DECOMPOSITION SPACES, INCIDENCE ALGEBRAS AND MÖBIUS INVERSION

IMMA GÁLVEZ-CARRILLO, JOACHIM KOCK, AND ANDREW TONKS

Abstract. We introduce the notion of decomposition space as a general framework for incidence algebras and Möbius inversion. A decomposition space is a simplicial ∞-groupoid satisfying an exactness condition weaker than the Segal condition, expressed in terms of generic and free maps in Delta. Just as the Segal condition expresses up-to-homotopy composition, the new condition expresses decomposition. We work as much as possible on the objective level of linear algebra with coefficients in ∞-groupoids, and develop the necessary homotopy linear algebra along the way. Independently of finiteness conditions, to any decomposition space there is associated an incidence (co)algebra (with coefficients in ∞-groupoids), and under a completeness condition (weaker than the Rezk condition) this incidence algebra is shown to satisfy a sign-free version of the Möbius inversion principle. Examples of decomposition spaces beyond Segal spaces are given by the Waldhausen S-construction of an abelian (or stable infinity) category. Their incidence algebras are various kinds of Hall algebras. Another class of examples are Schmitt restriction species. Imposing certain homotopy finiteness conditions yields the notion of Möbius decomposition space, covering the notion of Möbius category of Leroux (itself a common generalisation of locally finite posets (Rota et al.) and finite decomposition monoids (Cartier–Foata)), as well as many constructions of Dür, including the Faà di Bruno and Connes-Kreimer bialgebras. We take a functorial viewpoint throughout, emphasising conservative ULF functors, and show that most reduction procedures in the classical theory of incidence coalgebras are examples of this notion, and in particular that many are an example of decalage of decomposition spaces. Our main theorem concerns the Lawvere-Menni Hopf algebra of Möbius intervals, which contains the universal Möbius function (but does not come from a Möbius category): we establish that Möbius intervals (in the ∞-setting) form a decomposition space, and that it has the universal property also with respect to Möbius inversion in general decomposition spaces.

NOTE: The notion of decomposition space was arrived at independently by Dyckerhoff and Kapranov (arXiv:1212.3563) who call them unital 2-Segal spaces. Our theory is quite orthogonal to theirs: the definitions are different in spirit and appearance, and the theories differ in terms of motivation, examples and directions. For the few overlapping results ('decalage of decomposition is Segal' and 'Waldhausen's S is decomposition'), our approach seems generally simpler.

Date: 11 April, 2014.

The first author was partially supported by grants MTM2010-15831, MTM2010-20692, SGR1092-2009, and MTM2012-38122-C03-01 the second author by MTM2009-10359, MTM2010-20692, and SGR1092-2009, and the third author by MTM2010-15831, SGR119-2009.

Contents

-1. Introduction

Background and motivation

Leroux's notion of Möbius category [45] generalises at the same time locally finite posets (Rota [62]) and Cartier–Foata finite-decomposition monoids [10], the two classical settings for incidence algebras and Möbius inversion. An important advantage of having these classical theories on the same footing is that the appropriate class of functors (the conservative ULF functors (unique lifting of factorisations) (1.4)) connect different examples, and in particular give nice explanations of the process of reduction which is important in getting the most interesting algebras out of posets, a process that was sometimes rather ad hoc. As the most classical example of this process, the divisibility poset $(\mathbb{N}^\times, |)$ (considered as a category) admits a conservative ULF functor to the multiplicative monoid $(\mathbb{N}^\times, \times)$ (considered as a category with only one object). This functor induces a homomorphism of incidence coalgebras which is precisely the reduction map from the 'raw' incidence coalgebra of the divisibility poset to its reduced incidence coalgebra, which is isomorphic to the Cartier–Foata incidence coalgebra of the multiplicative monoid.

Shortly after Leroux's work, Dür [17] studied more involved categorical structures to extract further examples of incidence algebras and study their Möbius functions. In particular he realised what was later called the Connes–Kreimer Hopf algebra as the incidence coalgebra of a certain category of root-preserving forest embeddings, modulo the equivalence relation that identifies two root-preserving forest embeddings if their complement crowns are isomorphic forests. Another prominent example fitting Dür's formalism is the Faà di Bruno bialgebra, obtained in [30] from the category of surjections, which is however not a Möbius category.

Our work on Faà di Bruno formulae in bialgebras of trees [23] prompted us to look for a more general version of Leroux's theory, which would naturally realise the Faà di Bruno and Connes–Kreimer bialgebras as incidence coalgebras. A sequence of generalisations and simplifications of the theory led to the notion of decomposition space which is a main notion in this work.

The first abstraction step is to follow the objective method, pioneered in this context by Lawvere and Menni [43], working directly with the combinatorial objects rather than with numbers and functions on the vector spaces spanned by the objects, using linear algebra with coefficients in $\boldsymbol{Set}$. In the present work, the coefficients are ∞-groupoids. In the appendix we develop background needed in 'homotopy linear algebra' and homotopy cardinality, extending many results of Baez-Hoffnung-Walker [4] who worked with 1-groupoids. At the objective level, where all results and proofs are naturally bijective, finiteness

conditions do not play any essential role, since it is just as easy to handle infinite sets as finite ones. The price to pay is the absence of additive inverses: in particular, Möbius functions cannot exist in the usual form of an alternating sum. However, an equation expressing the Möbius inversion principle can be obtained by splitting into even and odd, and under the appropriate finiteness assumptions, one can pass from the objective level to the numerical level by taking cardinality; the even-odd split version of Möbius inversion then yields the usual form of an alternating sum.

There are two levels of finiteness conditions needed in order to take cardinality and arrive at algebraic (numerical) results: namely, just in order to obtain a numerical coalgebra, for each arrow f and for each $n \in \mathbb{N}$, there should be only finitely many decompositions of f into a chain of n arrows. Second, in order to obtain also Möbius inversion, the following additional finiteness condition is needed: for each arrow f, there is an upper bound on the number of non-identity arrows in a decomposition of f.

The importance of chains of arrows naturally suggests a simplicial viewpoint, regarding a category as a simplicial set via its nerve. Leroux's theory can be formulated in terms of simplicial sets, and many of the arguments then rely on certain simple pullback conditions, the first being the Segal condition which characterises categories among simplicial sets.

The fact that combinatorial objects typically have symmetries prompted the upgrade from sets to groupoids, in fact a substantial conceptual simplification [23]. This upgrade is straightforward, as long as the involved notions are taken in a correct homotopy sense: bijections of sets are replaced by equivalences of groupoids; the slices playing the role of vector spaces are homotopy slices, the pullbacks and fibres involved in the functors are homotopy pullbacks and homotopy fibres, and the sums are homotopy sums (i.e. colimits indexed by groupoids, just as classical sums are colimits indexed by sets). The passage to numbers and incidence algebras in the classical sense now goes via homotopy cardinality of groupoids. In this setting one may as well abandon also the strict notion of simplicial object in favour of a pseudo-functorial analogue. For example, the classifying space of $(\mathbb{B}, +, 0)$, the monoidal groupoid of finite sets and bijections under disjoint union, is actually only a pseudofunctor $\mathbf{B} : \Delta^{\mathrm{op}} \to \boldsymbol{Grpd}$. This level of abstraction allows us to state for example that the incidence algebra of $\mathbf{B}$ is the category of species with the Cauchy product (suggested as an exercise by Lawvere and Menni [43]).

While it is doable to handle all the 2-category theory involved to deal with groupoids, pseudo-functors, pseudo-natural isomorphisms, and so on, much conceptual clarity is obtained by passing immediately to ∞-groupoids: thanks to the monumental effort of Joyal [32], [33],

Lurie [49] and others, ∞-groupoids can now be handled efficiently, and at least at the elementary level we work on where all that is needed is some basic knowledge about (homotopy) pullbacks and (homotopy) sums, everything looks very much like the category of sets. So we work throughout with certain simplicial ∞-groupoids. Weak categories in ∞-groupoids are precisely Rezk complete Segal spaces [60]. Our theory at this level says that for any Rezk complete Segal category there is a natural incidence coalgebra defined with coefficients in ∞-groupoids, and that the objective sign-free Möbius inversion principle holds. To extract numerical coalgebras from this, some homotopy finiteness conditions must be imposed, and the passage to numbers is then via homotopy cardinality.

The final abstraction step, which becomes the starting point for the paper, is to notice that in fact neither the Segal condition nor the Rezk condition is needed in full in order to get a (co)associative (co)algebra and a Möbius inversion principle. Coassociativity follows from (in fact is essentially equivalent to) the *decomposition space axiom* (see 1.3 for the axiom, and the discussion at the beginning of Section 2 for its derivation from coassociativity): it is a simplicial ∞-groupoid sending generic/free pushout squares in Δ to pullbacks. Whereas the Segal condition is the expression of the ability to compose morphisms, the new condition is about the ability to decompose, which of course in general is easier to achieve than composability. In order to get the Möbius inversion principle (with coefficients in ∞-groupoids), a completeness condition is needed, but it is weaker than the Rezk axiom: it is enough that $s_0 : X_0 \to X_1$ is a monomorphism. Such simplicial ∞-groupoids we call *complete decomposition spaces*. Every Rezk complete Segal space is a complete decomposition space.

It is likely that all incidence (co)algebras can be realised directly (without imposing a reduction) as incidence (co)algebras of (complete) decomposition spaces. The decomposition space is found by analysing the reduction step. For example, Dür realises the q-binomial coalgebra as the reduced incidence coalgebra of the category of finite-dimensional vector spaces over a finite field and linear injections, by imposing the equivalence relation identifying two linear injections if their quotients are isomorphic. Trying to realise the reduced incidence coalgebra directly as a decomposition space immediately leads to the Waldhausen S-construction, which is a general class of examples: we show that for any abelian category or stable ∞-category, the Waldhausen S-construction is a decomposition space (which is not Segal). Under the appropriate finiteness conditions, the resulting incidence algebras contain the (derived) Hall algebras.

As another example we show that the Connes–Kreimer bialgebra is directly the incidence coalgebra of a decomposition space of combinatorial forests, without the need of reductions. This decomposition space

is not a Segal space. In fact we fit this example into a general class of examples of decomposition spaces, which includes also all Schmitt coalgebras of restriction species [64]. We introduce the notion of directed restriction species, a class of decomposition spaces that includes the Connes-Kreimer bialgebra of trees as well as related constructions with directed graphs.

The appropriate notion of morphism between decomposition spaces is that of conservative ULF functor. These induce coalgebra homomorphisms. Many relationships between incidence coalgebras, and in particular many of the reductions that play a central role in the classical theory (from Rota and Dür [17] to Schmitt [65]), are induced from conservative ULF functors. The simplicial viewpoint taken in this work reveals furthermore that most of these conservative ULF functors are actually instances of the notion of decalage, which goes back to Illusie [28]. Decalage is in fact an important ingredient in the theory to relate decomposition spaces to Segal spaces: we observe that the decalage of a decomposition space is a Segal space.

Our final example of a decomposition space constitutes our main theorem. Lawvere showed in the 1980s that there is a Hopf algebra of Möbius intervals which contains the universal Möbius function. The first published account is by Lawvere–Menni [43], where the objective method is first explored. More precisely, this Hopf algebra is obtained from the collection of all iso-classes of Möbius intervals, and features a canonical coalgebra homomorphism from any incidence coalgebra of a Möbius category X, defined by sending an arrow in X to its factorisation interval. Although this Hopf algebra is universal for incidence coalgebras of Möbius categories, it is not itself the incidence coalgebra of a Möbius category.

We show that it *is* a decomposition space. In fact, in order for this to work smoothly (and obtain the correct universal properties), we are forced now to work in ∞-groupoids — this is an important motivation for this abstraction step. We construct the decomposition space of all intervals, and establish that it is universal for decomposition spaces. This involves constructing homotopy-meaningful intervals from any given simplex in any given decomposition space. The main tools here are the universal property of pullbacks and certain factorisation systems on various ∞-categories related to decomposition spaces. The main factorisation system, the wide/cartesian factorisation system on the ∞-category of intervals, generalises the generic/free factorisation system on Δ which was the corner stone for our theory of decomposition spaces.

Throughout we have strived for deriving all results from elementary principles, such as pullbacks, factorisation systems and other universal constructions. It is also characteristic for our approach that we

are able to reduce many technical arguments to simplicial combinatorics. The main notions are formulated in terms of the generic/free factorisation system in Δ. To establish coassociativity we explore also the algebraist's Delta $\underline{\Delta}$ (including the empty ordinal) and establish and exploit a universal property of its twisted arrow category. As a general method for establishing functoriality in free maps, we study a certain category ∇ of convex correspondences in $\underline{\Delta}$. Finally, in order to construct the universal decomposition space of intervals, we study the category of finite strict intervals, yet another variation of the simplex category, related to it by an adjunction. These 'simplicial preliminaries' are likely to have applications also outside the theory of decomposition spaces.

Related work: 2-Segal spaces of Dyckerhoff and Kapranov

The notion of decomposition space was arrived at independently by Dyckerhoff and Kapranov [19]: a decomposition space is essentially the same thing as they call a unital 2-Segal space. We hasten to give them full credit for having arrived at the notion first. Unaware of their work, we arrived at the same notion from a very different path, and the theory we have developed for it is mostly orthogonal to theirs.

The definitions are different in appearance: the definition of decomposition space refers to preservation of certain pullbacks, whereas the definition of 2-Segal space (reproduced in 1.3.1 below) refers to triangulations of convex polygons. The coincidence of the notions was noticed by Mathieu Anel because two of the basic results are the same: specifically, the characterisation in terms of decalage and Segal spaces (our Theorem 1.5.5) and the result that the Waldhausen S-construction of a stable ∞-category is a decomposition space (our Theorem 4.6.9) were obtained independently (and first) in [19].

We were motivated by rather elementary aspects of combinatorics and quantum field theory, and our examples are all drawn from incidence algebras and Möbius inversion, whereas Dyckerhoff and Kapranov were motivated by representation theory, geometry, and homological algebra, and develop a theory with a much vaster range of examples in mind: in addition to Hall algebras and Hecke algebras they find cyclic bar construction, mapping class groups and surface geometry (see also [20] and [18]), construct a Quillen model structure and relate to topics of interest in higher category theory such as ∞-2-categories and operads.

In the end we think our contribution is just a little corner of a vast theory, but an important little corner, and we hope that our viewpoints and insights will prove useful also for the rest of the theory.

Related work on Möbius categories

Where incidence algebras and Möbius inversion are concerned, our work descends from Leroux et al. [45], [13], [46], Dür [17] and Lawvere-Menni [43]. There is a different notion of Möbius category, due to Haigh [26]. The two notions have been compared, and to some extent unified, by Leinster [44], who calls Leroux's Möbius inversion *fine* and Haigh's *coarse* (as it only depends on the underlying graph of the category). We should mention also the K-theoretic Möbius inversion for quasi-finite EI categories of Lück and collaborators [48], [22].

Summary by section

We proceed to summarise our results, section for section.

We begin in Section 0 with a minimal review of some elementary notions from the theory of ∞-categories. This is hardly more than a glossary, but it is our contention that it should be enough to render the paper accessible also to readers without prior experience with ∞-categories.

In Section 1, after a few preliminaries on simplicial objects and Segal spaces, we introduce the main notion of this work, decomposition spaces:

Definition. A simplicial space $X : \Delta^{\mathrm{op}} \to \boldsymbol{Grpd}$ is called a *decomposition space* when it takes generic/free pushouts in Δ to pullbacks.

We give a few equivalent pullback characterisations, and observe that every Segal space is a decomposition space. The relevant notion of morphism is that of conservative ULF functor (unique lifting of factorisations):

Definition. A simplicial map is called *ULF* if it is cartesian on generic face maps, and it is called *conservative* if cartesian on degeneracy maps. We write *cULF* for conservative and ULF.

After some variations, we come to decalage, and establish the following important relationship between Segal spaces and decalage:

Theorem 1.5.5. *A simplicial space X is a decomposition space if and only if both $\mathrm{Dec}_{\top}(X)$ and $\mathrm{Dec}_{\bot}(X)$ are Segal spaces, and the two comparison maps back to X are cULF.*

We also introduce the notion of *monoidal decomposition space*, as a monoid object in the monoidal ∞-category of decomposition spaces and cULF maps.

In Section 2 we establish that decomposition spaces induce coalgebras (with coefficients in ∞-groupoids), and that cULF maps induce coalgebra homomorphisms. We first explain how the decomposition space axioms follow directly from a naive notion of coassociativity. To establish coassociativity formally, we first need some more simplicial

preliminaries. In particular we introduce the twisted arrow category $\mathscr{D}$ of the category of finite ordinals, which is monoidal under *external sum*. We show that simplicial objects in a cartesian monoidal category can be characterised as monoidal functors on $\mathscr{D}$, and characterise decomposition spaces as those simplicial spaces whose extension to $\mathscr{D}$ preserves certain pullback squares. The coalgebra associated to a decomposition space X is the slice ∞-category $\boldsymbol{Grpd}_{/X_1}$, and its comultiplication map is given by the span

$$X_1 \overset{d_1}{\leftarrow} X_2 \overset{(d_2,d_0)}{\rightarrow} X_1 \times X_1.$$

The homotopy coassociativity of the incidence coalgebra is established in terms of the monoidal structure on $\mathscr{D}$. The incidence algebra of a monoidal decomposition space is naturally a *bialgebra*.

For any decomposition space, the linear dual of the comultiplication yields a convolution product in the *incidence algebra*. This contains, in particular, the *zeta functor* ζ, given by the span $X_1 \overset{=}{\leftarrow} X_1 \rightarrow 1$, and the counit ϵ, given by $X_1 \leftarrow X_0 \rightarrow 1$. In order to establish the Möbius inversion principle for the zeta functor, we need the following important condition.

Definition. A decomposition space X is *complete* when $s_0 : X_0 \rightarrow X_1$ is a monomorphism.

This condition together with the decomposition space axiom ensures that the notion of nondegenerate simplices is well-behaved. Let $\vec{X}_r \subset X_r$ denote the subspace of nondegenerate simplices, and consider the span $\Phi_r : \quad X_1 \leftarrow \vec{X}_r \rightarrow 1$. We can now establish the Möbius inversion principle (with hopefully self-explanatory notation):

Theorem 2.3.14. *For a complete decomposition space,*

$$\begin{aligned} \zeta * \Phi_{\text{even}} &= \epsilon + \zeta * \Phi_{\text{odd}}, \\ = \Phi_{\text{even}} * \zeta &= \epsilon + \Phi_{\text{odd}} * \zeta. \end{aligned}$$

In Section 3 we impose finiteness conditions in order to be able to take homotopy cardinality and obtain results at the numerical level of $\mathbb{Q}$-algebras.

Definition. A decomposition space X is called *locally finite* (3.1.1) when X_1 is locally finite and $s_0 : X_0 \rightarrow X_1$ and $d_1 : X_2 \rightarrow X_1$ are finite maps. It is called *Möbius* (3.2.1) when $\sum_r \vec{X}_r \rightarrow X_1$ is a finite map.

The condition 'locally finite' extends the notion of locally finite for posets. The condition ensures that the coalgebra structure descends to finite-groupoid coefficients, and hence, via homotopy cardinality, to $\mathbb{Q}$-algebras. In Section 3.1 we calculate the section coefficients (structure constants for the (co)multiplication) in some easy cases. The Möbius

condition is needed for the general Möbius inversion formula to descend to finite-groupoid coefficients and $\mathbb{Q}$-coefficients. We also note the following:

Proposition 3.2.2. *If a Möbius decomposition space is a Segal space, then it is Rezk complete.*

For a Möbius decomposition space, taking homotopy cardinality yields the following formula for the Möbius function (convolution inverse to the zeta function):

$$|\mu| = |\Phi_{\mathrm{even}}| - |\Phi_{\mathrm{odd}}|\,.$$

In Section 4 we give examples. The first batch of examples, similar to the binomial posets of Doubilet-Rota-Stanley [16] are straightforward, but serve to illustrate three points: the incidence algebra in question is realised directly from a decomposition space, without a reduction step; at the objective level, the convolution algebra is a monoidal structure of species (specifically: the usual Cauchy product of species, the shuffle product of $\mathbb{L}$-species, the Dirichlet product of arithmetic species, the Joyal-Street external product of q-species, and the Morrison 'Cauchy' product of q-species). In each of these cases, a power series representation results from taking cardinality.

The next class of examples include the Faà di Bruno bialgebra, the Butcher-Connes-Kreimer bialgebra of trees, with several variations, and similar structures on directed graphs (cf. Manchon [54] and Manin [56]). In Subsection 4.6 we come to an important class of examples, showing that the Waldhausen S-construction on an abelian category, or a stable ∞-category, is a decomposition space. We finish the section by computing the Möbius function in a few cases, and commenting on certain cancellations that occur in the process of taking cardinality, substantiating that these cancellations are not possible at the objective level (this is related to the distinction between bijections and natural bijections).

In Section 5 we show that Schmitt coalgebras of restriction species [64] (such as graphs, matroids, posets, etc.) come from decomposition spaces. We also introduce a new notion of *directed restriction species*. Whereas ordinary restriction species are presheaves of the category of finite sets and injections, directed restriction species are presheaves on the category of finite posets and convex inclusions. Examples covered by this notion are the Butcher-Connes-Kreimer bialgebra and the Manchon-Manin bialgebra of directed graphs. Both ordinary and directed restriction species are shown to be examples of a construction of decomposition spaces from what we call sesquicartesian fibrations, certain cocartesian fibrations over the category of finite ordinals that are also cartesian over convex maps.

In Section 6 we come to our main theorem, constructing a 'universal decomposition space', the decomposition space of intervals. The idea (due to Lawvere) is that to an arrow there is associated its category of factorisations, which is an interval. To set this up, we exploit factorisation systems and adjunctions derived from them, and start out in Subsection 6.1 with some general results about factorisation systems. Specifically we describe a situation in which a factorisation system lifts across an adjunction to produce a new factorisation system, and hence a new adjunction. Before coming to intervals in 6.3, we need flanked decomposition spaces (6.2): these are certain presheaves on the category Ξ of nonempty finite linear orders with a top and a bottom element. The ∞-category of flanked decomposition spaces features the important *wide/cartesian* factorisation system, where 'wide' is to be thought of as endpoint-preserving, and cartesian is like 'distance-preserving'. There is also the basic adjunction between decomposition spaces and flanked decomposition spaces, which in fact is the double dec construction. Intervals are first defined as certain flanked decomposition spaces which are contractible in degree -1 (this condition encodes an initial and a terminal object) (6.3.4), and via the basic adjunction we obtain the definitive ∞-category of intervals as a full subcategory of the ∞-category of complete decomposition spaces (6.4.1); it features the factorisation system wide/cULF (6.4.2), which extends the generic/free factorisation system on Δ (6.4.3). The factorisation-interval construction can now finally be described (Theorem 6.5.1) as a coreflection from complete decomposition spaces to intervals (or more precisely, on certain coslice categories). We show that every interval is a Segal space (6.2.17). The universal decomposition space U of intervals can finally (6.4.5) be defined very formally as a natural right fibration over Δ whose total space has objects wide interval maps from an ordinal. In plain words, U consists of subdivided intervals.

Theorem 6.4.7. *U is a complete decomposition space.*

The factorisation-interval construction yields a canonical functor $X \to U$, called the *classifying map*.

Theorem 6.5.2. *The classifying map is cULF.*

We conjecture that U is the terminal object in the ∞-category of complete decomposition spaces and cULF maps, and prove the following partial result:

Theorem 6.5.5. *For each complete decomposition space X, the space* $\mathrm{Map}_{\boldsymbol{cDcmp}^{\mathrm{cULF}}}(X, U)$ *is connected.*

We finish in Subsection 6.6 by imposing the Möbius condition, obtaining the corresponding finite results. A *Möbius interval* is an interval which is Möbius as a decomposition space. We show that every Möbius interval is a Rezk complete Segal space (6.6.6).

Our main theorem in this section is now:

Theorem 6.6.11. *The decomposition space of all Möbius intervals is Möbius.*

It follows that it admits a Möbius inversion formula with coefficients in finite ∞-groupoids or in $\mathbb{Q}$, and since every Möbius decomposition space admits a canonical cULF functor to it, we find that Möbius inversion in every incidence algebra (of a Möbius decomposition space) is induced from this master formula.

In the Appendix we develop what we need about homotopy linear algebra and homotopy cardinality. For the sake of flexibility (regarding what notions of finiteness the future will bring) we first work out the notions without finiteness conditions. The role of vector spaces is played by groupoid slices

$$\boldsymbol{Grpd}_{/S},$$

shown to be the homotopy-sum completion of S, and the role of linear maps is played by linear functors, i.e. given by pullback and lower-shriek along spans. We explain how to interpret scalar multiplication and sums (together: linear combinations), and how to expand these operations in coordinates. The canonical basis is given by the 'names', functors $\ulcorner x \urcorner : 1 \to S$. Groupoid slices and linear functors assemble into an ∞-category, which is monoidal closed. The tensor product is given by

$$\boldsymbol{Grpd}_{/S} \otimes \boldsymbol{Grpd}_{/T} = \boldsymbol{Grpd}_{/S\times T}.$$

In Subsection A.2 we get into the subtleties of finiteness conditions. An ∞-groupoid B is *locally finite* if at each base point b the homotopy groups $\pi_i(B, b)$ are finite for $i \geq 1$ and are trivial for i sufficiently large. It is called *finite* if furthermore it has only finitely many components. The cardinality of a finite ∞-groupoid is the sum (over the connected components) of the alternating product of the homotopy groups. We work out the basic properties of this notion.

For the ∞-groupoid version of linear algebra, we are strict about duality issues, developed in the setting of vector spaces and profinite-dimensional vector spaces (a brief review is in A.3.1). The role of vector spaces is played by finite-groupoid slices $\boldsymbol{grpd}_{/S}$ (where S is a locally finite ∞-groupoid), while the role of profinite-dimensional vector spaces is played by finite-presheaf categories $\boldsymbol{grpd}^S$. Linear maps are given by spans of *finite type*, meaning $S \xleftarrow{p} M \xrightarrow{q} T$ in which p is a finite map. Prolinear maps are given by spans of *profinite type*, where q is a finite map. In the end we have two ∞-categories: $\underrightarrow{\boldsymbol{lin}}$ whose objects are the finite-groupoid slices $\boldsymbol{grpd}_{/S}$ and whose mapping spaces are ∞-groupoids of finite-type spans, and the ∞-category $\underleftarrow{\boldsymbol{lin}}$ whose objects are finite-presheaf categories $\boldsymbol{grpd}^S$, and whose mapping spaces are ∞-groupoids of profinite-type spans.

Finally we follow Baez-Hoffnung-Walker [4] in defining cardinality in terms of a 'meta cardinality' functor, which induces cardinality notions in all slices. In our setting, this amounts to a functor

$$\begin{array}{rcl} \| \, \| : \underrightarrow{\boldsymbol{lin}} & \longrightarrow & \boldsymbol{Vect} \\ \boldsymbol{grpd}_{/S} & \longmapsto & \mathbb{Q}_{\pi_0 S} \end{array}$$

and a dual functor

$$\begin{array}{rcl} \| \, \| : \underleftarrow{\boldsymbol{lin}} & \longrightarrow & \underleftarrow{\boldsymbol{yect}} \\ \boldsymbol{grpd}^S & \longmapsto & \mathbb{Q}^{\pi_0 S}. \end{array}$$

For each fixed ∞-groupoid S, this gives an individual notion of cardinality $|\ | : \boldsymbol{grpd}_{/S} \to \mathbb{Q}_{\pi_0 S}$ (and dually $|\ | : \boldsymbol{grpd}^S \to \mathbb{Q}^{\pi_0 S}$), since vectors are just linear maps from the ground field.

The vector space $\mathbb{Q}_{\pi_0 S}$ is spanned by the elements $\delta_s := |\ulcorner s \urcorner|$. Dually, the profinite-dimensional vector space $\mathbb{Q}^{\pi_0 S}$ is spanned by the characteristic functions $\delta^t = \frac{|h^t|}{|\Omega(S,t)|}$ (the cardinality of the representable functors divided by the cardinality of the loop space).

Acknowledgments

We are indebted first of all to André Joyal, not only for his influence through the theory of species and the theory of quasi-categories, essential frameworks for our contribution, but also for his interest in our work, for many enlightening discussions and advice that helped shape it. We have also learned a lot from collaboration and discussions with David Gepner. Finally we thank Mathieu Anel, Kurusch Ebrahimi-Fard, Tom Leinster, and Fei Xu, for their interest and feedback.

0. Preliminaries on ∞-groupoids and ∞-categories

0.0.1. Groupoids and ∞-groupoids. Although most of our motivating examples can be naturally cast in the setting of 1-groupoids, we have chosen to work in the setting of ∞-groupoids. This is on one hand the natural generality of the theory, and on the other hand a considerable conceptual simplification: thanks to the monumental effort of Joyal [32], [33] and Lurie [49], the theory of ∞-categories has now reached a stage where it is just as workable (if not more) as the theory of 1-groupoids! The philosophy is that, modulo a few homotopy *caveats*, one is allowed to think as if working in the category of sets. A recent forceful vindication of this philosophy is Homotopy Type Theory [58], in which a syntax that resembles set theory is shown to be a powerful language for general homotopy types.

A recurrent theme in the present work is to upgrade combinatorial constructions from sets to ∞-groupoids. To this end the first step consists in understanding the construction in abstract terms, often in terms of pullbacks and sums, and then the second step consists in

copying over the construction to the ∞-setting. The ∞-category theory needed will be accordingly elementary, and it is our contention that it should be feasible to read this work without prior experience with ∞-groupoids or ∞-categories, simply by substituting the word 'set' for the word '∞-groupoid'. Even at the 0-level, our theory contributes interesting insight, revealing many constructions in the classical theory to be governed by very general principles proven useful also in other areas of mathematics.

The following short review of some basic aspects of ∞-categories should suffice for reading this paper, except the final Section 6, where some slightly more advanced machinery is used.

0.0.2. From posets to Rezk categories. A few remarks may be in order to relate these viewpoints with classical combinatorics. A 1-groupoid is the same thing as an ordinary groupoid, and a 0-groupoid is the same thing as a set. A (-1)-groupoid is the same thing as a truth value: up to equivalence there exist only two (-1)-groupoids, namely the contractible groupoid (a point) and the empty groupoid. A poset is essentially the same thing as a category in which all the mapping spaces are (-1)-groupoids. An ordinary category is a category in which all the mapping spaces are 0-groupoids. Hence the theory of incidence algebras of posets of Rota and collaborators can be seen as the (-1)-level of the theory. Cartier–Foata theory and Leroux theory take place at the 0-level. We shall see that in a sense the natural setting for combinatorics is the 1-level, since this level naturally takes into account that combinatorial structures can have symmetries. (From this viewpoint, it looks as if the classical theory compensates for working one level below the natural one by introducing reductions.) A Rezk category is a category whose mapping spaces are ∞-groupoids, and this is the level of generality at which we work.

0.0.3. ∞-categories and ∞-groupoids. By ∞-category we mean quasi-category [32]. These are simplicial sets satisfying the weak Kan condition: inner horns admit a filler. We refer to Joyal [32], [33] and Lurie [49]. The definition does not actually matter much in this work. The main point, Joyal's great insight, is that category theory can be generalised to quasi-categories, and that the results look the same, although to bootstrap the theory very different techniques are required. There are other implementations of ∞-categories, such as complete Segal spaces, see Bergner [8] for a survey. We will only use results that hold in all implementations, and for this reason we say ∞-category instead of referring explicitly to quasi-categories.

An ∞-groupoid is an ∞-category in which all morphisms are invertible. We often say *space* instead of ∞-groupoid, as they are a combinatorial substitute for topological spaces up to homotopy; for example, to each object x in an ∞-groupoid X, there are associated homotopy

groups $\pi_n(X, x)$ for $n > 0$. In terms of quasi-categories, ∞-groupoids are precisely Kan complexes, i.e. simplicial sets in which every horn, not just the inner ones, admits a filler.

∞-groupoids play the role analogous to sets in classical category theory. In particular, for any two objects x, y in an ∞-category $\mathscr{C}$ there is (instead of a hom set) a mapping space $\mathrm{Map}_{\mathscr{C}}(x, y)$ which is an ∞-groupoid. ∞-categories form a (large) ∞-category denoted $\boldsymbol{Cat}$. ∞-groupoids form a (large) ∞-category denoted $\boldsymbol{Grpd}$; it can be described explicitly as the coherent nerve of the (simplicially enriched) category of Kan complexes. Given two ∞-categories $\mathscr{D}$, $\mathscr{C}$, there is a functor ∞-category $\mathrm{Fun}(\mathscr{D}, \mathscr{C})$. Since $\mathscr{D}$ and $\mathscr{C}$ are objects in the ∞-category $\boldsymbol{Cat}$ we also have the ∞-groupoid $\mathrm{Map}_{\boldsymbol{Cat}}(\mathscr{D}, \mathscr{C})$, which can also be described as the maximal sub-∞-groupoid inside $\mathrm{Fun}(\mathscr{D}, \mathscr{C})$. When interpreting $\mathscr{D}$ and $\mathscr{C}$ as quasi-categories, and hence simplicial sets, this is just the simplicial mapping space.

0.0.4. Defining ∞-categories and sub-∞-categories. While in ordinary category theory one can define a category by saying what the objects and the arrows are (and how they compose), this from-scratch approach is more difficult for ∞-categories, as one would have to specify the simplicies in all dimensions and verify the filler condition (that is, describe the ∞-category as a quasi-category). In practice, ∞-categories are constructed from existing ones by general constructions that automatically guarantee that the result is again an ∞-category, although the construction typically uses universal properties in such a way that the resulting ∞-category is only defined up to equivalence. To specify a sub-∞-category of an ∞-category $\mathscr{C}$, it suffices to specify a subcategory of the homotopy category of $\mathscr{C}$ (i.e. the category whose hom sets are π_0 of the mapping spaces of $\mathscr{C}$), and then pull back along the components functor. What this amounts to in practice is to specify the objects (closed under equivalences) and specifying for each pair of objects x, y a full sub-∞-groupoid of the mapping space $\mathrm{Map}_{\mathscr{C}}(x, y)$, also closed under equivalences, and closed under composition.

0.0.5. Monomorphisms. A map of ∞-groupoids $f : X \to Y$ is a *monomorphism* when its fibres are (-1)-groupoids (i.e. are either empty or contractible). In other words, it is fully faithful as a functor: $\mathrm{Map}_X(a, b) \to \mathrm{Map}_Y(fa, fb)$ is an equivalence. In some respects, this notion behaves like for sets: for example, if f is a monomorphism, then there is a complement $Z := Y \setminus X$ such that $Y \simeq X + Z$. Hence a monomorphism is essentially an equivalence from X onto some connected components of Y. On the other hand, a crucial difference from sets to ∞-groupoids is that diagonal maps of ∞-groupoids are not in general monomorphisms. In fact $X \to X \times X$ is a monomorphism if and only if X is discrete (i.e. equivalent to a set).

0.0.6. Diagram categories and presheaves. Every 1-category is also a quasi-category via its nerve. In particular we have the ∞-category Δ of non-empty finite ordinals, and for each $n \geq 0$ the ∞-category $\Delta[n]$ which is the nerve of the linearly ordered set $\{0 \leq 1 \leq \cdots \leq n\}$. As an important example of a functor ∞-category, for a given ∞-category I, we have the ∞-category of presheaves $\mathrm{Fun}(I^{\mathrm{op}}, \boldsymbol{Grpd})$, and there is a Yoneda lemma that works as in the case of ordinary categories. In particular we have the ∞-category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \boldsymbol{Grpd})$ of simplicial ∞-groupoids, which will be one of our main objects of study.

Since arrows in an ∞-category do not compose on the nose (one can talk about 'a' composite, not 'the' composite), the 1-categorical notion of commutative diagram does not make sense. Commutative triangle in an ∞-category $\mathscr{C}$ means instead 'object in the functor ∞-category $\mathrm{Fun}(\Delta[2], \mathscr{C})$': the 2-dimensional face of $\Delta[2]$ is mapped to a 2-cell in $\mathscr{C}$ mediating between the composite of the 01 and 12 edges and the long edge 02. Similarly, 'commutative square' means object in the functor ∞-category $\mathrm{Fun}(\Delta[1] \times \Delta[1], \mathscr{C})$. In general, 'commutative diagram of shape I' means object in $\mathrm{Fun}(I, \mathscr{C})$, so when we say for example 'simplicial ∞-groupoid' it is not implied that the usual simplicial identities hold on the nose.

0.0.7. Adjoints, limits and colimits. There are notions of adjoint functors, limits and colimits, which behave in the same way as these notions in ordinary category theory, and are characterised by universal properties up to equivalence. For example, the singleton set $*$ (also denoted 1), or any contractible ∞-groupoid is a terminal object in $\boldsymbol{Grpd}$.

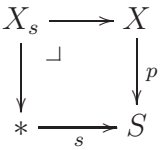
0.0.8. Pullbacks and fibres. Central to this work is the notion of pullback: given two morphisms of ∞-groupoids $X \to S \leftarrow Y$, there is a square

$$\begin{array}{ccc} X \times_S Y & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & S \end{array}$$

called the pullback, an example of a limit. It is defined via a universal property, as a terminal object in a certain auxiliary ∞-category consisting of squares with sides $X \to S \leftarrow Y$. All formal properties of pullbacks of sets carry over to ∞-groupoids.

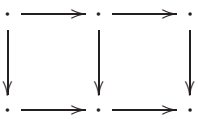
Given a morphism of ∞-groupoids, $p : X \to S$, and an object $s \in S$ (which in terms of quasi-categories can be thought of as a zero-simplex of S, but which more abstractly is encoded as a map $* \xrightarrow{s} S$ from the terminal ∞-groupoid $* = \Delta[0]$), the fibre of p over s is simply the

pullback



The following Lemma is used many times in our work. It is a straightforward extension of a familiar result in 1-category theory:

Lemma. 0.0.9. *If in a prism diagram of ∞-groupoids*



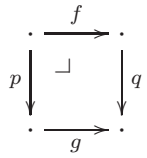
the outer rectangle and the right-hand square are pullbacks, then the left-hand square is a pullback.

A few remarks are in order: note that we talk about a prism, i.e. a $\Delta[1] \times \Delta[2]$-diagram. Although we have only drawn two of the squares of the prism, there is a third, whose horizontal sides are composites of the two indicated arrows. The triangles of the prism are not drawn either, because they are the fillers that exist by the axioms of quasi-categories. The proof follows the proof in the classical case, except that instead of saying 'given two arrows such and such, there exists a unique arrow making the diagram commute, etc.', one has to argue with equivalences of mapping spaces (or slice ∞-categories). See for example Lurie [49], Lemma 4.4.2.1 (for the dual case of pushouts).

0.0.10. Homotopy sums. In ordinary category theory, a colimit indexed by a discrete category (that is, a set) is the same thing as a sum (coproduct). For ∞-categories, the role of sets is played by ∞-groupoids. A colimit indexed by an ∞-groupoid is called a *homotopy sum*. In the case of 1-groupoids, these sums are ordinary sums weighted by inverses of symmetry factors. Their importance was stressed in [23]: by dealing with homotopy sums instead of ordinary sums, the formulae start to look very much like in the case of sets. For example, given a map of ∞-groupoids $X \to S$, we have that X is the homotopy sum of its fibres.

0.0.11. Slice categories and polynomial functors. Maps of ∞-groupoids with codomain S form the objects of a slice ∞-category $\boldsymbol{Grpd}_{/S}$, which behaves very much like slice categories in ordinary category theory. For example, for the terminal object $*$ we have $\boldsymbol{Grpd}_{/*} \simeq \boldsymbol{Grpd}$. Pullback along a morphism $f : T \to S$ defines an ∞-functor $f^* : \boldsymbol{Grpd}_{/S} \to \boldsymbol{Grpd}_{/T}$. This functor is right adjoint to the functor $f_! : \boldsymbol{Grpd}_{/T} \to \boldsymbol{Grpd}_{/S}$ given by post-composing with f. (This construction requires some care: as composition is not canonically defined, one has to choose composites. One can check that different choices yield

equivalent functors.) The following Beck-Chevalley rule (push-pull formula) [24] holds for ∞-groupoids: given a pullback square



there is a canonical equivalence of functors

$$p_! \circ f^* \simeq g^* \circ q_! . \tag{1}$$

0.0.12. Families. A map of ∞-groupoids $X \to S$ can be interpreted as a family of ∞-groupoids parametrised by S, namely the fibres X_s. Just as for sets, the same family can also be interpreted as a presheaf $S \to \boldsymbol{Grpd}$. Precisely, for each ∞-groupoid S, we have the fundamental equivalence

$$\boldsymbol{Grpd}_{/S} \xrightarrow{\sim} \mathrm{Fun}(S, \boldsymbol{Grpd}),$$

which takes a family $X \to S$ to the functor sending $s \mapsto X_s$. In the other direction, given a functor $F : S \to \boldsymbol{Grpd}$, its colimit is the total space of a family $X \to S$.

0.0.13. Symmetric monoidal ∞-categories. There is a notion of symmetric monoidal ∞-category, but it is technically more involved than the 1-category case, since in general higher coherence data has to be specified beyond the 1-categorical associator and Mac Lane pentagon condition. This theory has been developed in detail by Lurie [51, Ch.2], subsumed in the general theory of ∞-operads. In the present work, a few monoidal structures play an important role, but since they are directly induced by cartesian product, we have preferred to deal with them in an informal (and possibly not completely rigorous) way, with the same freedom as one deals with cartesian products in ordinary category theory. In these specific cases the formal treatment should not present any technical difficulties.

1. Decomposition spaces

1.1. Simplicial preliminaries

Our work relies heavily on simplicial machinery. We briefly review the notions needed, to establish conventions and notation.

1.1.1. The simplex category (the topologist's Delta). Recall that the 'simplex category' Δ is the category whose objects are the nonempty finite ordinals

$$[k] := \{0, 1, 2, \dots, k\},$$

and whose morphisms are the monotone maps. These are generated by the coface maps $d^i : [n-1] \to [n]$, which are the monotone injective functions for which $i \in [n]$ is not in the image, and codegeneracy maps

$s^i : [n+1] \to [n]$ which are monotone surjective functions for which $i \in [n]$ has a double preimage. We write $d^\perp := d^0$ and $d^\top := d^n$ for the outer coface maps.

1.1.2. Generic and free maps. The category Δ has a generic-free factorisation system.[1] A morphism of Δ is termed *generic*, and written $g : [m] \twoheadrightarrow [n]$ if it preserves end-points, $g(0) = 0$ and $g(m) = n$. A morphism is termed *free*, and written $f : [m] \rightarrowtail [n]$ if it is distance preserving, $f(i+1) = f(i)+1$ for $0 \le i \le m-1$. The generic maps are generated by the codegeneracy maps and the inner coface maps, and the free maps are generated by the outer coface maps.

The *amalgamated ordinal sum over* $[0]$ of two objects $[m]$ and $[n]$, denoted $[m] \pm [n]$, is given by the pushout of free maps

$$\begin{array}{ccc} [0] & \xrightarrow{(d^\top)^n} & [n] \\ {\scriptstyle (d^\perp)^m}\downarrow & & \downarrow{\scriptstyle (d^\perp)^m} \\ [m] & \xrightarrow[(d^\top)^n]{} & [m] \pm [n] = [m+n] \end{array} \tag{2}$$

This operation is not functorial on all maps in Δ, but on the subcategory Δ_{gen} of generic maps it is functorial and defines a monoidal structure on Δ_{gen} (dual to ordinal sum (cf. Lemma 2.1.2)).

The free maps $f : [n] \rightarrowtail [m]$ are precisely the maps that can be written

$$f : [n] \rightarrowtail [a] \pm [n] \pm [b].$$

Every generic map with source $[a] \pm [n] \pm [b]$ splits as

$$(\ [a] \xrightarrow{g_1} [a']\ ) \ \pm\ (\ [n] \xrightarrow{g} [k]\ ) \ \pm\ (\ [b] \xrightarrow{g_2} [b']\ )$$

With these observations we can be explicit about the generic-free factorisation:

Lemma. 1.1.3. *With notation as above, the generic-free factorisation of a free map f followed by a generic map $g_1 \pm g \pm g_2$ is given by*

$$\begin{array}{ccc} [n] & \xrightarrow{f} & [a] \pm [n] \pm [b] \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_1 \pm g \pm g_2} \\ [k] & \longrightarrow & [a'] \pm [k] \pm [b'] \end{array} \tag{3}$$

[1] The notions of generic and free are general notions in category theory (see [69], [70]; the notion goes back to [31]), and make sense for example whenever there is a cartesian monad on a presheaf category $\mathscr{C}$: in the Kleisli category, the free maps are those from $\mathscr{C}$, and the generic maps are those generated by the monad. In the case at hand Δ arises from the free-category monad on the category of directed graphs: Δ is the restriction of the Kleisli category to the subcategory of non-empty linear graphs.

1.1.4. Identity-extension squares. A square (3) in which g_1 and g_2 are identity maps is called an *identity-extension square*.

Lemma. 1.1.5. *Generic and free maps admit pushout along each other, and the resulting maps are again generic and free. In fact, generic-free pushouts are precisely the identity extension squares.*

$$\begin{array}{ccc} [n] & \rightarrowtail & [a] \pm [n] \pm [b] \\ \downarrow & & \downarrow \\ [k] & \rightarrowtail & [a] \pm [k] \pm [b] \end{array}$$

These pushouts are fundamental to this work. We will define decomposition spaces to be simplicial spaces $X : \Delta^{\mathrm{op}} \to \boldsymbol{Grpd}$ that send these pushouts to pullbacks.

The previous lemma has the following easy corollary.

Corollary. 1.1.6. *Every codegeneracy map is a pushout (along a free map) of $s^0 : [0] \to [1]$, and every generic coface maps is a pushout (along a free map) of $d^1 : [2] \to [1]$.*

1.2. Segal spaces

1.2.1. Simplicial ∞-groupoids. Our main object of study will be simplicial ∞-groupoids subject to various exactness conditions, all formulated in terms of pullbacks. More precisely we work in the functor ∞-category

$$\mathrm{Fun}(\Delta^{\mathrm{op}}, \boldsymbol{Grpd}),$$

whose vertices are functors from the ∞-category Δ^{op} to the ∞-category $\boldsymbol{Grpd}$. In particular, the simplicial identities for $X : \Delta^{\mathrm{op}} \to \boldsymbol{Grpd}$ are not strictly commutative squares; rather they are $\Delta[1] \times \Delta[1]$-diagrams in $\boldsymbol{Grpd}$, hence come equipped with a homotopy between the two ways around in the square. But this is precisely the setting for pullbacks.

Consider a simplicial ∞-groupoid $X : \Delta^{\mathrm{op}} \to \boldsymbol{Grpd}$. We recall the *Segal maps*

$$(\partial_{0,1}, \ldots, \partial_{r-1,r}) : X_r \longrightarrow X_1 \times_{X_0} \cdots \times_{X_0} X_1 \qquad r \geq 0.$$

where $\partial_{k-1,k} : X_r \to X_1$ is induced by map $[1] \rightarrowtail [r]$ sending 0,1 to $k-1, k$.

A *Segal space* is a simplicial ∞-groupoid satisfying the Segal condition, namely that the Segal maps should be equivalences.

Lemma. 1.2.2. *The following conditions are equivalent, for any simplicial ∞-groupoid X:*

(1) *X satisfies the Segal condition,*

$$X_r \xrightarrow{\simeq} X_1 \times_{X_0} \cdots \times_{X_0} X_1 \qquad r \geq 0.$$

(2) *The following square is a pullback for all* $p, q \geq r$

$$\begin{array}{ccc} X_{p-r+q} & \xrightarrow{{d_0}^{p-r}} & X_q \\ {\scriptstyle {d_{p+1}}^{q-r}}\downarrow & \lrcorner & \downarrow{\scriptstyle {d_{r+1}}^{q-r}} \\ X_p & \xrightarrow[{d_0}^{p-r}]{} & X_r \end{array}$$

(3) *The following square is a pullback for all* $n > 0$

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{d_\perp} & X_n \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_n & \xrightarrow[d_\perp]{} & X_{n-1} \end{array}$$

(4) *The following square is a pullback for all* $p, q \geq 0$

$$\begin{array}{ccc} X_{p+q} & \xrightarrow{{d_0}^{p}} & X_q \\ {\scriptstyle {d_{p+1}}^{q}}\downarrow & \lrcorner & \downarrow{\scriptstyle {d_1}^{q}} \\ X_p & \xrightarrow[{d_0}^{p}]{} & X_0 \end{array}$$

Proof. It is straightforward to show that the Segal condition implies (2). Now (3) and (4) are special cases of (2). Also (3) implies (2): the pullback in (2) is a composite of pullbacks of the type given in (3). Finally one shows inductively that (4) implies the Segal condition (1). □

A map $f : Y \to X$ of simplicial spaces is *cartesian* on an arrow $[n] \to [k]$ in Δ if the naturality square for f with respect to this arrow is a pullback.

Lemma. 1.2.3. *If $f : Y \to X$ is a map of simplicial spaces that is cartesian on outer face maps, and if X is a Segal space, then Y is a Segal space too.*

1.2.4. Rezk completeness. Let J denote the ∞-groupoid generated by one isomorphism $0 \to 1$; it can be seen as a combinatorial model of the infinity sphere $\Omega^\infty S^0$. A Segal space X is *Rezk complete* when the natural map (obtained by precomposing with $J \to *$)

$$\mathrm{Map}(*, X) \to \mathrm{Map}(J, X)$$

is an equivalence of ∞-groupoids. It means that the space of identity arrows is equivalent to the space of equivalences. (See [60], [8] and [37].) A Rezk complete Segal space is also called a *Rezk category*.

1.2.5. Ordinary nerve. Let $\mathscr{C}$ be a small 1-category. The *nerve* of $\mathscr{C}$ is the simplicial set

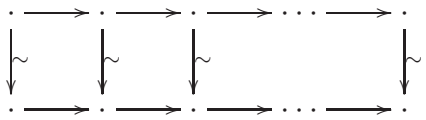
$$\begin{aligned} N\mathscr{C} : \Delta^{\mathrm{op}} &\longrightarrow \boldsymbol{Set} \\ [n] &\longmapsto \mathrm{Fun}([n], \mathscr{C}), \end{aligned}$$

where $\mathrm{Fun}([n], \mathscr{C})$ is the *set* of strings of n composable arrows. Subexamples of this are given by any poset or any monoid. The simplicial sets that arise like this are precisely those satisfying the Segal condition (which is strict in this context). If each set is regarded as a discrete ∞-groupoid, $N\mathscr{C}$ is thus a Segal space. In general it is not Rezk complete, since some object may have a nontrivial automorphism. As an example, suppose $\mathscr{C}$ is a one-object groupoid (i.e. a group). Then inside $(N\mathscr{C})_1$ the space of equivalences is the whole set $(N\mathscr{C})_1$. But the degeneracy map $s_0 : (N\mathscr{C})_0 \to (N\mathscr{C})_1$ is not an equivalence (unless the group is trivial).

1.2.6. The fat nerve of an essentially small 1-category. In most cases it is more interesting to consider the *fat nerve*, defined as the simplicial *groupoid*

$$\begin{aligned} X : \Delta^{\mathrm{op}} &\longrightarrow \boldsymbol{Grpd} \\ [k] &\longmapsto \mathrm{Map}(\Delta[k], \mathscr{C}), \end{aligned}$$

where $\mathrm{Map}(\Delta[k], \mathscr{C})$ is the mapping space, defined as the maximal subgroupoid of the functor category $\mathrm{Fun}(\Delta[k], \mathscr{C})$. In other words, $(\mathbf{N}\mathscr{C})_n$ is the groupoid whose objects are strings of n composable arrows in $\mathscr{C}$ and whose morphisms are connecting isos between such strings:



It is straightforward to check the Segal condition, remembering that the pullbacks involved are homotopy pullbacks. For instance, the pullback $X_1 \times_{X_0} X_1$ has as objects strings of 'weakly composable' arrows, in the sense that the target of the first arrow is isomorphic to the source of the second, and a comparison isomorphism is specified. The Segal map $X_2 \to X_1 \times_{X_0} X_1$ is the inclusion of the subgroupoid consisting of strictly composable pairs. But any weakly composable pair is isomorphic to a strictly composable pair, and the comparison isomorphism is unique, hence the inclusion $X_2 \hookrightarrow X_1 \times_{X_0} X_1$ is an equivalence. Furthermore, the fat nerve is Rezk complete. Indeed, it is easy to see that inside X_1, the equivalences are the invertible arrows of $\mathscr{C}$. But any invertible arrow is equivalent to an identity arrow.

Note that if $\mathscr{C}$ is a category with no non-trivial isomorphisms (e.g. any Möbius category in the sense of Leroux) then the fat nerve coincides with the ordinary nerve, and if $\mathscr{C}$ is just equivalent to such a category

then the fat nerve is level-wise equivalent to the ordinary nerve of any skeleton of $\mathscr{C}$.

1.2.7. Joyal-Tierney $t^!$ — the fat nerve of an ∞-category. The fat nerve construction is just a special case of the general construction $t^!$ of Joyal and Tierney [37], which is a functor from quasi-categories to complete Segal spaces, meaning specifically certain simplicial objects in the category of Kan complexes: given a quasi-category $\mathscr{C}$, the complete Segal space $t^!\mathscr{C}$ is given by

$$\begin{array}{rcl} \Delta^{\mathrm{op}} & \longrightarrow & \boldsymbol{Kan} \\ [n] & \longmapsto & \big[[k] \mapsto \boldsymbol{sSet}(\Delta[n] \times \Delta'[k], \mathscr{C})\big] \end{array}$$

where $\Delta'[k]$ denotes the groupoid freely generated by a string of k invertible arrows. They show that $t^!$ constitutes in fact a (right) Quillen equivalence between the simplicial sets with the Joyal model structure, and bisimplicial sets with the Rezk model structure.

Taking a more invariant viewpoint, talking about ∞-groupoids abstractly, the Joyal-Tierney $t^!$ functor associates to an ∞-category $\mathscr{C}$ the Rezk complete Segal space

$$\begin{array}{rcl} \Delta^{\mathrm{op}} & \longrightarrow & \boldsymbol{Grpd} \\ [n] & \longmapsto & \mathrm{Map}(\Delta[n], \mathscr{C}). \end{array}$$

1.2.8. Fat nerve of bicategories with only invertible 2-cells. From a bicategory $\mathscr{C}$ with only invertible 2-cells one can get a complete Segal bigroupoid by a construction analogous to the fat nerve. (In fact, this can be viewed as the $t^!$ construction applied to the so-called Duskin nerve of $\mathscr{C}$.) The *fat nerve* of a bicategory $\mathscr{C}$ is the simplicial bigroupoid

$$\begin{array}{rcl} \Delta^{\mathrm{op}} & \longrightarrow & \boldsymbol{2Grpd} \\ [n] & \longmapsto & \boldsymbol{PsFun}(\Delta[n], \mathscr{C}), \end{array}$$

the 2-groupoid of normalised pseudofunctors.

1.2.9. Monoidal groupoids. Important examples of the previous situation come from monoidal groupoids $(\mathscr{M}, \otimes, I)$. The fat nerve construction applied to the classifying space $B\mathscr{M}$ yields in this case a complete Segal bigroupoid, with zeroth space $B\mathscr{M}^{\mathrm{eq}}$, the classifying space of the full subcategory $\mathscr{M}^{\mathrm{eq}}$ spanned by the tensor-invertible objects.

The fat nerve construction can be simplified considerably in case $\mathscr{M}^{\mathrm{eq}}$ is contractible. This happens precisely when every tensor-invertible object is isomorphic to the unit object I and I admits no non-trivial automorphisms.

Proposition. 1.2.10. *If $(\mathscr{M}, \otimes, I)$ is a monoidal groupoid such that $\mathscr{M}^{\mathrm{eq}}$ is contractible, then the simplicial bigroupoid given by the classifying space is equivalent to the simplicial* 1*-groupoid*

$$\begin{array}{rcl} \Delta^{\mathrm{op}} & \longrightarrow & \boldsymbol{1\text{-}Grpd} \\ [n] & \longmapsto & \mathscr{M} \times \mathscr{M} \times \cdots \times \mathscr{M} =: \mathscr{M}^n. \end{array}$$

where the outer face maps project away an outer factor, the inner face maps tensor together two adjacent factors, and the degeneracy maps insert a neutral object.

We have omitted the proof, to avoid going into 2-category theory. (Note that the simplicial 1-groupoid that we obtain is not *strictly* simplicial, unless the monoidal structure is strict.)

Examples of monoidal groupoids satisfying the conditions of the Proposition are the monoidal groupoid $(\boldsymbol{FinSet}, +, 0)$ of finite sets and bijections or the monoidal groupoid $(\boldsymbol{Vect}, \oplus, \mathbf{0})$ of vector spaces and linear isomorphisms under direct sum. In contrast, the monoidal groupoid $(\boldsymbol{Vect}, \otimes, \Bbbk)$ of vector spaces and linear isomorphisms under tensor product is not of this kind, as the unit object has many automorphisms. The assignment $[n] \mapsto \boldsymbol{Vect}^{\otimes n}$ does constitute a Segal 1-groupoid, but it is not Rezk complete.

1.3. Decomposition spaces

Recall from Lemma 1.1.5 that generic and free maps in Δ admit pushouts along each other.

Definition. A *decomposition space* is a simplicial ∞-groupoid

$$X : \Delta^{\mathrm{op}} \to \boldsymbol{Grpd}$$

such that the image of any pushout diagram in Δ of a generic map g along a free map f is a pullback of ∞-groupoids,

$$X\left(\begin{array}{ccc} [p] & \xleftarrow{g'} & [m] \\ {\scriptstyle f'}\uparrow & \lrcorner & \uparrow{\scriptstyle f} \\ [q] & \xleftarrow[g]{} & [n] \end{array}\right) \quad = \quad \begin{array}{ccc} X_p & \xrightarrow{g'^*} & X_m \\ {\scriptstyle f'^*}\downarrow & \lrcorner & \downarrow{\scriptstyle f^*} \\ X_q & \xrightarrow[g^*]{} & X_n. \end{array}$$

Remark. 1.3.1. The notion of decomposition space can be seen as an abstraction of coalgebra, cf. Section 2 below: it is precisely the condition required to obtain a coassociative comultiplication on $\boldsymbol{Grpd}_{/X_1}$.

The notion is equivalent to the notion of unital (combinatorial) 2-Segal space introduced by Dyckerhoff and Kapranov [19] (their Definition 2.3.1, Definition 2.5.2, Definition 5.2.2, Remark 5.2.4). Briefly, their definition goes as follows. For any triangulation T of a convex polygon with n vertices, there is induced a simplicial subset $\Delta^T \subset \Delta[n]$. A simplicial space X is called 2-Segal if, for every triangulation T of every convex n-gon, the induced map $\mathrm{Map}(\Delta[n], X) \to \mathrm{Map}(\Delta^T, X)$ is

a weak homotopy equivalence. Unitality is defined in terms of pullback conditions involving degeneracy maps, similar to our (4) below. The equivalence between decomposition spaces and unital 2-Segal spaces follows from Proposition 2.3.2 of [19] which gives a pullback criterion for the 2-Segal condition.

1.3.2. Alternative formulations of the pullback condition. To verify the conditions of the definition, it will in fact be sufficient to check a smaller collection of squares. On the other hand, the definition will imply that many other squares of interest are pullbacks too. The formulation in terms of generic and free maps is preferred both for practical reasons and for its conceptual simplicity compared to the smaller or larger collections of squares.

Recall from Lemma 1.1.5 that the generic-free pushouts used in the definition are just the identity extension squares,

$$\begin{array}{ccc} [n] & \xrightarrow{\quad g \quad} & [k] \\ \downarrow & & \downarrow \\ [a]\pm[n]\pm[b] & \xrightarrow[\mathrm{id}\pm g\pm\mathrm{id}]{} & [a]\pm[k]\pm[b] \end{array}$$

Such a square can be written as a vertical composite of squares in which either $a = 1$ and $b = 0$, or vice-versa. In turn, since the generic map g is a composite of inner face maps $d^i : [m-1] \to [m]$ $(0 < i < m)$ and degeneracy maps $s^j : [m+1] \to [m]$, these squares are horizontal composites of pushouts of a single generic d^i or s^j along $d^\perp$ or $d^\top$. Thus, to check that X is a decomposition space, it is sufficient to check the following special cases are pullbacks, for $0 < i < n$:

$$\begin{array}{ccc} X_{1+n} & \xrightarrow{d_{1+i}} & X_n \\ {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[d_i]{} & X_{n-1}, \end{array} \qquad \begin{array}{ccc} X_{n+1} & \xrightarrow{d_i} & X_n \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_n & \xrightarrow[d_i]{} & X_{n-1}, \end{array}$$

$$\begin{array}{ccc} X_{1+n} & \xrightarrow{s_{1+j}} & X_{1+n+1} \\ {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[s_j]{} & X_{n+1}, \end{array} \qquad \begin{array}{ccc} X_{n+1} & \xrightarrow{s_j} & X_{n+1+1} \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_n & \xrightarrow[s_j]{} & X_{n+1}. \end{array} \tag{4}$$

In fact, we can be more economic: instead of checking all $0 < i < n$ it is enough to check all $n \geq 2$ and *some* $0 < i < n$, and instead of checking all $0 \leq j \leq n$ it is enough to check the case $j = n = 0$.

Proposition. 1.3.3. *A simplicial ∞-groupoid X is a decomposition space if and only if the following diagrams are homotopy pullbacks for*

all $n \geq 2$:

$$\begin{array}{ccc} X_1 & \xrightarrow{s_1} & X_2 \\ {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_0 & \xrightarrow[s_0]{} & X_1, \end{array} \qquad \begin{array}{ccc} X_1 & \xrightarrow{s_0} & X_2 \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_0 & \xrightarrow[s_0]{} & X_1. \end{array}$$

and the following diagrams are homotopy pullbacks for some choice of $i = i_n$, $0 < i < n$, for each $n \geq 2$:

$$\begin{array}{ccc} X_{1+n} & \xrightarrow{d_{1+i}} & X_n \\ {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[d_i]{} & X_{n-1}, \end{array} \qquad \begin{array}{ccc} X_{n+1} & \xrightarrow{d_i} & X_n \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_n & \xrightarrow[d_i]{} & X_{n-1}, \end{array}$$

Proof. To see the non-necessity of the other degeneracy cases, observe that for $n > 0$, every degeneracy map $s_j : X_n \to X_{n+1}$ is the section of an *inner* face map d_i (where $i = j$ or $i = j + 1$). Now in the diagram

$$\begin{array}{ccccc} X_{1+n} & \xrightarrow{s_{1+j}} & X_{1+n+1} & \xrightarrow{d_{1+i}} & X_{1+n} \\ {\scriptstyle d_\perp}\downarrow & & \downarrow{\scriptstyle d_\perp} & & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[s_j]{} & X_{n+1} & \xrightarrow[d_i]{} & X_n, \end{array}$$

the horizontal composites are identities, so the outer rectangle is a pullback, and the right-hand square is a pullback since it is one of cases outer face with inner face. Hence the left-hand square, by Lemma 0.0.9, is a pullback too. The case $s_0 : X_0 \to X_1$ is the only degeneracy map that is not the section of an inner face map, so we cannot eliminate the two cases involving this map. The non-necessity of the other inner-face-map cases is the content of the following lemma. □

Lemma. 1.3.4. *The following are equivalent for a simplicial ∞-groupoid X.*

(1) *For each $n \geq 2$, the following diagram is a pullback for all $0 < i < n$:*

$$\begin{array}{ccc} X_{1+n} & \xrightarrow{d_{1+i}} & X_n \\ {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[d_i]{} & X_{n-1}, \end{array} \qquad \left(\textit{resp.} \quad \begin{array}{ccc} X_{n+1} & \xrightarrow{d_i} & X_n \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_n & \xrightarrow[d_i]{} & X_{n-1}, \end{array} \right)$$

(2) *For each $n \geq 2$, the above diagram is a pullback for some $0 < i < n$.*

(3) *For each $n \geq 2$, the following diagram is a pullback:*

$$\begin{array}{ccc} X_{1+n} & \xrightarrow{{d_2}^{n-1}} & X_2 \\ {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[{d_1}^{n-1}]{} & X_1 \end{array} \qquad \left(\text{resp.} \quad \begin{array}{ccc} X_{n+1} & \xrightarrow{{d_1}^{n-1}} & X_2 \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_n & \xrightarrow[{d_1}^{n-1}]{} & X_1 \end{array} \right)$$

Proof. The hypothesised pullback in (2) is a special case of that in (1), and that in (3) is a horizontal composite of those in (2), since there is a unique generic map $[1] \to [n]$ in Δ for each n. The implication (3) $\Rightarrow$ (1) follows by Lemma 0.0.9 and the commutativity for $0 < i < n$ of the diagram

$$\begin{array}{ccccc} X_{1+n} & \xrightarrow{d_{1+i}} & X_n & \xrightarrow{{d_2}^{n-1}} & X_2 \\ {\scriptstyle d_\perp}\downarrow & \lrcorner & {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[d_i]{} & X_{n-1} & \xrightarrow[{d_1}^{n-1}]{} & X_1 \end{array}$$

Similarly for the 'resp.' case. □

Proposition. 1.3.5. *Any Segal space is a decomposition space.*

Proof. Let X be Segal space. In the diagram ($n \geq 2$)

$$\begin{array}{ccccc} X_{n+1} & \xrightarrow{d_n} & X_n & \xrightarrow{d_\top} & X_{n-1} \\ {\scriptstyle d_\perp}\downarrow & & {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[d_{n-1}]{} & X_{n-1} & \xrightarrow[d_\top]{} & X_{n-2}, \end{array}$$

since the horizontal composites are equal to $d_\top \circ d_\top$, both the outer rectangle and the right-hand square are pullbacks by the Segal condition (1.2.2 (3)). Hence the left-hand square is a pullback. This establishes the third pullback condition in Proposition 1.3.3. In the diagram

$$\begin{array}{ccccc} X_1 & \xrightarrow{s_1} & X_2 & \xrightarrow{d_\top} & X_1 \\ {\scriptstyle d_\perp}\downarrow & & {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_0 & \xrightarrow[s_0]{} & X_1 & \xrightarrow[d_\top]{} & X_0, \end{array}$$

since the horizontal composites are identities, the outer rectangle is a pullback, and the right-hand square is a pullback by the Segal condition. Hence the left-hand square is a pullback, establishing the first of the pullback conditions in Proposition 1.3.3. The remaining two conditions of Proposition 1.3.3, those involving $d_\top$ instead of $d_\perp$, are obtained similarly by interchanging the roles of $\perp$ and $\top$. □

Remark. 1.3.6. This result was also obtained by Dyckerhoff and Kapranov [19] (Propositions 2.3.3, 2.5.3, and 5.2.6).

Corollary 1.1.6 implies the following important property of decomposition spaces.

Lemma. 1.3.7. *In a decomposition space X, every generic face map is a pullback of $d_1 : X_2 \to X_1$, and every degeneracy map is a pullback of $s_0 : X_0 \to X_1$.*

Thus, even though the spaces in degree ≥ 2 are not fibre products of X_1 as in a Segal space, the higher generic face maps and degeneracies are determined by 'unit' and 'composition',

$$X_0 \xrightarrow{s_0} X_1 \xleftarrow{d_1} X_2.$$

In Δ^{op} there are more pullbacks than those between generic and free. Diagram (2) in 1.1.2 is a pullback in Δ^{op} that is not preserved by all decomposition spaces, though it is preserved by all Segal spaces. On the other hand, certain other pullbacks in Δ^{op} are preserved by general decomposition spaces. We call them colloquially 'bonus pullbacks':

Lemma. 1.3.8. *For a decomposition space X, the following squares are pullbacks:*

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{d_j} & X_n \\ {\scriptstyle s_i}\downarrow & \lrcorner & \downarrow{\scriptstyle s_i} \\ X_{n+2} & \xrightarrow[d_{j+1}]{} & X_{n+1} \end{array} \quad \text{for all } i<j, \quad \text{and} \quad \begin{array}{ccc} X_{n+1} & \xrightarrow{d_j} & X_n \\ {\scriptstyle s_{i+1}}\downarrow & \lrcorner & \downarrow{\scriptstyle s_i} \\ X_{n+2} & \xrightarrow[d_j]{} & X_{n+1} \end{array} \quad \text{for all } j \leq i.$$

Proof. We treat the case $i < j$; for the other case, interchange the roles of $\top$ and $\bot$. The case where $j = \top$ is clear since $d_\top$ is free, so from now on we assume that d_j is inner. The $i = 0$ case,

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{d_j} & X_n \\ {\scriptstyle s_0}\downarrow & \lrcorner & \downarrow{\scriptstyle s_0} \\ X_{n+2} & \xrightarrow[d_{j+1}]{} & X_{n+1} \end{array}$$

is seen to be a pullback by noting that s_0 is a section to the outer face map d_0, and applying the standard argument using Lemma 0.0.9. The square in the general case is obtained by base-change along iterates of $d_\bot$ of the square in the $i = 0$ case, and therefore it is again a pullback. □

Lemma. 1.3.9. *For a decomposition space X, the following squares are pullbacks for all $i < j$:*

$$\begin{array}{ccc} X_n & \xrightarrow{s_{j-1}} & X_{n+1} \\ {\scriptstyle s_i}\downarrow & \lrcorner & \downarrow{\scriptstyle s_i} \\ X_{n+1} & \xrightarrow[s_j]{} & X_{n+2} \end{array}$$

Proof. Just observe that s_j is a section to d_{j+1}, and apply the standard argument: if d_{j+1} is an outer face map then the square is a basic generic-free pullback; if d_{j+1} is inner, we can use instead the previous lemma. □

1.4. **Conservative ULF functors**

Definition. A simplicial map $F : Y \to X$ is called *ULF (unique lifting of factorisations)* if it is a cartesian natural transformation on generic face maps of Δ. It is called *conservative* if it is cartesian on degeneracy maps. It is called *cULF* if it is both cartesian and ULF.

Lemma. 1.4.1. *For a simplicial map $F : Y \to X$, the following are equivalent.*

(1) *F is cartesian on all generic maps (i.e. cULF).*
(2) *F is cartesian on every inner face map and on every degeneracy map.*
(3) *F is cartesian on every generic map of the form $[1] \to [n]$.*

Proof. That (1) implies (2) is trivial. The implication (2) $\Rightarrow$ (3) is easy since the generic map $[1] \to [n]$ factors as a sequence of inner face maps (or is a degeneracy map if $n = 0$). For the implication (3) $\Rightarrow$ (1), for a general generic map $[n] \to [m]$ observe that we have unique $[1] \to [n] \to [m]$ and since the ones starting in $[1]$ have cartesian component, also the map $[n] \to [m]$ must have cartesian component. □

Lemma. 1.4.2. *A simplicial map $F : Y \to X$ between decomposition spaces is cULF if and only if it is cartesian on the generic maps $[1] \to [2]$ and $[1] \to [0]$,*

$$\begin{array}{ccc} Y_1 & \longleftarrow & Y_2 \\ \downarrow & \llcorner & \downarrow \\ X_1 & \longleftarrow & X_2 \end{array} \qquad\qquad \begin{array}{ccc} Y_0 & \longrightarrow & Y_1 \\ \downarrow & \lrcorner & \downarrow \\ X_0 & \longrightarrow & X_1. \end{array}$$

Proof. Using Lemma 1.3.7, an exercise with pullbacks shows that F being cartesian on $s^0 : [1] \to [0]$ and $d^1 : [1] \to [2]$ implies it is cartesian on all other degeneracy and inner face maps. □

Remark. 1.4.3. The notion of cULF can be seen as an abstraction of coalgebra homomorphism, cf. 2.2.6 below: 'conservative' corresponds to unit preservation, 'ULF' corresponds to comultiplicativity.

In the special case where X and Y are fat nerves of 1-categories, then the condition that the square

$$\begin{array}{ccc} Y_0 & \longrightarrow & Y_1 \\ \downarrow & \lrcorner & \downarrow \\ X_0 & \longrightarrow & X_1 \end{array}$$

be a pullback is precisely the classical notion of conservative functor (i.e. if $f(a)$ is invertible then already a is invertible).

Similarly, the condition that the square

$$\begin{array}{ccc} Y_1 & \longleftarrow & Y_2 \\ \downarrow & \llcorner & \downarrow \\ X_1 & \longleftarrow & X_2 \end{array}$$

be a pullback is an up-to-isomorphism version of the classical notion of ULF functor, implicit already in Content–Lemay–Leroux [13], and perhaps made explicit first by Lawvere [42]; see Street [67] for the 2-categorical notion. In the case of the Möbius categories of Leroux, where there are no invertible arrows around, the two notions of ULF coincide. Furthermore, in this case, where identities are indecomposable, every ULF functor is conservative. This explains why the functor notion emphasised by Lawvere and Menni [43] (and Leinster [44]) is ULF rather than explicitly cULF.

Example. 1.4.4. Here is an example of a functor which is not cULF in Lawvere's sense (is not cULF on classical nerves), but which is cULF in the homotopical sense. Namely, let $\boldsymbol{OI}$ denote the category of finite ordered sets and monotone injections. Let $\boldsymbol{I}$ denote the category of finite sets and injections. The forgetful functor $\boldsymbol{OI} \to \boldsymbol{I}$ is not cULF in the classical sense, because the identity monotone map $\underline{2} \to \underline{2}$ admits a factorisation in $\boldsymbol{I}$ that does not lift to $\boldsymbol{OI}$, namely the factorisation into two nontrivial transpositions. However, it is cULF in our sense, as can easily be verified by checking that the square

$$\begin{array}{ccc} \boldsymbol{OI}_1 & \longleftarrow & \boldsymbol{OI}_2 \\ \downarrow & \llcorner & \downarrow \\ \boldsymbol{I}_1 & \longleftarrow & \boldsymbol{I}_2 \end{array}$$

is a pullback by computing the fibres of the horizontal maps over a given monotone injection.

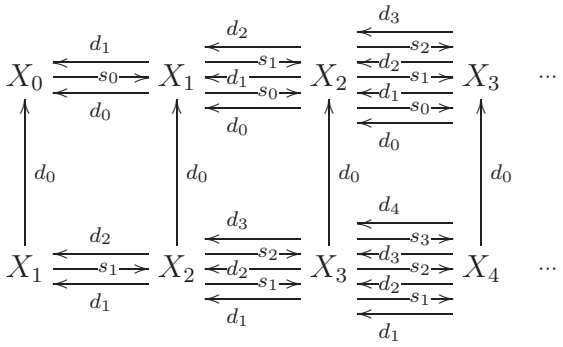
1.4.5. Right and left fibrations. A functor of Segal spaces $f : Y \to X$ is called a *right fibration* if it is cartesian on $d_\perp$ and on all generic maps, or a *left fibration* if it is cartesian on $d_\top$ and on generic maps. Here the condition on generic degeneracy maps is in fact a consequence of that on the face maps. These notions are most meaningful when the Segal spaces involved are Rezk complete.

Lemma. 1.4.6. *If X is a decomposition space and $f : Y \to X$ is cULF then also Y is a decomposition space.*

1.5. Decalage

1.5.1. Decalage. (See Illusie [28]). Given a simplicial space X as the top row in the following diagram, the *lower dec* $\mathrm{Dec}_\perp(X)$ is a new

simplicial space (bottom row in the diagram) obtained by deleting X_0 and shifting everything one place down, deleting also all d_0 face maps and all s_0 degeneracy maps. It comes equipped with a simplicial map $d_\perp : \mathrm{Dec}_\perp(X) \to X$ given by the original d_0:



Similarly, the upper dec, denoted $\mathrm{Dec}_\top(X)$ is obtained by instead deleting, in each degree, the last face map $d_\top$ and the last degeneracy map $s_\top$.

1.5.2. Decalage in terms of an adjunction. (See Lawvere [41].) The functor $\mathrm{Dec}_\perp$ can be described more conceptually as follows. There is an 'add-bottom' endofunctor $b : \Delta \to \Delta$, which sends $[k]$ to $[k+1]$ by adding a new bottom element. This is in fact a monad; the unit $\epsilon : \mathrm{Id} \Rightarrow b$ is given by the bottom coface map $d^\perp$. The lower dec is given by precomposition with b:

$$\mathrm{Dec}_\perp(X) = b^* X$$

Hence $\mathrm{Dec}_\perp$ is a comonad, and its counit is the bottom face map $d_\perp$.

Similarly, the upper dec is obtained from the 'add-top' monad on Δ. In Section 6 we shall exploit crucially the combination of the two comonads.

1.5.3. Slice interpretation. If X is the strict nerve of a category $\mathbb{C}$ then there is a close relationship between the upper dec and the slice construction. For the strict nerve, $X = N\mathbb{C}$, $\mathrm{Dec}_\top X$ is the disjoint union of all (the nerves of) the slice categories of $\mathbb{C}$:

$$\mathrm{Dec}_\top X = \sum_{x \in X_0} N(\mathbb{C}_{/x}).$$

(In general it is a homotopy sum.)

Any individual slice category can be extracted from the upper dec, by exploiting that the upper dec comes with a canonical augmentation given by (iterating) the bottom face map. The slice over an object x

is obtained by pulling back the upper dec along the name of x:

$$\begin{array}{ccc} 1 & \longleftarrow & N\mathbb{C}_{/x} \\ {\scriptstyle \ulcorner x \urcorner}\downarrow & \llcorner & \downarrow \\ X_0 & \xleftarrow[d_\perp]{} & \mathrm{Dec}_\top X \end{array}$$

There is a similar relationship between the lower dec and the coslices.

Proposition. 1.5.4. *If X is a decomposition space then $\mathrm{Dec}_\top(X)$ and $\mathrm{Dec}_\perp(X)$ are Segal spaces, and the maps $d_\top : \mathrm{Dec}_\top(X) \to X$ and $d_\perp : \mathrm{Dec}_\perp(X) \to X$ are cULF.*

Proof. We put $Y = \mathrm{Dec}_\top(X)$ and check the pullback condition 1.2.2 (3),

$$\begin{array}{ccc} Y_{n+1} & \xrightarrow{d_\perp} & Y_n \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ Y_n & \xrightarrow[d_\perp]{} & Y_{n-1}. \end{array}$$

This is the same as

$$\begin{array}{ccc} X_{n+2} & \xrightarrow{d_\perp} & X_{n+1} \\ {\scriptstyle d_{\top-1}}\downarrow & \lrcorner & \downarrow{\scriptstyle d_{\top-1}} \\ X_{n+1} & \xrightarrow[d_\perp]{} & Y_n \end{array}$$

and since now the horizontal face maps that with respect to Y were outer face maps, now become inner face maps in X, this square is one of the decomposition square axiom pullbacks. The cULF conditions says that the various $d_\top$ form pullbacks with all generic maps in X. But this follows from the decomposition space axiom for X. □

Theorem. 1.5.5. *For a simplicial ∞-groupoid $X : \Delta^{\mathrm{op}} \to \boldsymbol{Grpd}$, the following are equivalent*

1. *X is a decomposition space*
2. *both $\mathrm{Dec}_\top(X)$ and $\mathrm{Dec}_\perp(X)$ are Segal spaces, and the two comparison maps back to X are ULF and conservative.*
3. *both $\mathrm{Dec}_\top(X)$ and $\mathrm{Dec}_\perp(X)$ are Segal spaces, and the two comparison maps back to X are conservative.*
4. *both $\mathrm{Dec}_\top(X)$ and $\mathrm{Dec}_\perp(X)$ are Segal spaces, and the following squares are pullbacks:*

$$\begin{array}{ccc} X_1 & \xrightarrow{s_1} & X_2 \\ {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_0 & \xrightarrow[s_0]{} & X_1, \end{array} \qquad \begin{array}{ccc} X_1 & \xrightarrow{s_0} & X_2 \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_0 & \xrightarrow[s_0]{} & X_1. \end{array}$$

Proof. The implication (1) ⇒ (2) is just the preceding Proposition, and the implications (2) ⇒ (3) ⇒ (4) are specialisations. The implication (4) ⇒ (1) follows from Proposition 1.3.3. □

Remark. 1.5.6. Dyckerhoff and Kapranov [19] (Theorem 6.3.2) obtain the result that a simplicial space is 2-Segal (i.e. a decomposition space except that there are no conditions imposed on degeneracy maps) if and only if both dec's are Segal spaces.

Proposition. 1.5.7. *If $f : Y \to X$ is a conservative ULF functor between decomposition spaces, then $\mathrm{Dec}_\perp(f) : \mathrm{Dec}_\perp(Y) \to \mathrm{Dec}_\perp(X)$ is a right fibration of Segal spaces, cf. 1.4.5. Similarly, $\mathrm{Dec}_\top(f) : \mathrm{Dec}_\top(Y) \to \mathrm{Dec}_\top(X)$ is a left fibration.*

Proof. It is clear that if f is cULF then so is $\mathrm{Dec}_\perp(f)$. The further claim is that $\mathrm{Dec}_\perp(f)$ is also cartesian on d_0. But d_0 was originally a d_1, and in particular was generic, hence has cartesian component. □

1.6. **Monoidal decomposition spaces**

The ∞-category of decomposition spaces (as a full subcategory of simplicial ∞-groupoids), has finite products. Hence there is a symmetric monoidal structure on the ∞-category $\boldsymbol{Dcmp}^{\mathrm{cULF}}$ of decomposition spaces and cULF maps. We still denote this product as ×, although of course it is not the cartesian product in $\boldsymbol{Dcmp}^{\mathrm{cULF}}$.

Definition. A *monoidal decomposition space* is a monoid object (X, m, e) in $(\boldsymbol{Dcmp}^{\mathrm{cULF}}, \times, 1)$. A *monoidal functor* between monoidal decomposition spaces is a monoid homomorphism in $(\boldsymbol{Dcmp}^{\mathrm{cULF}}, \times, 1)$.

Example. 1.6.1. Recall that a category $\mathscr{E}$ with finite sums is *extensive* [9] when the natural functor $\mathscr{E}_{/A} \times \mathscr{E}_{/B} \to \mathscr{E}_{/A+B}$ is an equivalence. The fat nerve of an extensive 1-category is a monoidal decomposition space. The multiplication is given by taking sum, the neutral object by the initial object, and the extensive property ensures precisely that given a factorisation of a sum of maps, then each of the maps splits into sums of maps in a unique way.

Lemma. 1.6.2. *The dec of a monoidal decomposition space has again a natural monoidal structure, and the counit is a monoidal functor.*

2. Incidence (co)algebras and Möbius inversion

The goal in this section is to define a coalgebra (with ∞-groupoid coefficients) from any decomposition space. The following brief discussion explains the origin of the decomposition space axioms. For any simplicial space X, the span

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1$$

defines a linear functor, the *comultiplication*

$$\begin{array}{rcl} \Delta : \boldsymbol{Grpd}_{/X_1} & \longrightarrow & \boldsymbol{Grpd}_{/(X_1\times X_1)} \\ (S \xrightarrow{s} X_1) & \longmapsto & (d_2, d_0)_! \circ d_1^*(s). \end{array}$$

The desired coassociativity diagram (which should commute up to equivalence)

$$\begin{array}{ccc} \boldsymbol{Grpd}_{/X_1} & \xrightarrow{\Delta} & \boldsymbol{Grpd}_{/X_1\times X_1} \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta\times\mathrm{id}} \\ \boldsymbol{Grpd}_{/X_1\times X_1} & \xrightarrow[\mathrm{id}\times\Delta]{} & \boldsymbol{Grpd}_{/X_1\times X_1\times X_1} \end{array}$$

is induced by the spans in the outline of this diagram:

$$\begin{array}{ccccc} X_1 & \xleftarrow{d_1} & X_2 & \xrightarrow{(d_2,d_0)} & X_1\times X_1 \\ {\scriptstyle d_1}\uparrow & & {\scriptstyle d_1}\uparrow & \urcorner & \uparrow{\scriptstyle d_1\times\mathrm{id}} \\ X_2 & \xleftarrow{d_2} & X_3 & \xrightarrow[(d_3,d_0d_0)]{} & X_2\times X_1 \\ {\scriptstyle(d_2,d_0)}\downarrow & \llcorner & \downarrow{\scriptstyle(d_2^2,d_0)} & & \downarrow{\scriptstyle(d_2,d_0)\times\mathrm{id}} \\ X_1\times X_1 & \xleftarrow[\mathrm{id}\times d_1]{} & X_1\times X_2 & \xrightarrow[\mathrm{id}\times(d_2,d_0)]{} & X_1\times X_1\times X_1 \end{array}$$

Coassociativity will follow from Beck-Chevalley isomorphisms if the interior part of the diagram can be established, with pullbacks as indicated. Consider the upper right-hand square: it will be a pullback if and only if its composite with the first projection is a pullback:

$$\begin{array}{ccccc} X_2 & \xrightarrow{(d_2,d_0)} & X_1\times X_1 & \xrightarrow{\mathrm{pr}_1} & X_1 \\ {\scriptstyle d_1}\uparrow & \urcorner & {\scriptstyle d_1\times\mathrm{id}}\uparrow & \urcorner & \uparrow{\scriptstyle d_1} \\ X_3 & \xrightarrow[(d_3,d_0d_0)]{} & X_2\times X_1 & \xrightarrow[\mathrm{pr}_1]{} & X_2 \end{array}$$

But demanding the outer rectangle to be a pullback is precisely one of the basic decomposition space axioms. This argument is the origin of the decomposition space axioms.

Just finding an equivalence is not enough, though. Higher coherence has to be established, which will be accounted for by the full decomposition space axioms. To establish coassociativity in a strong homotopy sense we must deal on an equal footing with all 'reasonable' spans

$$\prod X_{n_j} \leftarrow \prod X_{m_j} \rightarrow \prod X_{k_i}$$

which could arise from composites of products of the comultiplication and counit. We therefore take a more abstract approach, relying on some more simplicial machinery. This also leads to another characterisation of decomposition spaces, and is actually of independent interest.

2.1. More simplicial preliminaries

2.1.1. The category $\underline{\Delta}$ of finite ordinals (the algebraist's Delta). We denote by $\underline{\Delta}$ the category of all finite ordinals (including the empty ordinal) and monotone maps. Clearly $\Delta \subset \underline{\Delta}$ (presheaves on $\underline{\Delta}$ are augmented simplicial sets), but this is not the most useful relationship between the two categories. We will thus use a different notation for the objects of $\underline{\Delta}$, given by their cardinality, with an underline:

$$\underline{n} = \{1, 2, \dots, n\}.$$

The category $\underline{\Delta}$ is monoidal under ordinal sum

$$\underline{m} + \underline{n} := \underline{m+n},$$

for which $\underline{0}$ is the neutral object.

The cofaces $d^i : \underline{n-1} \to \underline{n}$ and codegeneracies $s^i : \underline{n+1} \to \underline{n}$ in $\underline{\Delta}$ are, as usual, the injective and surjective monotone maps which skip and repeat the ith element, respectively, but note that now the index is $1 \leq i \leq n$.

Lemma. 2.1.2. *There is a canonical equivalence of monoidal categories (an isomorphism, if we consider the usual skeleta of these categories)*

$$\begin{array}{ccc} (\underline{\Delta}, +, \underline{0}) & \simeq & (\Delta_{\mathrm{gen}}^{\mathrm{op}}, \pm, [0]) \\ \underline{k} & \leftrightarrow & [k] \end{array}$$

Proof. The map from left to right sends $\underline{k} \in \underline{\Delta}$ to

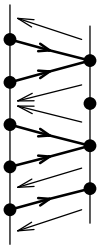
$$\mathrm{Hom}_{\underline{\Delta}}(\underline{k}, \underline{2}) \simeq [k] \in \Delta_{\mathrm{gen}}^{\mathrm{op}}.$$

The map in the other direction sends $[k]$ to the ordinal

$$\mathrm{Hom}_{\Delta_{\mathrm{gen}}}([k], [1]) \simeq \underline{k}.$$

In both cases, functoriality is given by precomposition. □

In both categories we can picture the objects as a line with some dots. The dots then represent the elements in $\underline{k}$, while the edges represent the elements in $[k]$; a map operates on the dots when considered a map in $\underline{\Delta}$ while it operates on the edges when considered a map in Δ_{gen}. Here is a picture of a certain map $\underline{5} \to \underline{4}$ in $\underline{\Delta}$ and of the corresponding map $[5] \leftarrow [4]$ in Δ_{gen}.



2.1.3. A twisted arrow category of $\underline{\Delta}$. Consider the category $\mathscr{D}$ whose objects are the arrows $\underline{n} \to \underline{k}$ of $\underline{\Delta}$ and whose morphisms (g, f) from $a : \underline{m} \to \underline{h}$ to $b : \underline{n} \to \underline{k}$ are commutative squares

$$\begin{array}{ccc} \underline{m} & \xrightarrow{g} & \underline{n} \\ {\scriptstyle a}\downarrow & {\scriptstyle (g,f)} & \downarrow{\scriptstyle b} \\ \underline{h} & \xleftarrow[f]{} & \underline{k}. \end{array} \tag{5}$$

That is, $\mathscr{D}^{\mathrm{op}}$ is the twisted arrow category [52, 6] of $\underline{\Delta}$.

There is a canonical factorisation system on $\mathscr{D}$: any morphism (5) factors uniquely as

$$\begin{array}{ccccc} \underline{m} & \xrightarrow{=} & \underline{m} & \xrightarrow{g} & \underline{n} \\ {\scriptstyle a=fbg}\downarrow & \varphi & {\scriptstyle bg}\downarrow & \gamma & \downarrow{\scriptstyle b} \\ \underline{h} & \xleftarrow[f]{} & \underline{k} & \xleftarrow[=]{} & \underline{k} \end{array}$$

The maps $\varphi = (\mathrm{id}, f) : fb \to b$ in the left-hand class of the factorisation system are termed *segalic*,

$$\begin{array}{ccc} \underline{m} & \xrightarrow{=} & \underline{m} \\ {\scriptstyle fb}\downarrow & \varphi & \downarrow{\scriptstyle b} \\ \underline{h} & \xleftarrow[f]{} & \underline{k}. \end{array} \tag{6}$$

The maps $\gamma = (g, \mathrm{id}) : bg \to b$ in the right-hand class are termed *ordinalic* and may be identified with maps in the slice categories $\underline{\Delta}_{/\underline{h}}$

$$\begin{array}{ccc} \underline{m} & \xrightarrow{g} & \underline{n} \\ {\scriptstyle bg}\downarrow & \gamma & \downarrow{\scriptstyle b} \\ \underline{h} & \xleftarrow[=]{} & \underline{h}. \end{array} \tag{7}$$

Observe that $\underline{\Delta}$ is isomorphic to the subcategory of objects with target $\underline{h} = \underline{1}$, termed the *connected objects* of $\mathscr{D}$,

$$\underline{\Delta} \xrightarrow{=} \underline{\Delta}_{/\underline{1}} \xrightarrow{\subseteq} \mathscr{D}. \tag{8}$$

The ordinal sum operation in $\underline{\Delta}$ induces a monoidal operation in $\mathscr{D}$: the *external sum* $(\underline{n} \to \underline{k}) \oplus (\underline{n}' \to \underline{k}')$ of objects in $\mathscr{D}$ is their ordinal sum $\underline{n} + \underline{n}' \to \underline{k} + \underline{k}'$ as morphisms in $\underline{\Delta}$. The neutral object is $\underline{0} \to \underline{0}$. The inclusion functor (8) is not monoidal, but it is easily seen to be oplax monoidal by means of the codiagonal map $\underline{1} + \underline{1} \to \underline{1}$.

Each object $\underline{m} \xrightarrow{a} \underline{k}$ of $\mathscr{D}$ is an external sum of connected objects,

$$a \;=\; a_1 \oplus a_2 \oplus \cdots \oplus a_k \;=\; \bigoplus_{i \in \underline{k}} \left(\underline{m_i} \xrightarrow{a_i} \underline{1} \right), \tag{9}$$

where $\underline{m_i}$ is (the cardinality of) the fibre of a over $i \in \underline{k}$.

Any segalic map (6) and any ordinalic map (7) in $\mathscr{D}$ may be written uniquely as external sums

$$\varphi \;=\; \varphi_1 \oplus \varphi_2 \oplus \cdots \oplus \varphi_h \;=\; \bigoplus_{j\in\underline{h}} \begin{pmatrix} \underline{m}_j & \xrightarrow{=} & \underline{m}_j \\ \downarrow & {\scriptstyle\varphi_j} & \downarrow{\scriptstyle b_j} \\ \underline{1} & \longleftarrow & \underline{k}_j \end{pmatrix} \tag{10}$$

$$\gamma \;=\; \gamma_1 \oplus \gamma_2 \oplus \cdots \oplus \gamma_h \;=\; \bigoplus_{j\in\underline{h}} \left(\underline{m}_j \xrightarrow{\gamma_j} \underline{n}_j\right) \tag{11}$$

where each γ_j is a map in $\underline{\Delta}_{/\underline{1}} = \underline{\Delta}$.

In fact $\mathscr{D}$ is a universal monoidal category in the following sense.

Proposition. 2.1.4. *For any cartesian category $(\mathscr{C}, \times, 1)$, there is an equivalence*

$$\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathscr{C}) \;\simeq\; \mathrm{Fun}^{\otimes}((\mathscr{D}, \oplus, 0), (\mathscr{C}, \times, 1))$$

between the categories of simplicial objects X in $\mathscr{C}$ and of monoidal functors $\overline{X} : \mathscr{D} \to \mathscr{C}$. The correspondence between X and $\overline{X}$ is determined by following properties.

(a) The functors $X : \Delta^{\mathrm{op}} \to \mathscr{C}$ and $\overline{X} : \mathscr{D} \to \mathscr{C}$ agree on the common subcategory $\Delta^{\mathrm{op}}_{\mathrm{gen}} \cong \underline{\Delta}$,

$$\begin{array}{ccccc} \Delta^{\mathrm{op}}_{\mathrm{gen}} & \hookrightarrow & \Delta^{\mathrm{op}} & & \\ \cong\downarrow & & & \searrow^{X} & \\ \underline{\Delta} & \hookrightarrow & \mathscr{D} & \xrightarrow{\overline{X}} & \mathscr{C}. \end{array}$$

(b) Let $(\underline{m} \xrightarrow{a} \underline{k}) = \bigoplus_i (\underline{m}_i \xrightarrow{a} \underline{1})$ be the external sum decomposition (9) of any object of $\mathscr{D}$, and denote by $f_i : [m_i] \rightarrowtail [m_1] \pm \cdots \pm [m_k] = [m]$ the canonical free map in Δ, for $i \in \underline{k}$. Then

$$\overline{X}\begin{pmatrix} \underline{m} & \xrightarrow{=} & \underline{m} \\ \downarrow & {\scriptstyle\varphi} & \downarrow{\scriptstyle a} \\ \underline{1} & \longleftarrow & \underline{k} \end{pmatrix} \;=\; (X(f_1), \ldots, X(f_k)) \;:\; X_m \longrightarrow \prod_{i\in\underline{k}} X_{m_i}$$

and each $X(f_i)$ is the composite of $\overline{X}(\varphi)$ with the projection to X_i.

Proof. Given $\overline{X}$, property (a) says that there is a unique way to define X on objects and generic maps. Conversely, given X, then for any object $a : \underline{m} \to \underline{k}$ in $\mathscr{D}$ we have

$$\overline{X}_a \;=\; \prod_{i\in\underline{k}} \overline{X}_{a_i} \;=\; \prod_{i\in\underline{k}} X_{m_i}$$

using (9), and for any ordinalic map γ we have

$$\overline{X}(\gamma) \;=\; \prod_{i\in\underline{k}} \overline{X}(\gamma_i) \;=\; \prod_{i\in\underline{k}} X(g_i)$$

using (11), where $g_i \in \Delta_{\mathrm{gen}}^{\mathrm{op}}$ corresponds to $\gamma_i \in \underline{\Delta}$.

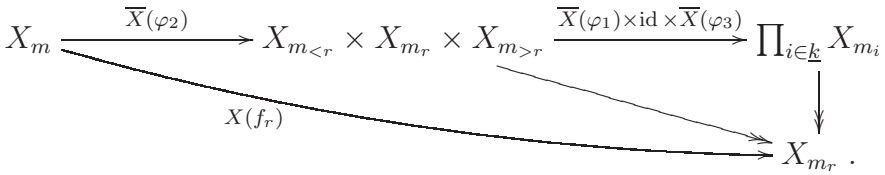
Thus we have a bijection between functors X defined on $\Delta_{\mathrm{gen}}^{\mathrm{op}}$ and functors $\overline{X}$ defined on the ordinalic subcategory of $\mathscr{D}$. Now we consider the free and segalic maps. Given $\overline{X}$, property (b) says that for any free map $f_r : [m_r] \to [m]$ we have

$$X(f_r) = \left(X_m \xrightarrow{\overline{X}(\varphi)} \prod_{i \in \underline{k}} X_{m_i} \twoheadrightarrow X_{m_r} \right)$$

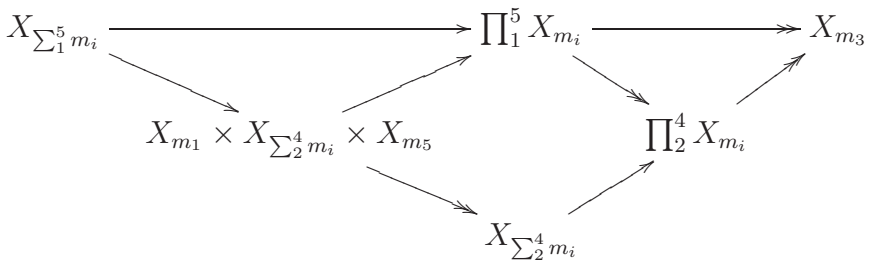
We may assume $k = 3$: given the factorisation

$$\varphi = \left(\begin{array}{ccccc} \underline{m} & \xrightarrow{=} & \underline{m}_{<r} + \underline{m}_r + \underline{m}_{>r} & \xrightarrow{=} & \sum_{i \in \underline{k}} m_i \\ \downarrow & \varphi_2 & \downarrow & \varphi_1 \oplus \mathrm{id} \oplus \varphi_3 & \downarrow \\ \underline{1} & \longleftarrow & \underline{3} & \longleftarrow & \underline{k} \end{array} \right)$$

one sees the value $X(f_r)$ is well defined from the following diagram

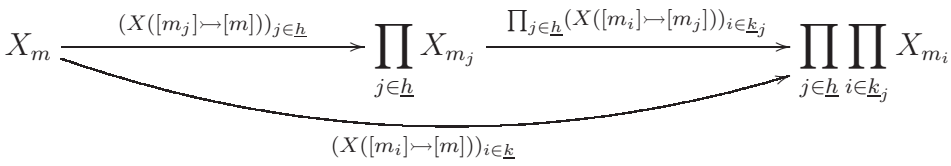


Functoriality of X on a composite of free maps, say $[m_3] \rightarrowtail [\sum_2^4 m_i] \rightarrowtail [\sum_1^5 m_i]$, now follows from the diagram



in which the first triangle commutes by functoriality of $\overline{X}$.

Conversely, given X, then property (b) says how to define $\overline{X}$ on segalic maps with connected domain and hence, by (10), on all segalic maps. Functoriality of $\overline{X}$ on a composite of segalic maps, say $(\mathrm{id}, \underline{1} \leftarrow \underline{h} \leftarrow \underline{k})$, follows from functoriality of X:



It remains only to check that the construction of $\overline{X}$ from X (and of X from $\overline{X}$) is well defined on composites of ordinalic followed by segalic

(free followed by generic) maps. One then has the mutually inverse equivalences required. Consider the factorisations in $\mathscr{D}$,

$$\begin{array}{ccccc} \underline{m} & \xrightarrow{=} & \underline{m} & \xrightarrow{g} & \underline{m}' \\ \downarrow & \varphi & \downarrow & \gamma & \downarrow \\ \underline{1} & \leftarrow & \underline{k} & \xleftarrow[=]{} & \underline{k} \end{array} \quad = \quad \begin{array}{ccccc} \underline{m} & \xrightarrow{g} & \underline{m}' & \xrightarrow{=} & \underline{m}' \\ \downarrow & \gamma' & \downarrow & \varphi' & \downarrow \\ \underline{1} & \xleftarrow[=]{} & \underline{1} & \leftarrow & \underline{k}. \end{array}$$

To show that $\overline{X}$ is well defined, we must show that the diagrams

$$\begin{array}{ccccc} X_m & \xrightarrow{\overline{X}(\varphi)=(X(f_1),\dots,X(f_k))} & \prod X_{m_i} & \longrightarrow\!\!\!\!\!\rightarrow & X_{m_r} \\ {\scriptstyle \overline{X}(\gamma')=X(\tilde{g})}\downarrow & & {\scriptstyle \overline{X}(\gamma')=\prod X(\tilde{g}_i)}\downarrow & & \downarrow{\scriptstyle X(\tilde{g}_r)} \\ X_{m'} & \xrightarrow[\overline{X}(\varphi')=(X(f'_1),\dots,X(f'_k))]{} & \prod X_{m'_i} & \longrightarrow\!\!\!\!\!\rightarrow & X_{m'_r}, \end{array}$$

commute for each r, where $\tilde{g}$, $\tilde{g}_i$ in Δ_{gen} correspond to g, g_i in $\underline{\Delta}$. This follows by functoriality of X, since $\tilde{g}$ restricted to m'_r is the corestriction of $\tilde{g}_r$. Finally we observe that this diagram, with $k = 3$ and $r = 2$, also serves to show that the construction of X from $\overline{X}$ is well defined on

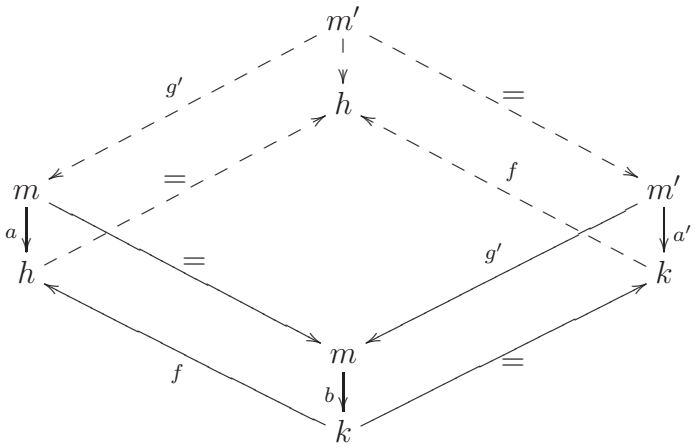
$$\begin{array}{ccc} [m_1 + m_2 + m_3] & \xleftarrow{f_2} & [m_2] \\ {\scriptstyle \tilde{g}}\uparrow & & \uparrow{\scriptstyle \tilde{g}_2} \\ [m'_1 + m'_2 + m'_3] & \xleftarrow[f'_2]{} & m'_2 \end{array}$$

□

Lemma. 2.1.5. *In the category $\mathscr{D}$, ordinalic and segalic maps admit pullback along each other, and the result is again maps of the same type.*

Proof. This is straightforward: in the diagram below, the map from a to b is segalic (given essentially by the bottom map f) and the map from a' to b is ordinalic (given essentially by the top map g'):

(12)



To construct the pullback, we are forced to repeat f and g', completing the squares with the corresponding identity maps. The connecting map in the resulting object is $fbg' : m' \to h$. It is clear from the presence of the four identity maps that this is a pullback. □

We now have the following important characterisation of decomposition spaces.

Proposition. 2.1.6. *A simplicial space $X : \Delta^{\mathrm{op}} \to \mathbf{Grpd}$ is a decomposition space if and only if its extension $\overline{X} : \mathscr{D} \to \mathbf{Grpd}$ preserves pullback squares of the kind described in 2.1.5.*

Proof. Since an ordinalic map is a sum, it can be decomposed into a sequence of maps in which each map has only one nontrivial summand. This means that a pullback diagram like (12) is a sum of diagrams of the form in which $\underline{h} = \underline{1}$. So to prove that these pullbacks are preserved, it is enough to treat the case $h = 1$. In this case, the map g' in the square is just a map in $\underline{\Delta}$, so it can be decomposed into face and degeneracy maps. The X-image is then a diagram of the form

$$\begin{array}{ccc} X_m & \longrightarrow & X_{m_1} \times \cdots \times X_{m_k} \\ \downarrow & & \downarrow \\ X_n & \longrightarrow & X_{n_1} \times \cdots \times X_{n_k}, \end{array}$$

where the map on the left is a face map or a degeneracy map. It follows that the map on the right is a product of maps in which all factors are identity maps except one, say the ith factor (which is again a face or a degeneracy map). Now whether or not this is a pullback can be checked on the projections onto the nontrivial factor:

$$\begin{array}{ccccc} X_m & \longrightarrow & X_{m_1} \times \cdots \times X_{m_k} & \longrightarrow & X_{m_i} \\ \downarrow & & \downarrow & & \downarrow \\ X_n & \longrightarrow & X_{n_1} \times \cdots \times X_{n_k} & \longrightarrow & X_{n_i} \end{array}$$

But by construction of $\overline{X}$, the composite horizontal maps are precisely free maps in the sense of the simplicial space X, and the vertical maps are precisely generic maps in the sense that it is an arbitrary map in $\underline{\Delta}$ and hence (in the other direction) a generic map in Δ, under the duality in 2.1.2. Since the right-hand square is always a pullback, by the standard pullback argument 0.0.9, the total square is a pullback (i.e. we have a decomposition space) if and only if the left-hand square is a pullback (i.e. the pullback condition on $\overline{X}$ is satisfied). □

2.2. Incidence coalgebras

2.2.1. Comultiplication and counit. For any decomposition space X, the diagram

$$X_1 \xleftarrow[d_1]{m_X} X_2 \xrightarrow[(d_2,d_0)]{p_X} X_1 \times X_1$$

defines a linear functor, the *comultiplication*

$$\begin{aligned} \Delta : \boldsymbol{Grpd}_{/X_1} &\longrightarrow \boldsymbol{Grpd}_{/(X_1\times X_1)} \\ (S \xrightarrow{s} X_1) &\longmapsto p_{X!} \circ m_X^*(s), \end{aligned}$$

and the diagram

$$X_1 \xleftarrow[s_0]{u_X} X_0 \xrightarrow{t_X} 1$$

defines a linear functor, the *counit*

$$\begin{aligned} \epsilon : \boldsymbol{Grpd}_{/X_1} &\longrightarrow \boldsymbol{Grpd} \\ (S \xrightarrow{s} X_1) &\longmapsto t_{X!} \circ u_X^*(s). \end{aligned}$$

We proceed to establish that this makes $\boldsymbol{Grpd}_{/X_1}$ a coassociative and counital coalgebra in a strong homotopy sense. We have more generally, for any $n \geq 0$, the generalised comultiplication maps

$$\Delta_n : \boldsymbol{Grpd}_{/X_1} \longrightarrow \boldsymbol{Grpd}_{/X_1\times\cdots\times X_1} \tag{13}$$

given by the spans

$$X_1 \leftarrow X_n \rightarrow X_1 \times \cdots \times X_1. \tag{14}$$

The case $n = 0$ is the counit map, and $n = 1$ gives the identity. The coassociativity will say that all combinations (composites and tensor products) of these agree whenever they have the same source and target. For this we exploit the category $\mathscr{D}$ introduced in 2.1, designed exactly to encode also cartesian powers of the various spaces X_k.

Definition. A *reasonable span* in $\mathscr{D}$ is a span $a \xleftarrow{g} m \xrightarrow{f} b$ in which g is ordinalic and f is segalic. Clearly the external sum of two reasonable spans is reasonable, and the composite of two reasonable spans is reasonable (by Lemma 2.1.5).

Let $X : \Delta^{\mathrm{op}} \to \boldsymbol{Grpd}$ be a fixed decomposition space, and interpret it also as a monoidal functor $\overline{X} : \mathscr{D} \to \boldsymbol{Grpd}$. A span in $\boldsymbol{Grpd}$ of the form

$$\overline{X}_a \leftarrow \overline{X}_m \rightarrow \overline{X}_b \tag{15}$$

is called reasonable if it is induced by a reasonable span in $\mathscr{D}$. Since $\overline{X}$ sends external sums to products, and sends pullbacks between ordinalic and segalic maps to pullbacks, it follows that reasonable spans like (15) are stable under products and under composition.

A linear map between slices of $\boldsymbol{Grpd}$ is called reasonable if it is given by a reasonable span like (15). That is, it is a pullback along a ordinalic map followed by a lowershriek along a segalic map.

The interest in these notions is of course that the generalised comultiplication maps Δ_n are reasonable, see (13,14) above.
In conclusion:

Lemma. 2.2.2. *Tensor products of reasonable linear maps are reasonable. For a decomposition space, composites of reasonable linear maps are reasonable.*

Proposition. 2.2.3. *Any reasonable linear map*

$$\boldsymbol{Grpd}_{/X_1} \longrightarrow \boldsymbol{Grpd}_{/X_1\times\cdots\times X_1}, \quad n\geq 0$$

is canonically equivalent to the nth comultiplication map.

Proof. We have to show that the only reasonable span of the form $X_1 \leftarrow \coprod X_{m_i} \to X_1\times\cdots\times X_1$ is (14). Indeed, the left leg must come from an ordinalic map, so since X_1 has only one factor, the middle object has also only one factor, i.e. is the image of $\underline{m}\to\underline{1}$. On the other hand, the right leg must be segalic, which forces $m=n$. □

Corollary. 2.2.4. *For a decomposition space X the comultiplication is coassociative and counital, and $\boldsymbol{Grpd}_{/X_1}$ is a strong homotopy comonoid.*

2.2.5. Coalgebra homomorphisms. The conservative ULF functors are important in our context in particular because they induce coalgebra homomorphisms:

Lemma. 2.2.6. *If $F: X\to Y$ is a conservative ULF map between decomposition spaces then $F_!: \boldsymbol{Grpd}_{/X_1}\to\boldsymbol{Grpd}_{/Y_1}$ is a coalgebra homomorphism.*

Proof. In the diagram

$$\begin{array}{ccccc} X_1 & \xleftarrow{g} & X_n & \xrightarrow{f} & X_1^n \\ {\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F_n} & & \downarrow{\scriptstyle F_1^n} \\ Y_1 & \xleftarrow[g']{} & Y_n & \xrightarrow[f']{} & Y_1^n \end{array}$$

the left-hand square is a pullback since F is conservative (case $n=0$) and ULF (cases $n>1$). Hence by the Beck-Chevalley condition we have an equivalence of functors $g'^*\circ F_{1!}\simeq F_{n!}\circ g^*$, and by postcomposing with $f'_!$ we arrive at the coalgebra homomorphism condition $\Delta'_n F_{1!}\cong F_{1!}^n\Delta_n$ □

Remark. 2.2.7. If Y is a Segal space, then the statement can be improved to an if-and-only-if statement.

Remark. 2.2.8. There is also a contravariant functoriality, but we shall not need it in this work. See Content–Lemay–Leroux [13, Prop. 5.6] and Leinster [44] for analysis of this.

2.2.9. Example. An important class of cULF maps are counits of decalage, cf. 1.5.4:

$$d_\perp : \mathrm{Dec}_\perp X \to X \qquad \text{and} \qquad d_\top : \mathrm{Dec}_\top X \to X.$$

We shall see that many coalgebra maps in the classical theory of incidence coalgebras, notably reduction maps, are induced from decalage in this way (4.1.1, 4.2.1, 4.1.3, 4.4.1, 4.3.1, 4.5.2 below).

2.2.10. Bialgebras. For a monoidal decomposition space as in 1.6 the resulting coalgebra is also a bialgebra. Indeed, the fact that the monoid multiplication is cULF means that it induces a coalgebra homomorphism, and similarly with the unit. Note that this notion of bialgebra is not symmetric: while the comultiplication is induced from internal, simplicial data in X, the multiplication is induced by extra structure (the monoid structure). In the applications, the monoid structure will typically be given by categorical sum, and hence is associative up to canonical isomorphisms, something that seems much stricter than the comultiplication.

Proposition. 2.2.11. *If $f : X \to Y$ is a cULF monoidal functor between monoidal decomposition spaces, then $f_! : \boldsymbol{Grpd}_{/X_1} \to \boldsymbol{Grpd}_{/Y_1}$ is a bialgebra homomorphism.*

2.3. Convolution product and Möbius inversion

Let X be a decomposition space. In this section we examine the incidence algebra $\boldsymbol{Grpd}^{X_1}$ which can be obtained from the incidence coalgebra $\boldsymbol{Grpd}_{/X_1}$ by taking the linear dual (see A.1.12).

2.3.1. Convolution. Consider two linear functors

$$F, G : \boldsymbol{Grpd}_{/X_1} \longrightarrow \boldsymbol{Grpd}$$

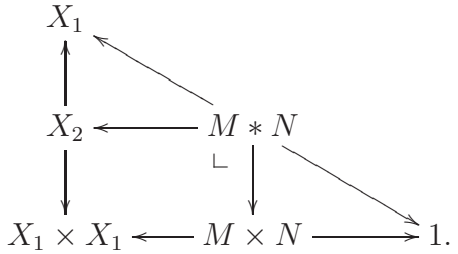
given by spans $X_1 \leftarrow M \to 1$ and $X_1 \leftarrow N \to 1$. Their tensor product $F \otimes G$ is then given by the span

$$X_1 \times X_1 \leftarrow M \times N \to 1.$$

Their *convolution* is the composite of $F \otimes G$ with the comultiplication,

$$F * G : \boldsymbol{Grpd}_{/X_1} \longrightarrow \boldsymbol{Grpd}_{/X_1} \otimes \boldsymbol{Grpd}_{/X_1} \longrightarrow \boldsymbol{Grpd},$$

given by the composite of spans



The neutral functor for the convolution product is ϵ.

2.3.2. The zeta functor. The *zeta functor*

$$\zeta : \boldsymbol{Grpd}_{/X_1} \to \boldsymbol{Grpd}$$

is the linear functor defined by the span

$$X_1 \xleftarrow{=} X_1 \to 1\,.$$

As an element of the linear dual (A.1.12), this corresponds to the terminal presheaf. We will see later that in the locally finite situation 3.1.1, upon taking the homotopy cardinality of the zeta functor one obtains the constant function 1 on $\pi_0 X_1$, that is, the classical zeta function.

It is clear from the definition of the convolution product that the kth convolution power of the zeta functor is given by

$$\zeta^k : \qquad X_1 \xleftarrow{g} X_k \to 1,$$

where $g : [1] \to [k]$ is the unique generic map in degree k.

We also introduce the following elements of the incidence algebra $\boldsymbol{Grpd}^{X_1}$: for each $a \in X_1$, let $(X_1)_{[a]}$ be the component of X_1 containing a, and let δ^a be the linear functor given by the span

$$\delta^a : \quad X_1 \leftarrow (X_1)_{[a]} \to 1,$$

We also have the representable functors

$$h^a := \mathrm{Map}(a, -) : X_1 \quad \longrightarrow \quad \boldsymbol{Grpd}$$

which viewed as linear functors $\boldsymbol{Grpd}_{/X_1} \to \boldsymbol{Grpd}$ are given by the spans

$$h^a : \quad X_1 \xleftarrow{\ulcorner a \urcorner} 1 \to 1.$$

Hence we have

$$\zeta = \sum_{a \in \pi_0 X_1} \delta^a = \int^a h^a.$$

We are interested in the invertibility of the zeta functor under the convolution product. Unfortunately, at the objective level it can practically *never* be convolution invertible, because the inverse μ should always be given by an alternating sum (cf. 2.3.14)

$$\mu = \Phi_{\mathrm{even}} - \Phi_{\mathrm{odd}}.$$

We have no minus sign available, but following the idea of Content–Lemay–Leroux [13], developed further by Lawvere–Menni [43], we establish that

$$\zeta * \Phi_{\mathrm{even}} = \epsilon + \zeta * \Phi_{\mathrm{odd}}.$$

In the category case (cf. [13] and [43]), Φ_{even} (resp. Φ_{odd}) are given by even-length (resp. odd-length) chains of non-identity arrows. (We keep the Φ-notation in honour of Content–Lemay–Leroux). In the general setting of decomposition spaces we cannot talk about arrows but we can still talk about non-degenerate simplices. In order for these to

behave well, it is necessary to impose a monomorphism condition (see 0.0.5)

2.3.3. Complete decomposition spaces. Call a decomposition space X *complete* if $s_0 : X_0 \to X_1$ is a monomorphism.

2.3.4. Discussion. It is clear that a Rezk complete Segal space is complete in the sense of 2.3.3. It makes sense also to state the Rezk completeness condition for decomposition spaces. We prefer the cheaper condition 2.3.3 for two reasons: first of all it is sufficient for its purpose, namely to ensure a meaningful notion of non-degenerate simplices. Second, it covers some important examples which are not Rezk complete. In particular, the classical nerve of a group is a complete decomposition space in the sense of 2.3.3, but is clearly not Rezk complete. The incidence algebra of the classical nerve of a group is the group algebra, certainly an example worth covering.

We will come eventually to the Möbius condition (3.2.1), which makes sense for complete decomposition spaces. We shall see that if a Möbius decomposition space is a Segal space then it is also Rezk complete (3.2.2).

Lemma. 2.3.5. *In a complete decomposition space, all degeneracy maps are monomorphisms.*

Proof. This follows from Lemma 1.3.7. □

An important motivating feature of the notion of complete decomposition space is that the notion of degeneracy is well behaved: we shall see that all issues about degeneracy can be settled in terms of the canonical projection map $X_n \to (X_1)^n$ sending a simplex to its principal edges. This feature is essential to construct (the odd and even parts of) the Möbius functor.

Throughout this section, X is a complete decomposition space.

2.3.6. Degenerate simplices. Consider the alphabet with three letters $\{0, 1, a\}$. Here 0 is to indicate degenerate edges $s_0(x) \in X_1$, the letter a denotes the edges specified to be non-degenerate, and 1 denotes the edges which are not specified to be degenerate or non-degenerate. More precisely we denote by $X_0 \subset X_1$ the full subgroupoid of degenerate edges, and denote by X_a the complement of $s_0 : X_0 \to X_1$, the full subgroupoid of *non-degenerate edges*, so that

$$X_1 = X_0 + X_a.$$

Now let w be a word of length n in the alphabet $\{0, 1, a\}$. Put

$$X^w := \prod_{i \in w} X_i \subset (X_1)^n.$$

This inclusion is full since $X_a \subset X$ is full by completeness. Denote by X_w the ∞-groupoid of n-simplices whose principal edges have the

types indicated in the word w, or more explicitly as full subgroupoid of X_n given by the pullback diagram,

$$\begin{array}{ccc} X_w & \longrightarrow & X_n \\ \downarrow & \lrcorner & \downarrow \\ X^w & \longrightarrow & (X_1)^n. \end{array} \tag{16}$$

Lemma. 2.3.7. *Let v, v' be words in the alphabet $\{0, 1, a\}$. Then*

$$X_{v1v'} = X_{v0v'} + X_{vav'}.$$

Proof. Consider the diagram

$$\begin{array}{ccccc} X_{v0v'} & \longrightarrow & X_{v1v'} & \longleftarrow & X_{vav'} \\ \downarrow & \lrcorner & \downarrow & \llcorner & \downarrow \\ X^{v0v'} & \longrightarrow & X^{v1v'} & \longleftarrow & X^{vav'} \end{array}$$

The two squares are pullbacks, by Lemma 0.0.9, since horizontal composition of either with the pullback square (16) for $w = v1v'$ gives again the pullback square (16), for $w = v0v'$ or $w = vav'$.

Since the bottom row is a sum diagram, it follows that the top row is also (since ∞-groupoids form a locally cartesian closed, and in particular extensive, ∞-category). □

2.3.8. Nondegenerate simplices. We put

$$\vec{X}_n = X_{a\cdots a} \subset X_n,$$

by construction a full subgroupoid of X_n.

Proposition. 2.3.9. *We have $\vec{X}_n = X_n \setminus \sum_{i=0}^{n} \mathrm{Im}(s_i)$.*

As usual in this paper, the notation Im refers to the homotopy notion, the essential image of the degeneracy functor.

Proof. From (16) it is clear that the image of $s_0 : X_{n-1} \to X_n$ is precisely $X_{01\cdots 1}$, and similarly with the other degeneracy maps. □

Lemma. 2.3.10. *If a complete decomposition space X is a Segal space, then $\vec{X}_n \subset X_n \simeq X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is the ∞-groupoid of strings of n composable non-degenerate arrows.*

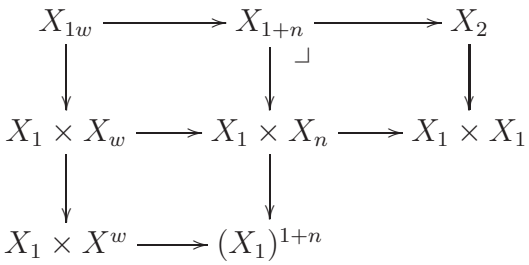
This follows immediately from (16). Note that if furthermore X is Rezk complete, we can say non-invertible instead of non-degenerate.

Lemma. 2.3.11. *For X a complete decomposition space, the square*

$$\begin{array}{ccc} X_{1w} & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ X_1 \times X_w & \longrightarrow & X_1 \times X_1 \end{array}$$

is a pullback.

Proof. Let n be the length of the word w. The square is the top rectangle in the diagram



The left-hand outer rectangle is a pullback by definition of X_{1w}. The bottom square is obviously a pullback, hence the top-left square is a pullback. But also the right-hand square is a pullback because X is a decomposition space. □

We define Φ_n to be the linear functor given by the span

$$X_1 \xleftarrow{m} \vec{X}_n \longrightarrow 1$$

If $n = 0$ then $\vec{X}_0 = X_0$ by convention and Φ_0 is given by the span

$$X_1 \xleftarrow{u} X_0 \longrightarrow 1.$$

That is, Φ_0 is the linear functor ϵ. Note that $\Phi_1 = \zeta - \epsilon$. The minus sign makes sense here, since X_0 (representing ϵ) is really a full subgroupoid of X_1 (representing ζ).

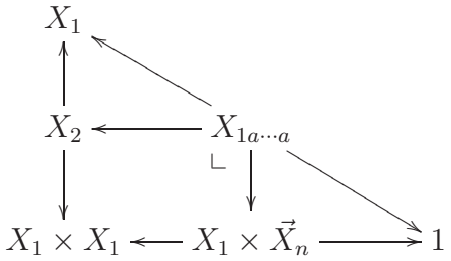
Lemma. 2.3.12. *We have $\Phi_n = (\Phi_1)^n = (\zeta - \epsilon)^n$, the nth convolution product of Φ_1 with itself.*

Proof. This follows from the definitions and (16). □

Proposition. 2.3.13. *The linear functors Φ_n satisfy*

$$\zeta * \Phi_n \quad = \quad \Phi_n + \Phi_{n+1} \quad = \quad \Phi_n * \zeta.$$

Proof. We can compute the convolution $\zeta * \Phi_n$ by Lemma 2.3.11 as



But Lemma 2.3.7 tells us that $X_{1a\cdots a} = X_{0a\cdots a} + X_{aa\cdots a} = \vec{X}_n + \vec{X}_{n+1}$, where the identification in the first summand is via s_0. This is an equivalence of ∞-groupoids over X_1 so the resulting span is $\Phi_r + \Phi_{r+1}$ as desired. The second identity claimed follows similarly from evident variations of the two lemmas. □

Put

$$\Phi_{\text{even}} := \sum_{n \text{ even}} \Phi_n, \qquad \Phi_{\text{odd}} := \sum_{n \text{ odd}} \Phi_n.$$

Theorem. 2.3.14. *For a complete decomposition space, the following Möbius inversion principle holds:*

$$\begin{aligned} \zeta * \Phi_{\text{even}} &= \epsilon + \zeta * \Phi_{\text{odd}}, \\ = \Phi_{\text{even}} * \zeta &= \epsilon + \Phi_{\text{odd}} * \zeta. \end{aligned}$$

Proof. This follows immediately from the proposition: all four linear functors are in fact equivalent to $\sum_{r\geq 0} \Phi_r$. □

2.3.15. Conservative ULF functors, compatibilities with degeneracy, and algebra homomorphisms. We finish this subsection observing that all the notions treated are well-behaved under cULF functors. Note first of all that by duality, cULF functors induce homomorphisms of incidence algebras.

Throughout the remainder of this section, X and Y are complete decomposition spaces. In fact if X is complete and $Y \to X$ is conservative, it follows that Y is also complete.

Lemma. 2.3.16. *If $f : Y \to X$ is conservative, then the following square is a pullback:*

$$\begin{array}{ccc} Y_1 & \longleftarrow & \vec{Y}_1 \\ \downarrow & \llcorner & \downarrow \\ X_1 & \longleftarrow & \vec{X}_1. \end{array}$$

Proof. This square is the complement of the pullback saying what conservative means. But it is general in extensive categories that in the situation

$$\begin{array}{ccccc} A' & \longrightarrow & A'+B' & \longleftarrow & B' \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & A+B & \longleftarrow & B, \end{array}$$

one square is a pullback if and only if the other is. □

Corollary. 2.3.17. *If $f : Y \to X$ is conservative, then for every word w, the following square is a pullback:*

$$\begin{array}{ccc} Y_n & \longleftarrow & Y_w \\ \downarrow & \llcorner & \downarrow \\ X_n & \longleftarrow & X_w. \end{array}$$

Proof. From 2.3.16 it follows that

$$\begin{array}{ccc} (Y_1)^n & \longleftarrow & Y^w \\ \downarrow & \llcorner & \downarrow \\ (X_1)^n & \longleftarrow & X^w \end{array}$$

is a pullback. Now the conclusion follows by a pullback argument with two squares of the form (16), using Lemma 0.0.9. □

Proposition. 2.3.18. *If $f : Y \to X$ is cULF, then the following square is a pullback:*

$$\begin{array}{ccc} Y_1 & \longleftarrow & \vec{Y}_n \\ \downarrow & \llcorner & \downarrow \\ X_1 & \longleftarrow & \vec{X}_n. \end{array}$$

Proof. Just compose the square of Corollary 2.3.17 (with $w = a \cdots a$) with the square

$$\begin{array}{ccc} Y_1 & \longleftarrow & Y_n \\ \downarrow & \llcorner & \downarrow \\ X_1 & \longleftarrow & X_n, \end{array}$$

which is a pullback since f is ULF. □

Corollary. 2.3.19. *If $f : Y \to X$ is cULF, then $f^*\Phi_n = \Phi_n$ for all $n \geq 0$.*

3. Locally finite decomposition spaces

We introduce two finiteness conditions: *locally finite* (3.1.1) is what is needed just for the coalgebra structure to have a cardinality. Secondly *Möbius* (3.2.1) is what is needed for also the Möbius inversion principle to descend to algebras over $\mathbb{Q}$.

3.1. Incidence (co)algebras and section coefficients

3.1.1. Locally finite decomposition spaces. A decomposition space $X : \Delta^{\mathrm{op}} \to \boldsymbol{Grpd}$ is called *locally finite* if X_1 is locally finite and both $s_0 : X_0 \to X_1$ and $d_1 : X_2 \to X_1$ are finite maps. This notion extends the notion of locally finite for posets.

From Lemma 1.3.7 and Lemma A.2.12 we immediately get:

Corollary. 3.1.2. *A decomposition space X is locally finite if and only if X_n is locally finite for every n, and $g : X_m \to X_n$ is finite for every generic map $g : [n] \to [m]$ in Δ.*

Corollary. 3.1.3. *If a decomposition space X is locally finite then so are $\mathrm{Dec}_\perp(X)$ and $\mathrm{Dec}_\top(X)$.*

3.1.4. Numerical incidence algebra. It follows from A.3.4 that for any locally finite decomposition space X, the comultiplication maps $n \geq 0$

$$\Delta_n : \boldsymbol{Grpd}_{/X_1} \quad \longrightarrow \quad \boldsymbol{Grpd}_{/X_1 \times X_1 \times \cdots \times X_1}$$

given by the spans

$$X_1 \xleftarrow{m} X_n \xrightarrow{p} X_1 \times X_1 \times \cdots \times X_1$$

restrict to linear functors

$$\Delta_n : \boldsymbol{grpd}_{/X_1} \quad \longrightarrow \quad \boldsymbol{grpd}_{/X_1 \times X_1 \times \cdots \times X_1}$$

Hence it makes sense to take cardinality of the linear functors

$$\boldsymbol{grpd} \xleftarrow{\epsilon} \boldsymbol{grpd}_{/X_1} \xrightarrow{\Delta} \boldsymbol{grpd}_{/X_1 \times X_1}$$

to obtain a coalgebra structure,

$$\mathbb{Q} \xleftarrow{|\epsilon|} \mathbb{Q}_{\pi_0 X_1} \xrightarrow{|\Delta|} \mathbb{Q}_{\pi_0 X_1} \otimes \mathbb{Q}_{\pi_0 X_1}$$

termed the *numerical incidence coalgebra* of X.

3.1.5. Morphisms. It is worth noticing that for *any* conservative ULF functor $F : Y \to X$, the induced coalgebra homomorphism $F_! : \boldsymbol{Grpd}_{/Y_1} \to \boldsymbol{Grpd}_{/X_1}$ restricts to a functor $\boldsymbol{grpd}_{/Y_1} \to \boldsymbol{grpd}_{/X_1}$. In other words, there are no further finiteness conditions to impose on morphisms.

3.1.6. Incidence bialgebras. If the locally finite decomposition space is monoidal, then the incidence coalgebra is in fact a *bialgebra*. Note that since the algebra structure in our setting is given simply by a lowershriek map, by the previous remark there are no finiteness conditions needed in order for it to descend to the numerical level.

We also have the notion of incidence *algebra*, defined as the (profinite-dimensional) linear dual of the incidence coalgebra. In the presence of a monoidal structure on the decomposition space, this causes a potential ambiguity regarding algebra structures. We make the convention that *incidence bialgebra* always refers to the incidence coalgebra with its extra multiplication.

3.1.7. Numerical convolution product. By duality, if X is locally finite, the convolution product descends to the profinite-dimensional vector space $\mathbb{Q}^{\pi_0 X_1}$ obtained by taking cardinality of $\boldsymbol{grpd}^{X_1}$. It follows from the general theory of homotopy linear algebra of Section A.1 (specifically A.7.1) that the cardinality of the convolution product is the linear dual of the cardinality of the multiplication. Since it is the same span that defines the comultiplication and the convolution product, it is also the exact same matrix that defines the cardinalities of these two maps. It follows that the structure constants for the convolution product (with respect to the pro-basis $\{\delta^x\}$) are the same as the

structure constants for the comultiplication (with respect to the basis δ_x). These are classically called the section coefficients, and we proceed to derive formulae for them in simple cases.

Throughout this subsection, let X be a locally finite decomposition space. The comultiplication at the objective level

$$\begin{array}{rcl} \boldsymbol{grpd}_{/X_1} & \longrightarrow & \boldsymbol{grpd}_{/X_1\times X_1} \\ \ulcorner f \urcorner & \longmapsto & \left[R_f : (X_2)_f \to X_2 \to X_1 \times X_1\right] \end{array}$$

yields a comultiplication of vector spaces by taking cardinality (remembering that $|\ulcorner f \urcorner| = \delta_f$):

$$\begin{array}{rcl} \mathbb{Q}_{\pi_0 X_1} & \longrightarrow & \mathbb{Q}_{\pi_0 X_1} \otimes \mathbb{Q}_{\pi_0 X_1} \\ \delta_f & \longmapsto & |R_f| \\ & = & \displaystyle\int^{(a,b)\in X_1\times X_1} |(X_2)_{f,a,b}|\, \delta_a \otimes \delta_b \\ & = & \displaystyle\sum_{a,b} \left|(X_1)_{[a]}\right| \left|(X_1)_{[b]}\right| |(X_2)_{f,a,b}|\, \delta_a \otimes \delta_b. \end{array}$$

where $(X_2)_{f,a,b}$ is the fibre over the three face maps. The integral sign is a sum weighted by homotopy groups. These weights together with the cardinality of the triple fibre are called the *section coefficients*, denoted

$$c^f_{a,b} := |(X_2)_{f,a,b}| \cdot \left|(X_1)_{[a]}\right| \left|(X_1)_{[b]}\right|.$$

In the case where X is a Segal space (and in particular, when X_0 is a 1-groupoid), we can be very explicit about the section coefficients. For a Segal space we have $X_2 \simeq X_1 \times_{X_0} X_1$, which helps to compute the fibre of $X_2 \to X_1 \times X_1$:

Lemma. 3.1.8. *The pullback*

$$\begin{array}{ccc} S & \longrightarrow & X_1 \times_{X_0} X_1 \\ \downarrow & \lrcorner & \downarrow \\ 1 & \xrightarrow[\ulcorner a,b \urcorner]{} & X_1 \times X_1 \end{array}$$

is given by

$$S \;=\; \begin{cases} \Omega(X_0, y) & \text{if } d_0 a \simeq y \simeq d_1 b \\ 0 & \text{else.} \end{cases}$$

Proof. We can compute the pullback as

$$\begin{array}{ccccc} S & \longrightarrow & X_1 \times_{X_0} X_1 & \longrightarrow & X_0 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{diag}} \\ 1 & \xrightarrow[\ulcorner a,b \urcorner]{} & X_1 \times X_1 & \xrightarrow[d_0 \times d_1]{} & X_0 \times X_0, \end{array}$$

and the result follows since in general

$$\begin{array}{ccc} A \times_C B & \longrightarrow & C \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{diag}} \\ A \times B & \longrightarrow & C \times C. \end{array}$$

□

Corollary. 3.1.9. *Suppose X is a Segal space, and that X_0 is a 1-groupoid. Given $a, b, f \in X_1$ such that $d_0 a \cong y \cong d_1 b$ and $ab = f$, then we have*

$$(X_2)_{f,a,b} = \Omega(X_0, y) \times \Omega(X_1, f).$$

Proof. In this case, since X_0 is a 1-groupoid, the fibres of the diagonal map $X_0 \to X_0 \times X_0$ are 0-groupoids. Thus the fibre of the previous lemma is the discrete space $\Omega(X_0, y)$. When now computing the fibre over f, we are taking that many copies of the loop space of f. □

Corollary. 3.1.10. *With notation as above, the section coefficients for a locally finite Segal 1-groupoid are*

$$c^{ab}_{a,b} = \frac{|\mathrm{Aut}(y)|\,|\mathrm{Aut}(ab)|}{|\mathrm{Aut}(a)|\,|\mathrm{Aut}(b)|}.$$

Coassociativity of the incidence coalgebra says that the section coefficients $\{c^{ab}_{a,b}\}$ form a *2-cocycle*,

$$c^{ab}_{a,b} c^{abc}_{ab,c} = c^{bc}_{b,c} c^{abc}_{a,bc}.$$

In fact this cocycle is cohomologically trivial, given by the coboundary of a 1-cochain,

$$c^{ab}_{a,b} = \partial(\phi)(a, b) = \phi(a)\phi(ab)^{-1}\phi(b),$$

In fact, if one fixes s, t such that $s + t = 1$, the 1-cochain may be taken to be

$$\phi(x \xrightarrow{a} y) = \frac{|\,\mathrm{Aut}(x)|^s |\,\mathrm{Aut}(y)|^t}{|\,\mathrm{Aut}(a)|}$$

.

3.1.11. 'Zeroth section coefficients': the counit. Let us also say a word about the zeroth section coefficients, i.e. the computation of the counit: the main case is when X is complete (in the sense that s_0 is a monomorphism). In this case, clearly we have

$$\epsilon(f) = \begin{cases} 1 & \text{if } f \text{ degenerate} \\ 0 & \text{else.} \end{cases}$$

If X is Rezk complete, the first condition is equivalent to being invertible.

The other easy case is when $X_0 = *$. In this case

$$\epsilon(f) = \begin{cases} \Omega(X_1, f) & \text{if } f \text{ degenerate} \\ 0 & \text{else.} \end{cases}$$

3.1.12. Example. The strict nerve of a 1-category $\mathscr{C}$ is a decomposition space which is discrete in each degree. The resulting coalgebra at the numerical level (assuming the due finiteness conditions) is the coalgebra of Content–Lemay–Leroux [13], and if the category is just a poset, that of Rota et al. [29].

For the fat nerve X of $\mathscr{C}$, we find

$$h^a * h^b = \begin{cases} \Omega(X_0, y)\, h^{ab} & \text{if } a \text{ and } b \text{ composable at } y \\ 0 & \text{else,} \end{cases}$$

as follows from 3.1.8. Note that the cardinality of the representable h^a is generally different from the canonical basis element δ^a.

3.1.13. Finite support. It is also interesting to consider the subalgebra of the incidence algebra consisting of functions with finite support, i.e. the full subcategory $\boldsymbol{grpd}^{X_1}_{fin.sup} \subset \boldsymbol{grpd}^{X_1}$, and numerically $\mathbb{Q}^{\pi_0 X_1}_{fin.sup} \subset \mathbb{Q}^{\pi_0 X_1}$. Of course we have canonical identifications $\boldsymbol{grpd}^{X_1}_{fin.sup} \simeq \boldsymbol{grpd}_{/X_1}$, as well as $\mathbb{Q}^{\pi_0 X_1}_{fin.sup} \simeq \mathbb{Q}_{\pi_0 X_1}$, but it is important to keep track of which side of duality we are on.

That the decomposition space is locally finite is not the appropriate condition for these subalgebras to exist. Instead the requirement is that X_1 be locally finite and the functor

$$X_2 \to X_1 \times X_1$$

be finite. (This is always the case for a locally finite Segal 1-groupoid, by Lemma 3.1.8.) Similarly, one can ask for the convolution unit to have finite support, which is to require $X_0 \to 1$ to be a finite map.

Dually, the same conditions ensure that comultiplication and counit extend from $\boldsymbol{grpd}_{/X_1}$ to $\boldsymbol{Grpd}^{rel.fin}_{/X_1}$, which numerically is some sort of vector space of summable infinite linear combinations.

3.1.14. Examples. If X is the strict nerve of a 1-category $\mathscr{C}$, then the finite-support convolution algebra is precisely the *category algebra* of $\mathscr{C}$. (For a finite category, of course the two notions coincide.)

Note that the convolution unit is

$$\epsilon = \sum_x \delta^{\mathrm{id}_x} = \begin{cases} 1 & \text{for id arrows} \\ 0 & \text{else,} \end{cases}$$

the sum of all indicator functions of identity arrows, so it will be finite if and only if the category has only finitely many objects.

In the case of the fat nerve of a 1-category, the finiteness condition for comultiplication is implied by the condition that every object has

a finite automorphism group (a condition implied by local finiteness). On the other hand, the convolution unit has finite support precisely when there is only a finite number of isoclasses of objects, already a more drastic condition. Note the 'category algebra' interpretation: compared to the usual category algebra there is a symmetry factor (cf. 3.1.12):

$$h^a * h^b = \begin{cases} \Omega(X_0, y)\, h^{ab} & \text{if } a \text{ and } b \text{ composable at } y \\ 0 & \text{else.} \end{cases}$$

Finally, the finite-support incidence algebras are important in the case of the Waldhausen S-construction: they are the Hall algebras, cf. 4.6 below. The finiteness conditions are then homological, namely finite Ext^0 and Ext^1.

3.2. Möbius inversion at the algebraic level

3.2.1. Möbius condition. A complete decomposition space X is called *Möbius* if the restricted composition map

$$\sum_r d_1{}^{r-1} : \sum_r \vec{X}_r \to X_1$$

is finite. In other words, the spans defining Φ_{even} and Φ_{odd} are of finite type, and hence descend to the finite groupoid-slices $\boldsymbol{grpd}_{/X_1}$.

If X is a Segal space, the Möbius condition says that for each arrow $a \in X_1$, the factorisations of a into non-degenerate $a_i \in \vec{X}_1$ have bounded length. In fact we have:

Proposition. 3.2.2. *If a Möbius decomposition space X is a Segal space, then it is Rezk complete.*

This extends the classical fact that a Möbius category in the sense of Leroux does not have non-identity invertible arrows [43, Lemma 2.4].

Proof. If X is not Rezk complete, then there exists a nondegenerate invertible arrow $a \in X_1$. Since for Segal spaces we have $\vec{X}_r \simeq \vec{X}_1 \times_{X_0} \cdots \times_{X_0} \vec{X}_1$ (by 2.3.10), we can use the arrow a and its inverse going back an forth any number of times to create nondegenerate simplices of any length. Hence $\sum \vec{X}_r \to X_1$ cannot be finite. □

Lemma. 3.2.3. *If a Möbius decomposition space X is a Segal space, then all its generic maps are finite.*

Proof. It is enough to show that $d_1 : X_2 \to X_1$ is finite. In the word notation of 2.3.6 we have altogether

$$X_2 = \vec{X}_2 + X_{0a} + X_{a0} + X_{00}.$$

Since X is Segal, we have $X_{0a} \simeq X_{a0} \simeq X_1$, and $X_{00} \simeq X_0$. The restriction to $\vec{X}_2$ is finite by the Möbius condition. The restriction to the next two pieces are clearly finite. The restriction to X_{00} coincides

with s_0, which is finite since X is complete as a decomposition space. Hence altogether d_1 is finite. □

Proposition. 3.2.4. *If $f : Y \to X$ is cULF and X is a Möbius decomposition space, then also Y is a Möbius decomposition space.*

Proof. We know that Y is a decomposition space by Lemma 1.4.6, and the cULF condition in fact ensures it is complete. It will furthermore be Möbius by Proposition 2.3.18. □

3.2.5. Möbius inversion at the algebraic level. The diagram $X_1 \xleftarrow{=} X_1 \longrightarrow 1$ defines the zeta functor (cf. 2.3.2), which as a presheaf is $\zeta = \int^t h^t$, the homotopy sum of the representables. Its cardinality is the usual zeta function in the incidence algebra $\mathbb{Q}^{\pi_0 X_1}$. The diagrams $X_1 \longleftarrow \vec{X}_r \longrightarrow 1$ define the Phi functors

$$\Phi_r : \boldsymbol{grpd}_{/X_1} \longrightarrow \boldsymbol{grpd},$$

with $\Phi_0 = \epsilon$. The cardinality of these functors are functions $|\zeta| : \pi_0(X_1) \to \mathbb{Q}$ and $|\Phi_r| : \pi_0(X_1) \to \mathbb{Q}$, elements in the incidence algebra $\mathbb{Q}^{\pi_0 X_1}$.

For a Möbius decomposition space X, we can take cardinality of the abstract Möbius inversion formula of 2.3.14 to get:

Theorem. 3.2.6. *If X is Möbius then the cardinality of the zeta functor $|\zeta| : \mathbb{Q}_{\pi_0 X_1} \to \mathbb{Q}$ is convolution invertible with inverse $|\mu| := |\Phi_{\text{even}}| - |\Phi_{\text{odd}}|$:*

$$|\zeta| * |\mu| = |\epsilon| = |\mu| * |\zeta| .$$

4. Examples

It is characteristic for the classical theory of incidence (co)algebras of posets that most often it is necessary to impose an equivalence relation on the set of intervals in order to arrive at the interesting 'reduced' (co)algebras. This equivalence relation may be simply isomorphism of posets, or equality of length of maximal chains as in binomial posets [16], or it may be more subtle order-compatible relations [17], [65]. Content, Lemay and Leroux [13] remarked that in some important cases the relationship between the original incidence coalgebra and the reduced one amounts to a conservative ULF functor, although they did not make this notion explicit. From our global simplicial viewpoint, we observe that very often these cULF functors arise from decalage, but often of a decomposition space which not a poset and perhaps not even a Segal space.

4.0.7. Decomposition spaces for the classical series. Classically, the most important incidence algebras are the power series representations. From the perspective of the objective method, these representations appear as cardinalities of various monoidal structures on species,

realised as incidence algbras with ∞-groupoid coefficients. We list six examples illustrating some of the various kinds of generating functions listed by Stanley [66] (see also Dür [17]).

(1) Ordinary generating functions, the zeta function being $\zeta(z) = \sum_{k\geq 0} z^k$. This comes from ordered sets and ordinal sum, and the incidence algebra is that of ordered species with the ordinary product.
(2) Exponential generating functions, the zeta function being $\zeta(z) = \sum_{k\geq 0} \frac{z^k}{k!}$. Objectively, there are two versions of this: one coming from the standard Cauchy product of species, and one coming from the shuffle product of $\mathbb{L}$-species (in the sense of [7]).
(3) Ordinary Dirichlet series, the zeta function being $\zeta(z) = \sum_{k>0} k^{-s}$. This comes from ordered sets with the cartesian product.
(4) 'Exponential' Dirichlet series, the zeta function being $\zeta(z) = \sum_{k>0} \frac{k^{-s}}{k!}$. This comes from the Dirichlet product of arithmetic species [3], also called the arithmetic product [53].
(5) q-exponential generating series, with zeta function $\zeta(z) = \sum_{k\geq 0} \frac{z^k}{[k]!}$. This comes from the Waldhausen S-construction on the category of finite vector spaces. The incidence algebra is that of q-species with a version of the external product of Joyal–Street [36].
(6) Some variation with zeta function $\zeta(z) = \sum_{k\geq 0} \frac{z^k}{\#\operatorname{Aut}(\mathbb{F}_q^k)}$, which arises from q-species with the 'Cauchy' product studied by Morrison [57].

Of these examples, only (1) and (3) have trivial section coefficients and come from a Möbius category in the sense of Leroux. We proceed to the details.

4.1. **Additive examples**

We start with several easy examples that serve to reiterate the importance of having incidence algebras of posets, monoids and monoidal ∞-groupoids on the same footing, with conservative ULF functors connecting them.

4.1.1. Linear orders and the additive monoid. Let $\mathbf{L}$ denote the nerve of the poset $(\mathbb{N}, \leq)$, and let $\mathbf{N}$ be the nerve of the additive monoid $(\mathbb{N}, +)$. Imposing the equivalence relation 'isomorphism of intervals' on the incidence coalgebra of $\mathbf{L}$ gives that of $\mathbf{N}$, and Content–Lemay–Leroux observed that this reduction is induced by a conservative ULF functor $r : \mathbf{L} \to \mathbf{N}$ sending $a \leq b$ to $b - a$. In fact we have:

Lemma. 4.1.2. *There is an isomorphism of simplicial sets*

$$\mathrm{Dec}_{\perp}(\mathbf{N}) \xrightarrow{\simeq} \mathbf{L}$$

given in degree k by

$$(x_0, \dots, x_k) \longmapsto [x_0 \leq x_0 + x_1 \leq \cdots \leq x_0 + \cdots + x_k]$$

and the conservative ULF functor r is isomorphic to the structure map

$$d_\perp : \mathrm{Dec}_\perp(\mathbf{N}) \to \mathbf{N}, \qquad (x_0, \dots, x_k) \mapsto (x_1, \dots, x_k).$$

The comultiplication on $\boldsymbol{Grpd}_{/\mathbf{N}_1}$ is given by

$$\Delta(\ulcorner n \urcorner) = \sum_{a+b=n} \ulcorner a \urcorner \otimes \ulcorner b \urcorner$$

and, taking cardinality, the classical incidence coalgebra is the vector space spanned by symbols δ_n with comultiplication $\Delta(\delta_n) = \sum_{a+b=n} \delta_a \otimes \delta_b$. The incidence algebra is the profinite-dimensional vector space spanned by the symbols δ^n with convolution product $\delta^a * \delta^b = \delta^{a+b}$, and is isomorphic to the ring of power series in one variable,

$$\begin{array}{rcl} \mathrm{Inc}\,\mathbf{N} & \xrightarrow{\simeq} & \mathbb{Q}[[z]] \\ \delta^n & \longmapsto & z^n \\ (\mathbb{N} \xrightarrow{f} \mathbb{Q}) & \longmapsto & \sum f(n)\, z^n. \end{array}$$

4.1.3. Injections and the monoidal ∞-groupoid of sets under sum. Let $\mathbf{I}$ be the nerve of the category of finite sets and injections, and let $\mathbf{B}$ be the nerve of the monoidal ∞-groupoid $(\mathbb{B}, +, 0)$ of finite sets and bijections, or of the corresponding 1-object bicategory (see Proposition 1.2.10). Dür [17] noted that imposing the equivalence relation 'having isomorphic complements' on the incidence coalgebra of $\mathbf{I}$ gives the binomial coalgebra. Again, we can see this reduction map as induced by a conservative ULF functor from a decalage:

Lemma. 4.1.4. *There is an equivalence of simplicial ∞-groupoids*

$$\mathrm{Dec}_\perp(\mathbf{B}) \xrightarrow{\simeq} \mathbf{I}$$

given in degree k by

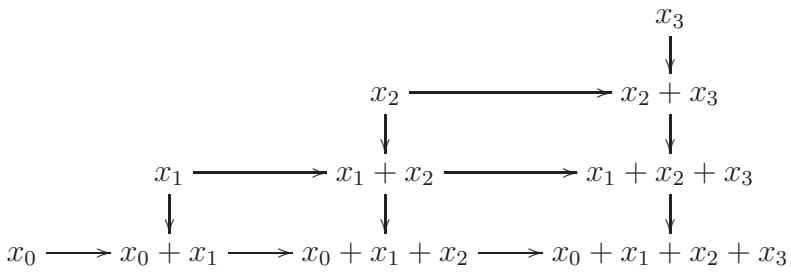
$$(x_0, \dots, x_k) \longmapsto [x_0 \subseteq x_0 + x_1 \subseteq \cdots \subseteq x_0 + \cdots + x_k]$$

and a conservative ULF functor $r : \mathbf{I} \to \mathbf{B}$ is given by

$$d_\perp : \mathrm{Dec}_\perp(\mathbf{B}) \to \mathbf{B}, \qquad (x_0, \dots, x_k) \mapsto (x_1, \dots, x_k).$$

The isomorphism may also be represented diagrammatically using diagrams reminiscent of those in Waldhausen's S-construction (cf. 4.6 below). As an example, both groupoids $\mathbf{I}_3$ and $\mathrm{Dec}_\perp(\mathbf{B})_3 = \mathbf{B}_4$ are

equivalent to the groupoid of diagrams



The face maps $d_i : \mathbf{I}_3 \to \mathbf{I}_2$ and $d_{i+1} : \mathbf{B}_4 \to \mathbf{B}_3$ both act by deleting the column beginning x_i and the row beginning x_{i+1}. In particular $d_\perp : \mathbf{I} \to \mathbf{B}$ deletes the bottom row, sending a string of injections to the sequence of successive complements (x_1, x_2, x_3). We will revisit this theme in the treatment of the Waldhausen S-construction.

From Lemma 1.6.2 and Proposition 2.2.11 we have:

Lemma. 4.1.5. *Both* $\mathbf{I}$ *and* $\mathbf{B}$ *are monoidal decomposition spaces under disjoint union, and* $\mathbf{I} \simeq \mathrm{Dec}_\perp(\mathbf{B}) \to \mathbf{B}$ *is a monoidal functor inducing a quotient of bialgebras* $\boldsymbol{Grpd}_{/\mathbf{I}_1} \to \boldsymbol{Grpd}_{/\mathbf{B}_1}$.

Formula 3.1.10 gives the comultiplication on $\boldsymbol{Grpd}_{/\mathbf{B}_1}$ as

$$\Delta(\ulcorner S \urcorner) \;=\; \sum_{A,B} \frac{\mathrm{Bij}(A+B,S)}{\mathrm{Aut}(A)\times\mathrm{Aut}(B)} \cdot \ulcorner A \urcorner \otimes \ulcorner B \urcorner \;=\; \sum_{\substack{A,B\subset S\\ A\cup B=S,\ A\cap B=\emptyset}} \ulcorner A \urcorner \otimes \ulcorner B \urcorner.$$

It follows that the convolution product on $\boldsymbol{Grpd}^{\mathbb{B}}$ is just the Cauchy product on groupoid-valued species

$$(F * G)[S] = \sum_{A+B=S} F[A] \times G[B].$$

For the representables, the formula says simply $h^A * h^B = h^{A+B}$.

The decomposition space $\mathbf{B}$ is locally finite, and taking cardinality gives the classical binomial coalgebra, spanned by symbols δ_n with

$$\Delta(\delta_n) = \sum_{a+b=n} \frac{n!}{a!\,b!}\, \delta_a \otimes \delta_b.$$

As a bialgebra we have $(\delta_1)^n = \delta_n$ and one recovers the comultiplication from $\Delta(\delta_n) = \big(\delta_0 \otimes \delta_1 + \delta_1 \otimes \delta_0\big)^n$.

Dually, the incidence algebra $\mathbb{Q}^{\pi_0\mathbf{B}}$ is the profinite-dimensional vector space spanned by symbols δ^n with convolution product

$$\delta^a * \delta^b = \frac{n!}{a!\,b!}\, \delta^{a+b},$$

This is isomorphic to the algebra $\mathbb{Q}[[z]]$, where δ^n corresponds to $z^n/n!$ and the cardinality of a species F corresponds to its exponential generating series.

4.1.6. Finite ordered sets, and the shuffle product of $\mathbb{L}$-species. Let **OI** denote (the fat nerve of) the category of finite ordered sets and monotone injections. This is the decalage of the decomposition space **Z** with $\mathbf{Z}_n = \mathbf{OI}_{/\underline{n}}$, the groupoid of arbitrary maps from a finite ordered set S to $\underline{n}$, or equivalently of n-shuffles of S. The incidence coalgebra of **Z** is the *shuffle coalgebra*. The section coefficients are the binomial coefficients, but on the objective level the convolution algebra is the shuffle product of $\mathbb{L}$-species (cf. [7]). This example will be subsumed in our theory of restriction species, developed in Section 5.

There is a map $\mathbf{Z} \to \mathbf{B}$ that takes an n-shuffle to the underlying n-tuple of subsets, and the decalage of this functor is the cULF functor $\mathbf{OI} \to \mathbf{I}$ given by forgetting the order, see Example 1.4.4.

Lemma. 4.1.7. *There is a commutative diagram of decomposition spaces and cULF functors,*

$$\begin{array}{ccccc} \mathbf{OI} & \xrightarrow{\simeq} & \mathrm{Dec}_\perp(\mathbf{Z}) & \xrightarrow{d_\perp} & \mathbf{Z} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{I} & \xrightarrow{\simeq} & \mathrm{Dec}_\perp(\mathbf{B}) & \xrightarrow{d_\perp} & \mathbf{B} \end{array}$$

Let A be a fixed set, an alphabet. The slice category $\underline{\Delta}^{\mathrm{inj}}_{/A}$ is the category of finite words (sequences) in A and subword inclusions (subsequences), cf. Lothaire [47] (see also Dür [17]). Again it is the decalage of the *A-coloured shuffle decomposition space* $\mathbf{Z}_A$ of A-words and complementary subword inclusions. More precisely, this space has in degree k the groupoid of A-words equipped with a non-necessarily-order-preserving map to $\underline{k}$. Precisely, the objects are spans

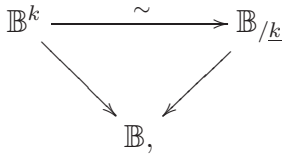
$$\underline{k} \leftarrow \underline{n} \to A.$$

The counit takes a subword inclusion to its complement word. This gives the Lothaire shuffle algebra of words. Again, it all amounts to observing that A-words admit a forgetful cULF functor to 1-words, which is just the decomposition space $\boldsymbol{Z}$ from before, and that this in turn admits a cULF functor to **B**.

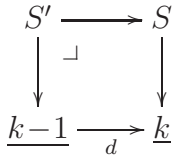
Note the difference between $\mathbf{Z}_A$ and the free monoid on A: the latter is like allowing only the trivial shuffles, where the subword inclusions are only concatenation inclusions. In terms of the structure maps $\underline{n} \to \underline{k}$, the free-monoid nerve allows only monotone maps, whereas the shuffle decomposition space allows arbitrary set maps.

4.1.8. Alternative, strict, version. The following strict version of **B** will come in handy in the treatment of restriction species in Section 5. First, an application of the Grothendieck construction gives an

equivalence of groupoids over $\mathbb{B}$,



that takes a k-tuple of finite sets to their disjoint union $\sum_{i\in\underline{k}} S_i$ with the obvious projection map to $\underline{k}$. Conversely, a map $S \to \underline{k}$ defines a the k-tuple $(S_1, \dots, S_k)$ by taking fibres. Contrary to the $(\mathbb{B}, +, 0)$, the groupoids $\mathbb{B}_{/\underline{k}}$ form a strict simplicial groupoid. The generic maps (generated by inner faces and degeneracies) are given by postcomposition of $S \to \underline{k}$ with the corresponding map $\underline{k} \to \underline{k'}$. The outer faces $d = d_\perp, d_\top : \mathbb{B}_{/\underline{k}} \to \mathbb{B}_{/\underline{k-1}}$ take $S \to \underline{k}$ to the pullback



The simplicial identities can be arranged to hold on the nose: the only subtlety is the pullback construction involved in defining the outer face maps, but these pullbacks can all be chosen in terms of subset inclusions. It is clear that the simplicial groupoid $\mathbb{B}_{/\underline{k}}$ is equivalent to the fat nerve of the classifying space of $\mathbb{B}$.

4.2. **Multiplicative examples**

4.2.1. Divisibility poset and multiplicative monoid. In analogy with 4.1.1, let $\mathbf{D}$ denote the nerve of the divisibility poset $(\mathbb{N}^\times, |)$, and let $\mathbf{M}$ be the nerve of the multiplicative monoid $(\mathbb{N}^\times, \cdot)$. Imposing the equivalence relation 'isomorphism of intervals' on the incidence coalgebra of $\mathbf{D}$ gives that of $\mathbf{M}$, and Content–Lemay–Leroux observed that this reduction is induced by a conservative ULF functor $r : \mathbf{D} \to \mathbf{M}$ sending $d|n$ to n/d. In fact we have:

Lemma. 4.2.2. *There is an isomorphism of simplicial sets*

$$\mathrm{Dec}_\perp(\mathbf{M}) \xrightarrow{\simeq} \mathbf{D}$$

given in degree k by

$$(x_0, x_1, \dots, x_k) \longmapsto [x_0|x_0x_1|\dots|x_0x_1\cdots x_k]$$

and the conservative ULF functor r is isomorphic to the structure map

$$d_\perp : \mathrm{Dec}_\perp(\mathbf{M}) \to \mathbf{M}, \qquad (x_0, \dots, x_k) \mapsto (x_1, \dots, x_k).$$

This example can be obtained from the example 4.1.1 directly, since $\mathbf{M} = \prod_p \mathbf{N}$ and $\mathbf{D} = \prod_p \mathbf{L}$, where the (weak) product is over all primes

p. Now $\mathrm{Dec}_\perp$ it is a right adjoint, so preserves products, and Lemma 4.2.2 follows from Lemma 4.1.1.

We can use the general formula 3.1.10: since there are no nontrivial automorphisms the convolution product is $\delta^m * \delta^n = \delta^{mn}$, and the incidence algebra is isomorphic to the Dirichlet algebra:

$$\begin{aligned} \mathrm{Inc}(\mathbf{D}) &\longrightarrow \mathbb{D} = \{\sum_{k>0} a_k k^{-s}\} \\ \delta^n &\longmapsto n^{-s} \\ f &\longmapsto \sum_{n>0} f(n) n^{-s}. \end{aligned}$$

4.2.3. Arithmetic species. The Dirichlet coalgebra (4.2.1) also has a fatter version: consider now instead the monoidal groupoid $(\mathbb{B}^\times, \times, 1)$ of non-empty finite sets under the cartesian product. It gives the classifying space $\mathbf{A}$ with $A_k := (\mathbb{B}^\times)^k$, where this time the inner face maps takes the cartesian product of two adjacent factors, and the outer face maps project away an outer factor.

The resulting coalgebra structure is

$$\Delta(S) = \sum_{A \times B \simeq S} A \otimes B.$$

Some care is due to interpret this correctly: the homotopy fibre over S is the groupoid whose objects are triples (A, B, ϕ) consisting of sets A and B equipped with a bijection $\phi : A \times B \xrightarrow{\sim} S$, and whose morphisms are pairs of isomorphisms $\alpha : A \xrightarrow{\sim} A'$, $\beta : B \xrightarrow{\sim} B'$ forming a commutative square with ϕ and ϕ'.

The corresponding incidence algebra $\boldsymbol{grpd}^{\mathbb{B}^\times}$ with the convolution product is the algebra of arithmetic species [3] under the Dirichlet product (called the arithmetic product of species by Maia and Méndez [53]).

The section coefficients are given directly by 3.1.10, and we find

$$\delta^m * \delta^n = \frac{(mn)!}{m!n!}\, \delta^{mn}$$

It follows that we can get an isomorphism with the Dirichlet algebra, namely

$$\begin{aligned} \mathrm{Inc}(\mathbf{A}) &\longrightarrow \mathbb{D} = \{\sum_{k>0} a_k k^{-s}\} \\ \delta^m &\longmapsto \frac{m^{-s}}{m!} \\ f &\mapsto \sum_{n>0} f(n) \frac{k^{-s}}{n!}; \end{aligned}$$

these are the 'exponential' (or modified) Dirichlet series (cf. Baez–Dolan [3].) So the incidence algebra zeta function in this setting is

$$\zeta = \sum_{k>0} \delta^k \mapsto \sum_{k>0} \frac{k^{-s}}{k!}$$

(which is not the usual Riemann zeta function).

4.3. **Linear examples**

The following classical examples lead us to classes of decomposition spaces which are not Segal spaces, namely Waldhausen's S-construction (4.6).

4.3.1. q-binomials: $\mathbb{F}_q$-vector spaces. Let $\mathbb{F}_q$ denote a finite field with q elements. Let $\mathbf{W}$ denote the fat nerve of the category ***vect*** of finite-dimensional $\mathbb{F}_q$-vector spaces and $\mathbb{F}_q$-linear injections. Impose the equivalence relation identifying two injections if their cokernels are isomorphic. This gives the q-binomial coalgebra (see Dür [17, 1.54]).

The same coalgebra can be obtained without reduction as follows. Put $\mathbf{V}_0 = *$, let $\mathbf{V}_1$ be the maximal groupoid of ***vect***, and let $\mathbf{V}_2$ be the groupoid of short exact sequences. The span

$$\begin{array}{ccccc} \mathbf{V}_1 & \longleftarrow & \mathbf{V}_2 & \longrightarrow & \mathbf{V}_1 \times \mathbf{V}_1 \\ E & \longleftarrow\!\shortmid & [E' \to E \to E''] & \longmapsto & (E', E'') \end{array}$$

(together with the span $\mathbf{V}_1 \leftarrow \mathbf{V}_0 \to 1$) defines a coalgebra on $\boldsymbol{grpd}_{/\mathbf{V}_1}$ which (after taking cardinality) is the q-binomial coalgebra, without further reduction. The groupoids and maps involved are part of a simplicial groupoid $\mathbf{V} : \Delta^{\mathrm{op}} \to \boldsymbol{Grpd}$, namely the Waldhausen S-construction of ***vect***, studied in more detail in the next section (4.6), where we'll see that this is a decomposition space but not a Segal space. The lower dec of $\mathbf{V}$ is naturally equivalent to the fat nerve $\mathbf{W}$ of the category of injections, and that the comparison map d_0 is the reduction map of Dür.

We calculate the section coefficients of $\mathbf{V}$. From Section 3.1 we have the following formula for the section coefficients (which is precisely the standard formula for the *Hall numbers*, as explained further in 4.6.11):

$$\frac{|\mathrm{SES}_{k,n,n-k}|}{\big|\mathrm{Aut}(\mathbb{F}_q^k)\big|\,\big|\mathrm{Aut}(\mathbb{F}_q^{n-k})\big|}.$$

Here $\mathrm{SES}_{k,n,n-k}$ is the groupoid of short exact sequence with specified vector spaces of dimensions k, n, and $n-k$. This is just a discrete space, and it has $(q-1)^n q^{\binom{k}{2}} q^{\binom{n-k}{2}} [n]!$ elements. Indeed, there are $(q-1)^k q^{\binom{k}{2}} \frac{[n]!}{[n-k]!}$ choices for the injection $\mathbb{F}_q^k \hookrightarrow \mathbb{F}_q^n$, and then $(q-1)^n q^{\binom{n}{2}} [n]!$ choices for identifying the cokernel with $\mathbb{F}_q^{n-k}$. Some q-yoga

yields altogether the q-binomials as section coefficients:

$$= \binom{n}{k}_q .$$

From this description we see that there is an isomorphism of algebras

$$\begin{array}{rcl} \mathrm{Inc}(\mathbf{V}) & \longrightarrow & \mathbb{Q}[[z]] \\ \delta^k & \longmapsto & \dfrac{z^k}{[k]!}. \end{array}$$

Clearly this algebra is commutative. However, an important new aspect is revealed on the objective level: here the convolution product is the external product of q-species of Joyal-Street [36]. They show (working with vector-space valued q-species), that this product has a natural non-trivial braiding (which of course reduces to commutativity upon taking cardinality).

4.3.2. Direct sums of $\mathbb{F}_q$-vector spaces and 'Cauchy' product of q-species. A coalgebra which is the q-analogue of $\mathbf{B}$ can be obtained from the classifying space of the monoidal groupoid $(\boldsymbol{vect}_{\mathbb{F}_q}, \oplus, 0)$ of finite-dimensional $\mathbb{F}_q$-vector spaces under direct sum. Comultiplication of a vector space V is the groupoid consisting of triples (A, B, ϕ) where ϕ is a linear isomorphism $A \oplus B \xrightarrow{\sim} V$. This groupoid projects to $\boldsymbol{vect} \times \boldsymbol{vect}$: the fibre over (A, B) is discrete of cardinality $|\mathrm{Aut}(V)|$, giving altogether the following section coefficient

$$\frac{\left|\mathrm{Aut}(\mathbb{F}_q^n)\right|}{\left|\mathrm{Aut}(\mathbb{F}_q^k)\right| \left|\mathrm{Aut}(\mathbb{F}_q^{n-k})\right|} = q^{k(n-k)} \binom{n}{k}_q .$$

At the objective level, this convolution product corresponds to the 'Cauchy' product of q-species in the sense of Morrison [57].

The resulting coalgebra is therefore, if we let δ_n denote the cardinality of the name of an n-dimensional vector space V:

$$\Delta(\delta_n) = \sum_{k \leq n} q^{k(n-k)} \binom{n}{k}_q \cdot \delta_k \otimes \delta_{n-k}.$$

Hence this one also has a power series representation, this time not with $\varphi(n) = [n]!$, but rather with $\varphi(n) = \#\,\mathrm{Aut}(\mathbb{F}_q^n)$.

4.4. Faà di Bruno bialgebra and variations

4.4.1. Faà di Bruno bialgebra. Classically (cf. Doubilet [15]) the Faà di Bruno bialgebra is constructed by imposing a *type-equivalence* relation on the incidence coalgebra of the poset $\mathbf{P}$ of all partitions of finite sets. Joyal [30] observed that it can also be realised directly from the category $\mathbf{S}$ of finite sets and surjections. (See also [23] for further development of this viewpoint.) Let $\mathbf{S}$ denote the fat nerve of the category of finite sets and surjections. That is, $\mathbf{S}_k$ is the groupoid of strings of k consecutive surjections.

A partition σ of a finite set X is encoded by the surjection $X \twoheadrightarrow S$, where S is the set of parts. Conversely, any surjection constitutes a partition of its domain. There is an equivalence of groupoids between partitions and surjections. Under this correspondence, if partition τ refines partition σ, then the corresponding surjections $X \twoheadrightarrow T$ and $X \twoheadrightarrow S$ fit into a sequence of surjections $X \twoheadrightarrow T \twoheadrightarrow S$. Hence we can write the partition poset nerve as having P_0 the groupoid of finite partitions (i.e. surjections), and P_k the groupoid of $k+1$ strings of surjections. Under this identification, the conservative ULF functor $F : \mathbf{P} \to \mathbf{S}$ simply deletes the first surjection in the string. Precisely, the partition-poset nerve is simply the decalage of the surjections nerve:

$$\mathbf{P} = \mathrm{Dec}_{\perp}(\mathbf{S}).$$

Finally note that the functor F is precisely reduction modulo type equivalence: recall that an interval $[\tau, \sigma]$ has *type* $1^{\lambda_1}2^{\lambda_2}\cdots$ if λ_k is the number of blocks of σ that consist of exactly k blocks of τ. Two intervals have the same type if and only if their images under $F : \mathbf{P} \to \mathbf{S}$ are isomorphic.

4.4.2. Faà di Bruno section coefficients. The category of finite sets and surjections is extensive. Hence the fat nerve $\mathbf{S}$ is a monoidal decomposition space (under $+$), hence the incidence coalgebra is a bialgebra. Note also that automatically the decalage of a monoidal decomposition space is monoidal, and the counit cULF. Hence the partition poset nerve is monoidal, and the reduction function a bialgebra homomorphism. Since $\mathbf{S}$ is monoidal, it is enough to describe the section coefficients on connected input. (A connected surjection is one with codomain 1.) Our general formula 3.1.10 gives

$$\Delta(\underline{n} \overset{f}{\twoheadrightarrow} \underline{1}) = \sum_{\substack{a:\underline{n}\to\underline{k}\\ b:\underline{k}\twoheadrightarrow\underline{1}}} \frac{\#\,\mathrm{Aut}(\underline{k}) \cdot \#\,\mathrm{Aut}(ab)}{\#\,\mathrm{Aut}(a) \cdot \#\,\mathrm{Aut}(b)} \ulcorner a \urcorner \otimes \ulcorner b \urcorner.$$

The order of the automorphism group of $\underline{k}$ and of a surjection $\underline{k} \twoheadrightarrow \underline{1}$ is $k!$, and for a general surjection $a : \underline{n} \twoheadrightarrow \underline{k}$ of type $1^{\lambda_1}2^{\lambda_2}\cdots$,

$$\#\,\mathrm{Aut}(a) \;=\; \prod_{j=1}^{\infty} \lambda_j!(j!)^{\lambda_j}$$

and hence

$$\Delta(\underline{n} \overset{f}{\twoheadrightarrow} \underline{1}) = \sum_{\substack{a:\underline{n}\to\underline{k}\\ b:\underline{k}\twoheadrightarrow\underline{1}}} \frac{n!}{\prod_{j=1}^{k} \lambda_j!(j!)^{\lambda_j}} \ulcorner a \urcorner \otimes \ulcorner b \urcorner.$$

The section coefficients, called the Faà di Bruno section coefficients, are the coefficients $\binom{n}{\lambda;k}$ of the Bell polynomials, cf. [21, (2.5)].

4.4.3. A decomposition space for the Faà di Bruno *Hopf* algebra. The Faà di Bruno *Hopf* algebra is obtained by further reduction,

classically stated as identifying two intervals in the partition poset if they are isomorphic as posets. This is equivalent to forgetting the value of λ_1. There is also a decomposition space that yields this Hopf algebra directly, obtained by quotienting the decomposition space $\mathbf{S}$ by the same equivalence relation. This means identifying two surjections (or sequences of composable surjections) if one is obtained from the other by taking disjoint union with a bijection. One may think of this as 'levelled forests modulo linear trees'. It is straightforward to check that this reduction respects the simplicial identities so as to define a simplicial groupoid, that it is a monoidal decomposition space, and that the quotient map from $\mathbf{S}$ is monoidal and cULF.

4.4.4. Ordered surjections. Let $\mathbf{OS}$ denote the fat nerve of the category of finite ordered set and monotone surjections. It is a monoidal decomposition space under ordinal sum. Hence to describe the resulting comultiplication, it is enough to say what happens to a connected ordered surjection, say $f : \underline{n} \twoheadrightarrow \underline{1}$, which we denote simply n: since there are no automorphisms around, we find

$$\Delta(n) = \sum_{k=1}^{n} \sum_{a} a \otimes k$$

where the second sum is over the $\binom{n-1}{k-1}$ possible surjections $a : n \twoheadrightarrow k$. This comultiplication has appeared in [5] and [25].

4.5. Graphs and trees

Various bialgebras of graphs and trees can be realised as incidence bialgebras of decomposition spaces which are not Segal. These examples will be subsumed in general classes of decomposition spaces, namely coming from restriction species, and the new notion of *directed restriction species* introduced in Section 5.

All the examples in this section are naturally bialgebras, with the monoidal structure given by disjoint union.

4.5.1. Graphs and restriction species. The following coalgebra of graphs seems to be due to Schmitt [65], §12. For a graph G with vertex set V (admitting multiple edges and loops), and a subset $U \subset V$, define $G|U$ to be the graph whose vertex set is U, and whose edges are those edges of G both of whose incident vertices belong to U. On the vector space spanned by isoclasses of graphs, define a comultiplication by the rule

$$\Delta(G) = \sum_{A+B=V} G|A \otimes G|B.$$

This coalgebra is the cardinality of the coalgebra of a decomposition space but not directly of a category. Indeed, define a simplicial groupoid with $\mathbf{G}_1$ the groupoid of graphs, and more generally let $\mathbf{G}_k$

be the groupoid of graphs with an ordered partition of the vertex set into k (possibly empty) parts. In particular, $\mathbf{G}_0$ is the contractible groupoid consisting only of the empty graph. The outer face maps delete the first or last part of the graph, and the inner face maps join adjacent parts. The degeneracy maps insert an empty part. It is clear that this is not a Segal space: a graph structure on a given set cannot be reconstructed from knowledge of the graph structure of the parts of the set, since chopping up the graph and restricting to the parts throws away all information about edges going from one part to another. One can easily check that it is a decomposition space. It is clear that the resulting coalgebra is Schmitt's coalgebra of graphs. Note that disjoint union of graphs makes this into a bialgebra too.

The graph example is typical for a big family of decomposition spaces, which can be treated uniformly, namely decomposition spaces of restriction species, in the sense of Schmitt [64] (see also [1]). We develop this theory further in Section 5.

4.5.2. Butcher-Connes-Kreimer Hopf algebra. Dür [17] (Ch.IV, §3) constructed what was later called the Connes-Kreimer Hopf algebra of rooted trees, after [11]: he starts with the notion of (combinatorial) tree (i.e. connected and simply connected graphs with a specified root vertex); then a forest is a disjoint union of rooted trees. He then considers the category of root-preserving inclusions of forests. A coalgebra is induced from this (in our language it is given by the simplicial groupoid $\mathbf{R}$, where $\mathbf{R}_k$ is the groupoid of strings of k root-preserving forest inclusions) but it is not the most interesting one. The Connes–Kreimer coalgebra is obtained by the reduction that identifies two root-preserving forest inclusions if their complement crowns are isomorphic forests.

We can obtain this coalgebra directly from a decomposition space: let $\mathbf{H}_1$ denote the groupoid of forests, and let $\mathbf{H}_2$ denote the groupoid of forests with an admissible cut. More generally, $\mathbf{H}_0$ is defined to be a point, and $\mathbf{H}_k$ is the groupoid of forests with $k-1$ compatible admissible cuts. These form a simplicial groupoid in which the inner face maps forget a cut, and the outer face maps projects away either the crown or the bottom layer (the part of the forest below the bottom cut). The notion of admissible cut is standard, see for example [12]. One convenient way to define what it means is to say that it is a root-preserving inclusion of forests: then the cut is interpreted as the division between the included forest and its complement. In this way we see that $\mathbf{H}_k$ is the groupoid of $k-1$ consecutive root-preserving inclusions.

There is a natural conservative ULF functor from $\mathbf{R}$ to $\mathbf{H}$: on $\mathbf{R}_1 \to \mathbf{H}_1$ it sends a root-preserving forest inclusion to its crown. More generally, on $\mathbf{R}_k \to \mathbf{H}_k$ it deletes the first inclusion in the string.

It is clear that $\mathbf{H}$ is not a Segal space: a tree with a cut cannot be reconstructed from its crown and its bottom tree, which is to say that

$\mathbf{H}_2$ is not equivalent to $\mathbf{H}_1 \times_{\mathbf{H}_0} \mathbf{H}_1$. It is straightforward to check that it *is* a decomposition space.

4.5.3. Operadic trees and P-trees. There is an important variation on the Connes-Kreimer Hopf algebra (but it is only a bialgebra): instead of considering combinatorial trees one considers operadic trees (i.e. trees with open incoming edges), or more generally P-trees for a finitary polynomial endofunctor P. For details on this setting, see [38], [40], [39] [23]; it suffices here to note that the notion covers planar trees, binary trees, effective trees, Feynman diagrams, etc.

There is a functor from operadic trees or P-trees to combinatorial trees which is *taking core* [40]: it amounts to shaving off all open-ended edges (and forgetting the P-decoration). This is a conservative ULF functor which realises the core bialgebra homomorphism from the bialgebra of operadic trees or P-trees to the Hopf algebra of combinatorial trees.

For operadic trees, when copying over the description of the nerve X where X_k is the groupoid of forests with $k-1$ compatible admissible cuts, there are two important differences: one is that X_0 is not just a point: it is the groupoid of node-less forests. The second is that unlike $\mathbf{H}$, this one is a Segal space: briefly this comes from the fact that the cuts do not throw away the edges cut, and hence there is enough data to reconstruct a tree with a cut from its bottom tree and crown by grafting. More precisely, the Segal maps $X_k \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$ simply return the layers seen in between the cuts. It is easy to see that this is an equivalence: given the layers separately, and a match of their boundaries, one can glue them together to reconstruct the original forest, up to isomorphism. In this sense the operadic-forest decomposition space is a 'category' with object set the set of edges. In this perspective, the combinatorial-forest decomposition space is obtained by throwing away the object information, i.e. the data governing the possible ways to compose. These two differences are crucial in the work on Green functions and Faà di Bruno formulae in [23].

4.5.4. Note about symmetries. It may be worth stressing here that one can *not* obtain the same bialgebra (either the combinatorial or the operadic) by taking isomorphism classes of each of the groupoids X_k: doing this would destroy symmetries that constitute an essential ingredient in the Connes–Kreimer bialgebra. Indeed, define a simplicial set Y in which $Y_k = \pi_0(X_k)$, the set of iso-classes of forests with k compatible admissible cuts. Consider the tree T

belonging to X_1. The fibre in X_2 is the (discrete) groupoid of all possible cuts in this tree:

The thing to notice here is that while the second and third cuts are isomorphic as abstract cuts, and therefore get identified in $Y_2 = \pi_0(X_2)$, this isomorphism is not vertical over the underlying tree T, so in the comultiplication formula at the groupoid level of X both cuts appear, and there is a total of 5 terms, whereas at the level of Y there will be only 4 terms. (Put in another way, the functor $X \to Y$ given by taking components is not cULF.)

It seems that there is no way to circumvent this discrepancy directly at the isoclass level: attempts involving ingenious decorations by natural numbers and actions by symmetric groups will almost certainly end up amounting to actually working at the groupoid level, and the conceptual clarity of the groupoid approach seems much preferable.

4.5.5. Free categories and free multicategories. Let G be a directed graph $E \rightrightarrows V$. Consider the polynomial endofunctor P given by $V \leftarrow E \overset{=}{\to} E \to V$. Then the groupoid of P-trees (4.5.3) (necessarily linear trees, since the middle map is an identity) is precisely (equivalent to) the set of arrows in the free category on G, and the decomposition space of P-trees described in 4.5.3 coincides with the nerve of this category.

More generally, for an arbitrary polynomial endofunctor P given by a diagram of sets $I \leftarrow E \to B \to I$, the groupoid of P-trees is the groupoid of operations of the free monad on P. Thinking of P as specifying a signature, we can equivalently think of P-trees as operations for the free (coloured) operad on that signature, or as the multi-arrows of the free multicategory on P regarded as a multigraph. To a multicategory there is associated a monoidal category [27], whose object set is the free monoid on the set of objects (colours). The decomposition space of P-trees is naturally identified with the nerve of the monoidal category associated to the multicategory of P-trees.

4.5.6. Directed graphs and free PROPs. These constructions readily generalise from trees to directed graphs (although the attractive polynomial interpretation does not). By a directed graph we understand a finite oriented graph with a certain number of open input edges, a certain number of open output edges, and prohibited to contain an oriented cycle. In particular, a directed graph has an underlying poset. The directed graphs form a groupoid G_1. We allow graphs without vertices, these form a groupoid G_0. Let G_2 denote the groupoid of directed graphs with an *admissible cut*: by this we mean a partition of the set of vertices into two disjoint parts: a poset filter F (i.e. an upward closed subset) and a poset ideal I (i.e. a downward closed subset). The edges connecting the two parts become the output edges of F and input edges of I; hence F and I become directed graphs again. Similarly, let

G_k denote the groupoid of directed graphs with $k-1$ compatible admissible cuts, just like we did for forests. It is clear that this defines a simplicial groupoid $\mathbf{G}$, easily verified to be a decomposition space and in fact a Segal space. The directed graphs form the set of operations of the free PROP with one generator in each input/output degree (m, n). The Segal space is the nerve of the associated monoidal category. The resulting coalgebra (in fact a bialgebra) has been studied in the context of Quantum Field Theory by Manchon [54]. Certain decorated directed graphs, and the resulting bialgebra have been studied by Manin [56], [55] in the theory of computation: his directed graphs are decorated by operations on partial recursive functions and switches. The decorating data is called a tensor scheme in [35], and the class of decorated graphs form the set of operations of the free (coloured) PROP on the tensor scheme. Again, the resulting decomposition space is naturally identified with the nerve of the associated monoidal category.

4.6. **Waldhausen S-construction**

4.6.1. Waldhausen S-construction of an abelian category. We follow Lurie [51, Subsection 1.2.2] for the account of Waldhausen S. For I a linearly ordered set, let $\mathrm{Ar}(I)$ denote the category of arrows in I: the objects are pairs of elements $i \leq j$ in I, and the morphisms are relations $(i, j) \leq (i', j')$ whenever $i \leq i'$ and $j \leq j'$. A *gap complex* in an abelian category $\mathscr{A}$ is a functor $F : N(\mathrm{Ar}(I)) \to \mathscr{A}$ such that

(1) For each $i \in I$, the object $F(i, i)$ is zero.
(2) For every $i \leq j \leq k$, the associated diagram

$$\begin{array}{ccc} 0 = F(j,j) & \rightarrowtail & F(j,k) \\ \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow \\ F(i,j) & \rightarrowtail & F(i,k) \end{array}$$

is a pushout (or equivalently a pullback).

Remark: since the pullback of a monomorphism is always a monomorphism, and the pushout of an epimorphism is always an epimorphism, it follows that automatically the horizontal maps are monomorphisms and the vertical maps are epimorphisms, as already indicated with the arrow typography. Altogether, it is just a fancy but very convenient way of saying 'short exact sequence' or '(co)fibration sequence'.

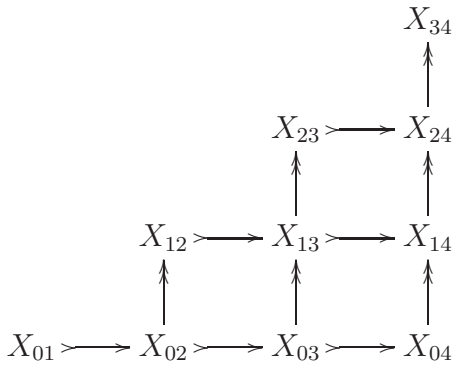
Let $\mathrm{Gap}(I, \mathscr{A})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Ar}(I), \mathscr{A})$ consisting of the gap complexes. This is a 1-category, since $\mathscr{A}$ was assumed to be an abelian 1-category.

The assignment

$$[n] \mapsto \mathrm{Gap}([n], \mathscr{A})^{\mathrm{eq}}$$

defines a simplicial space $S : \Delta^{\mathrm{op}} \to \boldsymbol{Grpd}$, which by definition is the Waldhausen S-construction on $\mathscr{A}$. Intuitively (or essentially), the

groupoid $\mathrm{Gap}([n], \mathscr{A})^{\mathrm{eq}}$ has as objects staircase diagrams like the following (picturing $n = 4$):



The face map d_i deletes all objects containing an i index. The degeneracy map s_i repeats the ith row and the ith column.

In particular $s_0(*) = 0$, and

$$d_0(A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n) = (A_2/A_1 \rightarrowtail \cdots \rightarrowtail A_n/A_1)$$
$$s_0(A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n) = (0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n)$$

The simplicial maps d_i, s_i for $i \geq 1$ are more straightforward: the simplicial set $\mathrm{Dec}_\perp(S\mathscr{A})$ is just the nerve of $\mathrm{mono}(\mathscr{A})$.

A string of composable monomorphisms $(A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n)$ determines, up to canonical isomorphism, short exact sequences $A_{ij} \rightarrowtail A_{ik} \twoheadrightarrow A_{jk} = A_{ij}/A_{ik}$ with $A_{0i} = A_i$. Hence the whole diagram can be reconstructed up to isomorphism from the bottom row. Similarly, since epimorphisms have uniquely determined kernels, the whole diagram can also be reconstructed from the last column.

Lemma. 4.6.2. *The projection $S_{n+1}\mathscr{A} \to \mathrm{Map}([n], \mathrm{mono}(\mathscr{A}))$ is a trivial Kan fibration. Similarly the projection $S_{n+1}\mathscr{A} \to \mathrm{Map}([n], \mathrm{epi}(\mathscr{A}))$.*

More precisely (with reference to the fat nerve):

Proposition. 4.6.3. *These equivalences assemble into levelwise simplicial equivalences*

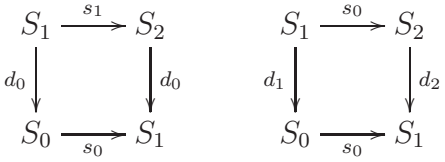
$$\mathrm{Dec}_\perp(S\mathscr{A}) \simeq N(\mathrm{mono}(\mathscr{A}))$$

$$\mathrm{Dec}_\top(S\mathscr{A}) \simeq N(\mathrm{epi}(\mathscr{A})).$$

Theorem. 4.6.4. *The Waldhausen S-construction of an abelian category $\mathscr{A}$ is a decomposition space.*

Proof. The previous proposition already implies that the two dec's of S are Segal spaces. By Theorem 1.5.5, it is therefore enough to establish

that the squares



are pullbacks. Note that we have $S_0 = *$ and $S_1 = \mathscr{A}^{\mathrm{iso}}$, and that $s_0 : S_0 \to S_1$ picks out the zero object, and since the zero object has no nontrivial automorphisms, this map is fully faithful. The map $d_0 : S_2 \to S_1$ sends a monomorphism to its quotient object. We need to compute the fibre over the zero object, but since s_0 is fully faithful, we are just asking for the full subgroupoid of S_2 consisting of those monomorphisms whose cokernel is zero. Clearly these are precisely the isos, so the fibre is just $\mathscr{A}^{\mathrm{iso}} = S_1$. The other pullback square is established similarly, but arguing with epimorphisms instead of monomorphisms. □

Remark. 4.6.5. Waldhausen's S-construction was designed for more general categories than abelian categories, namely what are now called Waldhausen categories, where the cofibrations play the role of the monomorphisms, but where there is no stand-in for the epimorphisms. The theorem does not generalise to Waldhausen categories in general, since in that case $\mathrm{Dec}_\top(S)$ is not necessarily a Segal space of any class of arrows.

4.6.6. Waldhausen S of a stable ∞-category. The same construction works in the ∞-setting, by considering stable ∞-categories instead of abelian categories. Let $\mathscr{A}$ be a stable ∞-category (see Lurie [51]). Just as in the abelian case, the assignment

$$[n] \mapsto \mathrm{Gap}([n], \mathscr{A})^{\mathrm{eq}}$$

defines a simplicial space $S : \Delta^{\mathrm{op}} \to \boldsymbol{Grpd}$, which by definition is the Waldhausen S-construction on $\mathscr{A}$. Note that in the case of a stable ∞-category, in contrast to the abelian case, every map can arise as either horizontal or vertical arrow in a gap complex. Hence the role of monomorphisms (cofibrations) is played by all maps, and the role of epimorphisms is also played by all maps.

Lemma. 4.6.7. *For each $k \in \mathbb{N}$, the two projection functors $S_{k+1}\mathscr{A} \to \mathrm{Map}(\Delta[k], \mathscr{A})$ are equivalences.*

From the description of the face and degeneracy maps, the following more precise result follows readily, comparing with the fat nerves:

Proposition. 4.6.8. *We have natural (levelwise) simplicial equivalences*

$$\mathrm{Dec}_\bot(S\mathscr{A}) \simeq N(\mathscr{A})$$
$$\mathrm{Dec}_\top(S\mathscr{A}) \simeq N(\mathscr{A}).$$

Theorem. 4.6.9. *Waldhausen's S-construction of a stable ∞-category $\mathscr{A}$ is a decomposition space.*

Proof. The proof is exactly the same as in the abelian case, relying on the following three fact:

(1) The dec's are Segal spaces.
(2) $s_0 : S_0 \to S_1$ is fully faithful.
(3) A map (playing the role of monomorphisms) is an equivalence if and only if its cofibre is the zero object, and a map (playing the role of epimorphism) is an equivalence if and only if its fibre is the zero object.

□

Remark. 4.6.10. This theorem was proved independently (and first) by Dyckerhoff and Kapranov [19], Theorem 7.3.3. They prove it more generally for exact ∞-categories, a notion they introduce. Their proof that Waldhausen's S-construction of an exact ∞-category is a decomposition space is somewhat more complicated than ours above. In particular their proof of unitality (the pullback condition on degeneracy maps) is technical and involves Quillen model structures on certain marked simplicial sets à la Lurie [49]. We do not wish to go into exact ∞-categories here, and refer instead the reader to [19], but we wish to point out that our simple proof above works as well for exact ∞-categories. This follows since the three points in the proof hold also for exact ∞-categories, which follows immediately from the definitions and basic results provided in [19], Sections 7.2 and 7.3.

4.6.11. Hall algebras. The finite-support incidence algebra of a decomposition space X was mentioned in 3.1.13. In order for it to admit a cardinality, the required assumption is that X_1 be locally finite, and that $X_2 \to X_1 \times X_1$ be a finite map. In the case of $X = S(\mathscr{A})$ for an abelian category $\mathscr{A}$, this translates into the condition that Ext^0 and Ext^1 be finite (which in practice means 'finite dimension over a finite field'). The finite-support incidence algebra in this case is the *Hall algebra* of $\mathscr{A}$ (cf. Ringel [61]; see also [63], although these sources twist the multiplication by the so-called Euler form).

For a stable ∞-category $\mathscr{A}$, with mapping spaces assumed to be locally finite (A.2.1), the finite-support incidence algebra of $S(\mathscr{A})$ is the *derived Hall algebra*. These were introduced by Toën [68] in the setting of dg-categories.

Hall algebras were one of the main motivations for Dyckerhoff and Kapranov [19] to introduce 2-Segal spaces. We refer to their work for development of this important topic.

4.7. Möbius functions and cancellation

We compute the Möbius functions in some of our examples. While the formula $\mu = \Phi_{\mathrm{even}} - \Phi_{\mathrm{odd}}$ seems to be the most general and uniform

expression of the Möbius function, it often not the most economical. At the numerical level, it is typically the case that much more practical expressions for the Möbius functions can be computed with different techniques. The $\Phi_{\text{even}} - \Phi_{\text{odd}}$ should not be dismissed on these grounds, though: it must be remembered that it constitutes a natural 'bijective' account, valid at the objective level, in contrast to many of the elegant cancellation-free expressions in the classical theory which are often the result of formal algebraic manipulations, typically power-series representations.

Comparison with the economical formulae raises the question whether these too can be realised at the objective level. This can be answered (in a few cases) by exhibiting an explicit cancellation between Φ_{even} and Φ_{odd}, which in turn may or may not be given by a *natural* bijection.

Once a more economical expression has been found for some Möbius decomposition space X, it can be transported back along any cULF functor $f : Y \to X$ to yield also more economical formulae for Y.

4.7.1. Natural numbers. For the decomposition space $\mathbf{N}$ (see 4.1.1), the incidence algebra is $\boldsymbol{grpd}^{\mathbb{N}}$, spanned by the representables h^n, and with convolution product

$$h^a * h^b = h^{a+b}.$$

To compute the Möbius functor, we have

$$\Phi_{\text{even}} = \sum_{r \text{ even}} (\mathbb{N} \smallsetminus \{0\})^r,$$

hence $\Phi_{\text{even}}(\underline{n})$ is the set of ordered compositions of the ordered set $\underline{n}$ into an even number of parts, or equivalently

$$\Phi_{\text{even}}(\underline{n}) = \{\underline{n} \twoheadrightarrow \underline{r} \mid r \text{ even } \},$$

the set of monotone surjections. In conclusion, with an abusive sign notation, the Möbius functor is

$$\mu(\underline{n}) = \sum_{r \geq 0} (-1)^r \{\underline{n} \twoheadrightarrow \underline{r}\}.$$

At the numerical level, this formula simplifies to

$$\mu(n) = \sum_{r \geq 0} (-1)^r \binom{n-1}{r-1} = \begin{cases} 1 & \text{for } n = 0 \\ -1 & \text{for } n = 1 \\ 0 & \text{else,} \end{cases}$$

(remembering that $\binom{-1}{-1} = 1$, and $\binom{k}{-1} = 0$ for $k \geq 0$).

On the other hand, since clearly the incidence algebra is isomorphic to the power series ring under the identification $|h^n| = \delta^n \leftrightarrow z^n \in \mathbb{Q}[[z]]$, and since the zeta function corresponds to the geometric series $\sum_n x^n = \frac{1}{1-x}$, we find that the Möbius function is $1 - x$. This corresponds the functor $\delta^0 - \delta^1$.

At the objective level, there is indeed a cancellation of ∞-groupoids taking place. It amounts to an equivalence of the Phi-groupoids restricted to $n \geq 2$:

$$\begin{array}{ccc} \Phi_{\mathrm{even}|r\geq 2} & \xrightarrow{\ \sim\ } & \Phi_{\mathrm{odd}|r\geq 2} \\ & \searrow \quad \swarrow & \\ & \mathbb{N}_{\geq 2} & \end{array}$$

which cancels out most of the stuff, leaving us with the much more economical Möbius function

$$\delta^0 - \delta^1$$

supported on $\mathbb{N}_{\leq 1}$. Since $\mathbb{N}$ is discrete, this equivalence (just a bijection) can be established fibrewise:

For each $n \geq 2$ there is a natural fibrewise bijection

$$\Phi_{\mathrm{even}}(n) \simeq \Phi_{\mathrm{odd}}(n).$$

To see this, encode the elements $(x_1, x_2, \ldots, x_k)$ in $\Phi_{\mathrm{even}}(n)$ as binary strings of length n and starting with 1 as follows: each coordinate x_i is represented as a string of length x_i whose first bit is 1 and whose other bits are 0, and all these strings are concatenated. In other words, thinking of the element $(x_1, x_2, \ldots, x_k)$ as a ordered partition of the ordered set n, in the binary representation the 1-entries mark the beginning of each part. (The binary strings must start with 1 since the first part must begin at the beginning.) For example, with $n = 8$, the element $(3, 2, 1, 1, 1) \in \Phi_{\mathrm{odd}}(8)$, is encoded as the binary string 10010111. Now the bijection between $\Phi_{\mathrm{even}}(n)$ and $\Phi_{\mathrm{odd}}(n)$ can be taken to simply flip the second bit in the binary representation. In the example, 10010111 is sent to 11010111, meaning that $(3, 2, 1, 1, 1) \in \Phi_{\mathrm{odd}}(8)$ is sent to $(1, 2, 2, 1, 1, 1) \in \Phi_{\mathrm{even}}(8)$. Because of this cancellation which occurs for $n \geq 2$ (we need the second bit in order to flip), the difference $\Phi_{\mathrm{even}} - \Phi_{\mathrm{odd}}$ is the same as $\delta_0 - \delta_1$, which is the cancellation-free formula.

The minimal solution $\delta^0 - \delta^1$ can also be checked immediately at the objective level to satisfy the defining equation for the Möbius function:

$$\zeta * \delta^0 = \zeta * \delta^1 + \delta^0$$

This equation says

$$\begin{array}{ccccc} \mathbb{N} \times \{0\} & = & (\mathbb{N} \times \{1\}) + \{0\} \\ {\scriptstyle \mathrm{add}}\downarrow & & {\scriptstyle \mathrm{add+incl}}\downarrow \\ \mathbb{N} & & \mathbb{N} \end{array}$$

In conclusion, the classical formula lifts to the objective level.

4.7.2. Finite sets and bijections. Already for the 'next' example, that of the monoidal groupoid $(\mathbb{B}, +, 0)$, whose incidence algebra is the algebra of species under the Cauchy convolution product (cf. 4.1.3), the situation is much more subtle.

Similarly to the previous example, we have $\Phi_r(S) = \mathrm{Surj}(S, \underline{r})$, but this time we are dealing with arbitrary surjections, as S is just an abstract set. Hence the Möbius functor is given by

$$\mu(S) = \sum_{r \geq 0} (-1)^r \,\mathrm{Surj}(S, \underline{r}).$$

Numerically, this is much more complicated than what is obtained from the observation that the incidence algebra, at the $\mathbb{Q}$-level, is just the power series algebra $\mathbb{Q}[[z]]$: since this time the zeta function is the exponential $\exp(z)$, the Möbius function is $\exp(-z)$, corresponding to

$$\mu(n) = (-1)^n.$$

The economical Möbius function suggests the existence of the following equivalence at the groupoid level:

$$\mu(S) = \int^r (-1)^r h^r(S) = \mathbb{B}_{\mathrm{even}}(S) - \mathbb{B}_{\mathrm{odd}}(S),$$

where

$$\mathbb{B}_{\mathrm{even}} = \sum_{r \text{ even}} \mathbb{B}_{[r]} \quad \text{and} \quad \mathbb{B}_{\mathrm{odd}} = \sum_{r \text{ odd}} \mathbb{B}_{[r]}$$

are the full subgroupoids of $\mathbb{B}$ consisting of the even and odd sets, respectively. However, it seems that such an equivalence is not possible, at least not over $\mathbb{B}$: while we are able to exhibit a bijective proof, this bijection is *not* natural, and hence does not assemble into a groupoid equivalence.

Proposition. 4.7.3. *For a fixed set S, there are monomorphisms $\mathbb{B}_{\mathrm{even}}(S) \hookrightarrow \Phi_{\mathrm{even}}(S)$ and $\mathbb{B}_{\mathrm{odd}}(S) \hookrightarrow \Phi_{\mathrm{odd}}(S)$, and a residual bijection*

$$\Phi_{\mathrm{even}}(S) - \mathbb{B}_{\mathrm{even}}(S) = \Phi_{\mathrm{odd}}(S) - \mathbb{B}_{\mathrm{odd}}(S).$$

This is not *natural in S, though, and hence does not constitute an isomorphism of species, only an equipotence of species.*

Corollary. 4.7.4. *For a fixed S there is a bijection*

$$\mu(S) \simeq \mathbb{B}_{\mathrm{even}}(S) - \mathbb{B}_{\mathrm{odd}}(S)$$

but it is not *natural in S.*

Proof of the Proposition. The map $\mathbb{B}_{\mathrm{even}} \to \mathbb{B}$ is a monomorphism, so for each set S of even cardinality there is a single element to subtract from $\Phi_{\mathrm{even}}(S)$. The groupoid Φ_{even} has as objects finite sets S equipped with a surjection $S \twoheadrightarrow \underline{k}$ for some even k. If S is itself of even cardinality n, then among such partitions there are $n!$ possible partitions into n parts. If there were given a total order on S, among these $n!$ n-block

partitions, there is one for which the order of S agrees with the order of the n parts. We would like to subtract that one and then establish the required bijection. This can be done fibrewise: over a given n-element set S, we can establish the bijection by choosing first a bijection $S \simeq \underline{n} = \{1, 2, \ldots, n\}$, the totally ordered set with n elements.

For each n, there is an explicit bijection

$$\{\textit{surjections } p : \underline{n} \twoheadrightarrow \underline{k} \mid k \textit{ even}, p \textit{ not the identity map}\}$$
$$\leftrightarrow$$
$$\{\textit{surjections } p : \underline{n} \twoheadrightarrow \underline{k} \mid k \textit{ odd}, p \textit{ not the identity map}\}$$

Indeed, define first the bijection on the subsets for which $p^{-1}(1) \neq \{1\}$, i.e. the element 1 is not alone in the first block. In this case the bijection goes as follows. If the elements 1 is alone in a block, join this block with the previous block. (There exists a previous block as we have excluded the case where 1 is alone in block 1.) If 1 is not alone in a block, separate out 1 to a block on its own, coming just after the original block. Example

$$(34, 1, 26, 5) \leftrightarrow (134, 2, 6, 5)$$

For the remaining case, where 1 is alone in the first block, we just leave it alone, and treat the remaining elements inductively, considering now the case where the element 2 is not alone in the second block. In the end, the only case not treated is the case where for each j, we have $p^{-1}(j) = \{j\}$, that is, each element is alone in the block with the same number. This is precisely the identity map excluded explicitly in the bijection. (Note that for each n, this case only appears on one of the sides of the bijection, as either n is even or n is odd.) □

In fact, already subtracting the groupoid $\mathbb{B}_{\text{even}}$ from Φ_{even} is not possible naturally. We would have first to find a monomorphism $\mathbb{B}_{\text{even}} \hookrightarrow \Phi_{\text{even}}$ over $\mathbb{B}$. But the automorphism group of an object $\underline{n} \in \mathbb{B}$ is $\mathfrak{S}_n$, whereas the automorphism group of any overlying object in Φ_{even} is a proper subgroup of $\mathfrak{S}_n$. In fact it is the subgroup of those permutations that are compatible with the surjection $\underline{n} \twoheadrightarrow \underline{k}$. So locally the fibration $\Phi_{\text{even}} \to \mathbb{B}$ is a group monomorphism, and hence it cannot have a section. So in conclusion, we cannot even realise $\mathbb{B}_{\text{even}}$ as a full subgroupoid in Φ_{even}, and hence it doesn't make sense to subtract it.

One may note that it is not logically necessary to be able to subtract the redundancies from Φ_{even} and Φ_{odd} in order to find the economical formula. It is enough to establish directly (by a separate proof) that the economical formula holds, by actually convoluting it with the zeta functor. At the object level the simplified Möbius function would be the groupoid

$$\mathbb{B}_{\text{even}} - \mathbb{B}_{\text{odd}}.$$

We might try to establish directly that

$$\zeta * \mathbb{B}_{\text{even}} = \zeta * \mathbb{B}_{\text{odd}} + \epsilon.$$

This should be a groupoid equivalence over $\mathbb{B}$. But again we can only establish this fibrewise. This time, however, rather than exploiting a non-natural total order, we can get away with a non-natural base-point. On the left-hand side, the fibre over an n-element set S, consists of an arbitrary set and an even set whose disjoint union is S. In other words, it suffices to give an even subset of S. Analogously, on the right-hand side, it amounts to giving an odd subset of S — or in the special case of $S = \emptyset$, we also have the possibility of giving that set, thanks to the summand ϵ. This is possible, non-naturally:

For a fixed nonempty set S, there is an explicit bijection between even subsets of S and odd subsets of S.

Indeed, fix an element $s \in S$. The bijection consists of adding s to the subset U if it does not belong to U, and removing it if it already belongs to U. Clearly this changes the parity of the set.

Again, since the bijection involves the choice of a basepoint, it seems impossible to lift it to a natural bijection.

4.7.5. Finite vector spaces. We calculate the Möbius function in the incidence algebra of the Waldhausen decomposition space of $\mathbb{F}_q$-vector spaces, cf. 4.3.1. In this case, Φ_r is the groupoid of strings of $r-1$ nontrivial injections. The fibre over V is the discrete groupoid of strings of $r-1$ nontrivial injections whose last space is V. This is precisely the set of nontrivial r-flags in V, i.e. flags for which the r consecutive codimensions are nonzero. In conclusion,

$$\mu(V) = \sum_{r=0}^{n} (-1)^r \{ \text{ nontrivial } r\text{-flags in } V\}.$$

(That's in principle a groupoid, but since we have fixed V, it is just a discrete groupoid: a flag inside a fixed vector space has no automorphisms.)

The number of flags with codimension sequence p is the q-multinomial coefficient

$$\binom{n}{p_1, p_2, \dots, p_r}_q.$$

In conclusion, at the numerical level we find

$$\mu(V) = \mu(n) = \sum_{r=0}^{n} (-1)^r \sum_{\substack{p_1+\cdots+p_r=n \\ p_i>0}} \binom{n}{p_1, p_2, \dots, p_r}_q.$$

On the other hand, it is classical that from the power-series representation (4.3.1) one gets the numerical Möbius function

$$\mu(n) = (-1)^n q^{\binom{n}{2}}.$$

While the equality of these two expressions can easily be established at the numerical level (for example via a zeta-polynomial argument, cf. below), we do not know of a objective interpretation of the expression $\mu(n) = (-1)^n q^{\binom{n}{2}}$. Realising the cancellation on the objective level would require first of all to being able to impose extra structure on V in such a way that among all nontrivial r-flags, there would be $q^{\binom{r}{2}}$ special ones!

4.7.6. Faà di Bruno. Recall (from 4.4.1) that the incidence coalgebra of the category of surjections is the Faà di Bruno coalgebra. Since this is a monoidal decomposition space, we have at our disposal the notion of multiplicative function, and these are determined by their values on the connected surjections. The multiplicative functions form a subalgebra of the incidence algebra, and clearly this subring contains both ζ and ϵ, and hence μ. It is therefore sufficient to calculate the Möbius function on connected surjections.

The general formula gives

$$\mu(\underline{n} \twoheadrightarrow \underline{1}) = \sum_{r=0}^{n} (-1)^n \boldsymbol{Tr}(n,r)$$

where $\boldsymbol{Tr}(n,r)$ is the (discrete) groupoid of n-leaf r-level trees with no trivial level (in fact, more precisely, strings of r nontrivial surjections composing to $n \twoheadrightarrow 1$).

On the other hand, classical theory (see Doubilet–Rota–Stanley [16]) gives the following 'connected Möbius function':

$$\mu(n) = (-1)^{n-1}(n-1)!.$$

In conjunction, the two expressions yield the following combinatorial identity:

$$(-1)^{n-1}(n-1)! \;=\; \sum_{r=0}^{n} (-1)^r \# \boldsymbol{Tr}(n,r).$$

We do not know how to realise the cancellation at the objective level. This would require developing first the theory of monoidal decomposition spaces and incidence bialgebras a bit further, a task we plan to take up in the near future.

4.7.7. Zeta polynomials. For a complete decomposition space X, we can write

$$X_r = \sum_w X_w = \sum_{k=0}^{r} \binom{r}{k} \vec{X}_k.$$

where w runs over the words of length r in the alphabet $\{0, a\}$ as in 2.3.6, and the binomial coefficient is an abusive shorthand for that many copies of $\vec{X}_k$, embedded disjointly into X_r by specific degeneracy

maps. Now we fibre over a fixed arrow $f \in X_1$, to obtain

$$(X_r)_f = \sum_{k=0}^{\infty} \binom{r}{k} (\vec{X}_k)_f,$$

where we have now allowed ourselves to sum to ∞.

The *'zeta polynomial'* of a decomposition space X is the function

$$\begin{array}{rcl} \zeta^r(f) : X_1 \times \mathbb{N} & \longrightarrow & \boldsymbol{Grpd} \\ (f, r) & \longmapsto & (X_r)_f \end{array}$$

assigning to each arrow f and length r the ∞-groupoid of r-simplices with long edge f. We don't actually know whether in general this is a polynomial in r, but when we know how to compute it, and it is a polynomial, then we can substitute $r = -1$ into it to find (assuming of course that X is a complete):

$$\zeta^{-1}(f) = \sum_{k=0}^{\infty} (-1)^k \Phi_k(f)$$

Hence $\zeta^{-1}(f) = \mu(f)$, as the notation suggests.

In some cases there is a polynomial formula for $\zeta^r(f)$. For example, in the case $X = (\mathbb{N}, +)$ we find $\zeta^r(n) = \binom{n+r-1}{n}$, and therefore $\mu(n) = \binom{n-2}{n}$, in agreement with the other calculations (of this trivial example). In the case $X = (\mathbb{B}, +)$, we find $\zeta^r(n) = r^n$, and therefore $\mu(n) = (-1)^n$ again.

Sometimes, even when a formula for $\zeta^r(n)$ cannot readily be found, the (-1)-value can be found by a power-series representation argument. For example in the case of the Waldhausen S of ***vect***, we have that $\zeta^r(n)$ is the set of r-flags of $\mathbb{F}_q^n$ (allowing trivial steps). We have

$$\zeta^r(n) = \sum_{\substack{p_1 + \cdots + p_r = n \\ p_i \geq 0}} \frac{[n]!}{[p_1]! \cdots [p_r]!},$$

and therefore

$$\sum_{n=0}^{\infty} \zeta^r(n) \frac{z^n}{[n]!} = \left(\sum_{n=0}^{\infty} \frac{z^n}{[n]!} \right)^r,$$

Now $\zeta^{-1}(n)$ can be read off as the nth coefficient in the inverted series $\left(\sum_{n=0}^{\infty} \frac{z^n}{[n]!}\right)^{-1}$. In the case at, hand, these coefficients are $(-1)^n q^{\binom{n}{2}}$, as we already saw.

Once a more economical Möbius function has been found for a decomposition space X, it can be exploited to yield more economical formulae for any decomposition space Y with a cULF functor to X. This is the content of the following obvious lemma:

Lemma. 4.7.8. *Suppose that for the complete decomposition space X we have found a Möbius inversion formula*

$$\zeta * \Psi_0 = \zeta * \Psi_1 + \epsilon.$$

Then for every decomposition space cULF over X, say $f : Y \to X$, we have the same formula

$$\zeta * f^*\Psi_0 = \zeta * f^*\Psi_1 + \epsilon$$

for Y.

4.7.9. Length. A *length functor* on a decomposition space X is a simplicial map from X to the nerve of $(\mathbb{N}, +)$. A length functor yields a grading on the incidence (co)algebra. In the rather special situation when this is cULF, the economical Möbius function formula

$$\mu = \delta^0 - \delta^1$$

for $(\mathbb{N}, +)$ induces the same formula for the Möbius functor of X. This is of course a very restrictive condition; in fact, for nerves of categories, this happens only for free categories on directed graphs (cf. Street [67]). For such categories, there is for each $n \in \mathbb{N}$ a linear span δ^n consisting of all the arrows of length n. In particular, δ^0 is the span $X_1 \leftarrow X_0 \to 1$ (the inclusion of the vertex set into the set of arrows), and δ^1 is the span $X_1 \leftarrow E \to 1$, the inclusion of the original set of edges into the set of all arrows. The simplest example is the free monoid on a set S, i.e. the monoid of words in the alphabet S. The economical Möbius function is then $\delta^0 - \delta^1$, where $\delta^1 = \sum_{s \in S} \delta^s$. In the power series ring, with a variable z_s for each letter $s \in S$, it is the series $1 - \sum_{s \in S} z_s$.

4.7.10. Decomposition spaces over B (4.1.3). Similarly, if a decomposition space X admits a cULF functor $\ell : X \to \mathbf{B}$ (which may be thought of as a 'length function with symmetries') then at the numerical level and at the objective level, locally for each object $S \in X_1$, we can pull back the compact Möbius 'functor' $\mu(S) = \sum_n (-1)^n h^n(S)$ from $\mathbf{B}$ to X, yielding the numerical Möbius function on X

$$\mu(f) = (-1)^{\ell(f)}.$$

An example of this is the coalgebra of graphs 4.5.1 of Schmitt [64]: the functor from the decomposition space of graphs to $\mathbf{B}$ which to a graph associates its vertex set is cULF. Hence the Möbius function for this decomposition space is

$$\mu(G) = (-1)^{\#V(G)}.$$

In fact this argument works for any restriction species.

5. Restriction species and directed restriction species

We show that restriction species and their associated coalgebras in the sense of Schmitt [64] are examples of decomposition spaces. Then we introduce the notion of *directed restriction species*, which covers various classical combinatorial coalgebras (such as for example the Connes-Kreimer bialgebra) and show that they also come from decomposition spaces. We unify the proofs of these results by giving a general construction of decomposition spaces from what we call sesquicartesian fibrations over the ordinal category $\underline{\Delta}$, involving covariant functoriality in all maps, and contravariant functoriality in convex inclusions.

The general construction can be viewed as follows. Since a monoid can be considered a one-object category, it yields in particular a decomposition space. Instead of regarding a monoid as a Segal space $X : \Delta^{\mathrm{op}} \to \boldsymbol{Grpd}$ with the property that $X_0 = 1$, monoids can be encoded as monoidal functors

$$(\underline{\Delta}, +, 0) \to (\boldsymbol{Grpd}, \times, 1),$$

and hence in particular are certain kinds of left fibrations $X \to \underline{\Delta}$. In this setting, a weaker structure than monoid is sufficient to obtain a decomposition space.

5.1. Restriction species (in the sense of Schmitt)

5.1.1. Restriction species. The notion of restriction species was introduced by Schmitt [64]: it is simply a presheaf on the category $\mathbb{I}$ of finite sets and injections. Compared to a classical species [30], a restriction species R is thus functorial not only on bijections but also on injections, meaning that a given structure on a set S induces also such a structure on every subset $A \subset S$ (denoted with a restriction bar):

$$\begin{array}{rcl} R[S] & \longrightarrow & R[A] \\ X & \longmapsto & X|A. \end{array}$$

The Schmitt construction associates to a restriction species $R : \mathbb{I}^{\mathrm{op}} \to \boldsymbol{Set}$ a coalgebra structure on the vector space spanned by the isoclasses of R-structures: the comultiplication is

$$\Delta(X) = \sum_{A+B=S} X|A \otimes X|B, \qquad X \in R[S],$$

and counit sending only the empty structures to 1.

A morphism of restriction species is just a natural transformation $R \Rightarrow R'$ of functors $\mathbb{I}^{\mathrm{op}} \to \boldsymbol{Set}$, i.e. for each finite set S a map $R[S] \to R'[S]$, natural in S. Since the summation in the comultiplication formula only involves the underlying sets, it is clear that a morphism of restriction species induces a coalgebra homomorphism.

A great many combinatorial coalgebras can be realised by the Schmitt construction (see [64] and also [1]). For example, graphs (4.5.1), matroids, posets, lattices, categories, etc., form restriction species and hence coalgebras.

5.1.2. Restriction species as decomposition spaces. Let $R : \mathbb{I}^{\mathrm{op}} \to \boldsymbol{Set}$ be a restriction species. It corresponds by the Grothendieck construction to a (discrete) right fibration

$$\mathbb{R} \to \mathbb{I},$$

where the total space $\mathbb{R}$ is the category of all R-structures and their structure-preserving injections. Precisely, a structure-preserving injection from $X \in R[S]$ to $X' \in R[S']$ consists of an injection of underlying sets $S \subset S'$ such that $X'|S = X$.

We construct a simplicial groupoid $\mathbf{R}$ where $\mathbf{R}_k$ is the groupoid of R-structures with an ordered partition of the underlying set into k (possibly empty) parts. Precisely, with reference to the strict version 4.1.8 of the finite-sets-and-bijections-nerve $\mathbf{B}$, we define $\mathbf{R}_k$ as the pullback

$$\mathbf{R}_k = \mathbb{B}_{/k} \times_{\mathbb{B}} \mathbb{R}^{\mathrm{iso}}.$$

The pullback construction delivers all the generic maps in $\mathbf{R}$, and so far the construction works for any species. To define also the free maps (i.e. outer face maps) we need the restriction structure on R: for example, the outer face map $d_\perp : \mathbb{B}_{/k} \to \mathbb{B}_{/k-1}$ is defined by sending $S \to \underline{k}$ to the pullback

$$\begin{array}{ccc} S' & \xrightarrow{\subset} & S \\ \downarrow & \lrcorner & \downarrow \\ \underline{k-1} & \longrightarrow & \underline{k}. \end{array}$$

Since $S' \hookrightarrow S$ is an injection, we can use functoriality of R (the fact that R is a restriction species) to get also the face map for $\mathbf{R}_k$. We shall formalise these constructions in 5.4.3. Note that by construction, as cULF over a decomposition space (the decomposition space $\mathbf{B}$ (cf. 4.1.3)), $\mathbf{R}$ is again a decomposition space.

Note that the subtlety in getting the free maps involves projecting away some parts of the underlying set. This means that maps lying over free maps are not vertical with respect to the projection down to $\mathbb{I}$. We shall develop theory to deal with this kind of problem.

A morphism of restriction species $R \to R'$ corresponds to a morphism of right fibrations $\mathbb{R} \to \mathbb{R}'$, and it is clear that the construction is functorial so as to induce a cULF functor of decomposition spaces.

Theorem. 5.1.3. *Given a restriction species R, the corresponding simplicial groupoid $\mathbf{R}$ is a decomposition space, and the (cardinality of the) associated coalgebra is the Schmitt coalgebra of R. A morphism*

of restriction species induces a cULF functor, whose cardinality is the coalgebra homomorphism resulting from the Schmitt construction.

We have already exploited (4.1.3) that lower dec of $\mathbf{B}$ is $\mathbf{I}$, the nerve of the category of injections $\mathbb{I}$. Similarly, it is straightforward to check that:

Lemma. 5.1.4. *The lower dec of the decomposition space of a restriction species $\mathbb{R}$ is the fat nerve of $\mathbb{R}$.*

5.1.5. Convex poset inclusions. Recall that a subposet $V \subset P$ is *convex* if $a, b \in V$ and $a \leq x \leq b$ imply $x \in V$. Let $\mathbb{C}$ denote the category of finite posets and convex poset inclusions.

An *ordered monotone partition* of a poset X is by definition a monotone map $X \to \underline{k}$ for $\underline{k} \in \underline{\Delta}$. Note that the fibres of such a map are convex subposets of X.

5.1.6. Directed restriction species. We introduce a new notion of directed restriction species, which is a generalisation of well-known constructions with lattices — see for example Schmitt [65] and also Figueroa and Gracia-Bondía [21].

A *directed restriction species* is by definition a functor

$$R : \mathbb{C}^{\mathrm{op}} \to \boldsymbol{Grpd},$$

or equivalently, by the Grothendieck construction, a right fibration $\mathbb{R} \to \mathbb{C}$. The idea is that the value on a poset S is the groupoid of all possible R-structures that have S as underlying poset. A morphism of directed restriction species is just a natural transformation.

Example. 5.1.7. The category of posets and convex inclusions is the terminal directed restriction species. Similarly there is a directed restriction species of lattices with convex inclusions, or categories with fully faithful cULF functors. (Note that a category has an underlying poset, namely by (-1)-truncation of all hom sets.) Rooted forests and convex maps form a directed restriction species. Similarly for directed graphs. In all these cases, there is a notion of underlying poset, which inherits the given structure from the ambient one. Note that in each case there is also a plain restriction species: in fact any subset of elements, convex or not, inherits the given structure.

5.1.8. Coalgebras from directed restriction species. Let R be any directed restriction species. An *admissible cut* of an object $X \in R[S]$ is by definition a monotone map from the underlying poset S to $\underline{2}$. That is, an admissible cut is an ordered monotone partition $A+B = S$. This agrees with the notion of admissible cut in Connes–Kreimer, and in related examples. Let $\mathbf{R}_2$ be the groupoid of R-structures with an admissible cut.

A coalgebra is defined by the rule

$$\Delta(X) = \sum_{A+B=S} X|A \otimes X|B, \qquad X \in R[S]. \tag{17}$$

Here the sum is over $\pi_0\mathbf{R}_2$, that is, all isomorphism classes of admissible cuts.

A special case of this construction is the Connes–Kreimer coalgebra of (combinatorial) trees (4.5.2). And also the Manchon–Manin coalgebra of directed graphs (4.5.6). Various examples of cobordism categories can also be envisioned.

5.1.9. Decomposition spaces from directed restriction species. If $\mathbb{R} \to \mathbb{C}$ is a directed restriction species, let $\mathbf{R}_k$ be the groupoid of R-structures on posets S with ordered monotone partitions into k possibly empty parts. In other words, $\mathbf{R}_2$ is the groupoid of R-structures with an admissible cut, and $\mathbf{R}_k$ is the groupoid of R-structures with $k-1$ compatible admissible cuts. The $\mathbf{R}_k$ form a simplicial groupoid. The functoriality in generic maps is clear, as these do not alter the underlying poset S. Functoriality in free maps comes from the structural restrictions, noting that free maps correspond to convex inclusions.

Theorem. 5.1.10. *The construction just outlined defines a decomposition space, whose incidence coalgebra coincides with Formula* (17)*. Morphisms of directed restriction species induce cULF functors and hence coalgebra homomorphisms.*

The theorem can be proved by a direct verification. The only subtlety is to establish functoriality in free maps of Δ. Rather than rendering this verification we prefer to take a rather abstract approach in the following subsections, establishing a general method for providing functoriality in free maps.

5.1.11. Decalage. Taking upper or lower dec of the decomposition space of a directed restriction species yields Segal spaces. The lower dec gives the (fat nerve of the) subcategory of $\mathbb{R}$ consisting of the maps that are order ideal inclusions (i.e. convex inclusions which are also downward closed). For example, in the case of the directed restriction species of forests, we get the category of forests and root-preserving inclusions of Dür [17]. Similarly, the upper dec yields the (fat nerve of the) subcategory of $\mathbb{R}$ consisting of the maps that are order filter inclusions (i.e. convex inclusions which are also upward closed).

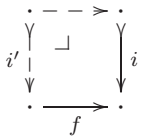
5.2. Further simplicial preliminaries

5.2.1. Finite ordinals. Recall that $\underline{\Delta}$ is the category whose objects are the finite (possibly empty) ordinals $\underline{k} := \{1, 2, \dots, k\}$, and whose arrows are the monotone maps. The distance-preserving maps in $\underline{\Delta}$ (which in the subcategory $\Delta \subset \underline{\Delta}$ we call 'free maps') are called *convex*: they are those $i : \underline{k'} \to \underline{k}$ such that $i(x+1) = i(x)+1$, for all $1 \leq x < k'$.

We denote the convex maps by arrows $\rightarrowtail$. Observe that the convex maps are just the canonical inclusions

$$f : \underline{n} \to \underline{a} + \underline{n} + \underline{b},$$

Lemma. 5.2.2. *Convex maps in $\underline{\Delta}$ admit basechange along any map. In other words, given the solid cospan consisting of f and i, with i convex,*



the pullback exists and i' is again convex.

5.2.3. Convex correspondences. Denote by ∇ the category of *convex correspondences* in $\underline{\Delta}$: the objects are those of $\underline{\Delta}$, and a morphism is a span

$$\underline{k}' \xleftarrow{i} \underline{k} \xrightarrow{f} \underline{n}$$

where i is convex. Composition of such spans is given by pullback, as allowed by the lemma. By construction, ∇ has a factorisation system in which the left-hand class (called *backward convex maps*) consists of spans of the form $\cdot \leftarrow \cdot \xrightarrow{=} \cdot$, and the right class (called *ordinalic*) consists of spans of the form $\cdot \xleftarrow{=} \cdot \rightarrow \cdot$; the right hand class forms of course a subcategory isomorphic to $\underline{\Delta}$. Note that ∇ has a zero object, namely $\underline{0}$. The zero maps are $\underline{n} \leftarrow \underline{0} \rightarrow \underline{k}$.

A map in ∇ can be understood as a monotone map, but defined possibly only on a certain middle convex part of an ordinal. The complement of the domain of definition consists of a bottom part and a top part. We can make such partial maps total by introducing new artificial bottom and top elements, and understand that the undefined parts are mapped there. Hence we are led to consider finite ordinals with a bottom and a top element:

5.2.4. Finite strict intervals. Let Ξ denote the category of *finite strict intervals* (cf. Joyal [34]): its objects are finite ordinals with a bottom and a top element required to be distinct, and the arrows are the monotone maps that also preserve bottom and top. We denote an object by the number of inner points, so as to write for example

$$\underline{k} := \{\bot, 1, 2, \ldots, k, \top\}.$$

(This naming convention is different from that we will use in Section 6, where our viewpoint on the same category is a bit different.)

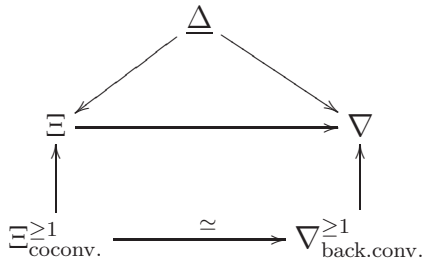
There is a canonical embedding

$$\underline{\Delta} \hookrightarrow \Xi$$

which to an ordinal adjoins a new bottom and a new top element. In particular the indexing convention is designed to reflect this embedding. Ξ has a factorisation system in which the left-hand class consists of maps for which the inverse image of every inner point is singleton (called *coconvex*), and whose right-hand class are the maps for which the inverse image of each of the outer points is singleton, in other words, they are the maps coming from $\underline{\Delta}$ (called ordinalic).

From the descriptions we see that the categories ∇ and Ξ are almost the same; the only difference is for maps factoring through $\underline{0}$: in ∇ each hom set $\mathrm{Hom}_{\nabla}(\underline{n}, \underline{k})$ contains exactly one such map, namely the zero map $\underline{n} \hookleftarrow \underline{0} \to \underline{0}$, whereas in $\mathrm{Hom}_{\Xi}(\underline{n}, \underline{k})$ there are $n+1$ maps through $\underline{0}$, depending on which elements map to top and bottom in the first step $\underline{n} \to \underline{0}$.

Lemma. 5.2.5. *There is a canonical functor $\Xi \to \nabla$, which is bijective on objects, and restricts to an isomorphism on the common subcategory $\underline{\Delta}$, and also restricts to an isomorphism $\Xi^{\geq 1}_{\mathrm{coconv.}} \xrightarrow{\sim} \nabla^{\geq 1}_{\mathrm{back.conv.}}$:*



All maps $\underline{n} \to \underline{0}$ in Ξ are sent to the zero map $\underline{n} \hookleftarrow \underline{0} \to \underline{0}$ in ∇.

The following is standard [34]:

Lemma. 5.2.6. *There is a canonical isomorphism of categories*

$$\Delta^{\mathrm{op}} \simeq \Xi$$

restricting to an isomorphism

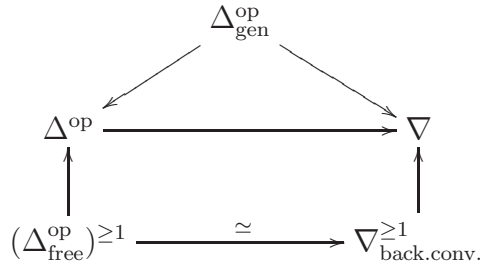
$$\Delta^{\mathrm{op}}_{\mathrm{gen}} \simeq \underline{\Delta}.$$

The generic maps in Δ correspond to the ordinalic maps in Ξ, and the free maps in Δ correspond to the coconvex in Ξ.

Combining these maps we get

Corollary. 5.2.7. *There is a canonical functor $\Delta^{\mathrm{op}} \to \nabla$, which is bijective on objects, and restricts to an isomorphism on the common subcategories $\Delta^{\mathrm{op}}_{\mathrm{gen}}$, takes the free maps to the backward-convex maps in ∇, restricting to an isomorphism $(\Delta^{\mathrm{op}}_{\mathrm{free}})^{\geq 1} \xrightarrow{\sim} \nabla^{\geq 1}_{\mathrm{back.conv.}}$, as indicated*

here:



All maps $[0] \to [n]$ *in* Δ *are sent to the zero map* $\underline{n} \leftarrow \underline{0} \to \underline{0}$ *in* ∇.

Corollary. 5.2.8. *A simplicial space* $X : \Delta^{\mathrm{op}} \to \boldsymbol{Grpd}$ *with* $X_0 = 1$ *can be realised from a* ∇*-diagram.*

Indeed, since X_0 is terminal, all the maps $X_n \to X_0$ coincide, so X factors through ∇.

5.2.9. Identity-extension squares. A square in $\underline{\Delta}$

$$\begin{array}{ccc} \underline{n} & \xrightarrow{j} & \underline{n'} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \underline{k} & \xrightarrow[i]{} & \underline{k'} \end{array}$$

in which the bottom map i is a convex map is called an *identity-extension square* if is it of the form

$$\begin{array}{ccc} \underline{n} & \xrightarrow{j} & \underline{a} + \underline{n} + \underline{b} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_a + f + \mathrm{id}_b} \\ \underline{k} & \xrightarrow[i]{} & \underline{a} + \underline{k} + \underline{b}. \end{array}$$

Lemma. 5.2.10. *An identity-extension square is both a pullback and a pushout.*

Lemma. 5.2.11. (1) *An identity-extension square is uniquely determined by* i *and* f.
(2) *An identity-extension square is uniquely determined by* j *and* f*, provided* $n > 0$.

Note a special case:

$$\begin{array}{ccc} \underline{0} & \longrightarrow & \underline{k'} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \underline{0} & \xrightarrow[i]{} & \underline{k'} \end{array}$$

is an identity-extension square, but there is more than one way to choose the a and b parts.

Recall from Lemma 1.1.5 that in Δ the pushout of a generic map along a free map is an iesq, and every iesq in which g is generic is such a pushout.

Proposition. 5.2.12. *Under the correspondence of Corollary 5.2.7, there is a bijection between the set of identity-extensions squares in $\underline{\Delta}$ and the set of identity-extension squares in Δ in which the vertical maps are generic*

$$\left\{ \begin{array}{ccc} \underline{n'} & \prec\!\!\!\!\longleftarrow & \underline{n} \\ \downarrow & & \downarrow \\ \underline{k'} & \prec\!\!\!\!\longleftarrow & \underline{k} \end{array} \quad \text{in } \underline{\Delta} \right\} \quad = \quad \left\{ \begin{array}{ccc} [n'] & \prec\!\!\!\!\longleftarrow & [n] \\ \uparrow & & \uparrow \\ [k'] & \prec\!\!\!\!\longleftarrow & [k] \end{array} \quad \text{in } \Delta \right\}$$

except in the case $k = 0$.

In the case $k = 0$, we necessarily have $n = 0$ and $n' = k'$, but there is not even a bijection on the bottom arrows.

Proof. The bijection is the composite of the three bijections

$$\left\{ \begin{array}{ccc} \underline{n'} & \prec\!\!\!\!\longleftarrow & \underline{n} \\ \downarrow & & \downarrow \\ \underline{k'} & \prec\!\!\!\!\longleftarrow & \underline{k} \end{array} \right\} = \left\{ \begin{array}{ccc} \underline{n'} & & \\ \downarrow & & \\ \underline{k'} & \prec\!\!\!\!\longleftarrow & \underline{k} \end{array} \right\} = \left\{ \begin{array}{ccc} [n'] & & \\ \uparrow & & \\ [k'] & \prec\!\!\!\!\longleftarrow & [k] \end{array} \right\} = \left\{ \begin{array}{ccc} [n'] & \prec\!\!\!\!\longleftarrow & [n] \\ \uparrow & & \uparrow \\ [k'] & \prec\!\!\!\!\longleftarrow & [k] \end{array} \right\}$$

where the first bijection is by Lemma 5.2.11 (1), the second is by Corollary 5.2.7 (here we use that $k \neq 0$), and the third is by Lemma 5.2.11 (2) restricted to the subcategory Δ. □

Proposition. 5.2.13. *If a covariant functor $M : \nabla \to \boldsymbol{Grpd}$ sends identity-extension squares to pullbacks then the composite*

$$\Delta^{\mathrm{op}} \to \nabla \to \boldsymbol{Grpd}$$

is a decomposition space.

Similarly:

Proposition. 5.2.14. *Let $u : M' \Rightarrow M : \nabla \to \boldsymbol{Grpd}$ be a natural transformation between functors that send identity-extension squares to pullbacks. If u is cartesian on arrows in $\underline{\Delta} \subset \nabla$, then it induces a cULF functor between decomposition spaces.*

5.2.15. Example: monoids. *A monoid viewed as a monoidal functor $(\underline{\Delta}, +, 0) \to (\boldsymbol{Grpd}, \times, 1)$ defines a ∇-space which satisfies iesq.* The contravariant functoriality on the convex maps is given as follows. The value on a convex map $\underline{n} \mapsto \underline{a} + \underline{n} + \underline{b}$ is simply the projection

$$X_{a+n+b} \simeq X_a \times X_n \times X_b \longrightarrow X_n,$$

where the first equivalence expresses that X is monoidal. The BC condition is implied by the iesq condition, which in turn is about projection too: for any identity-extension square

$$\begin{array}{ccc} \underline{a}+\underline{n}+\underline{b} & \xleftarrow{j} & \underline{n} \\ {\scriptstyle \mathrm{id}_a+f+\mathrm{id}_b=g}\downarrow & & \downarrow{\scriptstyle f} \\ \underline{a}+\underline{k}+\underline{b} & \xleftarrow[i]{} & \underline{k} \end{array}$$

the diagram

$$\begin{array}{ccc} X_{a+n+b} & \xrightarrow{j^*} & X_n \\ {\scriptstyle g_!}\downarrow & \lrcorner & \downarrow{\scriptstyle f_!} \\ X_{a+k+b} & \xrightarrow[i^*]{} & X_k \end{array}$$

is a pullback, since the upperstar functors are just projections.

5.2.16. Functors out of ∇. In view of the previous propositions, we are interested in defining functors out of ∇. By its construction as a category of spans, this amounts to defining a covariant functor on $\underline{\Delta}$ and a contravariant functor on $\underline{\Delta}_{\mathrm{convex}}$ which agree on objects, and such that for every pullback along a convex map the Beck–Chevalley condition holds. Better still, we can describe these as certain fibrations over $\underline{\Delta}$, called sesquicartesian fibrations, introduced in the next subsection. The fact that ∇ is not the whole bicategory of spans, and that the fibrations are similarly restricted, are just a minor distracting point. The essential points of the equivalence are well-understood and documented in the literature, as we proceed to explain.

The following technical result seems to be due to Hermida [27], with more detailed statement and proof given by Dawson-Paré-Pronk [14]. Our dependence on this result (which we don't quite know how to prove in the ∞-setting) means that the rest of this section should be interpreted only in 1-groupoids and 1-categories.

Proposition. 5.2.17. *Let $\mathbb{D}$ be a 1-category with pullbacks, and let $\mathscr{B}$ be a bicategory. The natural functor $\mathbb{D} \to \mathrm{Span}(\mathbb{D})$ induces an equivalence of categories*

$$\mathrm{Hom}(\mathrm{Span}(\mathbb{D}), \mathscr{B}) \simeq \mathrm{Sin}_{\mathrm{BC}}(\mathbb{D}, \mathscr{B}).$$

Here on the left we have pseudo-functors and pseudo-natural transformations, and on the right we have the category whose objects are sinister pseudofunctors satisfying the Beck-Chevalley condition (BC), and whose morphisms are the sinister pseudo-natural transformations. A pseudofunctor is *sinister* [14] if it sends all arrows to left adjoints, and it is *BC* if the image of any comma square has invertible mate. A *sinister pseudo-natural transformation* (between sinister pseudo-functors) is one whose naturality squares have invertible mate.

On the other hand, when $\mathscr{B} = \boldsymbol{Cat}$ we have:

Proposition. 5.2.18. *There is a natural equivalence of categories*

$$\mathrm{Sin}_{\mathrm{BC}}(\mathbb{D}, \boldsymbol{Cat}) \simeq \mathrm{Bicart}_{\mathrm{BC}}(\mathbb{D}).$$

Here on the right we have the category whose objects are bicartesian fibrations over $\mathbb{D}$ satisfying the Beck-Chevalley condition, and whose morphisms are functors over $\mathbb{D}$ preserving both cartesian and cocartesian arrows.

The proof of this result can be found (in the ∞-case) in Lurie [51], Proposition 6.2.3.17. Note however that Lurie does not consider the Beck-Chevalley condition (although he uses this name for something similar). More precisely he proves that bicartesian fibrations correspond to sinister functors and sinister transformations (called by him right-adjointable squares). It is clear though that the Beck-Chevalley condition goes on top of his result.

In the case at hand, the base category is $\underline{\Delta}$, but we only allow pullbacks along convex maps.

5.3. **Sesquicartesian fibrations**

A functor $X \to S$ is called a *bicartesian fibration* ([51], 6.2.3.1) when it is simultaneously a cartesian and a cocartesian fibration. We are interested in bicartesian fibrations over $\underline{\Delta}$, except that we only require the cartesianness over $\underline{\Delta}_{\mathrm{convex}}$. We call these *sesquicartesian fibrations*.

A sesquicartesian fibration $X \to \underline{\Delta}$ is said to have the *iesq property* if for every identity-extension square

$$\begin{array}{ccc} \underline{a}+\underline{n}+\underline{b} & \xleftarrow{j} & \underline{n} \\ {\scriptstyle \mathrm{id}_a+f+\mathrm{id}_b=g}\downarrow & & \downarrow{\scriptstyle f} \\ \underline{a}+\underline{k}+\underline{b} & \xleftarrow[i]{} & \underline{k} \end{array}$$

the diagram

$$\begin{array}{ccc} X_{a+n+b} & \xrightarrow{j^*} & X_n \\ {\scriptstyle g_!}\downarrow & \lrcorner & \downarrow{\scriptstyle f_!} \\ X_{a+k+b} & \xrightarrow[i^*]{} & X_k \end{array}$$

not only commutes (that's BC) but is furthermore a pullback.

Proposition. 5.3.1. *There is an equivalence of categories*

$$\mathrm{Hom}(\nabla, \boldsymbol{Cat}) \simeq \mathrm{Sesq}_{BC}(\underline{\Delta}),$$

under which the iesq conditions correspond to each other.

This is just a variation of the previous result.

So in order to construct nabla spaces satisfying the iesq property, we can construct sesquicartesian fibrations satisfying iesq, and then take maximal sub-groupoid.

5.3.2. Two-sided fibrations. Classically (the notion is due to Street), a *two-sided fibration* is a span of functors

$$\begin{array}{ccc} X & \xrightarrow{q} & T \\ {\scriptstyle p}\downarrow & & \\ S & & \end{array}$$

such that

— p is a cocartesian fibration whose p-cocartesian arrows are precisely the q-vertical arrows,

— q is a cartesian fibration whose q-cartesian arrows are precisely the p-vertical arrows

— for $x \in X$, an arrow f in S and g in T, the canonical map $f_! g^* x \to g^* f_!$ is an isomorphism.

In the setting of ∞-categories, Lurie [51], Section 2.4.7 (using the terminology 'bifibration') characterise two-sided fibrations as functors $X \to S \times T$ subject to a certain horn-filling condition, which among other technical advantages makes it clear that the notion is stable under base change $S' \times T' \to S \times T$. The classical axioms are derived from the horn-filling condition.

5.3.3. The category of arrows

$$\mathrm{Ar}(\mathscr{C}) \overset{(\mathrm{codom},\mathrm{dom})}{\longrightarrow} \mathscr{C} \times \mathscr{C}$$

is a two-sided fibration. Assuming that $\mathscr{C}$ has pullbacks, the codomain cocartesian fibration

$$\mathrm{Ar}(\mathscr{C}) \overset{\mathrm{codom}}{\to} \mathscr{C}$$

is a bicartesian fibration, and it satisfies BC.

5.3.4. Comma categories. Given functors

$$\begin{array}{ccc} & & B \\ & & {\scriptstyle G}\downarrow \\ A & \xrightarrow[F]{} & I \end{array}$$

the *comma category* $A{\downarrow}B$ is the category whose objects are triples (a, b, ϕ), where $a \in A$, $b \in B$, and $\phi : Fa \to Gb$. More formally it

is defined as the pullback two-sided fibration

$$\begin{array}{ccc} A{\downarrow}B & \longrightarrow & \mathrm{Ar}(I) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{codom},\mathrm{dom})} \\ B \times A & \xrightarrow[G\times F]{} & I \times I. \end{array}$$

Note that the factors come in the opposite order: $A{\downarrow}B \to B$ is the cocartesian fibration, and $A{\downarrow}B \to A$ the cartesian fibration.

Lemma. 5.3.5. *Given a two-sided fibration $X \to S \times T$, and let $R \to T$ be any map. Then the left-hand composite*

$$\begin{array}{ccc} X \times_T R & \longrightarrow & R \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{q} & T \\ {\scriptstyle p}\downarrow & & \\ S & & \end{array}$$

is a cocartesian fibration.

Proof. It is the pullback two-sided fibration of $X \to S \times T$ along $S \times R \to S \times T$. □

Corollary. 5.3.6. *In the situation of the previous lemma, if $X \to S$ is furthermore a bicartesian fibration and if $R \to T$ is a cartesian fibration, then the left-hand composite is a bicartesian fibration. If $X \to S$ satisfies BC, then so does the left-hand composite.*

We don't actually need this result, but rather the following more special case.

Lemma. 5.3.7. *If $X \to \underline{\Delta} \times T$ is a two-sided fibration such that $X \to \underline{\Delta}$ is a sesquicartesian fibration, then for any cartesian fibration $R \to T$, the left-hand composite in the diagram*

$$\begin{array}{ccc} X \times_T R & \longrightarrow & R \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & T \\ \downarrow & & \\ \underline{\Delta} & & \end{array}$$

is a sesquicartesian fibration. Furthermore, if $X \to \underline{\Delta}$ is iesq, then so is the left-hand composite.

Proof. The only non-trivial statement is about the iesq condition: given the pullback square

$$\begin{array}{ccc} X_{a+n+b} & \xrightarrow{j^*} & X_n \\ {\scriptstyle g_!}\downarrow & \lrcorner & \downarrow{\scriptstyle f_!} \\ X_{a+k+b} & \xrightarrow[i^*]{} & X_k \end{array}$$

expressing that $X \to S$ has the iesq property, the corresponding square for $X \times_T R \to S$ is simply obtained applying $- \times_T R$ to it, hence is again a pullback. □

5.4. **Decomposition spaces from sesquicartesian fibrations**

5.4.1. Restriction species and directed restriction species. Recall that a restriction species is a right fibration $\mathbb{R} \to \mathbb{I}$, where $\mathbb{I}$ is the category of finite sets and injections, and that a *directed restriction species* is a right fibration $\mathbb{R} \to \mathbb{C}$, where $\mathbb{C}$ denotes the category of posets and convex poset inclusions.

Proposition. 5.4.2. *The projection $\mathbb{I}{\downarrow}\underline{\Delta} \to \underline{\Delta}$ is an iesq sesquicartesian fibration.*

Proof. The comma category is taken over ***Set***. The objects of $\mathbb{I}{\downarrow}\underline{\Delta}$ are maps $S \to \underline{k}$, and the arrows are squares in ***Set***

$$\begin{array}{ccc} T & \longrightarrow & S \\ \downarrow & & \downarrow \\ \underline{n} & \longrightarrow & \underline{k} \end{array}$$

with $T \to S$ injective and $\underline{n} \to \underline{k}$ monotone. Just from being a comma category projection, $\mathbb{I}{\downarrow}\underline{\Delta} \to \underline{\Delta}$ is a cocartesian fibration. The cocartesian arrows are squares in ***Set*** of the form

$$\begin{array}{ccc} S & \xrightarrow{=} & S \\ \downarrow & & \downarrow \\ \underline{n} & \longrightarrow & \underline{k}. \end{array}$$

Over $\underline{\Delta}_{\text{convex}}$ it is also a cartesian fibration, as follows readily from the fact that the pullback lying over a convex map is injective: the cartesian arrows over a convex map are squares in ***Set*** of the form

$$\begin{array}{ccc} S' & \longrightarrow & S \\ \downarrow & \lrcorner & \downarrow \\ \underline{k'} & \rightarrowtail & \underline{k} \end{array}$$

Beck-Chevalley is a consequence of the iesq property. For the latter we need to check that given

$$\begin{array}{ccc} \underline{a}+\underline{n}+\underline{b} & \xleftarrow{j} & \underline{n} \\ {\scriptstyle \mathrm{id}_a+f+\mathrm{id}_b=g}\downarrow & & \downarrow{\scriptstyle f} \\ \underline{a}+\underline{k}+\underline{b} & \xleftarrow[i]{} & \underline{k} \end{array}$$

the resulting square

$$\begin{array}{ccc} \mathbb{I}_{/a+n+b} & \xrightarrow{j^*} & \mathbb{I}_{/n} \\ {\scriptstyle g_!}\downarrow & \lrcorner & \downarrow{\scriptstyle f_!} \\ \mathbb{I}_{/a+k+b} & \xrightarrow[i^*]{} & \mathbb{I}_{/k} \end{array}$$

is a pullback. But this is clear since $\mathbb{I}$ is an extensive category. □

Corollary. 5.4.3. *For any restriction species $\mathbb{R} \to \mathbb{I}$ the comma category projection $\mathbb{R}{\downarrow}\underline{\Delta} \to \underline{\Delta}$ is an iesq sesquicartesian fibration.*

Proof. This follows from Lemma 5.3.7. □

Proposition. 5.4.4. *The projection $\mathbb{C}{\downarrow}\underline{\Delta} \to \underline{\Delta}$ is an iesq sesquicartesian fibration.*

Proof. The comma category is taken over ***Poset***. The objects of $\mathbb{C}{\downarrow}\underline{\Delta}$ are poset maps $S \to \underline{k}$, and the arrows are squares in ***Poset***

$$\begin{array}{ccc} T & \longrightarrow & S \\ \downarrow & & \downarrow \\ \underline{n} & \longrightarrow & \underline{k} \end{array}$$

with $T \to S$ a convex poset inclusion and $\underline{n} \to \underline{k}$ a monotone map. Just from being a comma category projection, $\mathbb{C}{\downarrow}\underline{\Delta} \to \underline{\Delta}$ is a cocartesian fibration. The cocartesian arrows are squares in ***Poset*** of the form

$$\begin{array}{ccc} S & \xrightarrow{=} & S \\ \downarrow & & \downarrow \\ \underline{n} & \longrightarrow & \underline{k} \end{array}$$

Over $\underline{\Delta}_{\mathrm{convex}}$ it is also a cartesian fibration, as follows readily from the fact that the pullback lying over a convex map is a convex poset inclusion: the cartesian arrows over a convex map are squares in ***Poset*** of the form

$$\begin{array}{ccc} S' & \rightarrowtail & S \\ \downarrow & \lrcorner & \downarrow \\ \underline{k}' & \rightarrowtail & \underline{k} \end{array}$$

Beck-Chevalley is obvious from the fact that the cartesian arrows are pullback squares. Finally for the iesq property, here the argument is trickier than in the case of finite sets and injections. We need to check that given

$$\begin{array}{ccc} \underline{a}+\underline{n}+\underline{b} & \xleftarrow{j} & \underline{n} \\ {\scriptstyle \mathrm{id}_a+f+\mathrm{id}_b=g}\downarrow & & \downarrow{\scriptstyle f} \\ \underline{a}+\underline{k}+\underline{b} & \xleftarrow[i]{} & \underline{k} \end{array}$$

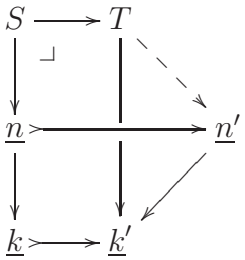
the resulting square

$$\begin{array}{ccc} \mathbb{C}_{/a+n+b} & \xrightarrow{j^*} & \mathbb{C}_{/n} \\ {\scriptstyle g_!}\downarrow & \lrcorner & \downarrow{\scriptstyle f_!} \\ \mathbb{C}_{/a+k+b} & \xrightarrow[i^*]{} & \mathbb{C}_{/k} \end{array}$$

is a pullback. This time it is not the case that $\mathbb{C}$ is extensive. Nevertheless, the iesq property is a direct check: an element in the pullback $\mathbb{C}_{/n} \times_{\mathbb{C}_{/k}} \mathbb{C}_{/k'}$ amounts of a diagram

$$\begin{array}{ccc} S & \longrightarrow & T \\ \downarrow & \lrcorner & \\ \underline{n} & & \downarrow \\ \downarrow & & \\ \underline{k} & \rightarrowtail & \underline{k}' \end{array}$$

Here the part $S \to \underline{n}$ is the element in $\mathbb{C}_{/n}$, and $T \to a+k+b$ is the element in $\mathbb{C}_{/k'}$, and saying that they have the same image in $\mathbb{C}_{/k}$ is to say that we have the pullback diagram. The claim is that given this diagram, there is a unique way to complete it to



Namely, on the element level T has three parts, namely the inverse images T_a, T_k and T_b. (We don't need to worry about the poset structure, since we already know all of T. The crucial thing is therefore that the covariant functoriality does not change the total space!). We now define $T \to n' = a+n+b$ as follows: we use $T_a \to a$ and $T_b \to b$ on the outer parts. On the middle part we know that $T_k = S$, so here we just use the map $S \to n$. □

Corollary. 5.4.5. *For any directed restriction species* $\mathbb{R} \to \mathbb{C}$ *the comma category projection* $\mathbb{R}{\downarrow}\underline{\Delta} \to \underline{\Delta}$ *is an iesq sesquicartesian fibration.*

Proof. This follows from Lemma 5.3.7. □

Now, by Propositions 5.2.13 and 5.3.1, iesq sesquicartesian fibrations over $\underline{\Delta}$ define decomposition spaces. The previous two corollaries therefore imply:

Corollary. 5.4.6. *Restriction species and directed restriction species define decomposition spaces.*

5.4.7. Towards decomposition categories. An iesq sesquicartesian fibration defines actually a decomposition *category*, not just a decomposition *space*. In fact we started this section observing that we are generalising the notion of monoid, which in our terminology includes monoidal groupoids. But many of our examples were actually monoidal categories, not just monoidal groupoids. It is therefore natural that the constructions meant to generalise these give actually simplicial diagrams in categories, not just in spaces or sets. We leave for another occasion the study of decomposition categories.

6. The decomposition space of Möbius intervals

Lawvere (in 1988, unpublished until Lawvere-Menni [43]) observed that there is a coalgebra (in fact a Hopf algebra) of isoclasses of Möbius intervals, which receives a canonical coalgebra homomorphism from any incidence coalgebra of a Möbius category. Furthermore, this Hopf algebra has Möbius inversion, and therefore Möbius inversion in all other incidence algebras (of Möbius categories) are induced from this master inversion formula.

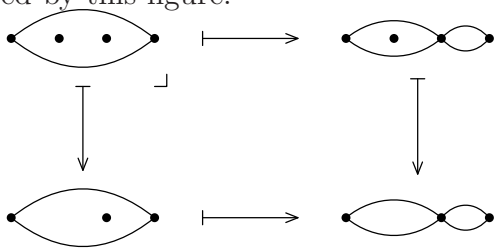
Here is the idea: a *Möbius interval* is a Möbius category (in the sense of Leroux) having an initial and a terminal object (not necessarily distinct). (It follows that it is actually a finite category.) An arrow $a : x \to y$ in a Möbius category $\mathbb{C}$ determines ([42]) a Möbius interval $I(a)$ (mimicking the identification of arrows and intervals in a poset), namely the category of factorisations of a: this category has an initial object 0 given by the factorisation id-followed-by-a, and a terminal object 1 given by the factorisation a-followed-by-id. There is a canonical conservative ULF functor from $I(a) \to \mathbb{C}$, sending 0 to x, sending 1 to y, and sending $0 \to 1$ to a. The longest arrow $0 \to 1$ in $I(a)$ has the same decomposition structure as a in $\mathbb{C}$, and hence the comultiplication of a can be computed inside $I(a)$.

Any collection of Möbius intervals closed under subintervals defines a coalgebra. It is an interesting integrability condition for such a collection to come from a single Möbius category. The Lawvere coalgebra

is simply the collection of *all* Möbius intervals (i.e. one from each isoclass).

Now, the coalgebra of Möbius intervals cannot be the coalgebra of a single Möbius Segal category, because such a Segal category X would have to have X_1 the space of all Möbius intervals, and X_2 the space of all subdivided Möbius intervals. But a Möbius interval with a subdivision (i.e. a midpoint) contains more information than the two parts of the subdivision: one from 0 to the midpoint, and one from the midpoint to 1. This shows that the Segal condition is not verified.

We shall prove that the space of all intervals *is* a decomposition space, as suggested by this figure:



While the idea is intuitively easy to grasp, a considerable amount of machinery is needed to construct the universal decomposition space and to get sufficient hold of its structural properties to prove the desired results about it. The main technicalities concern factorisation systems. We start with a subsection on general theory about factorisation systems, some results of which are already available in Lurie's book [49].

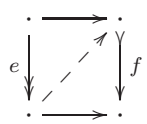
We then develop the theory of intervals, and construct the decomposition space of all intervals. We do it first without finiteness conditions, which we impose at the end.

6.1. **Factorisation systems and cartesian fibrations**

For background to this subsection, see Lurie [49], § 5.2.8.

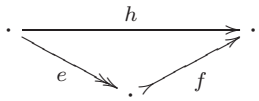
6.1.1. Factorisation systems. A *factorisation system* on an ∞-category $\mathscr{D}$ consists of two classes E and F of maps, that we shall depict as $\twoheadrightarrow$ and $\rightarrowtail$, such that

(1) The classes E and F are closed under equivalences.
(2) The classes E and F are orthogonal, $E \perp F$. That is, given $e \in E$ and $f \in F$, for every solid square



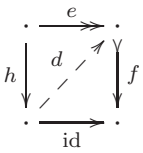
the space of fillers is contractible.

(3) Every map h admits a factorisation



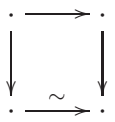
with $e \in E$ and $f \in F$.

(Note that in [49, Definition 5.2.8.8], the first condition is given as 'stability under formation of retracts'. In fact this stability follows from the three conditions above. Indeed, suppose $h \perp F$; factor $h = f \circ e$ as above. Since $h \perp f$, there is a diagonal filler in



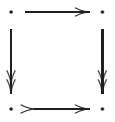
Now d belongs to ${}^{\perp}F$ since e and h do, and d belongs to $E^{\perp}$ since f and id do. Hence d is an equivalence, and therefore $h \in E$, by equivalence stability of E. Hence $E = {}^{\perp}F$, and is therefore closed under retracts. Similarly for F. It also follows that the two classes are closed under composition.)

6.1.2. Set-up. In this subsection, fix an ∞-category $\mathscr{D}$ with a factorisation system (E, F) as above. Let $\mathrm{Ar}(\mathscr{D}) = \mathrm{Fun}(\Delta[1], \mathscr{D})$, whose 0-simplices we depict vertically, then the domain projection $\mathrm{Ar}(\mathscr{D}) \to \mathscr{D}$ (induced by the inclusion $\{0\} \hookrightarrow \Delta[1]$) is a cartesian fibration; the cartesian arrows are the squares of the form

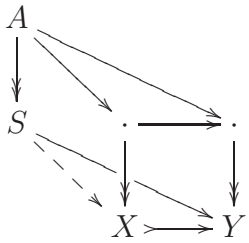


Let $\mathrm{Ar}^{E}(\mathscr{D}) \subset \mathrm{Ar}(\mathscr{D})$ denote the full subcategory spanned by the arrows in the left-hand class E.

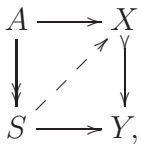
Lemma. 6.1.3. *The domain projection* $\mathrm{Ar}^{E}(\mathscr{D}) \to \mathscr{D}$ *is a cartesian fibration. The cartesian arrows in* $\mathrm{Ar}^{E}(\mathscr{D})$ *are given by squares of the form*



Proof. The essence of the argument is to provide uniquely the dashed arrow in



which amounts to filling



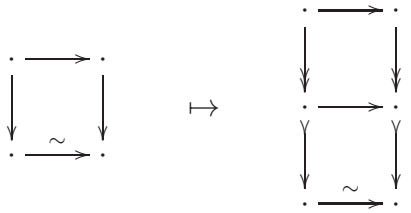
in turn uniquely fillable by orthogonality $E \perp F$. □

Lemma. 6.1.4. *The inclusion* $\mathrm{Ar}^E(\mathscr{D}) \to \mathrm{Ar}(\mathscr{D})$ *admits a right adjoint* w. *This right adjoint* $w : \mathrm{Ar}(\mathscr{D}) \to \mathrm{Ar}^E(\mathscr{D})$ *sends an arrow* a *to its* E*-factor. In other words, if* a *factors as* $a = f \circ e$ *then* $w(a) = e$.

Proof. This is dual to [49, 5.2.8.19]. □

Lemma. 6.1.5. *The right adjoint* w *sends cartesian arrows in* $\mathrm{Ar}(\mathscr{D})$ *to cartesian arrows in* $\mathrm{Ar}^E(\mathscr{D})$.

Proof. This can be seen from the factorisation:



The middle horizontal arrow is forced into F by the closure properties of right classes. □

Let $\mathrm{Fun}'(\Lambda^1_2, \mathscr{D}) = \mathrm{Ar}^E(\mathscr{D}) \times_{\mathscr{D}} \mathrm{Ar}^F(\mathscr{D})$ denote the ∞-category whose objects are pairs of composable arrows where the first arrow is in E and the second in F. Let $\mathrm{Fun}'(\Delta[2], \mathscr{D})$ denote the ∞-category of 2-simplices in $\mathscr{D}$ for which the two 'short' edges are in E and F respectively. The projection map $\mathrm{Fun}'(\Delta[2], \mathscr{D}) \to \mathrm{Fun}'(\Lambda^1_2, \mathscr{D})$ is always a trivial Kan fibration, just because $\mathscr{D}$ is an ∞-category.

Proposition. 6.1.6. *(*[49, 5.2.8.17]*.) The projection* $\mathrm{Fun}'(\Delta[2], \mathscr{D}) \to \mathrm{Fun}(\Delta[1], \mathscr{D})$ *induced by the long edge* $d_1 : [1] \to [2]$ *is a trivial Kan fibration.*

Corollary. 6.1.7. *There is an equivalence of ∞-categories*

$$\mathrm{Ar}(\mathscr{D}) \xrightarrow{\sim} \mathrm{Ar}^E(\mathscr{D}) \times_{\mathscr{D}} \mathrm{Ar}^F(\mathscr{D})$$

given by (E, F)-factoring an arrow.

Proof. Pick a section to the map in 6.1.6 and compose with the projection discussed just prior. □

Let x be an object in $\mathscr{D}$, and denote by $\mathscr{D}^E_{x/}$ the ∞-category of E-arrows out of x. More formally it is given by the pullback

$$\begin{array}{ccc} \mathscr{D}^E_{x/} & \longrightarrow & \mathrm{Ar}^E(\mathscr{D}) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{dom}} \\ * & \xrightarrow[\ulcorner x \urcorner]{} & \mathscr{D} \end{array}$$

Corollary. 6.1.8. *We have a pullback*

$$\begin{array}{ccc} \mathscr{D}_{x/} & \longrightarrow & \mathrm{Ar}^F(\mathscr{D}) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{dom}} \\ \mathscr{D}^E_{x/} & \longrightarrow & \mathscr{D} \end{array}$$

Proof. In the diagram

$$\begin{array}{ccccc} \mathscr{D}_{x/} & \longrightarrow & \mathrm{Ar}(\mathscr{D}) & \longrightarrow & \mathrm{Ar}^F(\mathscr{D}) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle w} & \lrcorner & \downarrow{\scriptstyle \mathrm{dom}} \\ \mathscr{D}^E_{x/} & \longrightarrow & \mathrm{Ar}^E(\mathscr{D}) & \xrightarrow[\mathrm{codom}]{} & \mathscr{D} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{dom}} & & \\ * & \xrightarrow[\ulcorner x \urcorner]{} & \mathscr{D} & & \end{array}$$

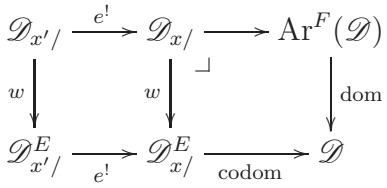
the right-hand square is a pullback by 6.1.7; the bottom square and the left-hand rectangle are clearly pullbacks, hence the top-left square is a pullback, and hence the top rectangle is too. □

Lemma. 6.1.9. *Let $e : x \to x'$ be an arrow in the class E. Then we have a pullback square*

$$\begin{array}{ccc} \mathscr{D}_{x'/} & \xrightarrow{e^!} & \mathscr{D}_{x/} \\ {\scriptstyle w}\downarrow & \lrcorner & \downarrow{\scriptstyle w} \\ \mathscr{D}^E_{x'/} & \xrightarrow[e^!]{} & \mathscr{D}^E_{x/} \end{array}$$

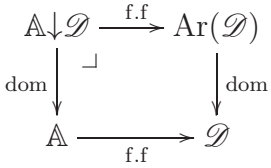
Here $e^!$ means 'precompose with e'.

Proof. In the diagram

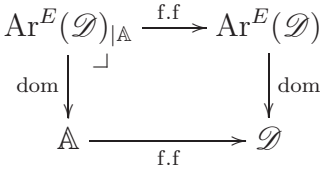


the functor $\mathscr{D}_{x/} \to \mathrm{Ar}^F(\mathscr{D})$ is 'taking F-factor'. Note that the horizontal composites are again 'taking F-factor' and codomain, respectively, since precomposing with an E-map does not change the F-factor. Since both the right-hand square and the rectangle are pullbacks by 6.1.8, the left-hand square is a pullback too. □

6.1.10. Restriction. We shall need a slight variation of these results. We continue the assumption that $\mathscr{D}$ is a ∞-category with a factorisation system (E, F). Given a full subcategory $\mathbb{A} \subset \mathscr{D}$, we denote by $\mathbb{A}{\downarrow}\mathscr{D}$ the 'comma category of arrows in $\mathscr{D}$ with domain in $\mathbb{A}$'. More precisely it is defined as the pullback

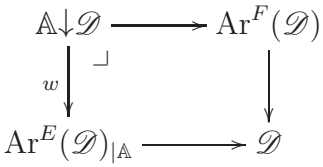


(This is dual to Artin gluing (cf. [24]).) The map $\mathbb{A}{\downarrow}\mathscr{D} \to \mathbb{A}$ is a cartesian fibration. Similarly, let $\mathrm{Ar}^E(\mathscr{D})_{|\mathbb{A}}$ denote the comma category of E-arrows with domain in $\mathbb{A}$, defined as the pullback



Again $\mathrm{Ar}^E(\mathscr{D})_{|\mathbb{A}} \to \mathbb{A}$ is a cartesian fibration (where the cartesian arrows are squares whose top part is in $\mathbb{A}$ and whose bottom horizontal arrow belongs to the class E). These two fibrations are just the restriction to $\mathbb{A}$ of the fibrations $\mathrm{Ar}(\mathscr{D}) \to \mathscr{D}$ and $\mathrm{Ar}^E(\mathscr{D}) \to \mathscr{D}$. Since the coreflection $\mathrm{Ar}(\mathscr{D}) \to \mathrm{Ar}^E(\mathscr{D})$ is vertical for the domain fibrations, it restricts to a coreflection $w : \mathbb{A}{\downarrow}\mathscr{D} \to \mathrm{Ar}^E(\mathscr{D})_{|\mathbb{A}}$.

Just as in the unrestricted situation (Corollary 6.1.7), we have a pullback square



saying that an arrow in $\mathscr{D}$ factors like before, also if it starts in an object in $\mathbb{A}$. Corollary 6.1.8 is the same in the restricted situation — just assume that x is an object in $\mathbb{A}$. Lemma 6.1.9 is also the same, just assume that $e : x' \to x$ is an E-arrow between $\mathbb{A}$-objects.

The following easy lemma expresses the general idea of extending a factorisation system.

Lemma. 6.1.11. *Given an adjunction $L : \mathscr{D} \rightleftarrows \mathscr{C} : R$ and given a factorisation system (E, F) on $\mathscr{D}$ with the properties*
— *RL preserves the class F;*
— *$R\epsilon$ belongs to F;*
consider the full subcategory $\widetilde{\mathscr{D}} \subset \mathscr{C}$ spanned by the image of L. (This can be viewed as the Kleisli category of the monad RL.) Then there is an induced factorisation system $(\widetilde{E}, \widetilde{F})$ on $\widetilde{\mathscr{D}} \subset \mathscr{C}$ with $\widetilde{E} := L(E)$ (saturated by equivalences), and $\widetilde{F} := R^{-1}F \cap \widetilde{\mathscr{D}}$.

Proof. It is clear that the classes $\widetilde{E}$ and $\widetilde{F}$ are closed under equivalences. The two classes are orthogonal: given $Le \in \widetilde{E}$ and $\tilde{f} \in \widetilde{F}$ we have $Le \perp \tilde{f}$ in the full subcategory $\widetilde{\mathscr{D}} \subset \mathscr{C}$ if and only if $e \perp R\tilde{f}$ in $\mathscr{D}$, and the latter is true since $R\tilde{f} \in F$ by definition of $\widetilde{F}$. Finally, every map $g : LA \to X$ in $\widetilde{\mathscr{D}}$ admits an $(\widetilde{E}, \widetilde{F})$-factorisation: indeed, it is transpose to a map $A \to RX$, which we simply (E, F)-factor in $\mathscr{D}$,

$$\begin{array}{ccc} A & \longrightarrow & RX, \\ & {\scriptstyle e}\searrow \quad \nearrow{\scriptstyle f} & \\ & D & \end{array}$$

and transpose back the factorisation (i.e. apply L and postcompose with the counit): g is now the composite

$$LA \xrightarrow{Le} LD \xrightarrow{Lf} LRX \xrightarrow{\epsilon} X,$$

where clearly $Le \in \widetilde{E}$, and we also have $\epsilon \circ Lf \in \widetilde{F}$ because of the two conditions imposed. □

6.1.12. Remarks. By general theory (6.1.4), having the factorisation system $(\widetilde{E}, \widetilde{F})$ implies the existence of a right adjoint to the inclusion

$$\mathrm{Ar}^{\widetilde{E}}(\widetilde{\mathscr{D}}) \longrightarrow \mathrm{Ar}(\widetilde{\mathscr{D}}).$$

This right adjoint returns the $\widetilde{E}$-factor of an arrow.

Inspection of the proof of 6.1.11 shows that we have the same factorisation property for other maps in $\mathscr{C}$ than those between objects in $\mathrm{Im}\, L$, namely giving up the requirement that the codomain should belong to $\mathrm{Im}\, L$: it is enough that the domain belongs to $\mathrm{Im}\, L$: *every map in $\mathscr{C}$ whose domain belongs to* $\mathrm{Im}\, L$ *factors as a map in $\widetilde{E}$ followed by a*

map in $\widetilde{F} := R^{-1}F$*, and we still have* $\widetilde{E} \perp \widetilde{F}$*, without restriction on the codomain in the right-hand class.* This result amounts to a coreflection:

Theorem. 6.1.13. *In the situation of Lemma 6.1.11, let* $\widetilde{\mathscr{D}}{\downarrow}\mathscr{C} \subset \mathrm{Ar}(\mathscr{C})$ *denote the* full *subcategory spanned by the maps with domain in* $\mathrm{Im}\, L$*. The inclusion functor*

$$\mathrm{Ar}^{\widetilde{E}}(\widetilde{\mathscr{D}}) \hookrightarrow \widetilde{\mathscr{D}}{\downarrow}\mathscr{C}$$

has a right adjoint, given by factoring any map with domain in $\mathrm{Im}\, L$ *and returning the* $\widetilde{E}$*-factor. Furthermore, the right adjoint preserves cartesian arrows (for the domain projections).*

Proof. Given that the factorisations exist as explained above, the proof now follows the proof of Lemma 5.2.8.18 in Lurie [49], using the dual of his Proposition 5.2.7.8. □

The following restricted version of these results will be useful.

Lemma. 6.1.14. *In the situation of Lemma 6.1.11, assume there is a full subcategory* $J : \mathbb{A} \hookrightarrow \mathscr{D}$ *such that*

— *All arrows in* $\mathbb{A}$ *belong to* E*.*

— *If an arrow in* $\mathscr{D}$ *has its domain in* $\mathbb{A}$*, then its* E*-factor also belongs to* $\mathbb{A}$*.*

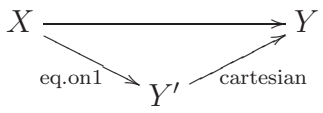
Consider the full subcategory $\widetilde{\mathbb{A}} \subset \mathscr{C}$ *spanned by the image of* LJ*. (This can be viewed as some kind of restricted Kleisli category.) Then there is induced a factorisation system* $(\widetilde{E}, \widetilde{F})$ *on* $\widetilde{\mathbb{A}} \subset \mathscr{C}$ *with* $\widetilde{E} := LJ(E)$ *(saturated by equivalences), and* $\widetilde{F} := R^{-1}F \cap \widetilde{\mathbb{A}}$*.*

Proof. The proof is the same as before. □

6.1.15. A basic factorisation system. Suppose $\mathscr{C}$ is any ∞-category, and $\mathscr{D}$ is an ∞-category with a terminal object 1. Then evaluation on 1 defines a cartesian fibration

$$ev_1 : \mathrm{Fun}(\mathscr{D}, \mathscr{C}) \to \mathscr{C}$$

for which the cartesian arrows are precisely the cartesian natural transformations. The vertical arrows are the natural transformations whose component at 1 is an equivalence. Hence the functor ∞-category has a factorisation system in which the left-hand class is the class of vertical natural transformations, and the right-hand class is the class of cartesian natural transformations:



Finally we shall need the following general result (not related to factorisation systems):

Lemma. 6.1.16. *Let $\mathscr{D}$ be any ∞-category. Then the functor*

$$\begin{aligned} F : \mathscr{D}^{\mathrm{op}} &\longrightarrow \boldsymbol{Grpd} \\ D &\longmapsto (\mathscr{D}_{D/})^{\mathrm{eq}}, \end{aligned}$$

corresponding to the right fibration $\mathrm{Ar}(\mathscr{D})^{\mathrm{cart}} \to \mathscr{D}$, preserves pullbacks.

Proof. Observe first that $F = \mathrm{colim}_{X \in \mathscr{D}^{\mathrm{eq}}} \mathrm{Map}(-, X)$, a homotopy sum of representables. Given now a pushout in $\mathscr{D}$,

$$\begin{array}{ccc} D & \longleftarrow & B \\ \uparrow & \lrcorner & \uparrow \\ A & \longleftarrow & C \end{array}$$

we compute, using the distributive law:

$$\begin{aligned} F(A \coprod_C B) &= \operatorname*{colim}_{X \in \mathscr{D}^{\mathrm{eq}}} \mathrm{Map}(A \coprod_C B, X) \\ &= \operatorname*{colim}_{X \in \mathscr{D}^{\mathrm{eq}}} \big(\mathrm{Map}(A, X) \times_{\mathrm{Map}(C,X)} \mathrm{Map}(B, X)\big) \\ &= \operatorname*{colim}_{X \in \mathscr{D}^{\mathrm{eq}}} \mathrm{Map}(A, X) \times_{\operatorname{colim} \mathrm{Map}(C,X)} \operatorname*{colim}_{X \in \mathscr{D}^{\mathrm{eq}}} \mathrm{Map}(B, X) \\ &= F(A) \times_{F(C)} F(B). \end{aligned}$$

□

6.2. **Flanked decomposition spaces**

6.2.1. Idea. The idea is that 'interval' should mean complete decomposition space (equipped) with both an initial and a terminal object. An object $x \in X_0$ is *initial* if the projection map $X_{x/} \to X$ is a levelwise equivalence. Here the *coslice* $X_{x/}$ is defined as the pullback of the lower dec $\mathrm{Dec}_\perp X$ along $1 \overset{\ulcorner x \urcorner}{\to} X_0$. Terminal objects are defined similarly with slices, i.e. pullbacks of the upper dec. It is not difficult to see (compare Proposition 6.2.17 below) that the existence of an initial or a terminal object forces X to be a Segal space.

While the intuition may be helpful, it is not obvious that the above definition of initial and terminal object should be meaningful for Segal spaces that are not Rezk complete. In any case, it turns out to be practical to approach the notion of interval from a more abstract viewpoint, which will allow us to get hold of various adjunctions and factorisation systems that are useful to prove things about intervals. We come to intervals in the next subsection. First we have to deal with flanked decomposition spaces.

6.2.2. The category Ξ of finite strict intervals. We denote by Ξ the category of finite strict intervals (cf. [34], see also 5 where we took a slightly different viewpoint), that is, a skeleton of the category

whose objects are nonempty finite linear orders with a bottom and a top element, required to be distinct, and whose arrows are the maps that preserve both the order and the bottom and top elements. We depict the objects as columns of dots, with the bottom and top dot white, then the maps are the order-preserving maps that send white dots to white dots, but are allowed to send black dots to white dots.

There is a forgetful functor $u : \Xi \to \Delta$ which forgets that there is anything special about the white dots, and just makes them black. This functor has a left adjoint $i : \Delta \to \Xi$ which to a linear order (column of black dots) adjoins a bottom and a top element (white dots).

Our indexing convention for Ξ follows the free functor i: the object in Ξ with k black dots (and two outer white dots) is denoted $[k-1]$. Hence the objects in Ξ are $[-1]$, $[0]$, $[1]$, etc. Note that $[-1]$ is an initial object in Ξ. The two functors can therefore be described on objects as $u([k]) = [k+2]$ and $i([k]) = [k]$, and the adjunction is given by the following isomorphism:

$$\Xi([n],[k]) = \Delta([n],[k+2]) \qquad n \geq 0, k \geq -1. \tag{18}$$

6.2.3. New outer degeneracy maps. Compared to Δ via the inclusion $i : \Delta \to \Xi$, the category Ξ has one extra coface map in Ξ, namely $[-1] \to [0]$. It also has, in each degree, two extra *outer codegeneracy maps*: $s^{\bot -1} : [n] \to [n-1]$ sends the bottom black dot to the bottom white dot, and $s^{\top +1} : [n] \to [n-1]$ sends the top black dot to the top white dot. (Both maps are otherwise bijective.)

6.2.4. Basic adjunction. The adjunction $i \dashv u$ induces an adjunction $i^* \dashv u^*$

$$\mathrm{Fun}(\Xi^{\mathrm{op}}, \boldsymbol{Grpd}) \underset{u^*}{\overset{i^*}{\rightleftarrows}} \mathrm{Fun}(\Delta^{\mathrm{op}}, \boldsymbol{Grpd})$$

which will play a central role in all the constructions in this section.

The functor i^* takes *underlying simplicial space*: concretely, applied to a Ξ^{op}-space A, the functor i^* deletes A_{-1} and removes all the extra outer degeneracy maps.

On the other hand, the functor u^*, applied to a simplicial space X, deletes X_0 and removes all outer face maps (and then reindexes).

The comonad

$$i^*u^* : \mathrm{Fun}(\Delta^{\mathrm{op}}, \boldsymbol{Grpd}) \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \boldsymbol{Grpd})$$

is precisely the double-dec construction $\mathrm{Dec}_\bot \mathrm{Dec}_\top$, and the counit of the adjunction is precisely the comparison map

$$\varepsilon_X = d_\top d_\bot : i^*u^*X = \mathrm{Dec}_\bot \mathrm{Dec}_\top X \longrightarrow X.$$

On the other hand, the monad

$$u^*i^* : \mathrm{Fun}(\Xi^{\mathrm{op}}, \boldsymbol{Grpd}) \to \mathrm{Fun}(\Xi^{\mathrm{op}}, \boldsymbol{Grpd})$$

is also a kind of double-dec, removing first the extra outer degeneracy maps, and then the outer face maps. The unit

$$\eta_A = s_{\perp-1}s_{\top+1} : A \to u^*i^*A$$

will also play an important role.

Lemma. 6.2.5. *If $f : Y \to X$ is a cULF map of simplicial spaces, then $u^*f : u^*Y \to u^*X$ is cartesian.*

Proof. The cULF condition on f says it is cartesian on 'everything' except outer face maps, which are thrown away when taking u^*f. □

Note that the converse is not always true: if u^*f is cartesian then f is ULF, but there is no information about $s_0 : Y_0 \to Y_1$, so we cannot conclude that f is conservative.

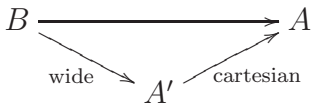
Dually:

Lemma. 6.2.6. *If a map of Ξ^{op}-spaces $g : B \to A$ is cartesian (or just cartesian on inner face and degeneracy maps), then $i^*g : i^*B \to i^*A$ is cartesian.*

6.2.7. Representables. The representables on Ξ we denote by $\Xi[-1]$, $\Xi[0]$, etc. By convention we will also denote the terminal presheaf on Ξ by $\Xi[-2]$, although it is not representable since we have chosen not to include $[-2]$ (a single white dot) in our definition of Ξ. Note that (18) says that i^* preserves representables:

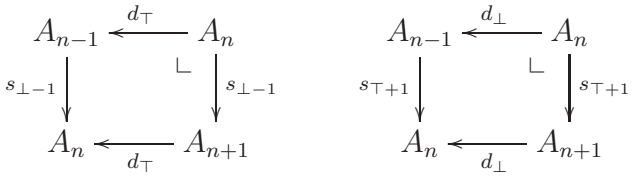
$$i^*(\Xi[k]) = \Delta[k+2], \qquad k \geq -1. \tag{19}$$

6.2.8. Wide/cartesian factorisation system. Call an arrow in $\mathrm{Fun}(\Xi^{\mathrm{op}}, \boldsymbol{Grpd})$ *wide* if its $[-1]$-component is an equivalence. Call an arrow *cartesian* if it is a cartesian natural transformation of Ξ^{op}-diagrams. By general theory (6.1.15) we have a factorisation system on $\mathrm{Fun}(\Xi^{\mathrm{op}}, \boldsymbol{Grpd})$ where the left-hand class is formed by the wide maps and the right-hand class consists of the cartesian maps. In concrete terms, given any map $B \to A$, since $[-1]$ is terminal in Ξ^{op}, one can pull back the whole diagram A along the map $B_{-1} \to A_{-1}$. The resulting Ξ^{op}-diagram A' is cartesian over A by construction, and by the universal property of the pullback it receives a map from B which is manifestly the identity in degree -1, hence wide.



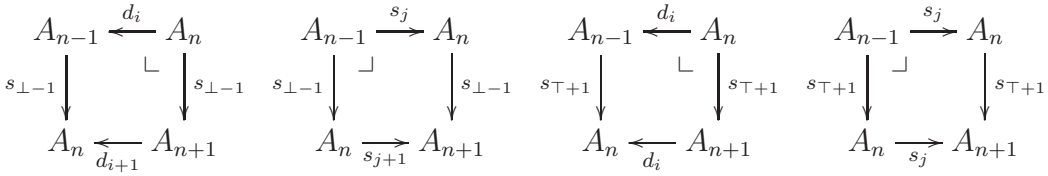
6.2.9. Flanked Ξ^{op}-spaces. A Ξ^{op}-space A is called *flanked* if the extra outer degeneracy maps form cartesian squares with opposite outer

face maps. Precisely, for $n \geq 0$



Here we have included the special extra face map $A_{-1} \leftarrow A_0$ both as a top face map and a bottom face map.

Lemma. 6.2.10. *('Bonus pullbacks' for flanked spaces.) In a flanked Ξ^{op}-space A, all the following squares are pullbacks:*



This is for all $n \geq 0$, and the running indices are $0 \leq i \leq n$ and $-1 \leq j \leq n$.

Proof. Easy argument with pullbacks, as in 1.3.8. □

Note that in the upper rows, all face or degeneracy maps are present, whereas in the lower rows, there is one map missing in each case. In particular, all the 'new' outer degeneracy maps appear as pullbacks of 'old' degeneracy maps.

6.2.11. Flanked decomposition spaces. By definition, a *flanked decomposition space* is a Ξ^{op}-space $A : \Xi^{\mathrm{op}} \to \boldsymbol{Grpd}$ that is flanked and whose underlying Δ^{op}-space i^*A is a decomposition space. Let $\boldsymbol{FD}$ denote the full subcategory of $\mathrm{Fun}(\Xi^{\mathrm{op}}, \boldsymbol{Grpd})$ spanned by the flanked decomposition spaces.

Lemma. 6.2.12. *If X is a decomposition space, then u^*X is a flanked decomposition space.*

Proof. The underlying simplicial space is clearly a decomposition space (in fact a Segal space), since all we have done is to throw away some outer face maps and reindex. The flanking condition comes from the 'bonus pullbacks' of X, cf. 1.3.9. □

It follows that the basic adjunction $i^* \dashv u^*$ restricts to an adjunction

$$i^* : \boldsymbol{FD} \rightleftarrows \boldsymbol{Dcmp} : u^*$$

between flanked decomposition spaces (certain Ξ^{op}-diagrams) and decomposition spaces.

Lemma. 6.2.13. *The counit*

$$\epsilon_X : i^*u^*X \to X$$

is cULF, when X is a decomposition space.

Proof. This was proved in Theorem 1.5.5. □

Lemma. 6.2.14. *The unit*

$$\eta_A : A \to u^*i^*A$$

is cartesian, when A is flanked.

Proof. The map η_A is given by $s_{\perp-1}$ followed by $s_{\top+1}$. The asserted pullbacks are precisely the 'bonus pullbacks' of Lemma 6.2.10. □

From Lemma 6.2.14 and Lemma 6.2.13 we get:

Corollary. 6.2.15. *The monad $u^*i^* : \boldsymbol{FD} \to \boldsymbol{FD}$ preserves cartesian maps.*

Lemma. 6.2.16. *$i^*A \to X$ is cULF in $\boldsymbol{Dcmp}$ if and only if the transpose $A \to u^*X$ is cartesian in $\boldsymbol{FD}$.*

Proof. This follows since the unit is cartesian (6.2.14), the counit is cULF (6.2.13), and u^* and i^* send those two classes to each other (6.2.5 and 6.2.6). □

Proposition. 6.2.17. *If A is a flanked decomposition space, then i^*A is a Segal space.*

Proof. Put $X = i^*A$. We have the maps

$$i^*A \xrightarrow{i^*\eta_A} i^*u^*i^*A = u^*i^*X \xrightarrow{\epsilon_X} X = i^*A$$

Now X is a decomposition space by assumption, so $i^*u^*X = \mathrm{Dec}_\perp \mathrm{Dec}_\top X$ is a Segal space and the counit is cULF (both statements by Theorem 1.5.5). On the other hand, since A is flanked, the unit η is cartesian by Lemma 6.2.14, hence $i^*\eta$ is cartesian by Lemma 6.2.6. Since i^*A is thus cartesian over a Segal space, it is itself a Segal space (1.2.3). □

Lemma. 6.2.18. *If $B \to A$ is a cartesian map of Ξ^{op}-spaces and A is a flanked decomposition space then so is B.*

Corollary. 6.2.19. *The wide/cartesian factorisation system restricts to a factorisation system on $\boldsymbol{FD}$.*

Lemma. 6.2.20. *The representable functors $\Xi[k]$ are flanked.*

Proof. Since the pullback squares required for a presheaf to be flanked are images of pushouts in Ξ, this follows since representable functors send colimits to limits. □

6.3. Intervals and the factorisation-interval construction

6.3.1. Complete Ξ^{op}-spaces. A Ξ^{op}-space is called *complete* if all degeneracy maps are monomorphisms. We are mostly interested in this notion for flanked decomposition spaces. In this case, if just $s_0 : A_0 \to A_1$ is a monomorphism, then all the degeneracy maps are monomorphisms. This follows because on the underlying decomposition space, we know from 2.3.5 that $s_0 : A_0 \to A_1$ being a monomorphism implies that all the simplicial degeneracy maps are monomorphisms, and by flanking we then deduce that also the new outer degeneracy maps are monomorphisms. Denote by $\boldsymbol{cFD} \subset \boldsymbol{FD}$ the full subcategory spanned by the complete flanked decomposition spaces.

It is clear that if X is a complete decomposition space, then u^*X is a complete flanked decomposition space, and if A is a complete flanked decomposition space then i^*A is a complete decomposition space. Hence the fundamental adjunction $i^* : \boldsymbol{FD} \rightleftarrows \boldsymbol{Dcmp} : u^*$ between flanked decomposition spaces and decomposition spaces restricts to an adjunction

$$i^* : \boldsymbol{cFD} \rightleftarrows \boldsymbol{cDcmp} : u^*$$

between complete flanked decomposition spaces and complete decomposition spaces.

Note that anything cartesian over a complete Ξ^{op}-space is again complete.

6.3.2. Reduced Ξ^{op}-spaces. A Ξ^{op}-space $A : \Xi^{\mathrm{op}} \to \boldsymbol{Grpd}$ is called *reduced* when $A[-1] \simeq *$.

Lemma. 6.3.3. *If $A \to B$ is a wide map of Ξ^{op}-spaces and A is reduced then B is reduced.*

6.3.4. Algebraic intervals. An *algebraic interval* is by definition a reduced complete flanked decomposition space. We denote by $\boldsymbol{aInt}$ the full subcategory of $\mathrm{Fun}(\Xi^{\mathrm{op}}, \boldsymbol{Grpd})$ spanned by the algebraic intervals. In other words, a *morphism of algebraic intervals* is just a natural transformation of functors $\Xi^{\mathrm{op}} \to \boldsymbol{Grpd}$. Note that the underlying decomposition space of an interval is always a Segal space.

Lemma. 6.3.5. *All representables $\Xi[k]$ are algebraic intervals (for $k \geq -1$), and also the terminal presheaf $\Xi[-2]$ is an algebraic interval.*

Proof. It is clear that all these presheaves are contractible in degree -1, and they are flanked by Lemma 6.2.20. It is also clear from (19) that their underlying simplicial spaces are complete decomposition spaces (they are even Rezk complete Segal spaces). □

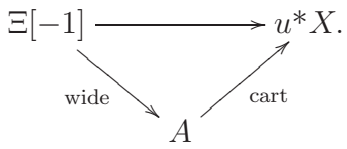
Lemma. 6.3.6. *$\Xi[-1]$ is an initial object in $\boldsymbol{aInt}$.*

Lemma. 6.3.7. *Every morphism in $\boldsymbol{aInt}$ is wide.*

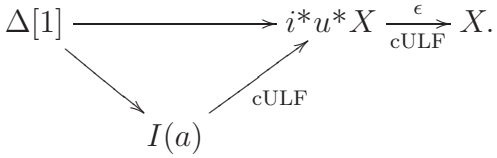
Corollary. 6.3.8. *If a morphism of algebraic intervals is cartesian, then it is an equivalence.*

6.3.9. The factorisation-interval construction. We now come to the important notion of factorisation interval $I(a)$ of a given arrow a in a decomposition space X. Intuitively, if X is just a category, the objects of $I(a)$ should be the two-step factorisations of a, with initial object id-followed-by-a and terminal object a-followed-by-id. The 1-cells should be arrows between such factorisations, or equivalently 3-step factorisations, and so on.

For a general (complete) decomposition space X, the idea can be formalised as follows. By Yoneda, to give an arrow $a \in X_1$ is to give $\Delta[1] \to X$ in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \boldsymbol{Grpd})$, or in the full subcategory $\boldsymbol{cDcmp}$. By adjunction, this is equivalent to giving $\Xi[-1] \to u^*X$ in $\boldsymbol{cFD}$. Now factor this map as a wide map followed by a cartesian map:



The object appearing in the middle is an algebraic interval since it is wide under $\Xi[-1]$ (6.3.3). By definition, the factorisation interval of a is $I(a) := i^*A$, equipped with a cULF map to X, as seen in the diagram



The map $\Delta[1] \to I(a)$ equips $I(a)$ with two endpoints, and a longest arrow between them. The cULF map $I(a) \to X$ sends the longest arrow of $I(a)$ to a.

More generally, by the same adjunction argument, given an k-simplex $\sigma : \Delta[k] \to X$ with long edge a, we get a k-subdivision of $I(a)$, i.e. a wide map $\Delta[k] \to I(a)$.

6.3.10. Remark. The factorisation-interval construction is due to Lawvere [42] in the case of a 1-category. In the case of a decomposition space, the construction shows, remarkably, that as far as comultiplication is concerned, any decomposition space is locally a Segal space, in the sense that the comultiplication of an arrow a may as well be performed inside $I(a)$, which is a Segal space by 6.2.17. So while there may be no global way to compose arrows even if their source and targets match, the *decompositions* that exist do compose again.

We proceed to formalise the factorisation-interval construction.

6.3.11. Coreflections. Inside the ∞-category of arrows $\mathrm{Ar}(\boldsymbol{cFD})$, denote by $\mathrm{Ar}^{\mathrm{w}}(\boldsymbol{cFD})$ the full subcategory spanned by the wide maps. The wide/cartesian factorisation system amounts to a coreflection

$$w : \mathrm{Ar}(\boldsymbol{cFD}) \longrightarrow \mathrm{Ar}^{\mathrm{w}}(\boldsymbol{cFD});$$

it sends an arrow $A \to B$ to its wide factor $A \to B'$, and in particular can be chosen to have A as domain again (6.1.4). In particular, for each algebraic interval $A \in \boldsymbol{aInt} \subset \boldsymbol{cFD}$, the adjunction restricts to an adjunction between coslice categories, with coreflection

$$w_A : \boldsymbol{cFD}_{A/} \longrightarrow \boldsymbol{cFD}^{\mathrm{w}}_{A/}.$$

The first ∞-category is that of flanked decomposition spaces under A, and the second ∞-category is that of flanked decomposition spaces with a wide map from A. Now, if a flanked decomposition space receives a wide map from an algebraic interval then it is itself an algebraic interval (6.3.3), and all maps of algebraic intervals are wide (6.3.7). So in the end the cosliced adjunction takes the form of the natural full inclusion functor

$$v_A : \boldsymbol{aInt}_{A/} \to \boldsymbol{cFD}_{A/}$$

and a right adjoint

$$w_A : \boldsymbol{cFD}_{A/} \to \boldsymbol{aInt}_{A/}.$$

6.3.12. Remark. These observations amount to saying that the functor $v : \boldsymbol{aInt} \to \boldsymbol{cFD}$ is a *colocal left adjoint*. This notion is dual to the important concept of local right adjoint [70].

We record the following obvious lemmas:

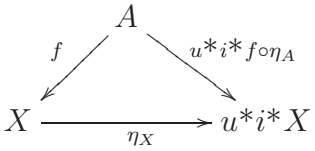
Lemma. 6.3.13. *The coreflection w sends cartesian maps to equivalences.*

Lemma. 6.3.14. *The counit is cartesian.*

6.3.15. Factorisation-interval as a comonad. We also have the basic adjunction $i^* \dashv u^*$ between complete decomposition spaces and complete flanked decomposition spaces. Applied to coslices over an algebraic interval A, and its underlying decomposition space $\underline{A} = i^*A$ we get the adjunction

$$L : \boldsymbol{cFD}_{A/} \rightleftarrows \boldsymbol{cDcmp}_{\underline{A}/} : R.$$

Here L is simply the functor i^*, while the right adjoint R is given by applying u^* and precomposing with the unit η_A. Note that the unit of this adjunction $L \dashv R$ at an object $f : A \to X$ is given by



We now combine the two adjunctions:

$$\boldsymbol{aInt}_{A/} \underset{w}{\overset{v}{\rightleftarrows}} \boldsymbol{cFD}_{A/} \underset{R}{\overset{L}{\rightleftarrows}} \boldsymbol{cDcmp}_{\underline{A}/}\,.$$

The factorisation-interval functor is the $\underline{A} = \Delta[k]$ instantiation:

$$I := L \circ v \circ w \circ R$$

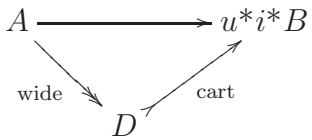
Indeed, this is precisely what we said in the construction, just phrased more functorially. It follows that the factorisation-interval construction is a comonad on $\boldsymbol{cDcmp}_{\underline{A}/}$.

Lemma. 6.3.16. *The composed counit is cULF.*

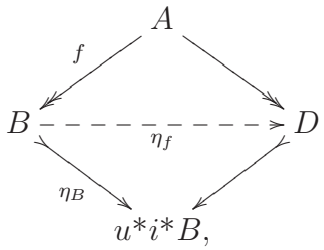
Proof. Follows readily from 6.2.13. □

Proposition. 6.3.17. *The composed unit $\eta : Id \Rightarrow w \circ R \circ L \circ v$ is an equivalence.*

Proof. The result of applying the four functors to an algebraic interval map $f : A \to B$ is the wide factor in



The unit on f sits in this diagram



where η_B is cartesian by 6.2.14. It follows now from orthogonality of the wide/cartesian factorisation system that η_f is an equivalence. □

Corollary. 6.3.18. *The functor $i^* \circ v : \boldsymbol{aInt} \to \boldsymbol{cDcmp}_{\Delta[1]/}$ is fully faithful.*

Proposition. 6.3.19. *I sends cULF maps to equivalences. In detail, for a cULF functor $F : Y \to X$ and any arrow $a \in Y_1$ we have a natural equivalence of intervals (and hence of underlying Segal spaces)*

$$I(a) \xrightarrow{\sim} I(Fa).$$

Proof. R sends cULF maps to cartesian maps, and w send cartesian maps to equivalences. □

Corollary. 6.3.20. *If X is an interval, with longest arrow $a \in X_1$, then $X \simeq I(a)$.*

Proposition. 6.3.21. *The composed functor*

$$\boldsymbol{aInt} \to \boldsymbol{cDcmp}_{\Delta[1]/} \to \boldsymbol{cDcmp}$$

is faithful (i.e. induces a monomorphism on mapping spaces).

Proof. Given two algebraic intervals A and B, denote by $f : \Delta[1] \to i^*A$ and $g : \Delta[1] \to i^*B$ the images in $\boldsymbol{cDcmp}_{\Delta[1]/}$. The claim is that the map

$$\mathrm{Map}_{\boldsymbol{aInt}}(A, B) \longrightarrow \mathrm{Map}_{\boldsymbol{cDcmp}_{\Delta[1]/}}(f, g) \longrightarrow \mathrm{Map}_{\boldsymbol{cDcmp}}(i^*A, i^*B)$$

is a monomorphism. We already know that the first part is an equivalence (by Corollary 6.3.18). The second map will be a monomorphism because of the special nature of f and g. We have a pullback diagram (mapping space fibre sequence for coslices):

$$\begin{array}{ccc} \mathrm{Map}_{\boldsymbol{cDcmp}_{\Delta[1]/}}(f,g) & \longrightarrow & \mathrm{Map}_{\boldsymbol{cDcmp}}(i^*A, i^*B) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{precomp}.f} \\ 1 & \xrightarrow[\ulcorner g \urcorner]{} & \mathrm{Map}_{\boldsymbol{cDcmp}}(\Delta[1], i^*B). \end{array}$$

Since $g : \Delta[1] \to i^*B$ is the image of the canonical map $\Xi[-1] \to B$, the map

$$1 \xrightarrow{\ulcorner g \urcorner} \mathrm{Map}_{\boldsymbol{cDcmp}}(\Delta[1], i^*B)$$

can be identified with

$$B_{-1} \xrightarrow{s_{\perp -1} s_{\top +1}} B_1,$$

which is a monomorphism since B is complete. It follows that the top map in the above pullback square is a monomorphism, as asserted. (Note the importance of completeness.) □

6.4. The decomposition space of intervals

6.4.1. Interval category as a full subcategory in *cDcmp*. We now invoke the general results about Kleisli categories (6.1.14). Let

$$\boldsymbol{Int} := \widetilde{\boldsymbol{aInt}}$$

denote the restricted Kleisli category for the adjunction

$$i^* : \boldsymbol{cFD} \rightleftarrows \boldsymbol{cDcmp} : u^* ,$$

as in 6.1.14. Hence $\boldsymbol{Int} \subset \boldsymbol{cDcmp}$ is the full subcategory of decomposition spaces underlying algebraic intervals. Say a map in $\boldsymbol{Int}$ is *wide* if it is the i^* image of a map in $\boldsymbol{aInt}$ (i.e. a wide map in $\boldsymbol{cFD}$).

Proposition. 6.4.2. *The wide maps as left-hand class and the cULF maps as right-hand class form a factorisation system on* $\boldsymbol{Int}$.

Proof. The wide/cartesian factorisation system on ***cFD*** is compatible with the adjunction $i^* \dashv u^*$ and the subcategory ***Int*** precisely as required to apply the general Lemma 6.1.14. Namely, we have:

— u^*i^* preserves cartesian maps by Corollary 6.2.15.

— $u^*\epsilon$ is cartesian by 6.2.5, since ϵ is cULF by 6.2.13.

— If $A \to B$ is wide, A an algebraic interval, then so is B, by 6.3.3.

The general Lemma 6.1.14 now tells us that there is a factorisation system on ***Int*** where the left-hand class are the maps of the form i^* of a wide map. The right-hand class of ***Int***, described by Lemma 6.1.14 as those maps f for which u^*f is cartesian, is seen by Lemma 6.2.16 to be precisely the cULF maps. □

We can also restrict the Kleisli category and the factorisation system to the category $\Xi+$ consisting of the representables together with the terminal object $\Xi[-2]$.

Lemma. 6.4.3. *The restriction of the Kleisli category to $\Xi+$ gives Δ, and the wide/cULF factorisation systems on* ***Int*** *restricts to the generic/free factorisation system on Δ.*

$$\begin{array}{ccccc} \Delta & \xrightarrow{\text{f.f.}} & \boldsymbol{Int} & \xrightarrow{\text{f.f.}} & \boldsymbol{cDcmp} \\ \uparrow & & \uparrow & & {\scriptstyle i^*}\uparrow\downarrow{\scriptstyle u^*} \\ \Xi+ & \longrightarrow & \boldsymbol{aInt} & \longrightarrow & \boldsymbol{cFD} \end{array}$$

Proof. By construction the objects are $[-2], [-1], [0], [1], \ldots$. By construction the hom spaces are

$$\begin{aligned} \mathrm{Map}_{\boldsymbol{Int}}(\Xi[k], \Xi[n]) &= \mathrm{Map}_{\boldsymbol{Dcmp}}(i^*\Xi[k], i^*\Xi[n]) \\ &= \mathrm{Map}_{\widehat{\Delta}}(\Delta[k+2], \Delta[n+2]) \\ &= \Delta([k+2], [n+2]). \end{aligned}$$

It is clear by the explicit description of i^* that it takes the maps in $\Xi+$ to the generic maps in Δ. On the other hand, it is clear that the cULF maps in Δ are the free maps. □

6.4.4. Arrow category and restriction to Δ. Let $\mathrm{Ar}^{\mathrm{w}}(\boldsymbol{Int}) \subset \mathrm{Ar}(\boldsymbol{Int})$ denote the full subcategory of the arrow category spanned by the wide maps. Recall (from 6.1.3) that $\mathrm{Ar}^{\mathrm{w}}(\boldsymbol{Int})$ is a cartesian fibration over ***Int*** via the domain projection. We now restrict this cartesian fibration to $\Delta \subset \boldsymbol{Int}$ as in 6.1.10:

$$\begin{array}{ccc} \mathrm{Ar}^{\mathrm{w}}(\boldsymbol{Int})_{|\Delta} & \xrightarrow{\text{f.f.}} & \mathrm{Ar}^{\mathrm{w}}(\boldsymbol{Int}) \\ {\scriptstyle \mathrm{dom}}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{dom}} \\ \Delta & \xrightarrow[\text{f.f.}]{} & \boldsymbol{Int} \end{array}$$

We put

$$\mathscr{U} := \mathrm{Ar}^{\mathrm{w}}(\boldsymbol{Int})_{|\Delta}.$$

$\mathscr{U} \to \Delta$ is the *Cartesian fibration of subdivided intervals*: the objects of $\mathscr{U}$ are the wide interval maps $\Delta[k] \twoheadrightarrow A$, which we think of as subdivided intervals. The arrows are commutative squares

$$\begin{array}{ccc} \Delta[k] & \longrightarrow & \Delta[n] \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

where the downwards maps are wide, and the rightwards maps are in Δ and in $\boldsymbol{cDcmp}$, respectively. (These cannot be realised in the world of Ξ^{op}-spaces, and the necessity of having them was the whole motivation for constructing $\boldsymbol{Int}$.) By 6.1.3, the cartesian maps are squares

$$\begin{array}{ccc} \Delta[k] & \longrightarrow & \Delta[n] \\ \downarrow & & \downarrow \\ A & \rightarrowtail & B. \end{array}$$

Hence, cartesian lift are performed by precomposing and then coreflecting (i.e. wide/cULF factorising and keeping only the wide part). For a fixed domain $\Delta[k]$, we have (in virtue of Proposition 6.3.21)

$$\boldsymbol{Int}^{\mathrm{w}}_{\Delta[k]/} \simeq \boldsymbol{aInt}_{\Xi[k-2]/}.$$

6.4.5. The decomposition space of intervals. The cartesian fibration $\mathscr{U} = \mathrm{Ar}^{\mathrm{w}}(\boldsymbol{Int})_{|\Delta} \to \Delta$ determines a right fibration, $U := \mathscr{U}^{\mathrm{cart}} = \mathrm{Ar}^{\mathrm{w}}(\boldsymbol{Int})^{\mathrm{cart}}_{|\Delta} \to \Delta$, and hence by straightening ([49], Ch.2) a simplicial ∞-groupoid

$$U : \Delta^{\mathrm{op}} \to \boldsymbol{Grpd}.$$

We shall not actually need the straightening, as it is more convenient to work directly with the right fibration $U \to \Delta$. Its fibre over $[k] \in \Delta$ is the ∞-groupoid U_k of k-subdivided intervals. That is, an interval $\underline{A}$ equipped with a wide map $\Delta[k] \twoheadrightarrow A$. Note that U_1 is equivalent to the ∞-groupoid $\boldsymbol{Int}^{\mathrm{eq}}$. Similarly, U_2 is equivalent to the ∞-groupoid of subdivided intervals, more precisely intervals with a wide map from $\Delta[2]$. Somewhat more exotic is U_0, the ∞-groupoid of intervals with a wide map from $\Delta[0]$. This means that the endpoints must coincide. This does not imply that the interval is trivial. For example, any category with a zero object provides an example of an object in U_0.

Among the generic maps in U, in each degree the unique map $g : U_r \to U_1$ consists in forgetting the subdivision. The space U also has the codomain projection $U \to \boldsymbol{Int}$. In particular we can describe the g-fibre over a given interval A:

Lemma. 6.4.6. *We have a pullback square*

$$\begin{array}{ccc} (A_r)_a & \longrightarrow & U_r \\ \downarrow & & \downarrow{\scriptstyle g} \\ * & \xrightarrow[\ulcorner A \urcorner]{} & U_1 \end{array}$$

where $a \in A_1$ denotes the longest edge.

Proof. Indeed, the fibre over a coslice is the mapping space, so the pullback is at first

$$\mathrm{Map}_{\mathrm{wide}}(\Delta[r], A)$$

But that's the full subgroupoid inside $\mathrm{Map}(\Delta[r], A) \simeq A_r$ consisting of the wide maps, but that means those whose restriction to the long edge is a. □

Theorem. 6.4.7. *The simplicial space $U : \Delta^{\mathrm{op}} \to \boldsymbol{Grpd}$ is a complete decomposition space.*

Proof. We first show it is a decomposition space. We need to show that for a generic-free pullback square in Δ^{op}, the image under U is a pullback:

$$\begin{array}{ccc} U_k & \xrightarrow{f'} & U_m \\ {\scriptstyle g'}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ U_n & \xrightarrow[f]{} & U_s \end{array}$$

This square is the outer rectangle in

$$\begin{array}{ccccccc} \boldsymbol{Int}^{\mathrm{w}}_{\Delta[k]/} & \xrightarrow{j} & \boldsymbol{Int}_{\Delta[k]/} & \xrightarrow{f'} & \boldsymbol{Int}_{\Delta[m]/} & \xrightarrow{w} & \boldsymbol{Int}^{\mathrm{w}}_{\Delta[m]/} \\ {\scriptstyle g'}\downarrow & & {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ \boldsymbol{Int}^{\mathrm{w}}_{\Delta[n]/} & \xrightarrow[j]{} & \boldsymbol{Int}_{\Delta[n]/} & \xrightarrow[f]{} & \boldsymbol{Int}_{\Delta[s]/} & \xrightarrow[w]{} & \boldsymbol{Int}^{\mathrm{w}}_{\Delta[s]/} \end{array}$$

(Here we have omitted taking maximal ∞-groupoids, but it doesn't affect the argument.) The first two squares consist in precomposing with the free maps f, f'. The result will no longer be a wide map, so in the middle columns we allow arbitrary maps. But the final step just applies the coreflection to take the wide part. Indeed this is how cartesian lifting goes in $\mathrm{Ar}^{\mathrm{w}}(\boldsymbol{Int})$. The first square is a pullback since j is fully faithful. The last square is a pullback since it is a special case of Lemma 6.1.9. The main point is the second square which is a pullback by Lemma 6.1.16 — this is where we use that the generic-free square in Δ^{op} is a pullback.

To establish that U is complete, we need to check that the map $U_0 \to U_1$ is a monomorphism. This map is just the forgetful functor

$$(\boldsymbol{Int}^{\mathrm{w}}_{*/})^{\mathrm{eq}} \to \boldsymbol{Int}^{\mathrm{eq}}.$$

The claim is that its fibres are empty or contractible. The fibre over an interval $\underline{A} = i^*A$ is

$$\mathrm{Map}_{\mathrm{wide}}(*, \underline{A}) = \mathrm{Map}_{\boldsymbol{aInt}}(\Xi[-2], A) = \mathrm{Map}_{\widehat{\Xi}}(\Xi[-2], A).$$

Note that in spite of the notation, $\Xi[-2]$ is not a representable: it is the terminal object, and it is hence the colimit of all the representables. It follows that $\mathrm{Map}_{\widehat{\Xi}}(\Xi[-2], A) = \lim A$. This is the limit of a cosimplicial diagram

$$\lim A \xrightarrow{e} * \rightrightarrows A_0 \cdots$$

In general the limiting map of a cosimplicial diagram does not have to be a monomorphism, but in this case it is, as all the coface maps (these are the degeneracy maps of A) are monomorphisms by completeness of A, and since A_{-1} is contractible. Since finally e is a monomorphism into the contractible space A_{-1}, the limit must be empty or contractible. Hence $U_0 \to U_1$ is a monomorphism, and therefore U is complete. □

6.5. **Universal property of** U

The refinements discussed in 6.1.12 now pay off to give us the following main result. Let $\boldsymbol{Int}{\downarrow}\boldsymbol{cDcmp}$ denote the comma category (as in 6.1.13). It is the full subcategory in $\mathrm{Ar}(\boldsymbol{cDcmp})$ spanned by the maps whose domain is in $\boldsymbol{Int}$. Let $\mathrm{Ar}^{\mathrm{w}}(\boldsymbol{Int})$ denote the full subcategory of $\mathrm{Ar}(\boldsymbol{Int})$ spanned by the wide maps. Recall (from 6.1.3) that both $\boldsymbol{Int}{\downarrow}\boldsymbol{cDcmp}$ and $\mathrm{Ar}^{\mathrm{w}}(\boldsymbol{Int})$ are cartesian fibrations over $\boldsymbol{Int}$ via the domain projections, and that the inclusion $\mathrm{Ar}^{\mathrm{w}}(\boldsymbol{Int}) \to \boldsymbol{Int}{\downarrow}\boldsymbol{cDcmp}$ commutes with the projections (but does not preserve cartesian arrows).

Theorem. 6.5.1. *The inclusion functor* $\mathrm{Ar}^{\mathrm{w}}(\boldsymbol{Int}) \hookrightarrow \boldsymbol{Int}{\downarrow}\boldsymbol{cDcmp}$ *has a right adjoint*

$$I : \boldsymbol{Int}{\downarrow}\boldsymbol{cDcmp} \to \mathrm{Ar}^{\mathrm{w}}(\boldsymbol{Int}),$$

which takes cartesian arrows to cartesian arrows.

Proof. We have already checked, in the proof of 6.4.2, that the conditions of the general Theorem 6.1.13 are satisfied by the adjunction $i^* \dashv u^*$ and the wide/cartesian factorisation system on $\boldsymbol{cFD}$. It remains to restrict this adjunction to the full subcategory $\boldsymbol{aInt} \subset \boldsymbol{cFD}$. □

Note that over an interval $\underline{A}$, the adjunction restricts to the adjunction of 6.3.15 as follows:

$$\begin{array}{ccc}
\boldsymbol{Int}^{\mathrm{w}}_{\underline{A}/} & \underset{I}{\rightleftarrows} & \boldsymbol{cDcmp}_{\underline{A}/} \\
\simeq \downarrow & & L \uparrow \downarrow R \\
\boldsymbol{aInt}_{A/} & \underset{w}{\overset{v}{\rightleftarrows}} & \boldsymbol{cFD}_{A/}
\end{array}$$

We now restrict these cartesian fibrations further to $\Delta \subset \boldsymbol{Int}$. We call the coreflection I, as it is the factorisation-interval construction:

$$\begin{array}{ccc} \mathscr{U} = \mathrm{Ar}^{\mathrm{w}}(\boldsymbol{Int})_{|\Delta} & \underset{I}{\rightleftarrows} & \Delta{\downarrow}\boldsymbol{cDcmp} \\ & {\scriptstyle \mathrm{dom}} \searrow \quad \swarrow {\scriptstyle \mathrm{dom}} & \\ & \Delta & \end{array}$$

The coreflection

$$I : \Delta{\downarrow}\boldsymbol{cDcmp} \to \mathscr{U}$$

is a morphism of cartesian fibrations over Δ (i.e. preserves cartesian arrows). Hence it induces a morphism of right fibrations

$$I : (\Delta{\downarrow}\boldsymbol{cDcmp})^{\mathrm{cart}} \to U.$$

Theorem. 6.5.2. *The morphism of right fibrations*

$$I : (\Delta{\downarrow}\boldsymbol{cDcmp})^{\mathrm{cart}} \to U$$

is cULF.

Proof. We need to establish that for the unique generic map $g : \Delta[1] \to \Delta[k]$, the following square is a pullback:

$$\begin{array}{ccc} \boldsymbol{cDcmp}_{\Delta[k]/} & \xrightarrow{\mathrm{pre}.g} & \boldsymbol{cDcmp}_{\Delta[1]/} \\ \downarrow{\scriptstyle I_k} & & \downarrow{\scriptstyle I_1} \\ \boldsymbol{Int}^{\mathrm{w}}_{\Delta[k]/} & \xrightarrow[\mathrm{pre}.g]{} & \boldsymbol{Int}^{\mathrm{w}}_{\Delta[1]/}. \end{array}$$

Here the functors I_1 and I_k are the coreflections of Theorem 6.5.1. We compute the fibres of the horizontal maps over a point $a : \Delta[1] \to X$. For the first row, the fibre is

$$\mathrm{Map}_{\boldsymbol{cDcmp}_{\Delta[1]/}}(g, a).$$

For the second row, the fibre is

$$\mathrm{Map}_{\boldsymbol{Int}^{\mathrm{w}}_{\Delta[1]/}}(g, I_1(a)).$$

But these two spaces are equivalent by the adjunction of Theorem 6.5.1. □

Inside $\Delta{\downarrow}\boldsymbol{cDcmp}$, we have the fibre over X, for the codomain fibration (which is a cocartesian fibration). This fibre is just $\Delta_{/X}$ the Grothendieck construction of the presheaf X. This fibre clearly includes into the cartesian part of $\Delta{\downarrow}\boldsymbol{cDcmp}$.

Lemma. 6.5.3. *The associated morphism of right fibrations*

$$\Delta_{/X} \to (\Delta{\downarrow}\boldsymbol{cDcmp})^{\mathrm{cart}}$$

is cULF.

Proof. For $g : \Delta[k] \to \Delta[1]$ the unique generic map in degree k, consider the diagram

$$\begin{array}{ccccc} \mathrm{Map}(\Delta[k], X) & \xrightarrow{\mathrm{pre}.g} & \mathrm{Map}(\Delta[1], X) & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle \ulcorner X \urcorner} \\ \boldsymbol{cDcmp}_{\Delta[k]/} & \xrightarrow[\mathrm{pre}.g]{} & \boldsymbol{cDcmp}_{\Delta[1]/} & \xrightarrow[\mathrm{codom}]{} & \boldsymbol{cDcmp}. \end{array}$$

The right-hand square and the outer rectangle are obviously pullbacks, as the fibres of coslices are the mapping spaces. Hence the left-hand square is a pullback, which is precisely to say that the vertical map is cULF. □

So altogether we have cULF map

$$\Delta_{/X} \to (\Delta{\downarrow}\boldsymbol{cDcmp})^{\mathrm{cart}} \to U,$$

or, by straightening, a cULF map of complete decomposition spaces

$$I : X \to U,$$

the *classifying map*. It takes a k-simplex in X to a k-subdivided interval, as already detailed in 6.3.

6.5.4. Conjecture. *U is a terminal object in the ∞-category $\boldsymbol{cDcmp}^{\mathrm{cULF}}$ of complete decomposition spaces and cULF maps.*

At the moment we are only able to prove the following weaker statement.

Theorem. 6.5.5. *For each complete decomposition space X, the space $\mathrm{Map}_{\boldsymbol{cDcmp}^{\mathrm{cULF}}}(X, U)$ is connected.*

Proof. Suppose $J : X \to U$ and $J' : X \to U$ are two cULF functors. As in the proof of Theorem 6.5.2, cULFness is equivalent to saying that we have a pullback

$$\begin{array}{ccc} \mathrm{Map}_{\boldsymbol{cDcmp}}(\Delta[k], X) & \xrightarrow{\mathrm{pre}.g} & \mathrm{Map}_{\boldsymbol{cDcmp}}(\Delta[1], X) \\ {\scriptstyle J_k}\downarrow & \lrcorner & \downarrow{\scriptstyle J_1} \\ \boldsymbol{Int}^{\mathrm{w}}_{\Delta[k]/} & \xrightarrow[\mathrm{pre}.g]{} & \boldsymbol{Int}^{\mathrm{w}}_{\Delta[1]/}. \end{array}$$

We therefore have equivalences between the fibres over a point $a : \Delta[1] \to X$:

$$\mathrm{Map}_{\boldsymbol{cDcmp}_{\Delta[1]/}}(g, a) \simeq \mathrm{Map}_{\boldsymbol{Int}^{\mathrm{w}}_{\Delta[1]/}}(g, J_1(a)).$$

But the second space is equivalent to $\mathrm{Map}_{\boldsymbol{Int}^{\mathrm{w}}}(\Delta[k], J_1(a))$. Since these equivalences hold also for J', we get

$$\mathrm{Map}_{\boldsymbol{Int}^{\mathrm{w}}}(\Delta[k], J_1(a)) \simeq \mathrm{Map}_{\boldsymbol{Int}^{\mathrm{w}}}(\Delta[k], J'_1(a)),$$

naturally in k. This is to say that $J_1(a)$ and $J'_1(a)$ are levelwise equivalent simplicial spaces. But a cULF map is determined by its

1-component, so J and J' are equivalent in the functor category. In particular, every object in $\mathrm{Map}^{\mathrm{cULF}}(X, U)$ is equivalent to the canonical I constructed in the previous theorems. □

6.6. **Möbius intervals and the universal Möbius function**

We finally impose the Möbius condition.

6.6.1. Nondegeneracy. Recall from 2.3.8 that for a complete decomposition space X we have

$$\vec{X}_r \subset X_r$$

the full subgroupoid of r-simplices none of whose principal edges are degenerate. These can also be described as the full subgroupoid

$$\vec{X}_r \simeq \mathrm{Map}_{\mathrm{nondegen}}(\Delta[r], X) \subset \mathrm{Map}(\Delta[r], X) \simeq X_r$$

consisting of the nondegenerate maps, i.e. maps for which the restriction to any principal edge $\Delta[1] \to \Delta[r]$ is non-degenerate.

Now assume that A is an interval. Inside

$$\mathrm{Map}_{\mathrm{nondegen}}(\Delta[r], A) \simeq \vec{A}_r$$

we can further require the maps to be wide. It is clear that this corresponds to considering only non-degenerate simplices whose longest edge is the longest edge $a \in A_1$:

Lemma. 6.6.2.

$$\mathrm{Map}_{\mathrm{wide+nondegen}}(\Delta[r], A) \simeq (\vec{A}_r)_a.$$

6.6.3. Nondegeneracy in U. In the case of $U : \Delta^{\mathrm{op}} \to \boldsymbol{Grpd}$, it is easy to describe the spaces $\vec{U}_r$. They consist of wide maps $\Delta[r] \to A$ for which none of the restrictions to principal edges $\Delta[1] \to A'$ are degenerate. In particular we can describe the fibre over a given interval A (in analogy with 6.4.6):

Lemma. 6.6.4. *We have a pullback square*

$$\begin{array}{ccc} (\vec{A}_r)_a & \longrightarrow & \vec{U}_r \\ \downarrow & & \downarrow \\ * & \xrightarrow[\ulcorner A \urcorner]{} & U_1. \end{array}$$

6.6.5. Möbius intervals. Recall (from 3.2.1) that a complete decomposition space X is called Möbius when the map

$$\sum \vec{X}_r \to X_1$$

is finite. A *Möbius interval* is an interval which is Möbius as a decomposition space.

Proposition. 6.6.6. *Any Möbius interval is a Rezk complete Segal space.*

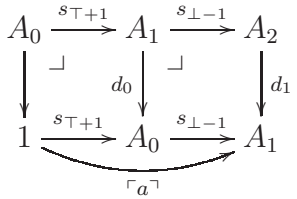
Proof. Just by being an interval it is a Segal space (by 6.2.17). Now the Möbius condition implies the Rezk condition by Proposition 3.2.2. □

Lemma. 6.6.7. *If X is a Möbius decomposition space, then for each $a \in X_1$, the interval $I(a)$ is a Möbius interval.*

Proof. We have a cULF map $I(a) \to X$. Hence by Proposition 3.2.4, $I(a)$ is again Möbius. □

Proposition. 6.6.8. *If A is a Möbius interval then for every r, the space A_r is finite.*

Proof. The squares



are pullbacks by the flanking condition 6.2.9 (the second is a bonus pullback, cf. 6.2.10). The bottom composite arrow picks out the long edge $a \in A_1$. (That the outer square is a pullback can be interpreted as saying that the 2-step factorisations of a are parametrised by their midpoint, which can be any point in A_0.) Since the generic maps of A are finite by Lemma 3.2.3, the map $d_1 : A_2 \to A_1$ is finite, hence the fibre A_0 is finite. The same argument works for arbitrary r, by replacing the top row by $A_r \to A_{r+1} \to A_{r+2}$, and letting the columns be d_0^r, d_0^r and d_1^r. □

(This can be seen as a homotopy version of [43] Lemma 2.3.)

Corollary. 6.6.9. *For a Möbius interval, the total space of all nondegenerate simplices $\sum_r \vec{A}_r$ is finite.*

Proof. This follows from the previous lemma and the definition of Möbius. □

6.6.10. The decomposition space of Möbius intervals. There is a decomposition space $\mathbf{MI} \subset U$ consisting of all Möbius intervals. In each degree, $\mathbf{MI}_k$ is the full subgroupoid of U_k consisting of the wide maps $\Delta[k] \to A$ for which A is Möbius.

Theorem. 6.6.11. *The decomposition space* $\mathbf{MI}$ *is Möbius.*

Proof. We need to prove that the map $\sum_r \vec{\mathbf{MI}}_r \to \mathbf{MI}_1$ is a finite map. So check out the fibre: fix a Möbius interval $A \in \mathbf{MI}_1$, and let $a \in A_1$ be the longest edge. From Lemma 6.6.4 we see that the fibre over A is $(\sum_r \vec{A}_r)_a = \sum_r (\vec{A}_r)_a$. But this is the fibre over $a \in A_1$ of the map $\sum_r \vec{A}_r \to A_1$, which is finite by the assumption that A is Möbius. □

Proposition. 6.6.12. *A decomposition space X is Möbius if and only if its classifying map factors through* $\mathbf{MI} \subset U$.

Proof. If the classifying map factors through $X \to \mathbf{MI}$, then X is cULF over a Möbius space, hence is itself Möbius. Conversely, if X is Möbius, then all its intervals are Möbius (6.6.7), hence clearly the image of the classifying map is contained in $\mathbf{MI}$. □

6.6.13. The universal Möbius function. The decomposition space of all intervals is complete, hence it has Möbius inversion in the abstract sense of 2.3.14. The Möbius function is the formal difference $\Phi_{\text{even}} - \Phi_{\text{odd}}$. Since every complete decomposition space X has a canonical cULF map to U, it follows that the Möbius function of X is induced from that of U. The latter can therefore be called the *universal Möbius function*.

The same reasoning works in the Möbius situation, and implies the existence of a universal Möbius function numerically. Namely, since $\mathbf{MI}$ is Möbius, its Möbius function ζ admits a cardinality $|\zeta|$:

Theorem. 6.6.14. *In the incidence algebra $\mathbb{Q}^{\pi_0\mathbf{MI}}$, the Möbius function $|\zeta| : \pi_0\mathbf{MI} \to \mathbb{Q}$ is invertible under convolution, and its inverse is the universal Möbius function*

$$|\mu| := |\Phi_{\text{even}}| - |\Phi_{\text{odd}}|\,.$$

The Möbius function in the (numerical) incidence algebra of any Möbius decomposition space is induced from this universal Möbius function via the classifying map.

Appendix A. Homotopy linear algebra and homotopy cardinality

A.0.0. Objective algebraic combinatorics. One may say that algebraic combinatorics is the study of combinatorial structures via algebraic objects associated to them. In the classical theory of Möbius inversion of Rota et al., the combinatorial objects are locally finite posets, and the associated algebraic structures are their incidence coalgebras and algebras, whose underlying vector spaces are freely generated by intervals in the poset. In our theory, decomposition spaces are viewed as a generalisation of the notion of poset. Instead of vector spaces to support the associated algebraic structures, we work with certain linear structures generated directly by the combinatorial objects (with coefficients in ∞-groupoids). This is the so-called 'objective method', advocated in particular by Lawvere and Schanuel (see Lawvere–Menni [43] for an explicit objective treatment of the Leroux theory of Möbius categories); the next level of objectivity is often called 'groupoidification', developed in particular by Baez, Hoffnung and Walker [4], where the scalars are (suitably finite) 1-groupoids. In the present work we take coefficients in ∞-groupoids, and hence incorporate homotopy theory.

At the same time, the abstract viewpoints forced upon us by this setting lead to some conceptual simplifications even at the 1-groupoid level.

A.0.1. Groupoid slices as vector spaces. To deal with algebraic structures at the objective level requires at least to be able to form sums (linear combinations). In analogy with taking the free vector space on a set, we can take the homotopy-sum completion of an ∞-groupoid S: this is (cf. A.1.6) the homotopy slice ∞-category $\boldsymbol{Grpd}_{/S}$, whose objects are groupoid maps $X \to S$. It stands in for the free vector space on a set $\pi_0 S$: just as a vector is a (finite) $\pi_0 S$-indexed family of scalars (namely its coordinates with respect to the basis), an object $X \to S$ in $\boldsymbol{Grpd}_{/S}$ is interpreted as S-indexed family of ∞-groupoids X_s, hence the fibre X_s plays the role of the sth coordinate.

The groupoid slices form an ∞-category in which the morphisms are the homotopy-sum preserving functors, the objective analogue of linear maps. They are given by spans of ∞-groupoids, i.e. doubly indexed families of ∞-groupoids, just as ordinary linear maps are given by matrices of numbers (once a basis has been chosen).

To really mimic vector spaces, where linear combinations are finite sums, we should require the total space X to be finite in a suitable sense (while the base is allowed to be infinite). Then one can take homotopy cardinality, and recover linear algebra over $\mathbb{Q}$. The finiteness conditions are needed to be able to take homotopy cardinality. However, as long as we are working at the objective level, it is not necessary to impose the finiteness conditions, and in fact, the theory can be developed more comfortably without them. Furthermore, the notion of homotopy cardinality is not the only notion of size: Euler characteristic and various multiplicative cohomology theories are other alternatives, and it is reasonable to expect that the future will reveal more comprehensive and unified notions of size and measures. For these reasons, we begin (A.1) with 'linear algebra' without finiteness conditions, and then proceed to incorporate finiteness conditions expressed in terms of homotopy groups.

A.0.2. Overview.

In Subsection A.1 we define the ∞-category $\boldsymbol{LIN}$ of groupoid slices and linear functors, without imposing any finiteness conditions.

For the finiteness conditions, the goals are:

(1) Define 'finite ∞-groupoid' and define homotopy cardinality of a finite ∞-groupoid.

(2) Define homotopy cardinality of 'finite' families (for example elements in the incidence coalgebras): if $x : X \to S$ is a family with X finite, its cardinality should be an element in the vector space $\mathbb{Q}_{\pi_0 S}$ freely generated by the set $\pi_0 S$.

(3) Define homotopy cardinality of finite presheaves (this is needed for the incidence algebras): these will take values in profinite-dimensional vector spaces.

To set this up uniformly, we follow Baez-Hoffnung-Walker [4] and define a cardinality *functor* from a certain ∞-category of finite slices and linear functors to vector spaces. From this 'meta cardinality', all the individual notions of cardinality of families and presheaves are induced, by observing that vectors are the same thing as linear maps from the ground field.

The 'linear' ∞-categories of groupoid slices are introduced as follows. There is an ∞-category $\boldsymbol{lin}$ whose objects are ∞-categories of the form $\boldsymbol{grpd}_{/\alpha}$ where α is a finite ∞-groupoid. The morphisms are finite spans $\alpha \leftarrow \mu \rightarrow \beta$. This ∞-category corresponds to the category $\boldsymbol{vect}$ of finite-dimensional vector spaces. We need infinite indexing, so the following two extensions are introduced.

There is an ∞-category $\underrightarrow{\boldsymbol{lin}}$ whose objects are ∞-categories of the form $\boldsymbol{grpd}_{/S}$ with S an 'arbitrary' ∞-groupoid, and whose morphisms are spans of finite type (i.e. the left leg has finite fibres). This ∞-category corresponds to the category $\underrightarrow{\boldsymbol{vect}}$ of general vector spaces (allowing infinite-dimensional ones).

Finally we have the ∞-category $\underleftarrow{\boldsymbol{lin}}$ whose objects are ∞-categories of the form $\boldsymbol{grpd}^S$ with S an 'arbitrary' ∞-groupoid, and whose morphisms are spans of profinite type (i.e. the right leg has finite fibres). This ∞-category corresponds to the category $\underleftarrow{\boldsymbol{vect}}$ of pro-finite-dimensional vector spaces.

Remark. To set up all this in order to define meta cardinality, it is actually only necessary to have 1-categories. This means that it is enough to consider equivalence classes of spans. However, although cardinality is a main motivation, we are equally interested in understanding how all this works at the objective level. This turns out to throw light on the deeper meaning of ind and pro, and actually to understand vector spaces better.

To actually introduce $\boldsymbol{lin}$, $\underrightarrow{\boldsymbol{lin}}$ and $\underleftarrow{\boldsymbol{lin}}$ as ∞-categories, we first 'extend scalars' from $\boldsymbol{grpd}$ to $\boldsymbol{Grpd}$, where there is more elbow room to perform the constructions. We work in the ambient ∞-category $\boldsymbol{LIN}$. So we define, as subcategories of $\boldsymbol{LIN}$: the ∞-category $\boldsymbol{Lin}$ consisting of $\boldsymbol{Grpd}_{/\alpha}$ and finite spans, the ∞-category $\underrightarrow{\boldsymbol{Lin}}$ consisting of $\boldsymbol{Grpd}_{/S}$ and spans of finite type, and the ∞-category $\underleftarrow{\boldsymbol{Lin}}$ consisting of $\boldsymbol{Grpd}^S$ and spans of profinite type. In the latter case, we can characterise the mapping spaces in terms of an attractive continuity condition.

The three ∞-categories constructed with $\boldsymbol{Grpd}$ coefficients are in fact equivalent to the three ∞-categories with $\boldsymbol{grpd}$ coefficients introduced heuristically.

There is a perfect pairing $\boldsymbol{grpd}_{/S} \times \boldsymbol{grpd}^S \to \boldsymbol{grpd}$, which upon taking cardinality yields the pairing $\mathbb{Q}_{\pi_0 S} \times \mathbb{Q}^{\pi_0 S} \to \mathbb{Q}$.

A.1. **Homotopy linear algebra without finiteness conditions**

A.1.1. Fundamental equivalence. Fundamental to many constructions and arguments is the canonical equivalence

$$\boldsymbol{Grpd}_{/S} \simeq \boldsymbol{Grpd}^S$$

which is the homotopy version of the equivalence $\boldsymbol{Set}_{/S} \simeq \boldsymbol{Set}^S$ (for S a set), expressing the two ways of encoding a family of sets $\{X_s \mid s \in S\}$: either regarding the members of the family as the fibres of a map $X \to S$, or as a parametrisation of sets $S \to \boldsymbol{Set}$. So the equivalence associates to an object $X \to S$ the functor $S^{\mathrm{op}} \to \boldsymbol{Grpd}$ that sends s to the ∞-groupoid X_s. The other direction is the Grothendieck construction, which works as follows: any presheaf $F : S \to \boldsymbol{Grpd}$ sits over the terminal presheaf $*$. The associated map is $\mathrm{colim}(F)$ sitting over $\mathrm{colim}(*)$. It remains to observe that $\mathrm{colim}(*)$ is equivalent to S itself. More formally, the Grothendieck construction equivalence is a consequence of a finer result, namely Lurie's straightening theorem. Lurie constructs ([49], Thm.2.1.2.2) a Quillen equivalence between the category of right fibrations over S and the category of (strict) simplicial presheaves on $\mathfrak{C}[S]$. Combining this result with the fact that simplicial presheaves on $\mathfrak{C}[S]$ is a model for the functor ∞-category $\mathrm{Fun}(S^{\mathrm{op}}, \boldsymbol{Grpd})$ (see [49], Proposition 5.1.1.1), the Grothendieck construction equivalence follows. Note that when S is just an ∞-groupoid (i.e. a Kan complex), $X \to S$ is a right fibration if and only if X itself is an ∞-groupoid. Hence altogether $\boldsymbol{Grpd}_{/S} \simeq \mathrm{Fun}(S^{\mathrm{op}}, \boldsymbol{Grpd})$, and since S^{op} is canonically equivalent to S (since it is just an ∞-groupoid), this establishes the fundamental equivalence from this fancier viewpoint.

A.1.2. Scalar multiplication and homotopy sums. The 'lowershriek' operation

$$f_! : \boldsymbol{Grpd}_{/I} \to \boldsymbol{Grpd}_{/J}$$

along a map $f : I \to J$ has two special cases, which play the role of scalar multiplication (tensoring with an ∞-groupoid) and vector addition (homotopy sums):

The ∞-category $\boldsymbol{Grpd}_{/I}$ is *tensored* over $\boldsymbol{Grpd}$. Given $S \in \boldsymbol{Grpd}$ and $g : X \to I$ in $\boldsymbol{Grpd}_{/I}$ we have

$$S \otimes g \;\; := \;\; g_!(S \times X \to X) : S \times X \to I \text{ in } \boldsymbol{Grpd}_{/I}.$$

It also has *homotopy sums*, by which we mean colimits indexed by an ∞-groupoid. The colimit of a functor $F : B \to \boldsymbol{Grpd}_{/I}$ is a special case of the lowershriek. Namely, the functor F corresponds by adjunction

to an object $g : X \to B \times I$ in $\boldsymbol{Grpd}_{/B\times I}$, and we have

$$\mathrm{colim}(F) = p_!(g)$$

where $p : B \times I \to I$ is the projection. We interpret this as the homotopy sum of the family $g : X \to B \times I$ with members

$$g_b : X_b \longrightarrow \{b\} \times I = I,$$

and we denote the homotopy sum by an integral sign:

$$\int^{b\in B} g_b \;:=\; p_! g \;\text{ in } \boldsymbol{Grpd}_{/I}. \tag{20}$$

Example. A.1.3. With $I = 1$, this gives the important formula

$$\int^{b\in B} X_b = X,$$

expressing the total space of $X \to B$ as the homotopy sum of its fibres.

Using the above, we can define the B-indexed *linear combination* of a family of vectors $g : X \to B \times I$ and scalars $f : S \to B$,

$$\int^{b\in B} S_b \otimes g_b = p_!(g_!(t)) \;: S \times_B X \to I \;\text{ in } \boldsymbol{Grpd}_{/I},$$

as illustrated in the first row of the following diagram

$$\begin{array}{ccccccc} S \times_B X & \xrightarrow{f'} & X & \xrightarrow{g} & B \times I & \xrightarrow{p} & I \\ \downarrow & \lrcorner & & \searrow & \downarrow{\scriptstyle q} & & \\ S & & \xrightarrow{\quad f \quad} & & B. & & \end{array} \tag{21}$$

Note that the members of the family $g_!(f')$ are just $(g_!(f'))_b = S_b \otimes g_b$.

A.1.4. Basis. In $\boldsymbol{Grpd}_{/S}$, the names $\ulcorner s \urcorner : 1 \to S$ play the role of a basis. Every object $X \to S$ can be written uniquely as a linear combination of basis elements, or by allowing repetition of the basis elements instead of scalar multiplication, as a homotopy sum of basis elements:

Lemma. A.1.5. *Given $f : S \to B$ in $\boldsymbol{Grpd}_{/B}$, we have*

$$f = \int^{s\in S} \ulcorner f(s) \urcorner = \int^{b\in B} S_b \otimes \ulcorner b \urcorner.$$

Proof. For the first expression, take as family $S \xrightarrow{(\mathrm{id},f)} S \times B$. Then the members of the family are the names $\ulcorner f(s) \urcorner$, and the formula follows from (20). For the second expression, take as family $g : B \xrightarrow{(\mathrm{id},\mathrm{id})} B \times B$, and as scalars $f : S \to B$ itself. Then the members of g are the names $\ulcorner b \urcorner$, and the scalars are S_b, and in (21) we have $p_!(g_!(f')) = f$ since pg and qg are the identity. □

The name $\ulcorner b \urcorner : 1 \to B$ corresponds under the Grothendieck construction to the representable functor

$$\begin{array}{ccc} B & \longrightarrow & \boldsymbol{Grpd} \\ x & \longmapsto & \mathrm{Map}(b, x) \end{array}$$

Thus, interpreted in presheaf category $\boldsymbol{Grpd}^B$, the Lemma is the standard result expressing any presheaf as a colimit of representables.

Proposition. A.1.6. *$\boldsymbol{Grpd}_{/S}$ is the homotopy-sum completion of S. Precisely, for $\mathscr{C}$ an ∞-category admitting homotopy sums, precomposition with the Yoneda embedding $S \to \boldsymbol{Grpd}_{/S}$ induces an equivalence of ∞-categories*

$$\mathrm{Fun}^{\int}(\boldsymbol{Grpd}_{/S}, \mathscr{C}) \xrightarrow{\simeq} \mathrm{Fun}(S, \mathscr{C}),$$

where the functor category on the left consists of homotopy-sum preserving functors.

Proof. Since every object in $\boldsymbol{Grpd}_{/S}$ can be written as a homotopy sum of names, to preserve homotopy sums is equivalent to preserving all colimits, so the natural inclusion $\mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{Grpd}_{/S}, \mathscr{C}) \to \mathrm{Fun}^{\int}(\boldsymbol{Grpd}_{/S}, \mathscr{C})$ is an equivalence. It is therefore enough to establish the equivalence

$$\mathrm{Fun}^{\mathrm{colim}}(\boldsymbol{Grpd}_{/S}, \mathscr{C}) \xrightarrow{\simeq} \mathrm{Fun}(S, \mathscr{C}).$$

In the case where $\mathscr{C}$ is cocomplete, this is true since $\boldsymbol{Grpd}_{/S} \simeq \mathrm{Fun}(S^{\mathrm{op}}, \boldsymbol{Grpd})$ is the colimit completion of S. The proof of this statement (Lurie [49], Theorem 5.1.5.6) goes as follows: it is enough to prove that left Kan extension of any functor $S \to \mathscr{C}$ along Yoneda exists and preserves colimits. Existence follows from [49, Lemma 4.3.2.13] since $\mathscr{C}$ is assumed cocomplete, and the fact that left Kan extensions preserve colimits [49, Lemma 5.1.5.5 (1)] is independent of the cocompleteness of $\mathscr{C}$. In our case $\mathscr{C}$ is not assumed to be cocomplete but only to admit homotopy sums. But since S is just an ∞-groupoid in our case, this is enough to apply Lemma 4.3.2.13 of [49] to guarantee the existence of the left Kan extension. □

A.1.7. Linear functors. A span

$$I \xleftarrow{p} M \xrightarrow{q} J$$

defines a *linear functor*

$$\boldsymbol{Grpd}_{/I} \xrightarrow{p^*} \boldsymbol{Grpd}_{/M} \xrightarrow{q_!} \boldsymbol{Grpd}_{/J}. \tag{22}$$

Lemma. A.1.8. *Linear functors preserve linear combinations.*

Proof. Suppose $\int^{b \in B} S_b \otimes g_b$ is the B-indexed linear combination of $f : S \to B$ and $g : X \to B \times I$ in $\boldsymbol{Grpd}_{/I}$. This is shown in the middle

row of the following diagram, and in the top row is shown the result of applying a linear functor L given by (22)

$$\begin{array}{lccccccccc}
L(\int^{b\in B} S_b \otimes g_b): & E & \xrightarrow{f''} & X' & \xrightarrow{g'} & B\times M & \longrightarrow & M & \xrightarrow{q} & J \\
 & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle B\times p} & \lrcorner & \downarrow{\scriptstyle p} & & \\
\int^{b\in B} S_b \otimes g_b: & S\times_B X & \xrightarrow{f'} & X & \xrightarrow{g} & B\times I & \longrightarrow & I & & \\
 & \downarrow & \lrcorner & \searrow & & \downarrow & & & & \\
 & S & & \xrightarrow{f} & & B & & & &
\end{array}$$

Now observe that f'' is the pullback of f along $X' \xrightarrow{g'} B\times M \to B$, and that the family $L(g)$ is given by $X' \xrightarrow{g'} B\times M \xrightarrow{B\times q} B\times J$. The result is now clear, since the first row of the diagram coincides with

$$\int^{b\in B} S_b \otimes L(g)_b: \quad E \xrightarrow{f''} X' \xrightarrow{g'} B\times M \xrightarrow{B\times q} B\times J \longrightarrow J$$

as required. □

A.1.9. Coordinates. Coming back to the span

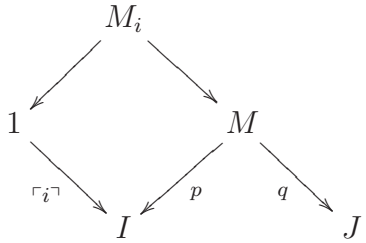
$$I \xleftarrow{p} M \xrightarrow{q} J$$

and the linear functor

$$q_! p^* : \boldsymbol{Grpd}_{/I} \longrightarrow \boldsymbol{Grpd}_{/J},$$

consider an element $\ulcorner i \urcorner : 1 \to I$. Then we have, by Lemma A.1.5,

$$q_! p^* \ulcorner i \urcorner = (M_i \to J) = \int^{j\in J} M_{i,j} \otimes \ulcorner j \urcorner$$



For a more general element $f : X \to I$ we have $f = \int^i X_i \otimes \ulcorner i \urcorner$ and so by homotopy linearity A.1.8

$$q_! p^* f = \int^{i,j} X_i \otimes M_{i,j} \otimes \ulcorner j \urcorner.$$

A.1.10. The ∞-category *LIN*. The linear functors can be characterised intrinsically as the left adjoint functors between slice categories, or equivalently (since slice categories are presentable), colimit preserving functors. This provides a slicker way of setting up the ∞-category we work in.

There is an ∞-category $\boldsymbol{Pr}^{\mathrm{L}}$, defined and studied in [49, Section 5.5.3], whose objects are the presentable ∞-categories, and whose morphisms are the left adjoint functors, or equivalently colimit-preserving functors. We define $\boldsymbol{LIN}$ to be the full subcategory of $\boldsymbol{Pr}^{\mathrm{L}}$ whose objects are the ∞-categories (equivalent to) $\boldsymbol{Grpd}_{/S}$. We call the functors *linear*. The mapping spaces in $\boldsymbol{LIN}$ are

$$\begin{aligned}\boldsymbol{LIN}(\boldsymbol{Grpd}_{/I}, \boldsymbol{Grpd}_{/J}) &= \mathrm{Fun}^{\mathrm{L}}(\boldsymbol{Grpd}_{/I}, \boldsymbol{Grpd}_{/J})^{\mathrm{eq}} \\ &\simeq \mathrm{Fun}^{\mathrm{L}}(\boldsymbol{Grpd}^{I}, \boldsymbol{Grpd}^{J})^{\mathrm{eq}} \\ &\simeq \mathrm{Fun}(I, \boldsymbol{Grpd}^{J})^{\mathrm{eq}} \\ &\simeq (\boldsymbol{Grpd}^{I\times J})^{\mathrm{eq}} \\ &\simeq (\boldsymbol{Grpd}_{/I\times J})^{\mathrm{eq}}.\end{aligned}$$

This shows in particular that the linear functors are given by spans. Concretely, tracing through the chain of equivalences, a span defines a left adjoint functor as described above in A.1.7. Composition in $\boldsymbol{LIN}$ is given by composing spans, i.e. taking a pullback. Showing this amounts to the Beck-Chevalley condition.

A.1.11. 'Internal hom'. The ∞-category $\boldsymbol{Pr}^{\mathrm{L}}$ has an internal hom: for two presentable ∞-categories $\mathscr{D}$ and $\mathscr{C}$, the functor ∞-category $\mathrm{Fun}^{\mathrm{L}}(\mathscr{D}, \mathscr{C})$ is again presentable. Also $\boldsymbol{LIN}$ has an internal hom:

$$\begin{aligned}\underline{\boldsymbol{LIN}}(\boldsymbol{Grpd}_{/I}, \boldsymbol{Grpd}_{/J}) &:= \mathrm{Fun}^{\mathrm{L}}(\boldsymbol{Grpd}_{/I}, \boldsymbol{Grpd}_{/J}) \\ &\simeq \mathrm{Fun}(I, \boldsymbol{Grpd}^{J}) \\ &\simeq \mathrm{Fun}(I \times J, \boldsymbol{Grpd}) \\ &\simeq \boldsymbol{Grpd}_{/I\times J}.\end{aligned}$$

The ∞-category $\boldsymbol{LIN}$ is monoidal closed, as a consequence of the fact that $\boldsymbol{Grpd}$ is cartesian closed: the tensor product is given by

$$\boldsymbol{Grpd}_{/I} \otimes \boldsymbol{Grpd}_{/J} = \boldsymbol{Grpd}_{/I\times J}$$

with neutral object $\boldsymbol{Grpd}$, and clearly

$$\boldsymbol{LIN}(\boldsymbol{Grpd}_{/I}\otimes\boldsymbol{Grpd}_{/J}, \boldsymbol{Grpd}_{/K}) \simeq \boldsymbol{LIN}(\boldsymbol{Grpd}_{/I}, \underline{\boldsymbol{LIN}}(\boldsymbol{Grpd}_{/J}, \boldsymbol{Grpd}_{/K}))$$

as both spaces are naturally equivalent to $(\boldsymbol{Grpd}_{/I\times J\times K})^{\mathrm{eq}}$. The tensor product can be characterised as universal recipient of linear functors, but we shall not need this fact.

A.1.12. The linear dual. 'Homming' into the neutral object defines a contravariant autoequivalence of $\boldsymbol{LIN}$:

$$\begin{aligned}\boldsymbol{LIN} &\longrightarrow \boldsymbol{LIN}^{\mathrm{op}} \\ \boldsymbol{Grpd}_{/S} &\longmapsto \underline{\boldsymbol{LIN}}(\boldsymbol{Grpd}_{/S}, \boldsymbol{Grpd}) \simeq \boldsymbol{Grpd}_{/S} \simeq \boldsymbol{Grpd}^{S}.\end{aligned}$$

Here there right-hand side should be considered the dual of $\boldsymbol{Grpd}_{/S}$. (Since our vector spaces are fully coordinatised, the difference between

a vector space and its dual is easily blurred. We will see a clearer difference when we come to the finiteness conditions, in which the dual of a 'vector space' $\boldsymbol{grpd}_{/S}$ is $\boldsymbol{grpd}^S$ which should rather be thought of as a pro-finite-dimensional vector space.)

A.1.13. Remark. It is clear that there is actually an ∞-2-category in play here, with the $\underline{\boldsymbol{LIN}}(\boldsymbol{Grpd}_{/S}, \boldsymbol{Grpd}_{/T})$ as hom ∞-categories. This can be described as a Rezk-category object in the 'distributor' $\boldsymbol{Cat}$, following the work of Barwick and Lurie [50]. Explicitly, let Λ_k denote the full subcategory of $\Delta_k \times \Delta_k$ consisting of the pairs (i, j) with $i + j \leq k$. These are the shapes of diagrams of k composable spans. They form a cosimplicial category. Define Sp_k to be the full subcategory of $\mathrm{Fun}(\Lambda_k, \boldsymbol{Grpd})$ consisting of those diagrams $S : \Lambda_k \to \boldsymbol{Grpd}$ for which for all $i' < i$ and $j' < j$ (with $i + j \leq k$) the square

$$\begin{array}{ccc} S_{i',j'} & \longrightarrow & S_{i,j'} \\ \downarrow & \lrcorner & \downarrow \\ S_{i',j} & \longrightarrow & S_{i,j} \end{array}$$

is a pullback. Then we claim that

$$\begin{array}{ccc} \Delta^{\mathrm{op}} & \longrightarrow & \boldsymbol{Cat} \\ [k] & \longmapsto & \mathrm{Sp}_k \end{array}$$

defines a Rezk-category object in $\boldsymbol{Cat}$ corresponding to $\underline{\boldsymbol{LIN}}$. We leave the claim unproved, as the result is not necessary for our purposes.

A.2. Cardinality of finite ∞-groupoids

A.2.1. Finite ∞-groupoids. An ∞-groupoid B is called *locally finite* if at each base point b the homotopy groups $\pi_i(B, b)$ are finite for $i \geq 1$ and are trivial for i sufficiently large. An ∞-groupoid is called *finite* if it is locally finite and has finitely many components. Note that B is locally finite iff it is a filtered colimit of finite ∞-groupoids. An example of a non locally finite ∞-groupoid is $B\mathbb{Z}$.

A.2.2. Cardinality. [2] The *(homotopy) cardinality* of a finite ∞-groupoid B is the nonnegative rational number given by the formula

$$|B| := \sum_{b \in \pi_0 B} \prod_{i>0} |\pi_i(B, b)|^{(-1)^i}.$$

Here the norm signs on the right refer to order of homotopy groups.

If G is a 1-groupoid, that is, an ∞-groupoid having trivial homotopy groups $\pi_i(G) = 0$ for $i > 1$, its cardinality is

$$|G| = \sum_{x \in \pi_0 G} \frac{1}{|\mathrm{Aut}_G(x)|}.$$

The notion and basic properties of homotopy cardinality have been around for a long time. See Baez–Dolan [2]. The first printed reference we know of is Quinn [59].

A.2.3. Remark. It is clear from the definition that a finite sum of finite ∞-groupoids is again finite, and that cardinality is compatible with finite sums:

$$\left|\sum_{i=1}^{n} X_i\right| = \sum_{i=1}^{n} |X_i| \,.$$

Lemma. A.2.4. *Suppose B is connected. Given a fibre sequence*

$$\begin{array}{ccc} F & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow \\ 1 & \longrightarrow & B, \end{array}$$

if two of the three spaces are finite then so is the third, and in that case

$$|E| \;=\; |F|\,|B| \,.$$

Proof. This follows from the homotopy long exact sequence of a fibre sequence. □

For $b \in B$, we denote by $B_{[b]}$ the connected component of B containing b. Thus an ∞-groupoid B is locally finite if and only if each connected component $B_{[b]}$ is finite.

Lemma. A.2.5. *Suppose B locally finite. Given a map $E \to B$, then E is finite if and only if all fibres E_b are finite, and are nonempty for only finitely many $b \in \pi_0 B$. In this situation,*

$$|E| \;=\; \sum_{b \in \pi_0(B)} |E_b| \left|B_{[b]}\right| .$$

Proof. Write E as the sum of the full fibres $E_{[b]}$, and apply Lemma A.2.4 to the fibrations $E_b \to E_{[b]} \to B_{[b]}$ for each $b \in \pi_0(B)$. Finally sum (A.2.3) over those $b \in \pi_0 B$ with non-empty E_b. □

Corollary. A.2.6. *Cardinality preserves (finite) products.*

Proof. Apply the previous lemma to a projection. □

A.2.7. Notation. Given any ∞-groupoid B and a function $q : \pi_0 B \to \mathbb{Q}$, we write

$$\int^{b \in B} q_b \;:=\; \sum_{b \in \pi_0 B} q_b \left|B_{[b]}\right|$$

if the sum is finite. Then the previous lemma says

$$|E| \;=\; \int^{b \in B} |E_b|$$

for any finite ∞-groupoid E and a map $E \to B$. Two important special cases are given by fibre products and loop spaces:

Lemma. A.2.8. *In the situation of a pullback*

$$\begin{array}{ccc} X \times_B Y & \longrightarrow & X \times Y \\ \downarrow & \lrcorner & \downarrow \\ B & \xrightarrow[diag]{} & B \times B, \end{array}$$

if X and Y are finite, and B is locally finite, then $X \times_B Y$ is finite and

$$|X \times_B Y| \;=\; \int^{b\in B} |X_b|\,|Y_b|\,.$$

Lemma. A.2.9. *In the situation of a loop space*

$$\begin{array}{ccc} \Omega(B,b) & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \ulcorner b \urcorner} \\ 1 & \xrightarrow[\ulcorner b \urcorner]{} & B_{[b]}\,. \end{array}$$

we have that B is locally finite if and only if each $\Omega(B,b)$ is finite, and in that case

$$|\Omega(B,b)| \cdot \left|B_{[b]}\right| = 1.$$

A.2.10. Finite maps. We say that a map $p : E \to B$ is *finite* if any pullback to a finite base X has finite total space X', as in the diagram

$$\begin{array}{ccc} X' & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow p \\ X & \xrightarrow[c]{} & B. \end{array} \tag{23}$$

Lemma. A.2.11. (1) *Pullbacks of finite maps are finite.*
(2) *A map $E \to B$ is finite if and only if each fibre E_b is finite.*

Proof. Statement (1), and one direction of (2), are clear. In the other direction, the map $X' \to X$ in the pullback diagram (23) has finite fibres $X'_x = E_{c(x)}$, so X finite implies X' finite by Lemma A.2.4. □

Lemma. A.2.12. *Suppose $p : E \to B$ has locally finite base.*

(1) *If p is finite then E is locally finite.*
(2) *If E is finite then p is finite.*

Proof. A full fibre $E_{[b]}$ of p is finite if and only if E_b is, by Lemma A.2.4. If each full fibre $E_{[b]}$ is finite, then each component $E_{[e]}$ is, and if E is finite then each full fibre is. □

Lemma. A.2.13. *B is locally finite iff each name $1 \to B$ is a finite map.*

A.3. **Finiteness conditions on groupoid slices**

In this subsection, after some motivation and background from linear algebra, we first explain the finiteness conditions imposed on slice categories in order to model vector spaces and pro-finite-dimensional vector spaces. Then afterwards we assemble all this into ∞-categories using more formal constructions.

A.3.1. Linear algebra rappels. There is a fundamental duality

$$\boldsymbol{Vect} \simeq \underleftarrow{\boldsymbol{vect}}^{\mathrm{op}}$$

between vector spaces and pro-finite-dimensional vector spaces: given any vector space V, the linear dual V^* is a pro-finite-dimensional vector space, and conversely, given a pro-finite-dimensional vector space, its continuous dual is a vector space. This equivalence is a formal consequence of the observation that the category $\boldsymbol{vect}$ of finite-dimensional vector spaces is self-dual: $\boldsymbol{vect} \simeq \boldsymbol{vect}^{\mathrm{op}}$, and the fact that $\boldsymbol{Vect} = \underrightarrow{\boldsymbol{vect}}$, the ind completion of $\boldsymbol{vect}$.

In the fully coordinatised situation typical to algebraic combinatorics, the vector space arises from a set S (typically an infinite set of isoclasses of combinatorial objects): the vector space is then

$$V = \mathbb{Q}_S = \left\{ \sum_{s \in S} c_s \, \delta_s \ : \ c_s \in \mathbb{Q} \text{ almost all zero} \right\},$$

the vector space with basis the symbols δ_s for each $s \in S$. The linear dual is then the function space $V^* = \mathbb{Q}^S$, having a canonical pro-basis consisting of the functions δ^s, taking the value 1 on s and 0 elsewhere.

Vectors in $\mathbb{Q}_S$ are finite linear combinations of the δ_s, and we represent a vector as an infinite column vector $\vec{v}$ with only finitely many non-zero entries. A linear map $f : \mathbb{Q}_S \to \mathbb{Q}_T$ is given by matrix multiplication

$$\vec{v} \mapsto A \cdot \vec{v}.$$

for A an infinite 2-dimensional matrix with T-many rows and S-many columns, and with the crucial property that it is *column finite*: in each column there are only finitely many non-zero entries. More generally, the matrix multiplication of two column-finite matrices makes sense and is again a column-finite matrix. The identity matrix is clearly column finite. A basis element δ_s is identified with the column vector all of whose entries are zero, except the one of index s.

On the other hand, elements in the function space $\mathbb{Q}^S$ are represented as infinite row vectors. A continuous linear map $\mathbb{Q}^T \to \mathbb{Q}^S$, dual to the linear map f, is represented by the *same matrix* A, but viewed now as sending a row vector $\vec{w}$ (indexed by T) to the matrix product $\vec{w} \cdot A$. Again the fact that A is column finite ensures that this matrix product is well defined.

There is a canonical perfect pairing

$$\begin{array}{rcl} \mathbb{Q}_S \times \mathbb{Q}^S & \longrightarrow & \mathbb{Q} \\ (\vec{v}, f) & \longmapsto & f(\vec{v}) \end{array}$$

given by evaluation. In matrix terms, it is just a matter of multiplying $f \cdot \vec{v}$.

A.3.2. Remark. In the theory of Möbius inversion, the incidence coalgebra is on the vector-space side of the duality: the coalgebra is the free vector space on some objects, and the formula for comultiplication is a finite sum, reflecting the fact that an object decomposes in finitely many ways. The incidence algebra is the linear dual, the profinite-dimensional vector space of functions on the objects. In many interesting cases the incidence algebra (a monoid object in a function space) restricts to a monoid in the space of functions with finite support, which can be regarded as a kind of Hall algebra. This happens under different finiteness conditions on the combinatorial structures. Note that the zeta function is not finitely supported (except in degenerate cases), and that Möbius inversion does not make sense in this context.

This duality has a very neat description in homotopy linear algebra. While the vector space $\mathbb{Q}_{\pi_0 S}$ is modelled by the ∞-category $\boldsymbol{grpd}_{/S}$, the function space $\mathbb{Q}^{\pi_0 S}$ is modelled by the ∞-category $\boldsymbol{grpd}^S$. The classical duality results from taking cardinality of a duality on the categorical level that we proceed to explain. For the most elegant definition of cardinality we first need to introduce the objective versions of $\boldsymbol{Vect}$ and $\underleftarrow{\boldsymbol{yect}}$.

Let $\boldsymbol{grpd} \subset \boldsymbol{Grpd}$ be the full subcategory spanned by the finite ∞-groupoids. For S any ∞-groupoid, let $\boldsymbol{grpd}_{/S}$ be the 'comma ∞-category' defined by the following pullback diagram of ∞-categories:

$$\begin{array}{ccc} \boldsymbol{grpd}_{/S} & \longrightarrow & \boldsymbol{Grpd}_{/S} \\ \downarrow & \lrcorner & \downarrow \\ \boldsymbol{grpd} & \longrightarrow & \boldsymbol{Grpd} \end{array}$$

Let S be a locally finite ∞-groupoid, and consider the following ∞-categories.

- denote by $\boldsymbol{grpd}^S$ full subcategory of $\boldsymbol{Grpd}^S$ spanned by the presheaves $S \to \boldsymbol{Grpd}$ whose image lies in $\boldsymbol{grpd}$, and
- denote by $\boldsymbol{Grpd}_{/S}^{\text{rel.fin.}}$ the full subcategory of $\boldsymbol{Grpd}_{/S}$ spanned by the finite maps $p : X \to S$.

Lemma. A.3.3. *The fundamental equivalence* $\boldsymbol{Grpd}^S \simeq \boldsymbol{Grpd}_{/S}$ *restricts to an equivalence*

$$\boldsymbol{grpd}^S \simeq \boldsymbol{Grpd}_{/S}^{rel.fin.}$$

Proof. The inclusions $\boldsymbol{grpd}_{/S} \subset \boldsymbol{Grpd}_{/S}$ and $\boldsymbol{grpd}^S \subset \boldsymbol{Grpd}^S$ are both full, and the objects characterising them correspond to each other under the fundamental equivalence because of Lemma A.2.11 (2). □

From the definition of finite map we have the following result.

Lemma. A.3.4. *For a span* $S \xleftarrow{p} M \xrightarrow{q} T$ *defining a linear map* $F : \boldsymbol{Grpd}_{/S} \to \boldsymbol{Grpd}_{/T}$*, the following are equivalent:*

(1) *p is finite,*
(2) *F restricts to*

$$\boldsymbol{grpd}_{/S} \xrightarrow{p^*} \boldsymbol{grpd}_{/M} \xrightarrow{q_!} \boldsymbol{grpd}_{/T}$$

(3) *F restricts to*

$$\boldsymbol{Grpd}_{/T}^{rel.fin.} \xrightarrow{q^*} \boldsymbol{Grpd}_{/M}^{rel.fin.} \xrightarrow{p_!} \boldsymbol{Grpd}_{/S}^{rel.fin.}$$

The ∞-category $\boldsymbol{grpd}_{/S}$ has finite homotopy sums: for I finite and $F : I \to \boldsymbol{grpd}_{/S}$ we have $\operatorname{colim} F = p_!(X \to I \times S)$, where $p : I \times S \to S$ is the projection. A family $X \to I \times S$ comes from some $F : I \to \boldsymbol{grpd}_{/S}$ and admits a homotopy sum in $\boldsymbol{grpd}_{/S}$ when for each $i \in I$, the partial fibre X_i is finite. Since already I was assumed finite, this is equivalent to having X finite.

The following is the finite version of Proposition A.1.6

Lemma. A.3.5. *The ∞-category* $\boldsymbol{grpd}_{/S}$ *is the finite homotopy sum-completion of S.*

A.4. Categories of linear maps with infinite-groupoid coefficients

Our main interest is in the linear ∞-categories with finite-groupoid coefficients, but it is technically simpler to introduce first the infinite-coefficients version of these ∞-categories, since they can be defined as subcategories in $\boldsymbol{LIN}$, and can be handled with the ease of presentable ∞-categories.

Recall that a span $(S \xleftarrow{p} M \xrightarrow{q} T)$ defines a linear functor

$$L : \boldsymbol{Grpd}_{/S} \xrightarrow{p^*} \boldsymbol{Grpd}_{/M} \xrightarrow{q_!} \boldsymbol{Grpd}_{/T}.$$

by pullback and postcomposition, as shown in the following diagram

$$\begin{array}{ccccc} L(x) : X' & \xrightarrow{p^*x} & M & \xrightarrow{q} & T \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p} & & \\ x : X & \longrightarrow & S. & & \end{array}$$

Let $\boldsymbol{Lin} \subset \boldsymbol{LIN}$ be the ∞-category whose objects are the slices $\boldsymbol{Grpd}_{/\sigma}$, with σ finite. Its morphisms are those linear functors between them which preserve finite objects. Clearly these are given by the spans of the form $\sigma \leftarrow \mu \rightarrow \tau$ where σ, τ and μ are finite. Note that there are equivalences of ∞-categories $\boldsymbol{Grpd}_{/\sigma} \simeq \boldsymbol{Grpd}^{\sigma}$ for each σ.

Let $\underrightarrow{\boldsymbol{Lin}}$ be the ∞-category whose objects are the slices $\boldsymbol{Grpd}_{/S}$ with S locally finite, and whose morphisms are the linear functors between them that preserve finite objects. These correspond to the spans of the form $S \xleftarrow{p} M \rightarrow T$ with p finite.

Let $\underleftarrow{\boldsymbol{Lin}}$ be the ∞-category whose objects are the presheaf categories $\boldsymbol{Grpd}^S$ with S locally finite, and whose morphisms are the continuous linear functors:

A linear functor $F : \boldsymbol{Grpd}^T \rightarrow \boldsymbol{Grpd}^S$ is called *continuous* when for all $\epsilon \subset S$ there exists $\delta \subset T$ and a factorisation

$$\begin{array}{ccc} \boldsymbol{Grpd}^T & \longrightarrow & \boldsymbol{Grpd}^{\delta} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F_\delta} \\ \boldsymbol{Grpd}^S & \longrightarrow & \boldsymbol{Grpd}^{\epsilon} \end{array}$$

where the horizontal maps are the projections of the canonical pro-structures.

Proposition. A.4.1. *For a linear functor $F : \boldsymbol{Grpd}^T \rightarrow \boldsymbol{Grpd}^S$ in $\boldsymbol{LIN}$, represented by a span*

$$S \xleftarrow{p} M \xrightarrow{q} T,$$

the following are equivalent.

(1) *The span is of finite type (i.e. p is a finite map).*
(2) *F is continuous.*

Proof. It is easy to see that if the span is of finite type then F is continuous: for any given finite $\epsilon \subset S$ with inclusion j, the pullback μ is finite, and we can take δ to be the essential full image of the composite $q \circ m$:

(24)
$$\begin{array}{ccccc} \epsilon & \xleftarrow{\bar{p}} & \mu & \xrightarrow{\bar{q}} & \delta \\ {\scriptstyle j}\downarrow & \llcorner & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle i} \\ S & \xleftarrow[p]{} & M & \xrightarrow[q]{} & T. \end{array}$$

Now by Beck-Chevalley,

$$j^* p_! q^* = \bar{p}_! m^* q^* = \bar{p}_! \bar{q}^* i^*$$

which is precisely the continuity condition.

Conversely, if the factorisation in the continuity diagram exists, let $\epsilon \leftarrow \mu \rightarrow \delta$ be the span (of finite ∞-groupoids) representing f_{δ_ϵ}. Then we have the outer rectangle of the diagram (24) and an isomorphism

$$j^* p_! q^* = \bar{p}_! \bar{q}^* i^*$$

Now a standard argument implies the existence of m completing the diagram: namely take the pullback of j and p, with the effect of interchanging the order of upperstar and lowershriek. Now both linear maps are of the form upperstars-followed-by-lowershriek, and by uniqueness of this representation, the said pullback must agree with μ and in particular is finite. Since this is true for every ϵ, this is precisely to say that p is a finite map. □

The continuity condition is precisely continuity for the pro-finite slice topology, as we proceed to explain. Every locally finite ∞-groupoid S is canonically the filtered colimit of its finite (full) subgroupoids:

$$S = \operatorname*{colim}_{\alpha \subset S} \alpha.$$

We use Greek letters here to denote finite ∞-groupoids. Similarly, $\boldsymbol{Grpd}^S$ is a cofiltered limit of ∞-categories $\boldsymbol{Grpd}^\alpha \simeq \boldsymbol{Grpd}_{/\alpha}$:

$$\boldsymbol{Grpd}^S = \lim_{\alpha \subset S} \boldsymbol{Grpd}^\alpha.$$

This leads to the following 'categorical' description of the mapping spaces:

$$\underleftarrow{\boldsymbol{Lin}}(\boldsymbol{Grpd}^T, \boldsymbol{Grpd}^S) := \lim_{\epsilon \subset S} \operatorname*{colim}_{\delta \subset T} \boldsymbol{Lin}(\boldsymbol{Grpd}^\delta, \boldsymbol{Grpd}^\epsilon).$$

A.5. Categories of linear maps with finite-groupoid coefficients

A.5.1. The ∞-category *lin*. We define ***lin*** to be the subcategory of $\widehat{\boldsymbol{Cat}}_\infty$ whose objects are those ∞-categories equivalent to $\boldsymbol{grpd}_{/\sigma}$ for some finite ∞-groupoid σ, and whose mapping spaces are the full subgroupoids of those of $\widehat{\boldsymbol{Cat}}_\infty$ given by the functors which are restrictions of functors in $\boldsymbol{Lin}(\boldsymbol{Grpd}_{/\sigma}, \boldsymbol{Grpd}_{/\tau})$. Note that the latter mapping space was exactly defined as those linear functors in ***LIN*** that preserved finite objects. Hence, by construction there is an equivalence of mapping spaces

$$\boldsymbol{lin}(\boldsymbol{grpd}_{/\sigma}, \boldsymbol{grpd}_{/\tau}) \simeq \boldsymbol{Lin}(\boldsymbol{Grpd}_{/\sigma}, \boldsymbol{Grpd}_{/\tau}),$$

and in particular, the mapping spaces are given by spans of finite ∞-groupoids. The maps can also be described as those functors that

preserve finite homotopy sums. By construction we have an equivalence of ∞-categories

$$\boldsymbol{lin} \simeq \boldsymbol{Lin}.$$

A.5.2. The ∞-category $\underrightarrow{\boldsymbol{lin}}$. Analogously, we define $\underrightarrow{\boldsymbol{lin}}$ to be the subcategory of $\widehat{\boldsymbol{Cat}}_\infty$, whose objects are the ∞-categories equivalent to $\boldsymbol{grpd}_{/S}$ for some locally finite ∞-groupoid S, and whose mapping spaces are the full subgroupoids of the mapping spaces of $\widehat{\boldsymbol{Cat}}_\infty$ given by the functors that are restrictions of functors in $\boldsymbol{Lin}(\boldsymbol{Grpd}_{/S}, \boldsymbol{Grpd}_{/T})$; in other words (by A.3.4), they are the ∞-groupoids of spans of finite type. Again by construction we have

$$\underrightarrow{\boldsymbol{lin}} \simeq \underrightarrow{\boldsymbol{Lin}}.$$

A.5.3. Categories of prolinear maps. We denote by $\underleftarrow{\boldsymbol{lin}}$ the ∞-category whose objects are the ∞-categories $\boldsymbol{grpd}^S$, where S is locally finite, and whose morphisms are restrictions of continuous linear functors. We have seen that the mapping spaces are given by spans of finite type:

$$\underleftarrow{\boldsymbol{lin}}(\boldsymbol{grpd}^T, \boldsymbol{grpd}^S) \;=\; \left\{ (T \xleftarrow{q} M \xrightarrow{p} S) \;:\; p \text{ finite} \right\}.$$

As in the ind case we have

$$\underleftarrow{\boldsymbol{lin}} \simeq \underleftarrow{\boldsymbol{Lin}},$$

and by combining the previous results we also find

$$\underleftarrow{\boldsymbol{lin}}(\boldsymbol{grpd}^T, \boldsymbol{grpd}^S) := \lim_{\epsilon \subset S} \operatorname*{colim}_{\delta \subset T} \boldsymbol{lin}(\boldsymbol{grpd}^\delta, \boldsymbol{grpd}^\epsilon).$$

A.5.4. Mapping categories. Just as $\widehat{\boldsymbol{Cat}}_\infty$ has internal mapping *categories* (of which the mapping spaces are the maximal subgroupoids), we also have internal mapping categories in $\boldsymbol{lin}$, denoted $\underline{\boldsymbol{lin}}$:

$$\underline{\boldsymbol{lin}}(\boldsymbol{grpd}_{/\sigma}, \boldsymbol{grpd}_{/\tau}) \simeq \boldsymbol{grpd}_{/\sigma\times\tau}.$$

Also $\underrightarrow{\boldsymbol{lin}}$ and $\underleftarrow{\boldsymbol{lin}}$ have mapping categories, but due to the finiteness conditions, they are not internal. Just as the mapping spaces are given (in each case) as ∞-groupoids of spans of finite type, the mapping categories are given as ∞-categories of spans of finite type. Denoting the mapping categories with underline, we content ourselves to record the important case of 'linear dual':

Proposition. A.5.5.

$$\begin{aligned} \underrightarrow{\underline{\boldsymbol{lin}}}(\boldsymbol{grpd}_{/S}, \boldsymbol{grpd}) &= \boldsymbol{grpd}^S \\ \underleftarrow{\underline{\boldsymbol{lin}}}(\boldsymbol{grpd}^T, \boldsymbol{grpd}) &= \boldsymbol{grpd}_{/T}. \end{aligned}$$

A.5.6. Remark. It is clear that the correct viewpoint here would be that there is altogether a 2-equivalence between the ∞-2-categories

$$\underrightarrow{\boldsymbol{lin}}^{\mathrm{op}} \cong \underleftarrow{\boldsymbol{lin}}$$

given on objects by $\boldsymbol{grpd}_{/S} \mapsto \boldsymbol{grpd}^S$, and by the identity on homs. It all comes formally from the ind-pro duality starting with the anti-equivalence

$$\boldsymbol{lin} \simeq \boldsymbol{lin}^{\mathrm{op}}.$$

Taking $S = 1$ we see that $\boldsymbol{grpd}$ is an object of both ∞-categories, and mapping into it gives the duality isomorphisms of Proposition A.5.5.

A.5.7. Monoidal structures. The ∞-category $\underrightarrow{\boldsymbol{lin}}$ has two monoidal structures: $\oplus$ and $\otimes$, where $\boldsymbol{grpd}_{/I} \oplus \boldsymbol{grpd}_{/J} = \boldsymbol{grpd}_{/I+J}$ and $\boldsymbol{grpd}_{/I} \otimes \boldsymbol{grpd}_{/J} = \boldsymbol{grpd}_{/I\times J}$. The neutral object for the first is clearly $\boldsymbol{grpd}_{/0} = 1$ and the neutral object for the second is $\boldsymbol{grpd}_{/1} = \boldsymbol{grpd}$. The tensor product distributes over the direct sum. The direct sum is both the categorical sum and the categorical product (i.e. is a biproduct). There is also the operation of infinite direct sum: it is the infinite categorical sum but not the infinite categorical product. (Just as it happens for vector spaces.)

Similarly, also the ∞-category $\underleftarrow{\boldsymbol{lin}}$ has two monoidal structures, $\oplus$ and $\otimes$, given as $\boldsymbol{grpd}^I \oplus \boldsymbol{grpd}^J = \boldsymbol{grpd}^{I+J}$ and $\boldsymbol{grpd}^I \otimes \boldsymbol{grpd}^J = \boldsymbol{grpd}^{I\times J}$. The $\otimes$ should be considered the analogue of a completed tensor product. Again $\oplus$ is both the categorical sum and the categorical product, and $\otimes$ distributes over $\oplus$. Again the structures allow infinite versions, but this times the infinite direct sum is a categorical infinite product but is not an infinite categorical sum.

(To see the difference between the role of infinite $\oplus_\alpha$ in $\underrightarrow{\boldsymbol{lin}}$ and in $\underleftarrow{\boldsymbol{lin}}$: in $\underleftarrow{\boldsymbol{lin}}$ there is a diagonal map $\boldsymbol{grpd}^I \to \oplus_\alpha \boldsymbol{grpd}^I = \boldsymbol{grpd}^{\sum_\alpha I}$ given by sending $X \to I$ to $\sum_\alpha X \to \sum_\alpha I$. This makes sense for a finite map $X \to I$, since the infinite sum of copies of that map is still finite, but it does not make sense in $\underrightarrow{\boldsymbol{lin}}$ since that $\sum_\alpha X$ is not finite. On the other hand, $\underrightarrow{\boldsymbol{lin}}$ sports a codiagonal $\oplus_\alpha \boldsymbol{grpd}_{/I} = \boldsymbol{grpd}_{/\sum_\alpha I} \to \boldsymbol{grpd}_{/I}$ given by sending $A \to \sum_\alpha I$ to the composite $A \to \sum_\alpha I \to I$ (where the second map is the codiagonal for the infinite sum of ∞-groupoids). Since X is finite it remains finite so there is no problem. In contrast this construction does not work in $\boldsymbol{lin}$: even if $A \to \sum_\alpha I$ is finite, $A \to \sum_\alpha I \to I$ will generally not be so.)

A.5.8. Summability. In algebraic combinatorics, the profinite stuff is often expressed in terms of notions of summability. We briefly digress to see the constructions from this angle.

For B a locally finite ∞-groupoid, a B-indexed family $g : E \to B \times I$ (as in A.1.2) is called *summable* if the composite $E \to B \times I \to I$ is

a finite map. The condition implies that in fact the members of the family were already finite maps. Indeed, with reference to the diagram

$$\begin{array}{ccccc} E_{b,i} & \longrightarrow & E_i & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \{b\} \times \{i\} & \longrightarrow & B \times \{i\} & \longrightarrow & B \times I \\ & & \downarrow & \lrcorner & \downarrow \\ & & \{i\} & \longrightarrow & I \end{array}$$

summability implies (by Lemma A.2.11.2) that each E_i is finite, and therefore (by Lemma A.2.13 since B is locally finite) we also conclude that each $E_{b,i}$ is finite, which is precisely to say that the members $g_b : E_b \to I$ are finite maps (cf. A.2.11.2 again). It thus makes sense to interpret the family as a family of objects in $\boldsymbol{Grpd}_{/I}^{rel.fin.}$. And finally we can say that a summable family is a family $g : E \to B \times I$ of finite maps $g_b : E_b \to I$, whose homotopy sum $p_!(g)$ is again a finite map. If I is finite, then the only summable families are the finite families (i.e. $E \to B \times I$ with E finite). A family $g : E \to B \times I$, given equivalently as a functor

$$F : B \to \boldsymbol{grpd}^I,$$

is summable if and only if it is a cofiltered limit of diagrams $F_\alpha : B \to \boldsymbol{grpd}^\alpha$ (with α finite).

It is easy to check that a map $q : M \to T$ (between locally finite ∞-groupoids) is finite if and only if for every finite map $f : X \to M$ we have that also $q_! f$ is finite. Hence we find

Lemma. A.5.9. *A span $I \xleftarrow{p} M \xrightarrow{q} J$ preserves summable families if and only if q is finite.*

A.6. **Duality**

We have a perfect pairing

$$\begin{array}{rcl} \boldsymbol{grpd}_{/S} \times \boldsymbol{grpd}^S & \longrightarrow & \boldsymbol{grpd} \\ (p, f) & \longmapsto & f(p) \end{array}$$

given by evaluation. In terms of spans, write the map-with-finite-total-space $p : X \to S$ as a finite span $1 \leftarrow X \xrightarrow{p} S$, and write the presheaf $f : S \to \boldsymbol{grpd}$ as the finite span $S \xleftarrow{f} F \to 1$, where F is the total space of the Grothendieck construction of f. (In other words, the functor F on S corresponds to a linear functor on $\boldsymbol{grpd}_{/S}$; write the representing span.) Then the evaluation is given by composing these two spans, and hence amounts just to taking the pullback of p and f.

The statements mean: for fixed $K \to S$ in $\boldsymbol{grpd}_{/S}$, the map

$$\begin{array}{rcl} \boldsymbol{grpd}^S & \longrightarrow & \boldsymbol{grpd} \\ f & \longmapsto & f(K) \end{array}$$

is prolinear. The resulting functor

$$\begin{array}{rcl} \boldsymbol{grpd}_{/S} & \longrightarrow & \underline{\underline{\boldsymbol{Lin}}}(\boldsymbol{grpd}^S, \boldsymbol{grpd}) \\ K & \longmapsto & [f \mapsto f(K)] \end{array}$$

is an equivalence of ∞-categories (cf. Proposition A.5.5).

Conversely, for fixed $X : S \to \boldsymbol{grpd}$ in $\boldsymbol{grpd}^S$, the map

$$\begin{array}{rcl} \boldsymbol{grpd}_{/S} & \longrightarrow & \boldsymbol{grpd} \\ K & \longmapsto & X(K) \end{array}$$

is linear. The resulting functor

$$\begin{array}{rcl} \boldsymbol{grpd}^S & \longrightarrow & \underline{\underline{\boldsymbol{Lin}}}(\boldsymbol{grpd}_{/S}, \boldsymbol{grpd}) \\ X & \longmapsto & X(K) \end{array}$$

is an equivalence of ∞-categories (cf. Proposition A.5.5).

A.6.1. Bases. Both $\boldsymbol{grpd}_{/S}$ and $\boldsymbol{grpd}^S$ feature a canonical basis, actually an essentially unique basis. The basis elements in $\boldsymbol{grpd}_{/S}$ are the names $\ulcorner s \urcorner : 1 \to S$: every map $p : X \to S$ can be written as a finite homotopy linear combination

$$p = \int^{s \in S} |X_s| \ \ulcorner s \urcorner.$$

Similarly, in $\boldsymbol{grpd}^S$, the representables $h^t := \mathrm{Map}(t, -)$ form a basis: every presheaf on S is a colimit and in fact a homotopy sum of such representables. These bases are dual to each other, except for a normalisation: if $p = \ulcorner s \urcorner$ and $f = h^t = \mathrm{Map}(t, -)$, then they pair to

$$\mathrm{Map}(t, s) \simeq \begin{cases} \Omega(S, s) & \text{if } t \simeq s \\ 0 & \text{else} \end{cases}$$

The fact that we obtain the loop space $\Omega(S, s)$ instead of 1 is actually a feature: we shall see below that upon taking cardinality we obtain the canonical pairing

$$\begin{array}{rcl} \mathbb{Q}_S \times \mathbb{Q}^S & \longrightarrow & \mathbb{Q} \\ (\delta_i, \delta^j) & \longmapsto & \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else.} \end{cases} \end{array}$$

A.7. **Cardinality as a functor**

A.7.1. Definition of cardinality. We define *meta cardinality* by the assignment

$$\| \quad \| : \underrightarrow{\boldsymbol{lin}} \to \boldsymbol{Vect}$$

defined on objects by

$$\|\boldsymbol{grpd}_{/T}\| := \mathbb{Q}_{\pi_0 T},$$

and on morphisms by taking a finite-type span $S \xleftarrow{p} M \xrightarrow{q} T$ to the linear map

$$\begin{array}{rcl} \mathbb{Q}_{\pi_0 S} & \longrightarrow & \mathbb{Q}_{\pi_0 T} \\ \delta_s & \longmapsto & \displaystyle\int^t |M_{s,t}|\,\delta_t = \sum_t \left|T_{[t]}\right| |M_{s,t}|\,\delta_t. \end{array}$$

That is, to the span M we assign the matrix $A_{t,s} := \left|T_{[t]}\right| |M_{s,t}|$, which is column finite since M is of finite type.

The idea of meta cardinality is that it is a global assignment which specialises to every slice ∞-category to define a relative cardinality, a cardinality of families, by the observation that families are special cases of spans, just as vectors can be identified with linear maps from the ground field. The idea of meta cardinality is due to Baez, Hoffnung and Walker [4] (although they are not to blame for our terminology).

Dually we define a meta cardinality

$$\| \quad \| : \underleftarrow{\boldsymbol{lin}} \to \underleftarrow{\boldsymbol{yect}}$$

defined on objects by

$$\|\boldsymbol{grpd}^S\| := \mathbb{Q}^{\pi_0 S},$$

and on morphisms by the *same* assignment of a matrix to a finite span as before.

Proposition. A.7.2. *The meta cardinality assignments just defined*

$$\| \quad \| : \underrightarrow{\boldsymbol{lin}} \to \boldsymbol{Vect}, \qquad\qquad \| \quad \| : \underleftarrow{\boldsymbol{lin}} \to \underleftarrow{\boldsymbol{yect}}$$

are functorial.

Proof. First observe that the functor is well defined on morphisms. Given a finite-type span $S \xleftarrow{p} M \xrightarrow{q} T$ defining linear functors $L : \boldsymbol{grpd}_{/S} \to \boldsymbol{grpd}_{/T}$ (as well as $L^\vee : \boldsymbol{grpd}^T \to \boldsymbol{grpd}^S$), the linear maps

$$\|L\| : \mathbb{Q}_{\pi_0 S} \longrightarrow \mathbb{Q}_{\pi_0 T}, \qquad\qquad \|L^\vee\| : \mathbb{Q}^{\pi_0 T} \longrightarrow \mathbb{Q}^{\pi_0 S}$$

are defined with respect to the given (pro-)bases by the matrix $\|L\|_{t,s} = |M_{s,t}| \left|T_{[t]}\right|$. That is:

$$\|L\| \left(\sum_{s \in \pi_0 S} c_s\, \delta_s \right) \;=\; \sum_{s,t} c_s\, |M_{s,t}| \left|T_{[t]}\right|\, \delta_t \;=\; \sum_{s \in \pi_0 S} c_s \int^{t \in T} |M_{s,t}|\, \delta_t\,,$$

and

$$\|L^\vee\| \left(\sum_{t \in \pi_0 T} c_t \, \delta^t \right) \;=\; \sum_{s,t} c_t \; |M_{s,t}| \, \big|T_{[t]}\big| \; \delta_s$$

In particular, we note

$$\|L^\vee\| \, (\delta^t) \;=\; \sum_{s} |M_{s,t}| \, \big|T_{[t]}\big| \; \delta_s \, .$$

The matrix $|M_{s,t}| \, \big|T_{[t]}\big|$ has finite entries and is column-finite: for each $s \in \pi_0 S$ the fibre M_s is finite so the map $M_s \to T$ is finite by Lemma A.2.12, and the fibres $M_{s,t}$ are non-empty for only finitely many $t \in \pi_0 T$. It is clear that equivalent spans define the same matrix, and the identity span $L = (S \leftarrow S \to S)$ gives the identity matrix: $\|L\|_{s_1,s_2} = 0$ if s_1, s_2 are in different components, and $\|L\|_{s,s} \;=\; |\Omega(S,s)| \, \big|S_{[s]}\big| \;=$ 1 by Lemma A.2.9. It remains to show that composition of spans corresponds to matrix product: for $L = (S \leftarrow M \to T)$, $L' = (T \leftarrow N \to U)$ we have

$$|(M \times_T N)_{s,u}| = \int^{t \in T} |M_{s,t} \times N_{t,u}| = \sum_{t \in \pi_0 T} |M_{s,t}| \, \big|T_{[t]}\big| \, |N_{t,u}|$$

and so $\|L'L\|_{u,s} = \displaystyle\sum_{t \in \pi_0 T} |M_{s,t}| \, \big|T_{[t]}\big| \, |N_{t,u}| \, \big|U_{[u]}\big| = \sum_{t \in \pi_0 T} \|L'\|_{u,t} \|L\|_{t,s}.$ □

A.7.3. Remark. The idea of defining cardinality as a functor is due to Baez, Hoffnung and Walker [4], in the setting of certain tame 1-groupoids. They work with a kind of 'square-integrability' instead of distinguishing between finite slices and pro-finite slices. This amounts roughly to studying Hilbert spaces on both sides of the duality instead of pairing vector spaces with profinite-dimensional vector spaces as we prefer to to. Finally, they bundle the symmetry factors into the fibres and use discrete sums, instead of using homotopy fibres and homotopy sums, as favoured in the present work.

A.7.4. Cardinality of families. As a consequence of this proposition we obtain for each locally finite ∞-groupoid T a notion of cardinality of T-indexed families. Let T be a locally finite ∞-groupoid and define the functor

$$|\ \ | : \boldsymbol{grpd}_{/T} \longrightarrow \big\|\boldsymbol{grpd}_{/T}\big\| = \mathbb{Q}_{\pi_0 T}, \qquad |x| \;:=\; \|L_x\| \; (\delta_1) \, .$$

Here $x : X \to T$ is an object of $\boldsymbol{grpd}_{/T}$ and $\|L_x\| : \mathbb{Q}_{\pi_0 1} \to \mathbb{Q}_{\pi_0 T}$ is induced by the linear functor L_x defined by the finite span $1 \leftarrow X \xrightarrow{x} T$. By the definition of $\|L\|$ in Proposition A.7.2, we can write

$$|x| \;\; = \;\; \sum_{t \in \pi_0 T} |X_t| \; |T_{[t]}| \; \delta_t \;\; = \;\; \int^{t \in T} |X_t| \; \delta_t$$

Lemma. A.7.5. *Let T be a locally finite ∞-groupoid.*

(1) *If T is connected, with $t \in T$, and $x : X \to T$ in $\mathbf{grpd}_{/T}$, then*

$$|x| \;=\; |X|\,\delta_t \quad\in\quad \mathbb{Q}_{\pi_0 T}.$$

(2) *The cardinality of $\ulcorner t \urcorner : 1 \to T$ in $\mathbf{grpd}_{/T}$ is the basis vector δ_t.*

Proof. (1) By definition, $|x| \;=\; |X_t|\,|T|\,\delta_t$, and by Lemma A.2.4, this is $|X|\,\delta_t$
(2) The fibre of $\ulcorner t \urcorner$ over t' is empty except when t, t' are in the same component, so we reduce to the case of connected T and apply (1). □

A.7.6. Cardinality of presheaves. We also obtain a notion of cardinality of presheaves: for each S, define

$$|\ \ | : \boldsymbol{grpd}^S \longrightarrow \left\|\boldsymbol{grpd}^S\right\| = \mathbb{Q}^{\pi_0 S}, \qquad |f| \;:=\; \|L_f\|.$$

Here $f : S \to \boldsymbol{grpd}$ is a presheaf, and L_f is its 'linearisation' $L_f : \boldsymbol{grpd}_{/S} \to \boldsymbol{grpd}$, which in turn is given by the span $S \leftarrow F \to 1$, where $F \to S$ is the Grothendieck construction of f. The meta cardinality of this span is then a linear map $\mathbb{Q}_{\pi_0 S} \to \mathbb{Q}_1$, or equivalently a pro-linear map $\mathbb{Q}^1 \to \mathbb{Q}^{\pi_0 S}$ — in either way interpreted as an element in $\mathbb{Q}^{\pi_0 S}$. In the first viewpoint, the linear map is

$$\begin{aligned} \mathbb{Q}_{\pi_0 S} &\longrightarrow \mathbb{Q}_1 \\ \delta_s &\longmapsto \int^1 |F_s|\,\delta_1 = |F_s|\,\delta_1 \end{aligned}$$

which is precisely the function

$$\begin{aligned} \pi_0 S &\longrightarrow \mathbb{Q} \\ s &\longmapsto |f(s)|\,. \end{aligned}$$

In the second viewpoint, it is the prolinear map

$$\begin{aligned} \mathbb{Q}^1 &\longrightarrow \mathbb{Q}^{\pi_0 S} \\ \delta_1 &\longmapsto \sum_s |F_s|\,\delta^s \end{aligned}$$

which of course also is the function $s \mapsto |f(s)|$.

In conclusion:

Proposition. A.7.7. *The cardinality of a presheaf $f : S \to \mathbf{grpd}$ is computed pointwise: $|f|$ is the function*

$$\begin{aligned} \pi_0 S &\longrightarrow \mathbb{Q} \\ s &\longmapsto |f(s)|\,. \end{aligned}$$

In other words, it is obtained by postcomposing with the basic homotopy cardinality.

Example. A.7.8. The cardinality of the terminal presheaf is the constant function 1. In incidence algebras, this says that the cardinality of the zeta functor 2.3.2 is the zeta function.

Example. A.7.9. The cardinality of the representable functor $h^t : S \to \boldsymbol{grpd}$ is

$$\begin{array}{rcl} \pi_0 S & \longrightarrow & \mathbb{Q} \\ s & \longmapsto & |\mathrm{Map}(t,s)| = \begin{cases} |\Omega(S,s)| & \text{if } t \simeq s \\ 0 & \text{else.} \end{cases} \end{array}$$

A.7.10. Remark. Note that under the Grothendieck-construction duality, $\boldsymbol{grpd}^S \simeq \boldsymbol{Grpd}^{relfin}_{/S}$, the representable presheaf h^s corresponds to $\ulcorner s \urcorner$, the name of s, which happens to belong also the subcategory $\boldsymbol{grpd}_{/S} \subset \boldsymbol{Grpd}^{relfin}_{/S}$, but that the cardinality of $h^s \in \boldsymbol{grpd}^S$ is *not* the same as the cardinality of $\ulcorner s \urcorner \in \boldsymbol{grpd}_{/S}$. This may seem confusing at first, but it is forced upon us by the choice of normalisation of the functor

$$\| \ \| : \underrightarrow{\boldsymbol{lin}} \to \boldsymbol{Vect}$$

which in turn looks very natural since the extra factor $\left|T_{[t]}\right|$ comes from an integral. A further feature of this apparent discrepancy is the following.

Proposition. A.7.11. *Cardinality of the canonical perfect pairing at the ∞-groupoid level yields precisely the perfect pairing on the vector-space level.*

Proof. We take cardinality of the perfect pairing

$$\begin{array}{rcl} \boldsymbol{grpd}_{/S} \times \boldsymbol{grpd}^S & \longrightarrow & \boldsymbol{grpd} \\ (p, f) & \longmapsto & f(p) \\ (\ulcorner s \urcorner, h^t) & \longmapsto & \begin{cases} \Omega(S,s) & \text{if } t \simeq s \\ 0 & \text{else} \end{cases} \end{array}$$

Since the cardinality of $\ulcorner s \urcorner$ is δ_s, while the cardinality of h^t is $|\Omega(S,t)|\,\delta^t$, the cardinality of the pairing becomes

$$(\delta_s, |\Omega(S,t)|\,\delta^t) \longmapsto \begin{cases} |\Omega(S,t)| & \text{if } t \simeq s \\ 0 & \text{else ,} \end{cases}$$

or equivalently:

$$(\delta_s, \delta^t) \longmapsto \begin{cases} 1 & \text{if } t \simeq s \\ 0 & \text{else ,} \end{cases}$$

as required. □

A.7.12. Remarks. The definition of meta cardinality involves a convention, namely to include the factor $\left|T_{[t]}\right|$. In fact, as observed by Baez–Hoffnung–Walker [4], other conventions are possible: for any exponents α_1 and α_2 with $\alpha_1 + \alpha_2 = 1$, it is possible to use the factor

$$\left|S_{[s]}\right|^{\alpha_1} \left|T_{[t]}\right|^{\alpha_2}.$$

They use $0+1$ in some cases and $1+0$ in other cases, according to what seems more practical. We think that these choices can be explained by which side of duality the constructions take place.

Our convention with the $\left|T_{[t]}\right|$ normalisation yields the 'correct' numbers in all the classical cases, like for example the standard Hall numbers when the decomposition space is the Waldhausen S-construction of a (suitably finite) abelian category, cf. 4.6.11, not to mention that the cardinality of the zeta functor (given by the span $S \xleftarrow{=} S \to 1$, or equivalently, by the terminal presheaf) is the zeta function (the constant function 1). The zeta function is the 'sum of everything', with no symmetry factors. A 'sum of everything', but with symmetry factors, appeared in our work [23] on the Faà di Bruno and Connes–Kreimer bialgebras, namely in the form of Green functions. That's on the other side of duality though (on the coalgebra side), and for this reason the important symmetry factors appear correctly.

References

[1] Marcelo Aguiar and Swapneel Mahajan. *Monoidal functors, species and Hopf algebras*, vol. 29 of CRM Monograph Series. American Mathematical Society, Providence, RI, 2010. With forewords by Kenneth Brown and Stephen Chase and André Joyal.

[2] John C. Baez and James Dolan. *From finite sets to Feynman diagrams*. In *Mathematics unlimited—2001 and beyond*, pp. 29–50. Springer, Berlin, 2001.

[3] John C. Baez and James Dolan. *Zeta functions*. http://ncatlab.org/johnbaez/Zeta+functions (2010).

[4] John C. Baez, Alexander E. Hoffnung, and Christopher D. Walker. *Higher dimensional algebra VII: groupoidification*. Theory Appl. Categ. **24** (2010), No. 18, 489–553. ArXiv:0908.4305.

[5] Hans-Joachim Baues. *The cobar construction as a Hopf algebra*. Invent. Math. **132** (1998), 467–489.

[6] Hans-Joachim Baues and Günther Wirsching. *Cohomology of small categories*. J. Pure Appl. Algebra **38** (1985), 187–211.

[7] François Bergeron, Gilbert Labelle, and Pierre Leroux. *Combinatorial species and tree-like structures*, vol. 67 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1998. Translated from the 1994 French original by Margaret Readdy, With a foreword by Gian-Carlo Rota.

[8] Julia E. Bergner. *A survey of* $(\infty, 1)$*-categories*. In *Towards higher categories*, vol. 152 of IMA Vol. Math. Appl., pp. 69–83. Springer, New York, 2010. ArXiv:math.AT/0610239.

[9] Aurelio Carboni, Stephen Lack, and Robert F. C. Walters. *Introduction to extensive and distributive categories*. J. Pure Appl. Algebra **84** (1993), 145–158.

[10] Pierre Cartier and Dominique Foata. *Problèmes combinatoires de commutation et réarrangements*. No. 85 in Lecture Notes in Mathematics. SpringerVerlag, Berlin, New York, 1969. Republished in the "books" section of the Séminaire Lotharingien de Combinatoire.

[11] Alain Connes and Dirk Kreimer. *Hopf algebras, renormalization and noncommutative geometry*. Comm. Math. Phys. **199** (1998), 203–242. ArXiv:hep-th/9808042.

[12] Alain Connes and Dirk Kreimer. *Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem*. Comm. Math. Phys. **210** (2000), 249–273. ArXiv:hep-th/9912092.

[13] Mireille Content, François Lemay, and Pierre Leroux. *Catégories de Möbius et fonctorialités: un cadre général pour l'inversion de Möbius*. J. Combin. Theory Ser. A **28** (1980), 169–190.

[14] Robert J. MacG. Dawson, Robert Paré, and Dorette A. Pronk. *Universal properties of Span*. Theory Appl. Categ. **13** (2004), No. 4, 61–85.

[15] Peter Doubilet. *A Hopf algebra arising from the lattice of partitions of a set*. J. Algebra **28** (1974), 127–132.

[16] Peter Doubilet, Gian-Carlo Rota, and Richard Stanley. *On the foundations of combinatorial theory. VI. The idea of generating function*. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory*, pp. 267–318. Univ. California Press, Berkeley, Calif., 1972.

[17] Arne Dür. *Möbius functions, incidence algebras and power series representations*, vol. 1202 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1986.

[18] Tobias Dyckerhoff and Mikhail Kapranov. *Crossed simplicial groups and structured surfaces*. Preprint, arXiv:1403.5799.

[19] Tobias Dyckerhoff and Mikhail Kapranov. *Higher Segal spaces I*. Preprint, arXiv:1212.3563.

[20] Tobias Dyckerhoff and Mikhail Kapranov. *Triangulated surfaces in triangulated categories*. Preprint, arXiv:1306.2545.

[21] Héctor Figueroa and José M. Gracia-Bondía. *Combinatorial Hopf algebras in quantum field theory. I*. Rev. Math. Phys. **17** (2005), 881–976. ArXiv:hep-th/0408145.

[22] Thomas M. Fiore, Wolfgang Lück, and Roman Sauer. *Finiteness obstructions and Euler characteristics of categories*. Adv. Math. **226** (2011), 2371–2469. ArXiv:0908.3417.

[23] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. *Groupoids and Faà di Bruno formulae for Green functions in bialgebras of trees*. Adv. Math. **254** (2014), 79–117. ArXiv:1207.6404.

[24] David Gepner and Joachim Kock. *Polynomial functors over infinity categories*. In preparation.

[25] A. B. Goncharov. *Galois symmetries of fundamental groupoids and noncommutative geometry*. Duke Math. J. **128** (2005), 209–284. ArXiv:math/0208144.

[26] John Haigh. *On the Möbius algebra and the Grothendieck ring of a finite category*. J. London Math. Soc. (2) **21** (1980), 81–92.

[27] Claudio Hermida. *Representable multicategories*. Adv. Math. **151** (2000), 164–225.

[28] Luc Illusie. *Complexe cotangent et déformations. I*. No. 239 in Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1971.

[29] Saj-nicole A. Joni and Gian-Carlo Rota. *Coalgebras and bialgebras in combinatorics*. Stud. Appl. Math. **61** (1979), 93–139.

[30] André Joyal. *Une théorie combinatoire des séries formelles*. Adv. in Math. **42** (1981), 1–82.

[31] André Joyal. *Foncteurs analytiques et espèces de structures*. In *Combinatoire énumérative (Montréal/Québec, 1985)*, vol. 1234 of Lecture Notes in Mathematics, pp. 126–159. Springer, Berlin, 1986.

[32] André Joyal. *Quasi-categories and Kan complexes*. J. Pure Appl. Algebra **175** (2002), 207–222.

[33] André Joyal. *The theory of quasi-categories*. In *Advanced Course on Simplicial Methods in Higher Categories, vol. II*, no. 45 in Quaderns. CRM, Barcelona, 2008. Available at http://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern45-2.pdf.

[34] André Joyal. *Disks, duality and Θ-categories*, September 1997.

[35] André Joyal and Ross Street. *The geometry of tensor calculus. I*. Adv. Math. **88** (1991), 55–112.

[36] André Joyal and Ross Street. *The category of representations of the general linear groups over a finite field*. J. Algebra **176** (1995), 908–946.

[37] André Joyal and Myles Tierney. *Quasi-categories vs Segal spaces*. In *Categories in algebra, geometry and mathematical physics*, vol. 431 of Contemp. Math., pp. 277–326. Amer. Math. Soc., Providence, RI, 2007. ArXiv:math/0607820.

[38] Joachim Kock. *Polynomial functors and trees*. Internat. Math. Res. Notices **2011** (2011), 609–673. ArXiv:0807.2874.

[39] Joachim Kock. *Data types with symmetries and polynomial functors over groupoids*. In *Proceedings of the 28th Conference on the Mathematical Foundations of Programming Semantics (Bath, 2012)*, vol. 286 of Electr. Notes in Theoret. Comp. Sci., pp. 351–365, 2012. ArXiv:1210.0828.

[40] Joachim Kock. *Categorification of Hopf algebras of rooted trees*. Cent. Eur. J. Math. **11** (2013), 401–422. ArXiv:1109.5785.

[41] F. William Lawvere. *Ordinal sums and equational doctrines*. In B. Eckmann, editor, *Seminar on triples and categorical homology theory, ETH 1966/67*, no. 80 in Lecture Notes in Mathematics, pp. 141–155. Springer-Verlag, New York, 1967. Reprinted in Repr. Theory Appl. Categ. 18 (2008) pp. 1–303. (Available from http://www.tac.mta.ca/tac/reprints/).

[42] F. William Lawvere. *State categories and response functors. Dedicated to Walter Noll*. Preprint (May 1986).

[43] F. William Lawvere and Matías Menni. *The Hopf algebra of Möbius intervals*. Theory Appl. Categ. **24** (2010), No. 10, 221–265.

[44] Tom Leinster. *Notions of Möbius inversion*. Bull. Belg. Math. Soc. **19** (2012), 911–935. ArXiv:1201.0413.

[45] Pierre Leroux. *Les catégories de Möbius*. Cahiers Topologie Géom. Différentielle **16** (1976), 280–282.

[46] Pierre Leroux. *The isomorphism problem for incidence algebras of Möbius categories*. Illinois J. Math. **26** (1982), 52–61.

[47] M. Lothaire. *Combinatorics on words*, vol. 17 of Encyclopedia of Mathematics and its Applications. Addison-Wesley Publishing Co., Reading, Mass., 1983. A collective work by Dominique Perrin, Jean Berstel, Christian Choffrut, Robert Cori, Dominique Foata, Jean Eric Pin, Guiseppe Pirillo, Christophe Reutenauer, Marcel-P. Schützenberger, Jacques Sakarovitch and Imre Simon, With a foreword by Roger Lyndon, Edited and with a preface by Perrin.

[48] Wolfgang Lück. *Transformation groups and algebraic K-theory*, vol. 1408 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1989. Mathematica Gottingensis.

[49] Jacob Lurie. *Higher topos theory*, vol. 170 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009. ArXiv:math/0608040.

[50] Jacob Lurie. $(\infty, 2)$-*categories and the Goodwillie calculus I*.
[51] Jacob Lurie. *Higher Algebra*. Available from http://www.math.harvard.edu/~lurie/, 2013.
[52] Saunders Mac Lane. *Categories for the working mathematician, second edition*. No. 5 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1998.
[53] Manuel Maia and Miguel Méndez. *On the arithmetic product of combinatorial species*. Discrete Math. **308** (2008), 5407–5427. ArXiv:math/0503436.
[54] Dominique Manchon. *On bialgebras and Hopf algebras of oriented graphs*. Confluentes Math. **4** (2012), 1240003, 10.
[55] Yuri I. Manin. *Renormalization and computation I: motivation and background*. Preprint, arXiv:0904.4921.
[56] Yuri I. Manin. *A course in mathematical logic for mathematicians*, vol. 53 of Graduate Texts in Mathematics. Springer, New York, second edition, 2010. Chapters I–VIII translated from the Russian by Neal Koblitz, With new chapters by Boris Zilber and the author.
[57] Kent E. Morrison. *An introduction to q-species*. Electron. J. Combin. **12** (2005), Research Paper 62, 15 pp. (electronic). ArXiv:math/0512052.
[58] The Univalent Foundations Program. Homotopy type theory: Univalent foundations of mathematics. Technical report, Institute for Advanced Study, 2013.
[59] Frank Quinn. *Lectures on axiomatic topological quantum field theory*. In *Geometry and quantum field theory (Park City, UT, 1991)*, pp. 323–453. Amer. Math. Soc., Providence, RI, 1995.
[60] Charles Rezk. *A model for the homotopy theory of homotopy theory*. Trans. Amer. Math. Soc. **353** (2001), 973–1007 (electronic).
[61] Claus Michael Ringel. *Hall algebras and quantum groups*. Invent. Math. **101** (1990), 583–591.
[62] Gian-Carlo Rota. *On the foundations of combinatorial theory. I. Theory of Möbius functions*. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2** (1964), 340–368 (1964).
[63] Olivier Schiffmann. *Lectures on Hall algebras*. ArXiv:math/0611617.
[64] William R. Schmitt. *Hopf algebras of combinatorial structures*. Canad. J. Math. **45** (1993), 412–428.
[65] William R. Schmitt. *Incidence Hopf algebras*. J. Pure Appl. Algebra **96** (1994), 299–330.
[66] Richard P. Stanley. *Generating functions*. In *Studies in combinatorics*, vol. 17 of MAA Stud. Math., pp. 100–141. Math. Assoc. America, Washington, D.C., 1978.
[67] Ross Street. *Categorical structures*. In *Handbook of algebra, Vol. 1*, pp. 529–577. North-Holland, Amsterdam, 1996.
[68] Bertrand Toën. *Derived Hall algebras*. Duke Math. J. **135** (2006), 587–615. ArXiv:math/0501343.
[69] Mark Weber. *Generic morphisms, parametric representations and weakly Cartesian monads*. Theory Appl. Categ. **13** (2004), 191–234 (electronic).
[70] Mark Weber. *Familial 2-functors and parametric right adjoints*. Theory Appl. Categ. **18** (2007), 665–732 (electronic).

Departament de Matemàtica Aplicada III, Universitat Politècnica de Catalunya, Escola d'Enginyeria de Terrassa, Carrer Colom 1, 08222 Terrassa (Barcelona), Spain
E-mail address: m.immaculada.galvez@upc.edu

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain
E-mail address: kock@mat.uab.cat

STORM, London Metropolitan University, 166–220 Holloway Road, London N7 8DB, UK
E-mail address: a.tonks@londonmet.ac.uk