldots, I_n)^t$ satisfies a closed system of ordinary differential equations (ODEs) which can be written in the following $n \times n$ matrix hypergeometric form:

$$(t\,\mathbb{I} - A)\dot{X}(t) = BX(t), \tag{4}$$

where $\mathbb{I}$ is the identity matrix, and A, B depend only on the coefficients c_j of G_t. The paper[14] also studied various properties of the solutions of the above systems.

On the other hand, in a series of integrable systems of classical mechanics and mathematical physics, in particular the Neumann system (see Sec. II), other families of hyperelliptic curves and Abelian integrals appear. As an illustration, consider first a family of genus g hyperelliptic curves of odd order

$$\Gamma_h = \{w^2 = (z - a_1)\cdots(z - a_{g+1})(z^g + h_1 z^{g-1} + \cdots + h_{g-1}z + h_g)\} \tag{5}$$

with the parameters $h_1, \ldots, h_g \in \mathbb{C}$. Here $a_1, \ldots, a_{g+1}$ are distinct constants. For generic values of h_i, the curves are 2-fold covering of $\mathbb{C} = \{z\}$ ramified at $z = a_1, \ldots, a_{g+1}$ and $\rho_1, \ldots, \rho_g$, the roots of the polynomial $P_g(z) = z^g + h_1 z^{g-1} + \cdots + h_{g-1}z + h_g$.

Choose the following canonical basis of g holomorphic differentials and g meromorphic differentials of the second kind on Γ_h:

$$\omega_i = \frac{z^{i-1}\,dz}{w}, \quad \omega_{g+i} = \frac{z^{g-1+i}\,dz}{w}, \qquad i = 1, \ldots, g.$$

For a cycle $\gamma \in H_1(\Gamma, C)$, the periods of the above differentials

$$J_1 = \oint_\gamma \omega_1, \quad \ldots, \quad J_{2g} = \oint_\gamma \omega_{2g} \tag{6}$$

become functions of the parameters $h_1, \ldots, h_g$ in (5) or of the roots $\rho_1, \ldots, \rho_g$.

Families of hyperelliptic curves Γ_h often appear in quadratures of integrable systems, for which $h_1, \ldots, h_g$ play the role of constants of motion. Certain linear combinations of the integrals $J_i(h)$ give *action* variables $\mathcal{I}_1(h), \ldots, \mathcal{I}_g(h)$, and knowledge of their properties is important in study of periodic solutions, in quantization, and in applications of the KAM theory to perturbations of the systems.

Note that J_i are not single-valued functions of h_i: when these parameters vary in such a way that one of the roots, say ρ_1, goes around a_i or $\rho_2, \ldots, \rho_g$, each integral J_i becomes a linear combination of $J_1, \ldots, J_{2g}$, i.e., undergoes a monodromy.

Following the classical theory of differential equations, the integrals $J_i = J_i(h_1, \ldots, h_g)$ are solutions of a systems of linear ODEs, with h_i being independent variables, called the *Picard–Fuchs* equations (see, e.g., Ref. 7):

$$\frac{\partial J}{\partial h_k} = M_k(h)\, J, \qquad J = (J_1, \ldots, J_{2g})^T, \quad k = 1, \ldots, g, \quad M_k \in GL(2g, \mathbb{C}). \tag{7}$$

They are natural generalizations of the Legendre equation (1) or (2).[9]

Due to the monodromy property, some of the components of $M_k(h)$ have poles when one of the roots ρ_i coincides with a_j or with the other roots.

The integrals J_i are transcendental functions of h_j and, as mentioned in several publications, instead of computing them numerically, in some cases it is less expensive to integrate numerically the above Picard–Fuchs equations, at least locally.

Following this idea, the authors of Ref. 4 derived differential equations for the periods J_i for any genus g, taking however, as an independent variable one of the roots ρ_i in (5), and not a constant of motion h_k. (Thus, they obtained the *Gauss–Manin* equations.)

A similar approach was followed in Refs. 2 and 3 to treat the actions of the Kovalevkaya top and the Jacobi problem on geodesics on a triaxial ellipsoid.

For another basis of meromorphic differentials on Γ_h, a similar system of Gauss–Manin equations was obtained in Ref. 6.

The only disadvantage of this approach is the dependence of all the constants h_k on any root ρ_i, which makes it difficult to study the properties of $J_i(h)$ as a function of one h_k, when all the other ones are fixed.

The choice of ρ_i instead of h_k was motivated in Ref. 4 by the observation that the Picard–Fuchs equations with the independent variables h_k become highly cumbersome even for the lowest non-trivial case $g = 2$.

The purpose of our note is to derive the Picard–Fuchs equations of type (7) for the case of the family of even order genus 2 curves

$$\Gamma_h = \{w^2 = (z - a_1)(z - a_2)(z - a_3)(z^3 + h_1 z + h_2)\},$$

which appear in quadratures of an integrable generalization of the Neumann system with a separable quartic potential, as described in Sec. II. The equations are written in a quite compact and symmetric form, suitable for possible applications.

In contrast to the odd order curves (5) and Eqs. (7), in our case the order of the Picard–Fuchs equations is 5, since they also include an Abelian integral of 3rd kind.

This observation is fully compatible with the result of Ref. 14 described by (3) and (4) since for $n = 5$ the dimension of the latter system is 5, however our system and the system (4) have different independent variables.

II. THE CLASSICAL NEUMANN SYSTEM AND ITS GENERALIZATIONS

Recall that the Neumann system describes the motion of a point on the unit sphere $S^{n-1} = \{\langle x, x\rangle = 1\}$, $x \in \mathbb{R}^n$ under the action of the quadratic potential $U = \langle x, Ax\rangle/2$, A being a diagonal matrix with constant eigenvalues $a_1, a_2, \dots, a_n$. The Hamiltonian of the problem has the form

$$H(x, y) = \frac{1}{2}(|y|^2|x|^2 - \langle y, x\rangle^2) + \frac{1}{2}\langle x, Ax\rangle,$$

where $p \in T_x S^{n-1}$ is the momentum (see, e.g., Refs. 10 and 11).

In Ref. 13, Neumann considered this problem in the case $n = 3$ and solved it completely in terms of theta-functions of 2 variables.

In the elliptic (spheroconical) coordinates $\lambda_1, \dots, \lambda_{n-1}$ on S^{n-1} such that

$$x_i^2 = \frac{(a_i - \lambda_1)\cdots(a_i - \lambda_{n-1})}{(a_i - a_1)\cdots(a_i - a_n)}, \qquad i = 1, \dots, n$$

and in the corresponding conjugated momenta $p_1, \dots, p_n$, the Hamiltonian takes a Stäckel form, and the system is reduced to the quadratures

$$\frac{\lambda_1^k d\lambda_1}{\sqrt{R(\lambda_1)}} + \cdots + \frac{\lambda_{n-1}^k d\lambda_{n-1}}{\sqrt{R(\lambda_{n-1})}} = \begin{cases} 0 & \text{if } k = 0, 1, \dots, n-2, \\ h_0 dt & \text{if } k = n-1, \end{cases} \qquad k = 0, 1, \dots, n-2, \tag{8}$$

$$R(\lambda) = \Phi(\lambda) P_{n-1}(\lambda), \quad \Phi(\lambda) = (\lambda - a_1)\cdots(\lambda - a_n),$$
$$P_{n-1}(\lambda) = \lambda^{n-1} + h_1\lambda^{n-2} + \cdots + h_{n-1} = (\lambda - \rho_1)\cdots(\lambda - \rho_{n-1}),$$

where $h_0, h_1, \ldots, h_{n-1}$ are constants of motion.

Here the differentials $\lambda^k d\lambda/\sqrt{R(\lambda)}$ can be regarded as holomorphic differentials on the genus $g = n - 1$ hyperelliptic curve $\Gamma_h = \{\mu^2 = \Phi(\lambda)P_{n-1}(\lambda)\}$, already described in (5).

By integrating the quadratures (8) and inverting the integrals, symmetric functions of the elliptic coordinates λ_j and, therefore, the Cartesian coordinates x_i, can be expressed in terms of theta-functions of u_k and, therefore, of the time t (see Ref. 12). The generic real invariant varieties are unions of n-1 dimensional tori $\mathbb{T}^{n-1}$. Moreover, the tori are real parts of complex Abelian varieties, which are isogeneous to the Jacobians of the curves, and the system is algebraically integrable (see Refs. 10 and 12).

On the other hand, as was shown in several publications (see, e.g., Refs. 5 and 15), the Neumann system admits a hierarchy of integrable generalizations, in which the quadratic potential $U(x) = \langle x, Ax\rangle/2$ is replaced by polynomial or rational potentials, which are all separable in the same elliptic coordinates. For all such generalizations, the dimension of the generic invariant tori is the same, $n - 1$. On the other hand, for a class of separable polynomial potentials of degree $2N$, the quadratures take the following form, which generalizes (8),

$$\frac{\lambda_1^k d\lambda_1}{\sqrt{\mathcal{R}(\lambda_1)}} + \cdots + \frac{\lambda_{n-1}^k\, d\lambda_{n-1}}{\sqrt{\mathcal{R}(\lambda_{n-1})}} = \begin{cases} 0 & \text{if} \quad k = 0, 1, \ldots, n-2, \\ h_0 dt & \text{if} \quad k = n-1, \end{cases} \qquad k = 0, 1, \ldots, n-2, \tag{9}$$

where now $\mathcal{R}(\lambda) = \Phi(\lambda)\mathcal{P}_{N+1}(\lambda)$,

$$\Phi(\lambda) = (\lambda - a_1)\cdots(\lambda - a_n), \quad \mathcal{P}_{N+1}(\lambda) = \lambda^{N+1} + h_1\lambda^{n-2} + \cdots + h_{n-1}.$$

The quadratures include n-1 holomorphic differentials on the hyperelliptic curve

$$\Gamma_h = \{\mu^2 = \Phi(\lambda)\,\mathcal{P}_{N+1}(\lambda)\}$$

of genus $g = [(n + N)/2]$. This implies that for the separable potentials of degree $2N > 4$, the genus of Γ_h is bigger than the dimension of the tori, and one can show that in this case the system is no more algebraic integrable.[1,18]

The action variables of the original and generalized Neumann systems are the periods of the Abelian integrals

$$\mathcal{J}_j(h_1, \ldots, h_{n-1}) = \frac{1}{2\pi}\oint_{\gamma_j} \frac{(\lambda^{N+1} + h_1\lambda^{n-2} + \cdots + h_{n-1})\, d\lambda}{\sqrt{\mathcal{R}(\lambda)}}, \qquad j = 1, \ldots, n-1,$$

γ_j being certain cycles on the Riemann surface Γ_h. Note that the functions $\mathcal{J}_j(h_1, \ldots, h_g)$ are also the frequencies of the angle variables on the tori $\mathbb{T}^{n-1}$. Then a solution to the Neumann system is periodic if and only if the quantities $\mathcal{J}_j$ are commensurable. So, knowledge of $\mathcal{J}_j(h)$ is important in describing periodic solutions of the system.

As follows from the above, the action variables $\mathcal{J}_j$ are linear combinations of the periods of the following basic g holomorphic and g meromorphic differentials on Γ_h:

$$J_k = \oint_\gamma \omega_k, \qquad \omega_s = \frac{\lambda^{s-1} d\lambda}{\sqrt{\mathcal{R}(\lambda)}}, \quad \omega_{g+s} = \frac{\lambda^{g+s-1} d\lambda}{\sqrt{\mathcal{R}(\lambda)}}, \quad s = 1, \ldots, g. \tag{10}$$

For the classical Neumann system with the quadratic potential ($N = 1$), the above $2g$ differentials satisfy the Picard–Fuch equations (7). However, for $N > 1$ this is not always true.

For concreteness, below we restrict ourselves to the simplest case $n = 3$ and the quartic separable potential ($N = 2$),

$$U(x) = \langle x, Ax\rangle^2 - 2\mathrm{Tr}A\langle x, Ax\rangle - \langle x, A^* x\rangle, \quad A^* = \det A\, A^{-1}.$$

In the elliptic coordinates, up to a constant, it reads

$$(\lambda_1^3 - \lambda_2^3)/(\lambda_1 - \lambda_2) = \lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2.$$

Then the quadratures (9) contain differentials on the genus 2 curve of order 6,

$$w^2 = (\lambda - a_1)(\lambda - a_2)(\lambda - a_3)\cdot(\lambda^3 + h_1\lambda + h_2), \tag{11}$$

whose compactification in $\mathbb{P}^2$ have 2 infinite points, which we denote by ∞_-, ∞_+.

The differentials (10) then are

$$\omega_1 = \frac{d\lambda}{w}, \quad \omega_2 = \frac{\lambda\, d\lambda}{w}, \quad \omega_3 = \frac{\lambda^2\, d\lambda}{w}, \quad \omega_4 = \frac{\lambda^3\, d\lambda}{w}.$$

One observes that, in contrast to ω_4, the differential ω_3 is meromorphic of the 3rd kind, i.e., it has a pair of simple poles at ∞_-, ∞_+, and that the corresponding periods $J_1, \dots, J_4$ do not form a closed system of differential equations with respect to the constants h_1 or h_2. It turns out that in this case the Picard–Fuchs equations must include also the period J_5 of the differential of the second kind $\omega_5 = \dfrac{\lambda^4\, d\lambda}{w}$.

III. THE PICARD–FUCHS EQUATIONS FOR GENUS 2 EVEN ORDER CURVES

To derive the Picard–Fuchs equations for the considered case, we first compute the derivatives of the integrals $J_1, \dots, J_5$ with respect to the roots ρ_α in (11). Namely, rewrite the genus 2 curve in the form

$$w^2 = R(\lambda), \quad R(\lambda) = (\lambda - e_1)(\lambda - e_2)(\lambda - e_3)(\lambda - e_4)(\lambda - e_5)(\lambda - e_6).$$

Like in several other publications (see, e.g., Ref. 5), we will use the following key relation:

$$A_j^{(k)} \frac{\partial}{\partial e_k}\left(\frac{\lambda^j}{w}\right) = \frac{a_j^{(k)}\lambda^4 + b_j^{(k)}\lambda^3 + c_j^{(k)}\lambda^2 + d_j^{(k)}\lambda + g_j^{(k)}}{w} - \frac{d}{d\lambda}\left(\frac{w}{\lambda - e_k}\right), \tag{12}$$
$$j = 0, 1, \dots, 4, \quad k = 1, \dots, 6,$$

where $A_j^{(k)}, a_j^{(k)}, \dots, g_j^{(k)}$ are functions of the branch points e_i only. Namely, let us write

$$R'(e_k) = \left.\frac{dR(\lambda)}{d\lambda}\right|_{\lambda = e_k} = e_k^5 + \Delta_1^{(k)} e_k^4 + \Delta_2^{(k)} e_k^3 + \Delta_3^{(k)} e_k^2 + \Delta_4^{(k)} e_k + \Delta_5^{(k)},$$

so that the coefficients $\Delta_i^{(k)}$ are elementary symmetric functions of $\{e_1, \dots, e_6\}\backslash e_k$ of degree i. In particular, $\Delta_1^{(1)} = -e_2 - e_3 - e_4 - e_5 - e_6,\ \Delta_5^{(1)} = -e_2e_3e_4e_5e_6$. Then comparing both sides of (12), we obtain

$$\begin{aligned}
&A_j^{(k)} = \frac{R'(e_k)}{e_k^j},\\
&a_0^{(k)} = a_1^{(k)} = a_2^{(k)} = a_3^{(k)} = a_4^{(k)} = A^{(k)} = 2,\\
&b_0^{(k)} = b_1^{(k)} = b_2^{(k)} = b_3^{(k)} = B^{(k)} = -\frac{1}{2}\left(e_k - 3\Delta_1^{(k)}\right), \quad b_4^{(k)} = B + \frac{R'(e_k)}{2e_k^4},\\
&c_0^{(k)} = c_1^{(k)} = c_2^{(k)} = C^{(k)} = -\frac{1}{2}(e_k^2 + e_k\Delta_1^{(k)} - 2\Delta_2^{(k)}),\\
&c_3^{(k)} = c_4^{(k)} = C^{(k)} + \frac{R'(e_k)}{2e_k^3},\\
&d_0^{(k)} = d_1^{(k)} = D^{(k)} = -\frac{1}{2}\left(e_k^3 + e_k^2\Delta_1^{(k)} + e_k\Delta_2^{(k)} - \Delta_3^{(k)}\right),\\
&d_2^{(k)} = d_3^{(k)} = d_4^{(k)} = D^{(k)} + \frac{R'(e_k)}{2e_k^2},\\
&g_0^{(k)} = G^{(k)} = -\frac{1}{2}\left(e_k^4 + e_k^3\Delta_1^{(k)} + e_k^2\Delta_2^{(k)} + e_k\Delta_3^{(k)}\right),\\
&g_1^{(k)} = g_2^{(k)} = g_3^{(k)} = g_4^{(k)} = G^{(k)} + \frac{R'(e_k)}{2e_k}.
\end{aligned} \tag{13}$$

Multiplying both sides of (12) by $d\lambda$, and using again the notation

$$\omega_1 = \frac{d\lambda}{w}, \quad \omega_2 = \frac{\lambda d\lambda}{w}, \quad \omega_3 = \frac{\lambda^2 d\lambda}{w}, \quad \omega_4 = \frac{\lambda^3 d\lambda}{w}, \quad \omega_5 = \frac{\lambda^4 d\lambda}{w}, \tag{14}$$

one gets

$$\frac{\partial}{\partial e_k}\omega_i = \frac{e_k^j}{R'(e_k)}\left(g_j^{(k)}\omega_1 + d_j^{(k)}\omega_2 + c_j^{(k)}\omega_3 + b_j^{(k)}\omega_4 + a_j^{(k)}\omega_5 + dF_k\right), \tag{15}$$

$$F_k = \frac{w}{\lambda - e_k}, \quad j = i-1, \quad i = 1, \dots, 5,$$

where $R'(e_1) = (e_1 - e_2)\cdots(e_1 - e_6)$, etc. Since

$$\frac{\partial}{\partial e_k}\left(\oint_\gamma \omega_i\right) = \oint_\gamma \frac{\partial}{\partial e_k}\omega_i,$$

and since dF_k is a differential of a meromorphic function of Γ_h, from (15) we obtain the following system for the vector of periods $J = (J_1, \dots, J_5)^t$,

$$2\frac{\partial J}{\partial e_k} = \mathcal{M}_k J, \qquad k = 1, \dots, 6, \tag{16}$$

$$\mathcal{M}_k = \frac{1}{R'(e_k)}\begin{pmatrix} 1 \\ e_k \\ e_k^2 \\ e_k^3 \\ e_k^4 \end{pmatrix}(G^{(k)}\, D^{(k)}\, C^{(k)}\, B^{(k)}\, A^{(k)}) + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ e_k & 1 & 0 & 0 & 0 \\ e_k^2 & e_k & 1 & 0 & 0 \\ e_k^3 & e_k^2 & e_k & 1 & 0 \end{pmatrix}, \tag{17}$$

with $G^{(k)}$, $D^{(k)}$, $C^{(k)}$, $B^{(k)}$, $A^{(k)}$ defined in (13). One observes that the right hand sides, as functions of e_k, may have only simple poles at $\{e_1, \dots, e_6\}\backslash e_k$, that is, the systems are of Fuchsian type.

The structure of the matrices $\mathcal{M}_k$ is similar to that of the Picard–Fuchs equations obtained in Refs. 4 and 6, however, not the same: the system (16) and (17) has an odd order.

Now, taking into account (11), we identify the roots ρ_1, ρ_2, ρ_3 with e_1, e_2, e_3, and the parameters a_1, a_2, a_3 with e_4, e_5, e_6, that is, we set

$$\begin{gathered} \lambda^3 + h_1\lambda + h_2 = (\lambda - \rho_1)(\lambda - \rho_2)(\lambda - \rho_3), \\ h_1 = \rho_1\rho_2 + \rho_1\rho_3 + \rho_2\rho_3, \quad h_2 = -\rho_1\rho_2\rho_3, \quad -\rho_1 - \rho_2 - \rho_3 := h_3 = 0. \end{gathered} \tag{18}$$

Then the following relation between the partial derivatives holds:

$$\begin{pmatrix} \dfrac{\partial J_i}{\partial \rho_1} \\ \dfrac{\partial J_i}{\partial \rho_2} \\ \dfrac{\partial J_i}{\partial \rho_3} \end{pmatrix} = \begin{pmatrix} 1 & \rho_2 + \rho_3 & -\rho_2\rho_3 \\ 1 & \rho_1 + \rho_3 & -\rho_1\rho_3 \\ 1 & \rho_2 + \rho_1, & -\rho_1\rho_2 \end{pmatrix}\begin{pmatrix} \dfrac{\partial J_i}{\partial h_3} \\ \dfrac{\partial J_i}{\partial h_1} \\ \dfrac{\partial J_i}{\partial h_2} \end{pmatrix}, \quad i = 1, \dots, 5,$$

and, therefore,

$$\begin{pmatrix} \dfrac{\partial J_i}{\partial h_3} \\ \dfrac{\partial J_i}{\partial h_1} \\ \dfrac{\partial J_i}{\partial h_2} \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} -\rho_1^2(\rho_2 - \rho_3) & \rho_2^2(\rho_1 - \rho_3) & -\rho_3^2(\rho_1 - \rho_2) \\ \rho_1(\rho_2 - \rho_3) & \rho_2(\rho_3 - \rho_1) & \rho_3(\rho_1 - \rho_2) \\ \rho_2 - \rho_3 & \rho_3 - \rho_1 & \rho_1 - \rho_2 \end{pmatrix}\begin{pmatrix} \dfrac{\partial J_i}{\partial \rho_1} \\ \dfrac{\partial J_i}{\partial \rho_2} \\ \dfrac{\partial J_i}{\partial \rho_3} \end{pmatrix}, \tag{19}$$

$$\Delta = (\rho_1 - \rho_2)(\rho_3 - \rho_1)(\rho_3 - \rho_2).$$

Now combining the above relations with Eqs. (16), and taking into account (13) and (18), we arrive at the following theorem.

Theorem 1. *The vector of periods $J = (J_1, \dots, J_5)^T$ of the differentials (14) of the even order curve (11) satisfies the equations*

$$2\frac{\partial J}{\partial h_1} = \mathcal{U}_1 J, \qquad 2\frac{\partial J}{\partial h_2} = \mathcal{U}_2 J, \tag{20}$$

$$\mathcal{U}_1 = \sum_{\alpha=1}^{3} \frac{1}{\Phi(\rho_\alpha)} \frac{\rho_\alpha}{(\rho_\alpha - \rho_\beta)^2(\rho_\alpha - \rho_\gamma)^2} \mathbf{S}_\alpha + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ h_3 & 1 & 0 & 0 & 0 \\ h_1 & h_3 & 1 & 0 & 0 \end{pmatrix},$$

$$\mathcal{U}_2 = \sum_{\alpha=1}^{3} \frac{1}{\Phi(\rho_\alpha)} \frac{1}{(\rho_\alpha - \rho_\beta)^2(\rho_\alpha - \rho_\gamma)^2} \mathbf{S}_\alpha + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ h_3 & 1 & 0 & 0 & 0 \end{pmatrix},$$

where

$$\Phi(\rho_\alpha) = (\rho_\alpha - a_1)(\rho_\alpha - a_2)(\rho_\alpha - a_3), \quad (\alpha, \beta, \gamma) = (1, 2, 3), \quad h_3 = 0,$$

$$\mathbf{S}_\alpha = \begin{pmatrix} 1 \\ \rho_\alpha \\ \rho_\alpha^2 \\ \rho_\alpha^3 \\ \rho_\alpha^4 \end{pmatrix} (G^{(\alpha)}\, D^{(\alpha)}\, C^{(\alpha)}\, B^{(\alpha)}\, A^{(\alpha)}),$$

$$A^{(\alpha)} = 2, \quad B^{(\alpha)} = -\frac{\rho_\alpha^2}{2} - \frac{3}{2}\rho_\alpha + \frac{3}{2}\sigma_1, \quad C^{(\alpha)} = h_1 - \frac{1}{2}\sigma_1\rho_\alpha + \sigma_2,$$

$$D^{(\alpha)} = \frac{1}{2}\left(-2\rho_\alpha^3 + \rho_\alpha^2\sigma_1 + h_2 - h_1\sigma_1 - \sigma_3\right), \quad G^{(\alpha)} = -\rho_\alpha\Phi(\rho_\alpha) + \frac{1}{2}\left(h_2\rho_\alpha - \sigma_3\rho_\alpha - h_2\sigma_1\right),$$

and $\sigma_1 = a_1 + a_2 + a_3, \sigma_2 = a_1a_2 + a_3a_1 + a_2a_3, \sigma_3 = a_1a_2a_3.$

The proof is direct and uses the identities

$$\rho_1^k(\rho_2 - \rho_3) + \rho_2^k(\rho_3 - \rho_1) + \rho_3^k(\rho_1 - \rho_2) = \begin{cases} 0, & k = 1 \\ -(\rho_1 - \rho_2)(\rho_3 - \rho_1)(\rho_3 - \rho_2), & k = 2 \\ (\rho_1 - \rho_2)(\rho_3 - \rho_1)(\rho_3 - \rho_2)h_3, & k = 3 \\ (\rho_1 - \rho_2)(\rho_3 - \rho_1)(\rho_3 - \rho_2)(h_3^2 - h_1), & k = 4. \end{cases}$$

Remark. The right hand sides of the systems (20) are symmetric functions of the roots e_1, e_2, e_3, hence, according to (18), they can be written in terms of the constants of motion h_1, h_2. However, the explicit expressions $\mathcal{U}_1(h_1, h_2), \mathcal{U}_2(h_1, h_2)$ appear to be too long to show here. We only mention that the components of $\mathcal{U}_1, \mathcal{U}_2$ have poles if and only if some of the roots e_1, e_2, e_3 coincide and, due to relations (18), these poles are of *first* order with respect to the corresponding increments of h_1, h_2. In other words, the systems (20) are of Fuchs type.

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