Similarities and differences between Success and Decisiveness

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Abstract. We consider binary voting systems in which a probability distribution over coalitions is known. In this broader context decisiveness is an extension of the Penrose-Banzhaf index and success an extension of the Rae index for simple games. Although decisiveness and success are conceptually different we analyze their numerical behavior. The main result provides necessary and sufficient conditions for the ordinal equivalence of them. Indeed, under anonymous probability distributions they become ordinally equivalent. Moreover, it is proved that for these distributions, decisiveness and success respect the strength of the seats, whereas luckiness reverses the order.

1 Introduction

In the classical model of voting simple games, modeled by (N, W), some 'power indices' have been introduced with the purpose of measuring the decisiveness of a player in the game. Shapley and Shubik's [21] interpretation of their index as the probability of being 'pivotal' in the making of a decision contributed to the association between 'power' and 'measure of decisiveness'. Penrose's [19], Banzhaf's [1] and Coleman's [5] indices are also evaluations of decisiveness. Since then many papers analyze interesting aspects of decisiveness (see e.g., Felsenthal and Machover [8] and [9], Turnovec et al. [23], Turnovec [22]).

An alternative view is the notion of satisfaction or success. That is, focusing in the likelihood of obtaining the result one votes for irrespective of whether one's vote is crucial for it or not. Rae [20] was the first to take an interest in a measure of success for symmetric simple games, or k-out-of-n games. Dubey and Shapley [7] suggest that the index can be generalized to any simple game and for any voter, leading to what can be referred to as the Rae index.

However, for any simple game there is an affine relationship between the Banzhaf index and the Rae index so that the two notions become quite similar.

Additionally to the given simple game (N, W), in this work we consider that a probability distribution over voters is known and that this probability distribution is independent. That is, we assume that we know – or at least have an estimate of – the probability of the voting intention for each voter that may arise. This information is captured in a vector $\mathbf{p} = (p_1, p_2, \ldots, p_n) \in (0,1)^n$ where p_i indicates the propensity to vote affirmatively for the proposed collective proposal. Although extreme estimations for p_i (either $p_i = 1$ or $p_i = 0$) are possible, we discard them since an infinitesimal error margin in the prediction seems more realistic. Clearly, if vector $\mathbf{p}$ is known then the probability of occurrence for each coalition is known. Thus, the model considered in this situation is represented by the triple $(N, W, \mathbf{p})$.

In this broader context Laruelle and Valenciano [15] (see also [16]) prove that success and decisiveness are related linearly for all game if and only if the (independent) probability distribution is anonymous and with equal inclination towards yes and no. Although success and decisiveness are conceptually different the main purpose of this work is to study when these notions have a similar behavior. A comparative study for independent and anonymous probability distributions for success and decisiveness is done in [11] for proper symmetric voting rules. It is proved that for success the higher the level of consensus required the lower the success of voters. Contrarily, decisiveness behaves differently when the common probability of acceptance is close to 1. Consequently, the different rankings between success and decisiveness depend on the behavior of decisiveness.

Other measures in the literature focus in different aspects to those of success and decisiveness: luck (which is also studied in this paper, see e.g. Holler and Packel [12] for a discussion), satisfaction (see e.g. Davis et al. [6] and [4]), or inclusiveness (see, König and Braüninger [13]).

The paper is organized as follows. In the remaining of this section we recall the main necessary definitions to follow the work. In Section 2 we introduce a Boolean treatment for simple games which admits a clear probabilistic interpretation and facilitates the further developments, we also incorporate a new simpler proof of an essential known result. Anonymous probability distributions are treated in Section 3 where we prove that success and decisiveness are ordinally equivalent if and only if the probability distribution is anonymous. For these distributions we also prove that luckiness shows an opposite order than success and decisiveness. Thus, while success and decisiveness respect the strength of the seats in the voting system, luckiness reverses the order.

1.1 The basic model (N, W)

The usual model for a voting scenario like the one described is a simple game, that is to say, a pair (N, W), where $N = \{1, 2, \ldots, n\}$ denotes the set of *voters*, and W is the set of *winning coalitions*, i.e., sets of voters whose favorable vote ensures the acceptation of the proposal. Subsets of N that are not in W are called losing coalitions, and it is assumed that:

1) $\emptyset$ is losing,
2) subsets of losing coalitions are again losing (monotonicity),
3) N is a winning coalition.

A winning coalition is *minimal* if each proper subset is a losing coalition. The set of minimal winning coalitions is usually denoted by W^m, and, because of monotonicity, it completely determines the game. A voter i is *null* if $i \notin S$ for all $S \in W^m$, i.e., it has no influence in the game. A simple game without null voters is called a *robust* simple game.

A useful tool for comparing the influence of two voters in a voting system is the *desirability relation*. Voter i is at least as desirable as voter j (written $i \succsim j$) in (N, W) if

$$S \cup \{j\} \in W \text{ implies } S \cup \{i\} \in W \quad \text{for all} \quad S \subseteq N \setminus \{i, j\};$$

if moreover $T \cup \{j\} \notin W$ but $T \cup \{i\} \in W$ for some $T \subseteq N \setminus \{i, j\}$ it is said that i is more desirable than j as a coalitional partner (and then write $i \succ j$). Of course, voters i and j are equally desirable (and write $i \approx j$) when $i \succsim j$ and $j \succsim i$. A game is called *complete* if $i \succsim j$ or $j \succsim i$ for all $i, j \in N$. Complete games give a total ranking of importance of the seats of voters and we will assume hereafter that in a complete game we have

$$1 \succsim 2 \succsim \cdots \succsim n.$$

Weighted games are examples of complete games. In a weighted game each player is a assigned a weight and the weight of a coalition is just the sum of the weights of the individuals that form it, and some preset quota is needed for passage. The winning coalitions are those whose weight equals or surpasses the preset quota. An arbitrary weighted representation is denoted

$$[q; w_1, w_2, \ldots, w_n]$$

where w_i is the weight of voter i and q is the quota.

A *symmetric* rule or *k-out-of-n-rule* is a game with n voters in which all minimal winning coalitions have size k. All players are therefore equally desirable and the game admits a weighted representation with quota k and weight 1 for all voters.

Loosely speaking, a *power index* is a function g which assigns to a simple game (N, W) a vector $g(N, W) \in \mathbb{R}^n$ where each component $g_i(N, W)$ is a measure for the ith voter in the simple game (N, W) according to g. As the game under analysis, (N, W), will always be clearly specified, we will write g instead of $g(N, W)$ hereafter. Although there are several power indices well recognized in this work we concentrate our attention in the Banzhaf index [1] (already anticipated by Penrose [19]) and the Rae index [20] since they are the most natural representatives of decisiveness and success respectively. Two forms of measuring power under different perspectives.

The 'raw' Banzhaf index for a voter i in (N, W) is given by the number of times in which i is *decisive in winning coalitions*, i.e., $i \in S \in W$ and $S \setminus \{i\} \notin W$. Dubey and Shapley [7] (see also Owen [18]) proposed the following normalization of the 'raw' Banzhaf index as the ratio:

$$Bz_i = \frac{\text{number of winning coalitions in which } i \text{ is decisive}}{2^{n-1}} \tag{1}$$

where the denominator counts the number of coalitions containing i. Expression (1) is equivalent to

$$Bz_i = \frac{\text{number of coalitions in which } i \text{ is decisive}}{2^n} \tag{2}$$

in which i is *decisive* in a coalition S if either $i \in S \in W$ and $S \setminus \{i\} \notin W$ or $i \notin S \notin W$ and $(S \cup \{i\}) \in W$.

The Rae index for a voter i in (N, W) is given by the proportion of coalitions in which the state (either winning or losing) of the coalition coincides with the vote of the voter.

$$Rae_i = \frac{|\{S : i \in S \in W\}| + |\{S : i \notin S \notin W\}|}{2^n} \tag{3}$$

The relation between these two indices is given by

$$Rae_i = 0.5 + 0.5Bz_i \tag{4}$$

1.2 The extended model (N, W, p)

Let $\mathbf{P}_N$ denote the set of all distributions of probabilities over 2^N in which each voter i independently votes 'yes' with probability p_i and 'no' with probability $1 - p_i$. The probability of coalition S to be formed is then given by

$$P(S) = \prod_{i \in S} p_i \prod_{i \in N \setminus S} (1 - p_i) \tag{5}$$

Thus, an element of $\mathbf{P}_N$ is determined by $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and the probability of acceptance of the submitted proposal is given by

$$\sum_{S:S \in W} P(S) = \sum_{S:S \in W} \prod_{i \in S} p_i \prod_{i \in N \setminus S} (1 - p_i)$$

Let's consider now the natural extensions of the Banzhaf and Rae indices in the more general context $(N, W, \mathbf{p})$.

Definition 1. *Let $(N, W, \mathbf{p})$ be a simple game with probability distribution $\mathbf{p} \in \mathbf{P}_N$ over coalitions, and let $i \in N$:*

i) Voter i's success is the probability that i is successful:

$$\Omega_i(\mathbf{p}) = P(i \text{ is successful}) = \sum_{S:i \in S \in W} P(S) + \sum_{S:i \notin S \notin W} P(S) \tag{6}$$

ii) Voter i's decisiveness is the probability that i is decisive:

$$\Phi_i(\mathbf{p}) = P(i \text{ is decisive}) = \sum_{\substack{S \,:\, i \in S \in W \\ S \setminus i \notin W}} P(S) + \sum_{\substack{S \,:\, i \notin S \notin W \\ S \cup i \in W}} P(S) \tag{7}$$

Decisiveness only depends on the other voters' behaviour, not his/her own, since voter i's decisiveness can be written as

$$\Phi_i(\mathbf{p}) = \sum_{\substack{S \,:\, i \in S \in W \\ S \setminus i \notin W}} (P(S) + P(S \setminus \{i\})).$$

and, for each S, $P(S) + P(S \setminus \{i\})$ is the probability of all voters in $S \setminus \{i\}$ voting 'yes' and those in $N \setminus S$ voting 'no'. Since voter i's success depends on all voters' behavior and voter i's decisiveness depends only on the other voters' behavior, there is no way to derive one of these notions from the other, and the only relations in general are the obvious:

$$\Phi_i(\mathbf{p}) \leq \Omega_i(\mathbf{p}),$$

and Barry's ([2] and [3]) equation: 'Success' = 'Decisiveness'+ 'Luck', which remains valid in this more general context:

$$\Omega_i(\mathbf{p}) = \Phi_i(\mathbf{p}) + \Lambda_i(\mathbf{p}), \tag{8}$$

wherein $\Lambda_i(\mathbf{p})$ denotes voter i's 'luck' or probability of being 'lucky', that is:

$$\Lambda_i(\mathbf{p}) = \sum_{\substack{S \,:\, i \,\in\, S \,\in\, W \\ S \setminus i \,\in\, W}} P(S) + \sum_{\substack{S \,:\, i \,\notin\, S \,\notin\, W \\ S \cup i \,\notin\, W}} P(S)$$

2 Another view of the two models

For the sake of getting a more efficient description in the developments and proofs of the work we consider a Boolean context which turns out to be equivalent to the classical model presented.

2.1 The basic model (N, χ)

Assume that both the game and the actions of voters are binary, that is, the voters can only be in one of two possible states: voting 'yes' or voting 'no' for the issue at hand, and, after the vote has concluded, the result of the game is either to 'pass' the proposal or to 'defeat' it. Let the binary variable x_i indicate the action of voter i for $i = 1, 2, \ldots, n$:

$$x_i = \begin{cases} 1 & \text{if voter } i \text{ votes 'yes'} \\ 0 & \text{if voter } i \text{ votes 'no'}. \end{cases}$$

Then, vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ represents the states of all voters and is in one-to-one correspondence with the coalition $S = \{i \in N : x_i = 1\}$. Let χ represent the state of the game, and

$$\chi = \begin{cases} 1 & \text{if the proposal is accepted} \\ 0 & \text{if the the proposal is rejected.} \end{cases}$$

That is, the state of the game is a deterministic function of the states of voters. Thus, it can be written as

$$\chi = \chi(\mathbf{x}) = \sum_{S : S \in W} \prod_{i \in S} x_i \prod_{i \in N \setminus S} (1 - x_i) \tag{9}$$

where $\chi(\mathbf{x})$ is called the multilinear extension of the game (Owen [17]). Each game (N, W) is completely determined by a unique multilinear extension (N, χ) in which:

1) $\chi(\mathbf{0}) = \chi(0, 0, \ldots, 0) = 0$,
2) χ is a non-decreasing function in each variable x_i, and
3) $\chi(\mathbf{1}) = \chi(1, 1, \ldots, 1) = 1$.

As χ and W are in one-to-one correspondence, we will indistinctively use from now on the simple game given by (N, χ) or its multilinear extension.

Example 1 (k-out-of-n rule or symmetric voting rule). A proposal is passed in a k-out-of-n game if and only if at least k of the n voters vote affirmatively. The symmetric voting rule is given by

$$\chi(\mathbf{x}) = 1 \text{ if and only if } \sum_{i=1}^{n} x_i \geq k.$$

Note that for n odd and $k = (n+1)/2$, it is the simple majority rule, while when $k = n$, it is the unanimity rule which can also be expressed as

$$\chi(\mathbf{x}) = \prod_{i=1}^{n} x_i = \min\{x_1, x_2, \ldots, x_n\}.$$

For any $i = 1, 2, \ldots, n$ the following equation, which may easily be deduced from (9), can be used for the multilinear function:

$$\chi(\mathbf{x}) = x_i \chi(1_i, \mathbf{x}) + (1 - x_i)\chi(0_i, \mathbf{x}) \tag{10}$$

where $(\cdot_i, \mathbf{x}) = (x_1, x_2, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)$.

Let's going to describe the Banzhaf and the Rae indices by using χ. Let

$$m_i = \sum_{(1_i, \mathbf{x})} [\chi(1_i, \mathbf{x}) - \chi(0_i, \mathbf{x})]$$

where the summand over $(1_i, \mathbf{x})$ covers the set of all coalitions that contain voter i and m_i counts the number of winning coalitions which contain i and are decisive for it, and

$$m'_i = \sum_{(0_i, \mathbf{x})} [\chi(1_i, \mathbf{x}) - \chi(0_i, \mathbf{x})]$$

where the summand over $(0_i, \mathbf{x})$ covers the set of all coalitions that does not contain voter i and m'_i counts the number of losing coalitions which do not contain i and are decisive for it. It is clear that

$$m_i = m'_i.$$

Therefore, both m_i and m'_i coincide with the raw Banzhaf index (i.e., the numerator in (1)) which can be written in this context as

$$Bz_i = \frac{m_i}{2^{n-1}}$$

which is obviously equivalent to

$$Bz_i = \frac{2m_i}{2^n} = \frac{m_i + m'_i}{2^n} = \frac{\text{total decisiveness for } i}{\text{total number of coalitions}}$$

which coincides with the formula in (2).

The raw Rae index (numerator in (3)) corresponds to

$$|\{(1_i, \mathbf{x}) : \chi(1_i, \mathbf{x}) = 1\}| + |\{(0_i, \mathbf{x}) : \chi(0_i, \mathbf{x}) = 0\}|$$

which after normalization by the total number of coalitions 2^n converts into the Rae index.

$$Rae_i = \frac{|\{(1_i, \mathbf{x}) : \chi(1_i, \mathbf{x}) = 1\}| + |\{(0_i, \mathbf{x}) : \chi(0_i, \mathbf{x}) = 0\}|}{2^n}$$

2.2 The extended model $(N, \chi, \mathbf{p})$

Let's consider a particular vote. Let X_i be a binary random variable, and

$$X_i = \begin{cases} 1 & \text{if voter } i \text{ votes 'yes'} \\ 0 & \text{if voter } i \text{ votes 'no'}, \end{cases}$$

with $P\{X_i = 1\} = p_i = \mathbb{E}(X_i)$, for $i = 1, 2, \ldots, n$, where $\mathbb{E}(\cdot)$ denotes the expected value of the random variable and p_i is the expected probability that voter i votes affirmatively according to some external observer and at a given time before the vote takes place. Assume that the random variables X_i, $i = 1, 2, \ldots, n$, are mutually statistically independent. Introducing $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, the expectation to pass the proposal submitted to vote in (N, χ) is given by

$$f(\mathbf{p}) = P(\{\chi(\mathbf{X}) = 1\}) = \mathbb{E}(\chi(\mathbf{X}))$$

which is a function of the expected probabilities of the voters.

Example 2 (Example 1 continued). The unanimity rule $\chi(\mathbf{x}) = \prod_{i=1}^{n} x_i$ in Example 1 has

$$f(\mathbf{p}) = \prod_{i=1}^{n} p_i$$

as the expected probability to pass the proposal.

If $p_1 = p_2 = \cdots = p_n = p$ the k-out-of-n game, in which $\chi(\mathbf{x}) = 1$ if and only if $\sum_{i=1}^{n} x_i \geq k$ has expected probability

$$f(p) = \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

to pass the proposal.

Note that $f(\mathbf{p})$ is a multilinear in each p_i. Thus, when $p_1 = p_2 = \cdots = p_n = p$, $f(p)$ is a polynomial function in p.

Proposition 1. *Let $f(\mathbf{p})$ be the expected value to pass the proposal in a simple game. Then, for $i = 1, 2, \ldots, n$,*

$$f(\mathbf{p}) = p_i f(1_i, \mathbf{p}) + (1 - p_i) f(0_i, \mathbf{p}) \tag{11}$$

where $(\cdot_i, \mathbf{p}) = (p_1, p_2, \ldots, p_{i-1}, \cdot, p_{i+1}, \ldots, p_n)$. Moreover, $f(\mathbf{p})$ is strictly increasing in each p_i (for $0 < p_i < 1$) if the simple game is robust and it is non-decreasing for simple games in general.

Proof. Using decomposition (10) and the independence of voters,

$$f(\mathbf{p}) = \mathbb{E}(\chi(\mathbf{X})) = \mathbb{E}(X_i)\mathbb{E}(\chi(1_i, \mathbf{X})) + (1 - \mathbb{E}(X_i))(\mathbb{E}(\chi(0_i, \mathbf{X})).$$

Equation (11) follows immediately. From equation (11),

$$\frac{\partial f(\mathbf{p})}{\partial p_i} = f(1_i, \mathbf{p}) - f(0_i, \mathbf{p}) \tag{12}$$

so that

$$\frac{\partial f(\mathbf{p})}{\partial p_i} = \mathbb{E}(\chi(1_i, \mathbf{X})) - \mathbb{E}(\chi(0_i, \mathbf{X}))$$

Because χ is nondecreasing, then $\chi(1_i, \mathbf{x}) - \chi(0_i, \mathbf{x}) \geq 0$. In addition, $\chi(1_i, \mathbf{y}) - \chi(0_i, \mathbf{y}) = 1$ for some $\mathbf{y}$ because each voter is non-null in robust simple games. Since $0 < p_i < 1$ for all $i = 1, 2, \ldots, n$, $\mathbf{y}$ has a positive probability of occurring. Thus, $\mathbb{E}(\chi(1_i, \mathbf{X})) - \mathbb{E}(\chi(0_i, \mathbf{X})) > 0$, and the monotonicity result follows. If the simple game is not robust, it has at least a null voter, i, for whom $\mathbb{E}(\chi(1_i, \mathbf{X})) - \mathbb{E}(\chi(0_i, \mathbf{X})) = 0$, and the second monotonicity result follows.

We may now formulate convenient expressions for decisiveness and success in definition 1.

Proposition 2 (Decisiveness and success measures in the context $(N, \chi, \mathbf{p})$).

1. *$\Phi_i(\mathbf{p}) = \frac{\partial f(\mathbf{p})}{\partial p_i} = f(1_i, \mathbf{p}) - f(0_i, \mathbf{p})$,*
2. *$\Omega_i(\mathbf{p}) = p_i f(1_i, \mathbf{p}) + (1 - p_i)(1 - f(0_i, \mathbf{p}))$.*

Proof. 1. From equation (7) in definition 1 and equation (5) it follows

$$\begin{aligned}
\Phi_i(\mathbf{p}) &= \sum_{\substack{S\,:\, i \,\in\, S \,\in\, W \\ S \setminus i \,\notin\, W}} P(S) + \sum_{\substack{S\,:\, i \,\notin\, S \,\notin\, W \\ S \cup i \,\in\, W}} P(S) \\
&= \sum_{\substack{S\,:\, j \,\in\, S \setminus i,\, S \,\in\, W \\ S \setminus i \,\notin\, W}} (p_i + (1 - p_i)) \left(\prod_{j \in S \setminus i} p_j \prod_{j \in N \setminus S} (1 - p_j) \right) \\
&= \sum_{\substack{S\,:\, j \,\in\, S \setminus i,\, S \,\in\, W \\ S \setminus i \,\notin\, W}} \prod_{j \in S \setminus i} p_j \prod_{j \in N \setminus S} (1 - p_j)
\end{aligned}$$

On the other hand

$$f(1_i,\mathbf{p}) = \sum_{\substack{S\,:\,i\,\in\,S\,\in\,W\\ S\,\setminus\,i\,\notin\,W}} \prod_{j\in S\setminus i} p_j \prod_{j\in N\setminus S} (1-p_j) + \sum_{\substack{S\,:\,i\,\in\,S\,\in\,W\\ S\,\setminus\,i\,\in\,W}} \prod_{j\in S\setminus i} p_j \prod_{j\in N\setminus S} (1-p_j)$$

and

$$f(0_i,\mathbf{p}) = \sum_{\substack{S\,:\,i\,\notin\,S,\\ S\,\in\,W}} \prod_{j\in S\setminus i} p_j \prod_{j\in (N\setminus S)\setminus i} (1-p_j) = \sum_{\substack{S\,:\,i\,\in\,S\,\in\,W\\ S\,\setminus\,i\,\in\,W}} \prod_{j\in S\setminus i} p_j \prod_{j\in N\setminus S} (1-p_j)$$

Thus, the second term in $f(1_i,\mathbf{p})$ simplifies with $f(0_i,\mathbf{p})$ in $f(1_i,\mathbf{p})-f(0_i,\mathbf{p})$ and therefore $\Phi_i(\mathbf{p})$ coincides with $f(1_i,\mathbf{p})-f(0_i,\mathbf{p})$.

2. From equation (6) in definition 1 it is clear that success is

$$\Omega_i(\mathbf{p}) = P(\{X_i = 1\} \cap \{\chi(\mathbf{X}) = 1\}) + P(\{X_i = 0\} \cap \{\chi(\mathbf{X}) = 0\})$$

as we assume independent probability distributions it is clear that the first term is $p_i f(1_i,\mathbf{p})$ since p_i is a common factor in all the addends of $f(1_i,\mathbf{p})$, while the second term coincides with $(1-p_i)(1-f(0_i,\mathbf{p}))$ since $1-p_i$ is a common factor of all addends in $1-f(0_i,\mathbf{p})$ and this latter expression is the probability that the proposal does not pass with voter i voting against it.

2.3 The special case of anonymous and equally inclined probability distributions

A probability distribution is *anonymous* if the probability of a coalition depends only on the number of 'yes'-voters, that is, $P(S) = P(R)$ whenever $|S| = |R|$. Hence, an anonymous distribution in $\mathbf{P}_N$ is given by $P(S) = p^s(1-p)^{n-s}$ for all $S \subseteq N$ with $s = |S|$ and $\mathbf{p} = p = (p, p, \ldots, p)$.

An anonymous probability distribution *with equal inclination* towards 'yes' and 'no' is given by

$$p^*(S) = \frac{1}{2^n} \quad \text{for all coalitions } S \subseteq N.$$

where $\mathbf{p} = p^* = (1/2, 1/2, \ldots, 1/2)$. This is equivalent to assuming that each voter, independently of the others, votes 'yes' with probability 1/2, and votes 'no' with probability 1/2.

As the basic model (N, W) corresponds to the model $(N, W, \mathbf{p})$ for anonymous and equal inclination probability vector $\mathbf{p} = p^*$ it follows from previous proposition 2 that

$$\Phi_i(p^*) = Bz_i \quad \text{and} \quad \Omega_i(p^*) = Rae_i.$$

since

$$\Phi_i(p^*) = f(1_i, p^*) - f(0_i, p^*), \text{ and}$$
$$\Omega_i(p^*) = 0.5 + 0.5[\, f(1_i, p^*) - f(0_i, p^*)\,].$$

which leads to the well-known affine relationship (4).

The approach followed above of this section 2 allows an easy treatment of some known but also some new properties. Indeed, the next result was already proved by Laruelle et al. [14] by using double induction in the proof. We provide here an alternative shorter proof.

Theorem 1. *$\Omega_i(\mathbf{p}) = 0.5 + 0.5\Phi_i(\mathbf{p})$ holds for all game (N, χ) if and only if $\mathbf{p}$ is anonymous and equally inclined (i.e., $\mathbf{p} = p^*$).*

Proof. The implication from right to left is obvious. For the other implication, assume that $\Omega_i(\mathbf{p}) = 0.5 + 0.5\Phi_i(\mathbf{p})$ holds for all game. By substituting the expressions of $\Omega_i(\mathbf{p})$ and $\Phi_i(\mathbf{p})$ in proposition 2:

$\Omega_i(\mathbf{p}) = p_i f(1_i, \mathbf{p}) + (1 - p_i)(1 - f(0_i, \mathbf{p})$ and $\Phi_i(\mathbf{p}) = f(1_i, \mathbf{p}) - f(0_i, \mathbf{p})$ it follows:

$$(p_i - 0.5)[f(1_i, \mathbf{p}) + f(0_i, \mathbf{p}) - 1] = 0.$$

Thus either $p_i = 0.5$ or $f(1_i, \mathbf{p}) + f(0_i, \mathbf{p}) - 1 = 0$ for all game.

But for the unanimity rule[1] it holds: $f(1_i, \mathbf{p}) = \prod_{j \neq i} p_j$ and $f(0_i, \mathbf{p}) = 0$, thus $f(1_i, \mathbf{p}) + f(0_i, \mathbf{p}) - 1 < 0$ and therefore for all $i \in N$ it must be $p_i = 0.5$.

Thus, for anonymous and equally inclined probability distributions, success and decisiveness are related in an affine relation but this strong similarity is exclusive for p^*.

3 A study for anonymous probability distributions

The formulas stated in the following lemma will be used in this work.

Lemma 1. *Let i, j be different voters in N. Then,*

a) $$\Phi_i(\mathbf{p}) - \Phi_j(\mathbf{p}) = (p_j - p_i)[f(1_i, 1_j, \mathbf{p}) + f(0_i, 0_j, \mathbf{p}) - f(1_i, 0_j, \mathbf{p}) - f(0_i, 1_j, \mathbf{p})] + [f(1_i, 0_j, \mathbf{p}) - f(0_i, 1_j, \mathbf{p})]$$

b) $$\Omega_i(\mathbf{p}) - \Omega_j(\mathbf{p}) = 2p_i(1 - p_j)f(1_i, 0_j, \mathbf{p}) - 2p_j(1 - p_i)f(0_i, 1_j, \mathbf{p}) + p_j - p_i$$

Proof. Part *a*) is obtained by using (11) and (12) in Definition 1-ii). Part *a*) is obtained by using proposition 2-2) in Definition 1-i).

Definition 2. *Two power indices g and h in the context $(N\chi, \mathbf{p})$ are* ordinally equivalent *if they rank voters equally, i.e., for all $i, j \in N$:*

$$g_i > g_j \iff h_i > h_j$$

which implies that $g_i = g_j$ if and only if $h_i = h_j$.

[1] Some other rules can also be considered instead.

Of course any power index gives a total ranking for players. We say that the rankings given by g and h are *opposite* if they rank voters in an opposite way, i.e., for all $i, j \in N$:

$$g_i > g_j \iff h_i < h_j$$

which implies that $g_i = g_j$ if and only if $h_i = h_j$.

As a consequence of the next result we deduce that, if the game has more than two voters, then success and decisiveness are ordinally equivalent but for non-anonymous probability distributions these rankings can always be different.

Theorem 2. *Let i, j be different voters in N. Then,*

i) If $p_i = p_j$ then $[\ \Omega_i(\mathbf{p}) > \Omega_j(\mathbf{p}) \iff \Phi_i(\mathbf{p}) > \Phi_j(\mathbf{p})\ ]$ for any game χ.

ii) If $N = \{i, j\}$ then $[\ \Omega_i(\mathbf{p}) > \Omega_j(\mathbf{p}) \iff \Phi_i(\mathbf{p}) > \Phi_j(\mathbf{p})\ ]$ for any game χ.

iii) If $p_i \neq p_j$ and there is a voter $k \neq i, j$ with $p_k \neq 1/2$, then we can find a game χ such that $[\Omega_i(\mathbf{p}) - \Omega_j(\mathbf{p})]\,[\Phi_i(\mathbf{p}) - \Phi_j(\mathbf{p})] < 0$.

iv) If $p_i \neq p_j$ and for any voter $k \neq i, j$ it is $p_k = 1/2$, then we can find a game χ such that $\Phi_i(\mathbf{p}) \neq \Phi_j(\mathbf{p})$ and $\Omega_i(\mathbf{p}) = \Omega_j(\mathbf{p})$.

Proof. *i)* If $p_i = p_j = p$ then from lemma 1 it is

$$\Phi_i(\mathbf{p}) - \Phi_j(\mathbf{p}) = f(1_i, 0_j, \mathbf{p}) - f(0_i, 1_j, \mathbf{p})$$

$$\Omega_i(\mathbf{p}) - \Omega_j(\mathbf{p}) = 2p(1-p)[f(1_i, 0_j, \mathbf{p}) - f(0_i, 1_j, \mathbf{p})]$$

for any game χ. The fact that $0 < p < 1$ leads to the conclusion.

ii) If $N = \{i, j\}$ then there are only four possible games, and we see that in all of them the statement is true.

- If $f(\mathbf{p}) = p_i$ it is $\Phi_i = 1 > \Phi_j = 0$ and $\Omega_i = 1 > \Omega_j = 1 - p_i - p_j + 2p_ip_j$.
- If $f(\mathbf{p}) = p_j$ it is $\Phi_i = 0 < \Phi_j = 1$ and $\Omega_i = 1 - p_i - p_j + 2p_ip_j < \Omega_j = 1$.
- If $f(\mathbf{p}) = p_ip_j$ it is $\Phi = (p_j, p_i)$ and $\Omega = (1 - p_i + p_ip_j, 1 - p_j + p_ip_j)$.
- If $f(\mathbf{p}) = p_i + p_j - p_ip_j$ it is $\Phi = (1 - p_j, 1 - p_i)$ and $\Omega = (1 - p_j + p_ip_j, 1 - p_i + p_ip_j)$.

iii) Assume, without loss of generality, that $p_i < p_j$, and let $k \neq i, j$ be such that $p_k \neq 1/2$.

- If $p_k < 1/2$ then consider the game χ with $f(\mathbf{p}) = p_ip_k + p_jp_k - p_ip_jp_k$. In this case it is $f(1_i, 1_j, \mathbf{p}) = p_k$, $f(0_i, 0_j, \mathbf{p}) = 0$ and $f(1_i, 0_j, \mathbf{p}) = f(0_i, 1_j, \mathbf{p}) = p_k$. Thus, from Lemma 1,

$$\Phi_i(\mathbf{p}) - \Phi_j(\mathbf{p}) = (p_j - p_i)(-p_k) < 0$$

$$\Omega_i(\mathbf{p}) - \Omega_j(\mathbf{p}) = (p_j - p_i)(-2p_k + 1) > 0$$

- If $p_k > 1/2$ then consider the gameχ with $f(\mathbf{p}) = p_k + p_i p_j - p_i p_j p_k$. In this case it is $f(1_i, 1_j, \mathbf{p}) = 1$ and $f(0_i, 0_j, \mathbf{p}) = f(1_i, 0_j, \mathbf{p}) = f(0_i, 1_j, \mathbf{p}) = p_k$. Thus, from Lemma 1,

$$\Phi_i(\mathbf{p}) - \Phi_j(\mathbf{p}) = (p_j - p_i)(1 - p_k) > 0$$

$$\Omega_i(\mathbf{p}) - \Omega_j(\mathbf{p}) = (p_j - p_i)(1 - 2p_k) < 0$$

iv) Taking any $k \neq i, j$, the game χ with $f(\mathbf{p}) = p_k + p_i p_j - p_i p_j p_k$ verifies that

$$\Phi_i(\mathbf{p}) - \Phi_j(\mathbf{p}) = (p_j - p_i)/2 \text{ and } \Omega_i(\mathbf{p}) - \Omega_j(\mathbf{p}) = 0$$

and the statement is proved.

The next Corollary shows that although success and decisiveness are conceptually different they rank voters in the same way for anonymous probability distributions.

Corollary 1. *Let* $n \geq 3$. *Then,*

$$[\Phi_i(\mathbf{p}) > \Phi_j(\mathbf{p}) \;\Leftrightarrow\; \Omega_i(\mathbf{p}) > \Omega_j(\mathbf{p})] \text{ for all } \chi,\ i, j \in N \iff \mathbf{p} \text{ is anonymous.}$$

Proof. From Theorem 2-*i*) it is clear that if all components of $\mathbf{p}$ coincide ($\mathbf{p} = p$) then $\Phi_i(p) > \Phi_j(p)$ if and only if $\Omega_i(p) > \Omega_j(p)$, for any game χ and for any different elements i, j in N.

Conversely, if $p_i \neq p_j$ for some components $i, j \in N$ then, from Theorem 2-*iii),iv)* we can find a game χ for which $\Phi_i(\mathbf{p}) > \Phi_j(\mathbf{p})$ but $\Omega_i(\mathbf{p}) \ngtr \Omega_j(\mathbf{p})$.

Corollary 1 states that, if the probability distribution is anonymous, i.e., coincident for all voters, then success and decisiveness rank them in the same way independently of the game considered and this only occurs for these probability distributions. Thus, for anonymous probability distributions both measures are ordinally equivalent and therefore no matter, from the ordinal point of view, which measure we take, since both measures give the same ranking. Although both measures are conceptually different and evaluate different aspects of power the rankings they produce are the same.

But for non-anonymous probability distributions we can always get different rankings for the two measures. If one considers, as we do, that the two indices measure different aspects of the game it is then relevant to compute both for having a good understanding of the problem.

In the next subsection let's investigate a little deeper about the ordinal equivalence between success and decisiveness for anonymous probability distributions.

3.1 The antagonistic behavior of luckiness in front of success and decisiveness

Corollary 1 showed that success and decisiveness rank voters equally for all game if the probability distribution is anonymous. However which is the effect of luckiness? Barry's equation (8) is of course a useful tool to clarify this posed question.

From lemma 1 and Barry's equation (8) it easily follows the next lemma.

Lemma 2.

$$\Lambda_i(\mathbf{p}) - \Lambda_j(\mathbf{p}) = 2[p_i f(1_i, 0_j, \mathbf{p}) - p_j f(0_i, 1_j, \mathbf{p})] + (p_i - p_j)(b-1) - a(1 + 2p_i p_j)$$

where: $a = f(1_i, 0_j, \mathbf{p}) - f(0_i, 1_j, \mathbf{p})$ *and*
$b = \frac{\partial^2 f}{\partial p_i \partial p_j} = [f(1_i, 1_j, \mathbf{p}) + f(0_i, 0_j, \mathbf{p}) - f(1_i, 0_j, \mathbf{p}) - f(0_i, 1_j, \mathbf{p})].$

Note that a and b are functions of the probabilities of voters different from i and j. Thus, they do not depend on p_i and p_j.

Proposition 3. *Let i, j be different voters in N. If $p_i = p_j$ then*

$$\Lambda_i(\mathbf{p}) > \Lambda_j(\mathbf{p}) \iff \Phi_i(\mathbf{p}) < \Phi_j(\mathbf{p}) \iff \Omega_i(\mathbf{p}) < \Omega_j(\mathbf{p}).$$

Proof. Because of corollary 1 we only need to prove the first equivalence. As $p_i = p_j$, then by lemma 1: $\Phi_i(\mathbf{p}) - \Phi_j(\mathbf{p}) = a$ and by lemma 2: $\Lambda_i(\mathbf{p}) - \Lambda_j(\mathbf{p}) = a \cdot (-2p^2 + 2p - 1)$ where $p = p_i = p_j$. But $-2p^2 + 2p - 1 < 0$ for all $p \in (0, 1)$, which concludes the proof.

Next corollary follows from proposition 3.

Corollary 2. *For anonymous probability distributions ($\mathbf{p} = p = (p, \ldots, p)$)*

$$\Lambda_i(p) > \Lambda_j(p) \iff \Phi_i(p) < \Phi_j(p) \iff \Omega_i(p) < \Omega_j(p).$$

That is, the ranks given by luckiness and decisiveness (or luckiness and success) are opposite under anonymous probability distributions. Hence, concerning rankings and for anonymous probability distributions, the opposite effect that luckiness show compared with decisiveness has no additive effect on success, since it does not alter the ranking given by decisiveness, that is:

- if $\Omega_i(p) - \Omega_j(p) > 0$ then $\Phi_i(p) - \Phi_j(p) \geq \Omega_i(p) - \Omega_j(p)$.
- if $\Omega_i(p) - \Omega_j(p) < 0$ then $\Phi_i(p) - \Phi_j(p) \leq \Omega_i(p) - \Omega_j(p)$.

In summary, for anonymous probability distributions the net effect of luckiness over success is negligible (at least from the ordinal point of view) when it is compared with decisiveness.

3.2 The importance of the seats in a voting rule

When the probability distribution is anonymous, it seems intuitive that those voters that occupy a stronger seat should be more powerful for different reasonable power measures. A tool to compare the strength of the seats of two voters is the desirability relation. Freixas and Pons prove in [10] the following.

Lemma 3. *Let* $(N, \chi, \mathbf{p})$, $i, j \in N$. *Then*

1. $i \succsim j \quad \Leftrightarrow \quad f(1_i, 0_j, \mathbf{p}) - f(0_i, 1_j, \mathbf{p}) \geq 0$ *for all* $\mathbf{p}$.
2. $i \succ j \quad \Leftrightarrow \quad f(1_i, 0_j, \mathbf{p}) - f(0_i, 1_j, \mathbf{p}) > 0$ *for all* $\mathbf{p}$.

Thus from lemmas 1, 2, 3 it follows the next corollary.

Corollary 3.

1. *If* $i \succsim j$ *and* $p_i = p_j$, *then* $\Phi_i(\mathbf{p}) \geq \Phi_j(\mathbf{p})$,
2. *If* $i \succ j$ *and* $p_i = p_j$, *then* $\Phi_i(\mathbf{p}) > \Phi_j(\mathbf{p})$,
3. *If* $i \succsim j$ *and* $p_i = p_j$, *then* $\Omega_i(\mathbf{p}) \geq \Omega_j(\mathbf{p})$,
4. *If* $i \succ j$ *and* $p_i = p_j$, *then* $\Omega_i(\mathbf{p}) > \Omega_j(\mathbf{p})$,
5. *If* $i \succsim j$ *and* $p_i = p_j$, *then* $\Lambda_i(\mathbf{p}) \leq \Lambda_j(\mathbf{p})$,
6. *If* $i \succ j$ *and* $p_i = p_j$, *then* $\Lambda_i(\mathbf{p}) < \Lambda_j(\mathbf{p})$.

Thus, for anonymous probability distributions the strategic part of the model $(N, \chi, \mathbf{p})$, which is captured by vector $\mathbf{p} = p = (p, \ldots, p)$, is neutral since all players have a common probability. Thus what really matters is then the strength of the seats for players. Decisiveness and success are ordinally equivalent to the desirability relation, while Luck is opposite to it. That is, the less significant is the seat of a voter the most lucky the voter is. Next result follows from the previous corollary.

Corollary 4. *In a complete game with ranking* $1 \succsim 2 \succsim \cdots \succsim n$ *and with anonymous probability distribution* (N, χ, p) *it holds:*

1. $\Phi_1(p) \geq \Phi_2(p) \geq \cdots \geq \Phi_n(p)$,
2. $\Omega_1(p) \geq \Omega_2(p) \geq \cdots \geq \Omega_n(p)$,
3. $\Lambda_1(p) \leq \Lambda_2(p) \leq \cdots \leq \Lambda_n(p)$.

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