TABLE II
CONDITION 6)

k	$\{i_1, \ldots, i_k\}$	i_{k+1}	Inequality (3)
1	$\{1\}$	2	$\left\Vert \frac{s+1}{1.4s+1.5} \right\Vert_\infty = 0.714 < 1$
		3	$\left\Vert \frac{3s+3}{3.7s+4} \right\Vert_\infty = 0.811 < 1$
	$\{2\}$	1	$\left\Vert \frac{s+1}{2.8s+3} \right\Vert_\infty = 0.357 < 1$
		3	$\left\Vert \frac{3s+3}{4.4s+5} \right\Vert_\infty = 0.682 < 1$
	$\{3\}$	1	$\left\Vert \frac{s+1}{3.7s+4} \right\Vert_\infty = 0.270 < 1$
		2	$\left\Vert \frac{s+1}{2.2s+2.5} \right\Vert_\infty = 0.455 < 1$
2	$\{1,2\}$	3	$\left\Vert \frac{8.4s+9}{9.8s+11} \right\Vert_\infty = 0.857 < 1$
	$\{1,3\}$	2	$\left\Vert \frac{7.4s+8}{9.8s+11} \right\Vert_\infty = 0.755 < 1$
	$\{2,3\}$	1	$\left\Vert \frac{4.4s+5}{9.8s+11} \right\Vert_\infty = 0.455 < 1$

ii) Conditions in Theorem 1.
Conditions 1), 2), and 4) are obvious.
Conditions 3) and 6): see Tables I and II, respectively.
Condition 5)

$$\left(I - \tilde{V}_1 D\right)\tilde{X}_2 = \left(I - \tilde{V}_2 D\right)\tilde{X}_1 = \frac{70.56s^3 + 245.8s^2 + 285s + 110}{2(92s+100)(s+1)^2}$$

$$\left(I - \tilde{V}_1 D\right)\tilde{X}_3 = \left(I - \tilde{V}_3 D\right)\tilde{X}_1 = -\frac{61.74s^3 + 226.1s^2 + 274s + 110}{2(92s+100)(s+1)^2}.$$

iii) Transfer function of the total compensator set

$$C_{all} = \frac{\tilde{V}_1^{-1}\tilde{X}_1 + \tilde{V}_2^{-1}\tilde{X}_2 + \tilde{V}_3^{-1}\tilde{X}_3}{\tilde{V}_1^{-1}\tilde{Y}_1 + \tilde{V}_2^{-1}\tilde{Y}_2 + \tilde{V}_3^{-1}\tilde{Y}_3 - 2} = -\frac{(s+1)(901.6s^2 + 2090s + 1210)}{1803.2s^3 + 8688s^2 + 12\,870s + 6050} = -\frac{s+1}{2s+5}.$$

V. CONCLUSION

This note has presented a new idea of advanced cooperative order in a compensator set, and has shown its achievability in the multicompensator configuration. The idea can be formulated in other multicompensator configurations, e.g., the parallel one and the decentralized one. The discussion in such cases is the future work.

REFERENCES

[1] J. P. Corfmat and A. S. Morse, "Decentralized control of linear multivariable systems," *Automatica*, vol. 12, pp. 479–495, 1976.

[2] K. Mori, K. Sano, and H. Ihara, "Autonomous controllability of decentralized system aiming at fault-tolerance," in *Proc. 8th IFAC World Congr.*, Kyoto, Japan, 1981, pp. 1833–1839.

[3] M. Vidyasagar and N. Viswanadham, "Reliable stabilization using a multi-controller configuration," *Automatica*, vol. 21, pp. 599–602, 1985.

[4] K. D. Minto and R. Ravi, "New results on the multi-controller scheme for the reliable control of linear plants," in *Proc. Amer. Control Conf.*, Boston, MA, 1991, pp. 615–619.

[5] A. N. Gündeş, "Reliable stabilization of linear plants using a two-controller configuration," *Syst. Control Lett.*, vol. 23, pp. 297–304, 1994.

[6] N. Sebe and T. Kitamori, "Reliable stabilization based on a multi-compensator configuration," in *Proc. 12th IFAC World Congr.*, vol. 4, Sydney, Australia, 1993, pp. 5–8.

[7] K. Suyama, "Reliable load-sharing control," in *Proc. 33rd IEEE Conf. Decision Control*, Orlando, FL, 1994, pp. 1542–1547.

Preservation of Controllability–Observability in Expanded Systems

Lubomír Bakule, José Rodellar, Josep M. Rossell, and Pere Rubió

***Abstract*—The result contributed by this note is that controllability–observability of an original continuous-time LTI dynamic system can always be simultaneously preserved in expanded systems within the Inclusion Principle when using block structured complementary matrices. This new structure offers more degrees of freedom for the selection of specific complementary matrices than well known used cases, such as aggregations and restrictions, which enable such preservation only in certain special cases. A complete unrestricted transmission of these qualitative properties from the original controllable–observable system to its expansion is a basic requirement on the expansion/contraction process, mainly when controllers/observers are designed in expanded systems to be consequently contracted for implementation in initially given systems. An original system composed of two overlapped subsystems is adopted as a general prototype case. A numerical example is supplied.**

***Index Terms*—Controllability, inclusion principle, large-scale systems, observability, overlapping decomposition.**

I. INTRODUCTION

A large variety of real world systems consists of subsystems sharing common parts. For either structural or computational reasons it is often convenient to construct decentralized controllers/observers by using overlapping information sets. There exist control problems in different areas such as traffic problems, flexible structures or power systems, where this is a particularly effective way to proceed. This has motivated to formalize a general mathematical setting which has been named as the Inclusion Principle (IP). Essentially, this principle defines a framework for two dynamic systems with different dimensions, in which solutions of the system with larger dimension include solutions of the system with smaller dimension. Both systems are related through linear transformations (expansions and contractions) that have

Manuscript received October 3, 2000; revised January 22, 2001. Recommended by Associate Editor F. M. Callier. This work was supported in part by the MEC under Grant DGESIC-SAB1999-0114, by the ASCR under Grant A2075802, by the CICYT under Grant TAP99-1079-C03-02, and by the UPC under Grant PR99-08.

L. Bakule is with the Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, 182 08 Prague 8, Czech Republic (e-mail: bakule@utia.cas.cz).

J. Rodellar is with the Department of Applied Mathematics III, Universitat Politècnica de Catalunya (UPC), 08034-Barcelona, Spain (e-mail: jose.rodellar@upc.es).

J. M. Rossell and P. Rubió are with the Department of Applied Mathematics III, Universitat Politècnica de Catalunya (UPC), 08240-Manresa, Spain (e-mail: josep.maria.rossell@upc.es; pere.rubio@upc.es).

Publisher Item Identifier S 0018-9286(01)06600-4.

the freedom of the selection of the so-called complementary matrices. One of the basic research issues within the IP is the question whether structural properties of the systems are transmitted or not from the original system to its expansion when applying the expansion/contraction framework. Such transmission is a major practical issue when using this framework for control design. In fact, an original system with strongly coupled subsystems is usually expanded to a larger system with weakly coupled subsystems. Then, controllers/observers are usually designed for the expanded system using well-known weak coupling control design methods and subsequently contracted for the implementation in the initial system. In this context, the importance of the transmission of controllability/observability into expanded systems without any restrictions is clear. All previous results enable such transmission only partially because of the usage of unstructured complementary matrices given in the forms of aggregations or restrictions. The motivating reason of this note is to overcome such restriction by proving that there are block structured complementary matrices ensuring always the simultaneous transmission of controllability and observability.

A. Relevant References and Outline of the Paper

The IP has been developed by Šiljak and his co-workers [4], [5], [9]. The conditions given in previous works [4]–[7], [9] on the complementary matrices have a fundamental, implicit nature, in the sense that they have the form of matrix products from which it is not easy to select specific values for these matrices. In fact, only two particular forms of these matrices, corresponding to aggregations and restrictions, have been commonly adopted in the literature for numerical computations [1], [6], [9], [13]. A new characterization of the complementary matrices has been recently presented in [2], [3], [8], which gives a more explicit way for their selection and which includes aggregations and restrictions as particular cases. It relies on a new constructive way of approaching the concept of canonical form within the IP previously proposed in [5], [9]. Overlapping control/observer design strategies can be found for instance in [10], [12], [14]. The result by Malinowski *et al.* [7] states that, when using well known particular forms of complementary matrices such that aggregations and restrictions, an original controllable–observable system transmits these properties only in certain special cases. Particularly the expanded systems become either controllable or observable but not simultaneously controllable–observable. In this note, it is shown that expanded system can always preserve controllability–observability, provided that the original system holds both properties, by using the new characterization of the complementary matrices [2], [3]. To the author's knowledge, no other results are available in the literature ensuring that both properties are always simultaneously transmitted when expanding a controllable–observable initial system.

The paper is organized as follows. Section II states the problem formulation, first including necessary preliminaries on the expansion/contraction scheme and the complementary matrices. Section III presents the main result on the preservation of controllability–observability in expanded systems, using a prototype case with an original system composed of two overlapped subsystems for two different selections of block structured complementary matrices. However, the extension of this case to any number of overlapped subsystems is straightforward. Section IV presents an illustrative numerical example.

II. Problem Formulation

A. Preliminaries

Consider a pair of linear systems

$$\mathbf{S}:\quad \begin{aligned}\dot{x} &= Ax + Bu,\\ y &= Cx,\end{aligned} \qquad \tilde{\mathbf{S}}:\quad \begin{aligned}\dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}\tilde{u},\\ \tilde{y} &= \tilde{C}\tilde{x}\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^{\mathbf{n}}$, $u(t) \in \mathbb{R}^{\mathbf{m}}$, $y(t) \in \mathbb{R}^{\mathbf{l}}$ are the state, input, output of $\mathbf{S}$ at time t, and $\tilde{x}(t) \in \mathbb{R}^{\tilde{\mathbf{n}}}$, $\tilde{u}(t) \in \mathbb{R}^{\tilde{\mathbf{m}}}$, $\tilde{y}(t) \in \mathbb{R}^{\tilde{\mathbf{l}}}$ are those ones of $\tilde{\mathbf{S}}$. A, B, C and $\tilde{A}, \tilde{B}, \tilde{C}$ are constant matrices of dimensions $n\times n$, $n\times m$, $l\times n$ and $\tilde{n}\times\tilde{n}$, $\tilde{n}\times\tilde{m}$, $\tilde{l}\times\tilde{n}$, respectively. Suppose that the dimensions of the state, input, output vectors x, u, y of $\mathbf{S}$ are smaller than (or at most equal to) those of $\tilde{x}, \tilde{u}, \tilde{y}$ of $\tilde{\mathbf{S}}$. Denote $x(t; x_0, u)$ and $y[x(t)]$ the state behavior and the corresponding output of $\mathbf{S}$ for a fixed input $u(t)$ and for an initial state $x(0) = x_0$, respectively. Similar notations $\tilde{x}(t; \tilde{x}_0, \tilde{u})$ and $\tilde{y}[\tilde{x}(t)]$ are used for the state behavior and output of system $\tilde{\mathbf{S}}$. The systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$ are related by the transformations $\tilde{x} = Vx$, $x = U\tilde{x}$, $\tilde{u} = Ru$, $u = Q\tilde{u}$, $\tilde{y} = Ty$, $y = S\tilde{y}$, where V, R and T are constant matrices of appropriate dimensions and full-column ranks. U, Q and S are constant matrices of appropriate dimensions and full row ranks satisfying the relations $UV = I_n$, $QR = I_m$, $ST = I_l$, where I_n, I_m, I_l are identity matrices of indicated dimensions.

Definition 1 (Inclusion Principle): We say that the system $\tilde{\mathbf{S}}$ includes the system $\mathbf{S}$, that is $\tilde{\mathbf{S}} \supset \mathbf{S}$, if there exists a quadruplet (U, V, R, S) such that, for any initial state x_0 and any fixed input $u(t)$ of $\mathbf{S}$, the choice $\tilde{x}_0 = Vx_0$ and $\tilde{u}(t) = Ru(t)$ for all $t \geq 0$ of the initial state $\tilde{x}_0$ and input $\tilde{u}(t)$ of the system $\tilde{\mathbf{S}}$ implies $x(t; x_0, u) = U\tilde{x}(t; \tilde{x}_0, \tilde{u})$ and $y[x(t)] = S\tilde{y}[\tilde{x}(t)]$, for all $t \geq 0$.

If $\tilde{\mathbf{S}} \supset \mathbf{S}$, then $\tilde{\mathbf{S}}$ is said to be an *expansion* of $\mathbf{S}$ and $\mathbf{S}$ is a *contraction* of $\tilde{\mathbf{S}}$.

The matrices $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ can be expressed as

$$\tilde{A} = VAU + M \quad \tilde{B} = VBQ + N \quad \tilde{C} = TCU + L \tag{2}$$

where M, N and L are *complementary matrices* of appropriate dimensions. Usually, the transformations V, R, T are set *a priori* to define structural relations between the state, control and output variables in both systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$. Given these transformations, the choice of the complementary matrices M, N, L offers degrees of freedom to build an expanded system $\tilde{\mathbf{S}}$ from an original system $\mathbf{S}$ to meet certain specifications. In fact, the motivation of this note lies in the exploitation of such degrees of freedom to build controllable–observable expanded systems.

For $\tilde{\mathbf{S}}$ to be an expansion of $\mathbf{S}$, a set of conditions on M, N and L is required, which is provided by the following theorem [2, Th. 1.4].

Theorem 1: The system $\tilde{\mathbf{S}}$ is an expansion of the system $\mathbf{S}$ if and only if $UM^iV = 0$, $UM^{i-1}NR = 0$, $SLM^{i-1}V = 0$ and $SLM^{i-1}NR = 0$ for all $i = 1, 2, \ldots, \tilde{n}$.

Consider the system $\mathbf{S}$ given in (1) such that

$$A = (A_{ij}) \quad B = (B_{ij}) \quad C = (C_{ij}), \qquad i, j = 1, 2, 3, \tag{3}$$

where A_{ii}, B_{ii}, C_{ii} have dimensions $n_i \times n_i$, $n_i \times m_i$, $l_i \times n_i$, respectively, with $n = n_1 + n_2 + n_3$, $m = m_1 + m_2 + m_3$ and $l = l_1 + l_2 + l_3$. Suppose that $\mathbf{S}$ in (3) is composed of two overlapped subsystems

$$\begin{aligned}\mathbf{S_1}:&\quad \{(A_{ij}), (B_{ij}), (C_{ij}), \qquad i, j = 1, 2\}\\ \mathbf{S_2}:&\quad \{(A_{ij}), (B_{ij}), (C_{ij}), \qquad i, j = 2, 3\}\end{aligned} \tag{4}$$

where overlappings appear in A_{22}, B_{22} and C_{22}.

Consider the following transformation matrices to implement the expansion/contraction procedure:

$$V = \begin{pmatrix} I_{n_1} & 0 & 0\\ 0 & I_{n_2} & 0\\ 0 & I_{n_2} & 0\\ 0 & 0 & I_{n_3}\end{pmatrix} \quad R = \begin{pmatrix} I_{m_1} & 0 & 0\\ 0 & I_{m_2} & 0\\ 0 & I_{m_2} & 0\\ 0 & 0 & I_{m_3}\end{pmatrix}$$

$$T = \begin{pmatrix} I_{l_1} & 0 & 0\\ 0 & I_{l_2} & 0\\ 0 & I_{l_2} & 0\\ 0 & 0 & I_{l_3}\end{pmatrix}. \tag{5}$$

The system in (3) and (4) with the transformations (5) has been extensively adopted in the literature as prototype case [2]–[5], [7], [9], [13]. It considers an initial system with a given overlapping structure in states, control and outputs (4), which can be expanded by (5) to a system with weaker interconnections in which the state vector x_2, the control vector u_2 and the output vector y_2 appear repeated in $\tilde{x}^t = (x_1^t, x_2^t, x_2^t, x_3^t)$, $\tilde{u}^t = (u_1^t, u_2^t, u_2^t, u_3^t)$ and $\tilde{y}^t = (y_1^t, y_2^t, y_2^t, y_3^t)$, respectively. The corresponding matrix dimensions for the expanded system are then $\tilde{n} = n_1 + 2n_2 + n_3$, $\tilde{m} = m_1 + 2m_2 + m_3$ and $\tilde{l} = l_1 + 2\ l_2 + l_3$, respectively. The practical interest of this case is that it is simple enough to alleviate notations but the results derived for it can be easily generalized for any number of interconnected overlapping subsystems. It will be also used throughout this note. For this case, the form of the complementary matrices is given by the following theorem [2, Th. 3.9].

Theorem 2: Consider the systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$ given in (1) with the structure in (3) and the transformation matrices V, R and T given in (5). Then, $\tilde{\mathbf{S}}$ includes $\mathbf{S}$ if and only if the following conditions are satisfied:

$$\left.\begin{array}{l} \begin{pmatrix} M_{12} \\ M_{23}+M_{33} \\ M_{42} \end{pmatrix} (M_{22}+M_{33})^{i-2} \\ (M_{21} \quad M_{22}+M_{23} \quad M_{24}) = 0 \\ \begin{pmatrix} M_{12} \\ M_{23}+M_{33} \\ M_{42} \end{pmatrix} (M_{22}+M_{33})^{i-2} \\ (N_{21} \quad N_{22}+N_{23} \quad N_{24}) = 0 \\ \begin{pmatrix} L_{12} \\ L_{23}+L_{33} \\ L_{42} \end{pmatrix} (M_{22}+M_{33})^{i-2} \\ (M_{21} \quad M_{22}+M_{23} \quad M_{24}) = 0 \end{array}\right\} \text{for all } i = 2, \ldots, \tilde{n}$$
$$\left.\begin{array}{l} \begin{pmatrix} L_{12} \\ L_{23}+L_{33} \\ L_{42} \end{pmatrix} (M_{22}+M_{33})^{i-2} \\ (N_{21} \quad N_{22}+N_{23} \quad N_{24}) = 0 \end{array}\right\} \begin{array}{l}\text{for all } i = 2, \ldots \\ \tilde{n}+1, \end{array} \tag{6}$$

where

$$M = \begin{pmatrix} 0 & M_{12} & -M_{12} & 0 \\ M_{21} & M_{22} & M_{23} & M_{24} \\ -M_{21} & -(M_{22}+M_{23}+M_{33}) & M_{33} & -M_{24} \\ 0 & M_{42} & -M_{42} & 0 \end{pmatrix} \tag{7}$$

and N, L have the same structure as the matrix M. □

Substituting (5) and (7) into (2), we get the form of the expanded system.

Corollary 1: Suppose that Theorem 2 holds. Then, the matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ are shown in (8) at the bottom of the page. The matrices $\tilde{B}$ and $\tilde{C}$ have the same structure as $\tilde{A}$ when substituting A_{ij} by B_{ij}, M_{ij} by N_{ij} and A_{ij} by C_{ij}, M_{ij} by L_{ij}, $i, j = 1, \ldots, 4$, respectively. □

B. The Problem

Since Theorem 1, which gives the essential conditions on the complementary matrices, involves powers of full matrices M, N and L, only a few simple standard forms of these matrices have been commonly used in practice up to now [1], [4], [6], [9], [13], such as those ones corresponding to aggregations and restrictions. Concerning the transmission of controllability–observability from the original system to its expansions, these forms can ensure the preservation of these properties only in certain special cases [7]. Particularly, in [7] it is shown that a system $\mathbf{S}$ with the prototype structure (3) and (4) and the expansion/contraction transformations given in (5) can maintain either controllability or observability in it expanded form when using the standard complementary matrices, but both properties do not hold simultaneously. The motivation of this work lies in the practical interest in removing restrictions in the transmission of controllability/observability, mainly when controllers/observers are designed in expanded spaces by using well known weak coupling control design methods for their consequent contraction to initially given systems. The generalized procedure for selection of complementary matrices proposed in [2], [3], [8] offers an appropriate tool for this purpose. In this context, we formulate the problem.

The Problem: Consider the pair of linear systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$ in (1), with the initial system $\mathbf{S}$ having the overlapping structure given by (3) and (4) and the expanded system $\tilde{\mathbf{S}}$ given by Corollary 1, obtained through the transformations in (5) and the complementary matrices whose structure is characterized in Theorem 2. Prove that there always exist specific complementary matrices such that the controllability–observability of the expanded system are simultaneously guaranteed provided that the original system holds these properties.

III. Main Results

Theorem 2 gives us a block structure for the complementary matrices as well as the conditions to be satisfied by the blocks to guarantee the IP. Observing these conditions we may identify two broad classes from (6), that is: *a)* when the matrices inside the left brackets in (6) are zero, and *b)* when the matrices inside the right brackets in (6) are zero. The case *a)* results in the following complementary matrices:

$$\begin{array}{lll} M_{12} = 0, & M_{23}+M_{33} = 0, & M_{42} = 0 \\ L_{12} = 0, & L_{23}+L_{33} = 0, & L_{42} = 0. \end{array} \tag{9}$$

The case *b)* results in the following complementary matrices:

$$\begin{array}{lll} M_{21} = 0, & M_{22}+M_{23} = 0, & M_{24} = 0 \\ N_{21} = 0, & N_{22}+N_{23} = 0, & N_{24} = 0. \end{array} \tag{10}$$

The cases *a)* and *b)* include aggregations and restrictions, respectively.

In this section, we present the results on controllability and observability corresponding with the cases (9) and (10). The well known Hautus Lemma [11, Lemma 3.3.7] is used as a test. It asserts that the

$$\tilde{A} = \begin{pmatrix} A_{11} & \frac{1}{2}A_{12}+M_{12} & \frac{1}{2}A_{12}-M_{12} & A_{13} \\ A_{21}+M_{21} & \frac{1}{2}A_{22}+M_{22} & \frac{1}{2}A_{22}+M_{23} & A_{23}+M_{24} \\ A_{21}-M_{21} & \frac{1}{2}A_{22}-(M_{22}+M_{23}+M_{33}) & \frac{1}{2}A_{22}+M_{33} & A_{23}-M_{24} \\ A_{31} & \frac{1}{2}A_{32}+M_{42} & \frac{1}{2}A_{32}-M_{42} & A_{33} \end{pmatrix}. \tag{8}$$

system $\tilde{\mathbf{S}}$ in (1) is controllable and observable if and only if the controllability matrix $\tilde{H}^c_{(\tilde{A},\tilde{B})} = (\tilde{A} - \lambda I_{\tilde{n}} | \tilde{B})$ and the observability matrix

$$\tilde{H}^o_{(\tilde{C},\tilde{A})} = \begin{pmatrix} \tilde{A} - \lambda I_{\tilde{n}} \\ \hline \tilde{C} \end{pmatrix}$$

have rank $\tilde{n}$ for all $\lambda \in \mathbb{C}$.

Theorem 3: Consider the systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$ given in (1), where the initial system $\mathbf{S}$ is controllable–observable and the expanded system $\tilde{\mathbf{S}}$ satisfies Corollary 1. Assume that the complementary submatrices satisfy (9). Then, there always exist submatrices M_{ij}, N_{ij}, L_{ij} ensuring that $\tilde{\mathbf{S}}$ is controllable–observable.

Proof: Consider the matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and M, N, L given by Theorem 2 and Corollary 1. For simplicity, we will keep the notation $\tilde{H}^c_{(\tilde{A},\tilde{B})}$ and $\tilde{H}^o_{(\tilde{C},\tilde{A})}$ for all the matrices obtained by manipulating rows and columns of the controllability and observability matrices within their linear combinations, since their ranks are not altered.

Controllability: Let us consider the controllability matrix $\tilde{H}^c_{(\tilde{A},\tilde{B})} = (\tilde{A} - \lambda I_{\tilde{n}} | \tilde{B})$ for the system $\tilde{\mathbf{S}}$ with the matrices given in Corollary 1 and using the complementary matrices given in (9). By doing simple linear combinations of rows and columns (several additions and subtractions only), the controllability matrix can be transformed into the form as shown in (11) at the bottom of the page. Let us choose M_{21}, M_{22} and M_{24} such that $A_{21} - M_{21} = 0$, $A_{22} - 2M_{22} = 0$ and $A_{23} - M_{24} = 0$. Then, denote the matrix (11) as

$$\tilde{H}^c_{(\tilde{A},\tilde{B})} = \left(\begin{array}{c|c|c|c} A - \lambda I_n & B & 0 & P_1 \\ \hline 0 & P_2 & \frac{1}{2}A_{22} + M_{33} - \lambda I_{n_2} & P_3 \end{array}\right).$$

We shall consider two cases: *i)* when λ is an eigenvalue of A, and *ii)* otherwise.

Case i): Suppose that λ is an eigenvalue of A. Select the columns of the matrix $(A - \lambda I_n | B)$ that give rank n for a given eigenvalue λ of A. This is possible because we assume that the initial system (A, B) is controllable. Denote this new matrix as P_4. The corresponding columns of the matrices (0) and P_2 form a matrix that is denoted by P_5. The remaining non selected columns of the matrix $(A - \lambda I_n | B)$ together with the corresponding columns of the matrices (0) and P_2 are joined to the matrices P_1 and P_3, respectively, in order to form two blocks of matrices denoted as P_6 and P_7, respectively. Thus

$$\text{rank } \tilde{H}^c_{(\tilde{A},\tilde{B})} = \text{rank} \left(\begin{array}{c|c|c} P_4 & 0 & P_6 \\ \hline P_5 & \frac{1}{2}A_{22} + M_{33} - \lambda I_{n_2} & P_7 \end{array}\right).$$

Denote $X = (1/2)A_{22} + M_{33}$. Select the matrix X with all their eigenvalues distinct and simultaneously different from the eigenvalues of A. Thus, rank $P_4 = n$ and rank $((1/2)A_{22} + M_{33} - \lambda I_{n_2}) =$ rank $(X - \lambda I_{n_2}) = n_2$ for all λ. Therefore, rank $\tilde{H}^c_{(\tilde{A},\tilde{B})} = n + n_2 = \tilde{n}$ for all eigenvalues λ of A.

Case ii): Suppose that λ is not an eigenvalue of A, but it is an eigenvalue of X. We proceed as in the above case *i)*, but now $P_4 = (A - \lambda I_n)$ with rank n, $P_5 = 0$, $P_6 = (B | P_1)$ and $P_7 = (P_2 | P_3)$. Thus

$$\text{rank } \tilde{H}^c_{(\tilde{A},\tilde{B})} = \text{rank} \left(\begin{array}{c|c|c} P_4 & 0 & P_6 \\ \hline 0 & X - \lambda I_{n_2} & P_7 \end{array}\right).$$

In this case, rank $(X - \lambda I_{n_2}) = n_2 - 1$ for all eigenvalues λ of X. The $n_2 - 1$ independent columns of $(X - \lambda I_{n_2})$ are not the same for all λ. We need to get only one column of the matrix P_7 such that its substitution into the matrix $(X - \lambda I_{n_2})$ gives rank $(X - \lambda I_{n_2}) = n_2$. Obviously, it is always possible because the matrices N_{ij} are completely free. For instance, if the matrix X is selected as a diagonal matrix with the above imposed conditions, any column vector of $P_3 = (N_{22} + N_{23} + 2N_{33})$ with nonzero values can be chosen. Therefore, rank $\tilde{H}^c_{(\tilde{A},\tilde{B})} = n + n_2 = \tilde{n}$.

If λ is not an eigenvalue of X, considering the matrix

$$\left(\begin{array}{c|c|c} P_4 & 0 & P_6 \\ \hline 0 & X - \lambda I_{n_2} & P_7 \end{array}\right)$$

we obtain rank $(X - \lambda I_{n_2}) = n_2$ and so that rank $\tilde{H}^c_{(\tilde{A},\tilde{B})} = n + n_2 = \tilde{n}$.

Therefore, the selection of $M_{21} = A_{21}$, $M_{22} = (1/2)A_{22}$ and $M_{24} = A_{23}$, together with the appropriate selection of matrix M_{33} and one column of P_7 through choosing matrices N_{ij}, guarantee the controllability.

Observability: The corresponding observability matrix

$$\tilde{H}^o_{(\tilde{C},\tilde{A})} = \begin{pmatrix} \tilde{A} - \lambda I_{\tilde{n}} \\ \hline \tilde{C} \end{pmatrix}$$

in $\tilde{\mathbf{S}}$ with the above selected complementary submatrices has the form as shown in (12) at the bottom of the next page. Denote this matrix

$$\tilde{H}^o_{(\tilde{C},\tilde{A})} = \left(\begin{array}{c|c} A - \lambda I_n & 0 \\ \hline C & 0 \\ \hline Q_1 & -L_{22} - L_{33} \\ \hline Q_2 & X - \lambda I_{n_2} \end{array}\right).$$

$$\tilde{H}^c_{(\tilde{A},\tilde{B})} = \left(\begin{array}{ccc|ccc|c|c} A_{11} - \lambda I_{n_1} & A_{12} & A_{13} & B_{11} & B_{12} & B_{13} & 0 & -2N_{12} \\ A_{21} & A_{22} - \lambda I_{n_2} & A_{23} & B_{21} & B_{22} & B_{23} & 0 & N_{23} + N_{33} \\ A_{31} & A_{32} & A_{33} - \lambda I_{n_3} & B_{31} & B_{32} & B_{33} & 0 & -2N_{42} \\ \hline A_{21} - M_{21} & A_{22} - 2M_{22} & A_{23} - M_{24} & B_{21} - N_{21} & B_{22} - N_{22} - N_{23} & B_{23} - N_{24} & M_{22} + M_{33} - \lambda I_{n_2} & N_{22} + N_{23} + 2N_{33} \end{array}\right). \quad (11)$$

We shall consider two cases: *i)* when λ is an eigenvalue of A, and *ii)* otherwise.

Case i): Suppose that λ is an eigenvalue of A. Because we assume that the initial system (C, A) is observable, select the rows of the matrix $\left(\frac{A-\lambda I_n}{C}\right)$ resulting in the rank n for a given eigenvalue λ of A. Denote this new matrix as Q_3. The remaining nonselected rows of the matrix $\left(\frac{A-\lambda I_n}{C}\right)$ are joined to the matrix Q_1 in order to form the blocks of matrices denoted by Q_4 and $\left(\begin{smallmatrix}0\\ -L_{22}-L_{33}\end{smallmatrix}\right)$, respectively. Then,

$$\text{rank}\ \tilde{H}^o_{(\tilde{C},\tilde{A})} = \text{rank}\begin{pmatrix} Q_3 & 0 \\ \hline Q_4 & \begin{pmatrix} 0 \\ -L_{22}-L_{33}\end{pmatrix} \\ \hline Q_2 & X-\lambda I_{n_2}\end{pmatrix}.$$

Selecting the matrix $X = (1/2)A_{22} + M_{33}$ as in the case of controllability, then rank $(X - \lambda I_{n_2}) = n_2$. Thus, rank $\tilde{H}^o_{(\tilde{C},\tilde{A})} = n + n_2 = \tilde{n}$.

Case ii): Suppose that λ is not an eigenvalue of A, but it is an eigenvalue of X. Consider the above observability matrix

$$\tilde{H}^o_{(\tilde{C},\tilde{A})} = \begin{pmatrix} A-\lambda I_n & 0 \\ \hline C & 0 \\ \hline Q_1 & -L_{22}-L_{33} \\ \hline Q_2 & X-\lambda I_{n_2}\end{pmatrix}.$$

In this case, rank $(A - \lambda I_n) = n$ and rank $(X - \lambda I_{n_2}) = n_2 - 1$ for all λ. The $n_2 - 1$ independent rows of $(X - \lambda I_{n_2})$ are not the same for all λ. We need to choose only one row of the matrix $(Q_1 | -L_{22} - L_{33})$ to be substituted into $(Q_2 | X - \lambda I_{n_2})$ such that rank $(X - \lambda I_{n_2}) = n_2$. Because the matrices L_{22} and L_{33} are completely free, it is always possible to select these matrices such that one row of $(-L_{22} - L_{33})$ has, for instance, all its values different from zero. Thus, rank $\tilde{H}^o_{(\tilde{C},\tilde{A})} = n + n_2 = \tilde{n}$.

If λ is not an eigenvalue of X, rank $(X - \lambda I_{n_2}) = n_2$ and then rank $\tilde{H}^o_{(\tilde{C},\tilde{A})} = n + n_2 = \tilde{n}$.

Therefore, an appropriate selection of L_{22} and L_{33} guarantees the observability. Consequently, the expanded system is always controllable–observable. □

Consider now the case (10). It results in the following theorem.

Theorem 4: Consider the systems **S** and $\tilde{\mathbf{S}}$ given in (1), where the initial system **S** is controllable–observable and the expanded system $\tilde{\mathbf{S}}$ satisfies Corollary 1. Assume that the complementary submatrices satisfy (10). Then, there always exist submatrices M_{ij}, N_{ij}, L_{ij} ensuring that $\tilde{\mathbf{S}}$ is controllable–observable.

Proof: It proceeds in a completely analogous way as the proof of Theorem 3. □

IV. EXAMPLE

A. Objective

Consider the system **S** given in (1) with the specific matrices

$$A = \begin{pmatrix} -1 & 1 & 1 & 1 \\ \hline 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ \hline 0 & 2 & 1 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 1 & -1 \\ \hline 1 & 0 & 0 \\ 1 & 0 & 0 \\ \hline -1 & 1 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \end{pmatrix}. \tag{13}$$

(A, B) is controllable and (C, A) is observable. The objective is to illustrate the simultaneous transmission of controllability–observability from the original system (13) to its expansion by using the generalized selection of complementary matrices presented in Section III and compare it with the partial transmission of these properties when considering standard forms of complementary matrices corresponding to aggregations and restrictions.

B. Results

Case a): The system $\tilde{\mathbf{S}}$ expanded by using an aggregation. Choose a typical matrix M used in the literature [7], [9] for this case given by

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ A_{21} & \frac{1}{2}A_{22} & -\frac{1}{2}A_{22} & -A_{23} \\ -A_{21} & -\frac{1}{2}A_{22} & \frac{1}{2}A_{22} & A_{23} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrices N and L have the same structure as M substituting A_{ij} by B_{ij} and C_{ij}, respectively. The corresponding controllability and observability matrices are shown in (14) and (15) at the bottom of the next page. For $\lambda = -1$, rank $\tilde{H}^c_{(\tilde{A},\tilde{B})}$ = rank $\tilde{H}^o_{(\tilde{C},\tilde{A})} = 5$, while

$$\tilde{H}^o_{(\tilde{C},\tilde{A})} = \begin{pmatrix} A_{11}-\lambda I_{n_1} & A_{12} & A_{13} & 0 \\ A_{21} & A_{22}-\lambda I_{n_2} & A_{23} & 0 \\ A_{31} & A_{32} & A_{33}-\lambda I_{n_3} & 0 \\ \hline C_{11} & C_{12} & C_{13} & 0 \\ C_{21} & C_{22} & C_{23} & 0 \\ C_{31} & C_{32} & C_{33} & 0 \\ \hline C_{21}-L_{21} & C_{22}+2L_{33} & C_{23}-L_{24} & -L_{22}-L_{33} \\ \hline 2A_{21} & A_{22}-2M_{33} & 2A_{23} & X-\lambda I_{n_2} \end{pmatrix}. \tag{12}$$

$\tilde{n} = 6$. Therefore, the expanded system $\tilde{\mathbf{S}}$ is neither controllable nor observable when considering aggregation.

Case b): The system $\tilde{\mathbf{S}}$ expanded by using a restriction. In this case, another frequently used choice of the matrix M [7], [9] is given by

$$M = \begin{pmatrix} 0 & \frac{1}{2}A_{12} & -\frac{1}{2}A_{12} & 0 \\ 0 & \frac{1}{2}A_{22} & -\frac{1}{2}A_{22} & 0 \\ 0 & -\frac{1}{2}A_{22} & \frac{1}{2}A_{22} & 0 \\ 0 & -\frac{1}{2}A_{32} & \frac{1}{2}A_{32} & 0 \end{pmatrix}.$$

The matrices N and L have the same structure as M substituting A_{ij} by B_{ij} and C_{ij}, respectively. The corresponding expanded controllability matrix and observability matrix are as shown in (16) and (17) at the bottom of the page. Rank $\tilde{H}^{c}_{(\tilde{A},\,\tilde{B})} = \text{rank } \tilde{H}^{o}_{(\tilde{C},\,\tilde{A})} = 5$ for $\lambda = -1$ also in this case. Thus, the system $\tilde{\mathbf{S}}$ is neither controllable nor observable when using restriction.

Case c): The system $\tilde{\mathbf{S}}$ expanded by using the proposed method. We may select, for instance, the blocks of matrices M_{ij}, N_{ij}, L_{ij},

$$\tilde{H}^{c}_{(\tilde{A},\,\tilde{B})} = (\tilde{A} - \lambda I_{\tilde{n}} | \tilde{B}) = \left(\begin{array}{cccccc|cccc} -1-\lambda & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & \frac{1}{2} & \frac{1}{2} & -1 \\ 0 & -1-\lambda & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & -1-\lambda & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1-\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1-\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 1 & \frac{1}{2} & -1-\lambda & -1 & \frac{1}{2} & \frac{1}{2} & 2 \end{array}\right) \tag{14}$$

$$\tilde{H}^{o}_{(\tilde{C},\,\tilde{A})} = \left(\frac{\tilde{A} - \lambda I_{\tilde{n}}}{\tilde{C}}\right) = \left(\begin{array}{cccccc} -1-\lambda & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & -1-\lambda & 0 & 0 & 0 & 0 \\ 0 & 1 & -1-\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & -1-\lambda & 0 & 0 \\ 0 & 0 & 0 & 1 & -1-\lambda & 0 \\ 0 & 1 & \frac{1}{2} & 1 & \frac{1}{2} & -1-\lambda \\ \hline 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{array}\right). \tag{15}$$

$$\tilde{H}^{c}_{(\tilde{A},\,\tilde{B})} = \left(\tilde{A} - \lambda I_{\tilde{n}} | \tilde{B}\right) = \left(\begin{array}{cccccc|cccc} -1-\lambda & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & -1-\lambda & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1-\lambda & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1-\lambda & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1-\lambda & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & -1-\lambda & -1 & 0 & 1 & 2 \end{array}\right) \tag{16}$$

$$\tilde{H}^{o}_{(\tilde{C},\,\tilde{A})} = \left(\frac{\tilde{A} - \lambda I_{\tilde{n}}}{\tilde{C}}\right) = \left(\begin{array}{cccccc} -1-\lambda & 1 & 1 & 0 & 0 & 1 \\ 0 & -1-\lambda & 0 & 0 & 0 & 0 \\ 0 & 1 & -1-\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & -1-\lambda & 0 & 0 \\ 0 & 0 & 0 & 1 & -1-\lambda & 0 \\ 0 & 0 & 0 & 2 & 1 & -1-\lambda \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right). \tag{17}$$

$$\tilde{H}^{c}_{(\tilde{A},\,\tilde{B})} = \left(\tilde{A} - \lambda I_{\tilde{n}} | \tilde{B}\right) = \left(\begin{array}{cccccc|cccc} -1-\lambda & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & \frac{3}{2} & -\frac{1}{2} & -1 \\ 0 & -1-\lambda & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1-\lambda & 1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2-\lambda & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & -3-\lambda & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 1 & \frac{1}{2} & -1-\lambda & -1 & \frac{1}{2} & \frac{1}{2} & 2 \end{array}\right) \tag{20}$$

$$\tilde{H}^{o}_{(\tilde{C},\,\tilde{A})} = \begin{pmatrix} \tilde{A} - \lambda I_{\tilde{n}} \\ \hline \tilde{C} \end{pmatrix} = \left(\begin{array}{cccccc} -1-\lambda & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & -1-\lambda & 0 & 1 & 0 & 0 \\ 0 & 1 & -1-\lambda & 1 & 2 & 0 \\ 0 & 0 & 0 & -2-\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & -3-\lambda & 0 \\ 0 & 1 & \frac{1}{2} & 1 & \frac{1}{2} & -1-\lambda \\ \hline 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 2 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{array}\right). \tag{21}$$

$i, j = 1, \ldots, 4$, of M, N and L as follows:

$$M = \left(\begin{array}{c|cc|cc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -\frac{1}{2} & 0 & \frac{3}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} & 0 \\ \hline 0 & \frac{1}{2} & 0 & -\frac{3}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{5}{2} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$N = \left(\begin{array}{c|c|c|c} 0 & 1 & -1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & -1 & 1 & -1 \\ 0 & -1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \tag{18}$$

$$L = \left(\begin{array}{c|cc|cc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 2 & 0 \\ \hline -1 & 0 & 0 & -1 & -2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{19}$$

These matrices satisfy the conditions $A_{21} - M_{21} = 0$, $A_{22} - 2M_{22} = 0$, $A_{23} - M_{24} = 0$ required in the proof of Theorem 3. In this case, the free matrix X has been selected as $X = \left(\begin{smallmatrix} -2 & 0 \\ 0 & -3 \end{smallmatrix}\right)$. The eigenvalues $\{-2, -3\}$ are distinct from $\{-1\}$, which is the eigenvalue of matrix A with multiplicity 4. The controllability matrix and observability matrix in $\tilde{\mathbf{S}}$ are then as shown in (20) and (21) at the top of the page. We can observe that now rank $\tilde{H}^{c}_{(\tilde{A},\,\tilde{B})} =$ rank $\tilde{H}^{o}_{(\tilde{C},\,\tilde{A})} = 6$ for all $\lambda \in \mathbb{C}$. Thus, by selecting appropriate complementary matrices M, N and L it is possible to achieve an expanded system $\tilde{\mathbf{S}}$ simultaneously controllable–observable.

V. Conclusion

This note contributes to the result that complementary matrices always exist in the expansion/contraction process transmitting simultaneously controllability/observability from the original continuous-time LTI system to its expanded system provided the original system possesses these properties. This result is based on a generalized selection procedure of complementary matrices derived recently, which offers more degrees of freedom than standard selections such as in aggregations and restrictions. Constructive proofs of such complete transmission of controllability–observability have been given for two different classes of complementary matrices and a numerical example has been presented to illustrate it in comparison with the partial transmission using other standard matrices. Thereby, this note r shows that a previous result, which stated that an expanded system could not preserve controllability and observability simultaneously, is not valid. One of the important practical issues of the Inclusion Principle is the design of controllers/observers in expanded systems, which are then contracted for the implementation in original given systems. Ensuring a complete transmission of controllability/observability contributes in removing restrictions on such a design.

References

[1] L. Bakule and J. Rodellar, “Decentralized control and overlapping decomposition of mechanical systems. Part 1: System decomposition. Part 2: Decentralized stabilization,” *Int. J. Control*, vol. 61, no. 3, pp. 559–587, 1995.

[2] L. Bakule, J. Rodellar, and J. M. Rossell, “Structure of expansion–contraction matrices in the inclusion principle for dynamic systems,” *SIAM J. Matrix Anal. Appl.*, vol. 21, no. 4, pp. 1136–1155, 2000.

[3] ——, “Generalized selection of complementary matrices in the inclusion principle,” *IEEE Trans. Automat. Contr.*, vol. 45, pp. 1237–1243, June 2000.

[4] M. Ikeda and D. D. Šiljak, “Overlapping decompositions, expansions and contractions of dynamic systems,” *Large Scale Syst.*, vol. 1, pp. 29–38, 1980.

[5] M. Ikeda, D. D. Šiljak, and D. E. White, “An inclusion principle for dynamic systems,” *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 244–249, Mar. 1984.

[6] H. Ito, "Overlapping decomposition for multirate decentralized control," in *Preprints 8th IFAC/IFORS/IMACS/IFIP Symp. Large Scale Systems: Theory and Applications*, vol. 1, Patras, Greece, 1980, pp. 259–264.

[7] K. Malinowski and M. Singh, "Controllability and observability of expanded systems with overlapping decompositions," *Automatica*, vol. 21, pp. 203–208, 1985.

[8] J. M. Rossell, "Contribution to decentralized control of large-scale systems via overlapping models," Ph.D. dissertation (in Spanish), Tech. Univ. Catalunya, Barcelona, Spain, 1998.

[9] D. D. Šiljak, *Decentralized Control of Complex Systems*. New York: Academic, 1991.

[10] D. D. Šiljak, S. M. Mladenović, and S. Stanković, "Overlapping decentralized observation and control of a platoon of vehicles," in *Proc. Amer. Control Conf.*, San Diego, CA, 1999, pp. 4522–4526.

[11] E. D. Sontag, *Mathematical Control Theory*. New York: Springer-Verlag, 1990.

[12] S. S. Stanković, S. Mladenovíč, and D. D. Šiljak, "Headway control of a platoon of vehicles based on the inclusion principle," in *Complex Dynamic Systems with Incomplete Information*. Aachen, Germany: Shaker Verlag, 1998, pp. 153–167.

[13] S. S. Stanković, X. B. Chen, and D. D. Šiljak, "Stochastic inclusion principle applied to decentralized control automatic generation control," *Int. J. Control*, vol. 72, no. 3, pp. 276–288, 1999.

[14] S. S. Stanković and D. D. Šiljak, "Contractibility of overlapping decentralized control," in *Proc. Amer. Control Conf.*, Chicago, IL, 2000, pp. 811–818.

An Improved Closed-Loop Stability Related Measure for Finite-Precision Digital Controller Realizations

J. Wu, S. Chen, G. Li, R. H. Istepanian, and J. Chu

***Abstract*—The pole-sensitivity approach is employed to investigate the stability issue of the discrete-time control system, where a digital controller, implemented with finite word length (FWL), is used. A new stability related measure is derived, which is more accurate in estimating the closed-loop stability robustness of an FWL implemented controller than some existing measures for the pole-sensitivity analysis. This improved stability measure thus provides a better criterion to find the optimal realizations for a generic controller structure that includes output-feedback and observer-based controllers. A numerical example is used to verify the theoretical analysis and to illustrate the design procedure.**

***Index Terms*—Closed-loop stability, digital controller, finite word length, optimization.**

I. Introduction

The current controller design methodology often assumes that the controller is implemented exactly, even though in reality a control law can only be realized in finite precision. It is well-known that a designed stable control system may achieve a lower than predicted performance or even become unstable when the controller is implemented with a finite-precision device. It has been noted that a controller design can be implemented with different realizations and that the FWL effect on the closed-loop stability depends on the controller realization structure. This property can be utilized to select controller realization in order to improve the robustness of closed-loop stability under controller perturbations. Currently, two approaches exist for determining the optimal controller realizations under different criteria, namely pole-sensitivity measures [1]–[5] and complex stability radius measures [6], [7].

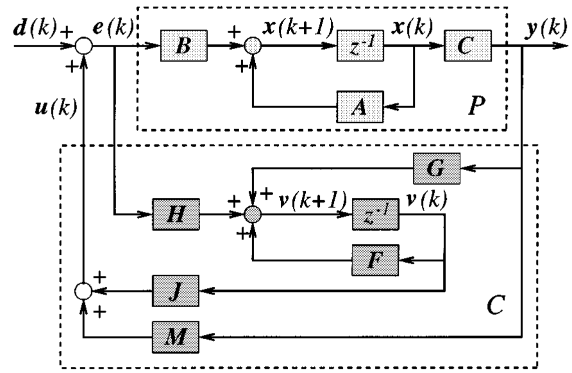

Fig. 1. Discrete-time closed-loop system with a generic digital controller.

In the first approach, the pole sensitivity measures based on a 2-norm [2] and a 1-norm [3] are used to quantify the FWL effect, leading to a nonconvex and nonsmooth optimization problem in finding an optimal FWL controller realization. Efficient global optimization techniques to solve for this optimization problem are readily available [4], [5], [8]. Fialho and Georgiou [7] used the complex stability radius measure to formulate an optimal FWL controller realization problem that can be represented as a special H_∞ norm minimization problem and solved for with the method of linear matrix inequality [9], [10]. In this second approach, the FWL perturbations are assumed to be complex-valued. Although this assumption is somewhat artificial, the approach has certain attractive features and requires further investigation.

The contribution of this note is twofold. First, a generic controller structure is considered that includes output-feedback and observer-based controllers. Second, adopting the pole-sensitivity approach, a new stability related measure is proposed for the unified controller structure and an optimization procedure is developed to find the optimal controller realization that maximizes this new measure. Through theoretical analysis and numerical results, it is shown that this improved measure is less conservative in estimating the FWL closed-loop stability robustness of a controller realization than the existing pole-sensitivity measures of [2], [3].

II. Problem Formulation

Consider the discrete-time closed-loop control system depicted in Fig. 1, where the linear time-invariant plant P is described by

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{e}(k) \\ \mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) \end{cases} \quad (1)$$

which is completely state controllable and observable with $\mathbf{A} \in \mathcal{R}^{n\times n}$, $\mathbf{B} \in \mathcal{R}^{n\times p}$ and $\mathbf{C} \in \mathcal{R}^{q\times n}$; and the digital controller C is described by

$$\begin{cases} \mathbf{v}(k+1) = \mathbf{F}\mathbf{v}(k) + \mathbf{G}\mathbf{y}(k) + \mathbf{H}\mathbf{e}(k) \\ \mathbf{u}(k) = \mathbf{J}\mathbf{v}(k) + \mathbf{M}\mathbf{y}(k) \end{cases} \quad (2)$$

Manuscript received May 15, 2000; revised October 5, 2000 and January 30, 2001. Recommended by Associate Editor T. Chen. The work of the first two authors was supported by the UK Royal Society under a KC Wong fellowship (RL/ART/CN/XFI/KCW/11949).

J. Wu and J. Chu are with the National Laboratory of Industrial Control Technology, Zhejiang University, Hangzhou, 310027, P. R. China.

S. Chen is with the Department of Electronics and Computer Science, University of Southampton, Southampton SO17 1BJ, U.K. (e-mail: sqc@ecs.soton.ac.uk).

G. Li is with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore.

R. H. Istepanian is with the Department of Electronic and Computer Engineering, Brunel University, Uxbridge, Middlesex UB8 3PH, UK.

Publisher Item Identifier S 0018-9286(01)06599-0.