

HIGHER-ORDER MECHANICS: VARIATIONAL PRINCIPLES AND OTHER TOPICS

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ABSTRACT. After reviewing the Lagrangian-Hamiltonian unified formalism (i.e., the Skinner-Rusk formalism) for higher-order (non-autonomous) dynamical systems, we state a unified geometrical version of the Variational Principles which allows us to derive the Lagrangian and Hamiltonian equations for these kinds of systems. Then, the standard Lagrangian and Hamiltonian formulations of these principles and the corresponding dynamical equations are recovered from this unified framework..

1. Introduction. Higher-order systems appear in many models in theoretical and mathematical physics, such as in the mathematical description of relativistic particles with spin, string theories, gravitation, Podolsky's generalization of electromagnetism and others. They also appear in some problems of fluid mechanics, electric networks and classical physics, and in numerical models arising from the geometric discretization of first-order dynamical systems that preserve their inherent geometric structures (see [20, 21] for a long but non-exhaustive list of references).

In these kinds of systems and, in particular, in higher-order mechanics, the dynamics have explicit dependence on accelerations or higher-order derivatives of the generalized coordinates of position. So, for Lagrangian systems, if the Lagrangian function depends on derivatives of order k , the corresponding Euler-Lagrange equations are of order $2k$. Thus, the geometric descriptions of these systems use higher-order tangent and jet bundles as the main tool (see, for instance, [5, 9, 10, 17, 18, 19, 23]).

Furthermore, a generalization of the Lagrangian and Hamiltonian formalisms of first-order autonomous mechanical systems exists that compresses them into a single formalism: the *Skinner-Rusk* or *Lagrangian-Hamiltonian unified formalism*, proposed by R. Skinner and R. Rusk for first-order autonomous mechanical systems [24]. It was generalized to non-autonomous dynamical systems, control systems, first-order classical field theories and higher-order classical field theories [2, 3, 7, 12]. The generalization of the Skinner-Rusk unified formalism for higher-order mechanical systems has been developed in recent papers [6, 20, 21].

The aim of this paper is twofold: first to review this unified formalism for higher-order mechanical systems and second to state the variational principles for higher-order systems and derive the higher-order Euler-Lagrange and Hamilton equations

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using this unified framework. These geometric variational techniques are based on those introduced for first-order field theories in [1, 13, 14, 15]. Our study is made for non-autonomous higher-order mechanical systems (the autonomous case can thereby be obtained by using trivial bundles and removing the time-dependence).

In particular, we start by introducing some basic geometrical background in Section 2, and then reviewing the construction of the framework for the unified Lagrangian-Hamiltonian formalism for non-autonomous higher-order systems (developed in [21]) in Section 3. The main contributions of the paper begin in Section 4, where we establish the variational principle and use it to derive the higher-order equations for the Lagrangian-Hamiltonian unified formalism, which are written in several equivalent geometric ways. Then, in Section 5, these equations are analyzed in detail, showing how they compress not only the dynamical evolution equations but also the equations of the Legendre-Ostrogradsky transformation connecting the Lagrangian and Hamiltonian formalism, which appear as compatibility and consistency conditions for the equations. Other relevant results are presented in Sections 6 and 7, where first we recover the generalization to higher-order systems of the *Hamilton Variational Principle* of the Lagrangian formalism and the *Hamilton-Jacobi Variational Principle* of the Hamiltonian formalism, and then the higher-order Euler-Lagrange and the Hamilton equations. All these results are obtained in a straightforward way from this unified formalism. Finally, some conclusions and further research on these topics are discussed in Section 8.

All the manifolds are real, second countable and C^∞ . The maps and the structures are assumed to be C^∞ . Sum over repeated indices is understood.

2. Higher-order jet bundles over \mathbb{R} . (See [23] for details on jet bundles and [21] for details and proofs on the unified formalism).

Let $E \xrightarrow{\pi} \mathbb{R}$ be a fiber bundle with $\dim E = n + 1$, and let $\eta \in \Omega^1(\mathbb{R})$ be the canonical volume form in \mathbb{R} . If $k \in \mathbb{N}$, the k th order jet bundle of the projection π , $J^k\pi$, is the $((k + 1)n + 1)$ -dimensional manifold of the k -jets of sections $\phi \in \Gamma(\pi)$. A point in $J^k\pi$ is denoted by $j_x^k\phi$, where $\phi \in \Gamma(\pi)$ is any representative of the equivalence class and $x \in \mathbb{R}$. We have the following natural projections: if $r \leq k$,

$$\begin{array}{ccc} \pi_r^k : J^k\pi & \longrightarrow & J^r\pi & \quad & \pi^k : J^k\pi & \longrightarrow & E & \quad & \bar{\pi}^k = \pi \circ \pi^k : J^k\pi & \longrightarrow & \mathbb{R} \\ j_x^k\phi & \longmapsto & j_x^r\phi & \quad & j_x^k\phi & \longmapsto & \phi(x) & \quad & j_x^k\phi & \longmapsto & x \end{array}$$

Notice that $\pi_0^k = \pi^k$, where $J^0\pi$ is canonically identified with E , and $\pi_k^k = \text{Id}_{J^k\pi}$. Furthermore, if $\phi \in \Gamma(\pi)$ is a section of π , we also denote the canonical lifting of ϕ to $J^k\pi$ by $j^k\phi \in \Gamma(\bar{\pi}^k)$.

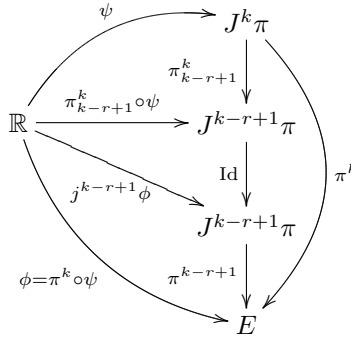
Let t be the global coordinate in \mathbb{R} such that $\eta = dt$, and (t, q_0^A) , $1 \leq A \leq n$, local coordinates in E adapted to the bundle structure. Then, natural coordinates in $J^k\pi$ are $(t, q_0^A, q_1^A, \dots, q_k^A) \equiv (t, q_i^A)$, with $q_0^A = \phi^A$, $q_i^A = \frac{d^i \phi^A}{dt^i}$. Using these coordinates, the local expressions of the natural projections are

$$\pi_r^k(t, q_i^A) = (t, q_j^A) \quad , \quad \pi^k(t, q_i^A) = (t, q_0^A) \quad , \quad \bar{\pi}^k(t, q_i^A) = (t) \quad .$$

Remark 1. In a more general situation, we can replace \mathbb{R} with any 1-dimensional manifold M . In this context, the only difference is that, in general, we do not have a global coordinate. Nevertheless, M is orientable and parallelizable and all our results hold.

A section $\psi \in \Gamma(\bar{\pi}^k)$ is *holonomic of type r* , $1 \leq r \leq k$, if $j^{k-r+1}\phi = \pi_{k-r+1}^k \circ \psi$, where $\phi = \pi^k \circ \psi \in \Gamma(\pi)$; that is, the section $\pi_{k-r+1}^k \circ \psi$ is the lifting of a section

of π to $J^{k-r+1}\pi$. In particular, a section ψ is *holonomic of type 1* if $j^k\phi = \psi$; that is, ψ is the canonical k -jet lifting of a section $\phi \in \Gamma(\pi)$, where $\phi = \pi^k \circ \psi$. A vector field $X \in \mathfrak{X}(J^k\pi)$ is a *semispray of type r* if every integral section of X is holonomic of type r . Throughout this paper, sections that are holonomic of type 1 are simply called *holonomic*.



In natural coordinates, the local expression of a holonomic section of type r is

$$\psi(t) = (t, q_0^A, q_1^A, \dots, q_{k-r+1}^A, \psi_{k-r+2}^A, \dots, \psi_k^A) .$$

Thus, the local expression of a semispray of type r is

$$X = \frac{\partial}{\partial t} + q_1^A \frac{\partial}{\partial q_0^A} + \dots + q_{k-r+1}^A \frac{\partial}{\partial q_{k-r}^A} + X_{k-r+1}^A \frac{\partial}{\partial q_{k-r+1}^A} + \dots + X_k^A \frac{\partial}{\partial q_k^A} .$$

3. Lagrangian-Hamiltonian unified formalism. Let $\pi: E \rightarrow \mathbb{R}$ be the configuration bundle of a k th order dynamical system, with $\dim E = n + 1$. The *higher-order extended jet-momentum bundle* and the *higher-order restricted jet-momentum bundle* are

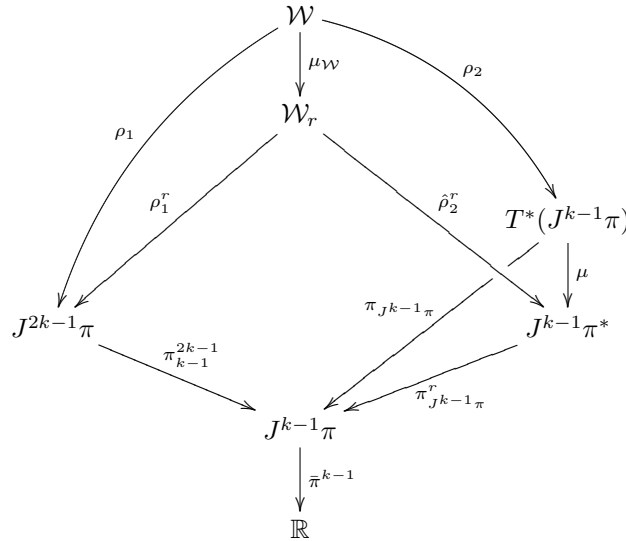
$$\mathcal{W} = J^{2k-1}\pi \times_{J^{k-1}\pi} T^*(J^{k-1}\pi) \quad ; \quad \mathcal{W}_r = J^{2k-1}\pi \times_{J^{k-1}\pi} J^{k-1}\pi^* ,$$

where $J^{k-1}\pi^* = T^*(J^{k-1}\pi)/(\bar{\pi}^{k-1})^*T^*\mathbb{R}$.

(Observe that $\dim T^*(J^{k-1}\pi) = 2kn + 2 > 2kn + 1 = \dim J^{2k-1}\pi = \dim J^{k-1}\pi^*$). These bundles are endowed with the canonical projections

$$\begin{aligned} \rho_1: \mathcal{W} &\rightarrow J^{2k-1}\pi \quad ; \quad \rho_2: \mathcal{W} \rightarrow T^*(J^{k-1}\pi) \\ \rho_{J^{k-1}\pi}: \mathcal{W} &\rightarrow J^{k-1}\pi \quad ; \quad \rho_{\mathbb{R}}: \mathcal{W} \rightarrow \mathbb{R} \\ \rho_1^r: \mathcal{W}_r &\rightarrow J^{2k-1}\pi \quad ; \quad \hat{\rho}_2^r: \mathcal{W}_r \rightarrow J^{k-1}\pi^* \\ \rho_{J^{k-1}\pi}^r: \mathcal{W}_r &\rightarrow J^{k-1}\pi \quad ; \quad \rho_{\mathbb{R}}^r: \mathcal{W}_r \rightarrow \mathbb{R} . \end{aligned}$$

In addition, the natural quotient map $\mu: T^*(J^{k-1}\pi) \rightarrow J^{k-1}\pi^*$ induces a natural projection $\mu_{\mathcal{W}}: \mathcal{W} \rightarrow \mathcal{W}_r$.



If (t, q_0^A) are local coordinates in E adapted to the bundle structure, the induced coordinates in all these bundles are

$$\begin{aligned} J^{2k-1}\pi &: (t, q_0^A, \dots, q_{k-1}^A, q_k^A, \dots, q_{2k-1}^A) \equiv (t, q_i^A, q_j^A). \\ T^*(J^{k-1}\pi) &: (t, q_0^A, \dots, q_{k-1}^A, p, p_A^0, \dots, p_A^{k-1}) \equiv (t, q_i^A, p, p_A^i). \\ J^{k-1}\pi^* &: (t, q_0^A, \dots, q_{k-1}^A, p_A^0, \dots, p_A^{k-1}) \equiv (t, q_i^A, p_A^i). \\ \mathcal{W} &: (t, q_0^A, \dots, q_{k-1}^A, q_k^A, \dots, q_{2k-1}^A, p, p_A^0, \dots, p_A^{k-1}) \equiv (t, q_i^A, q_j^A, p, p_A^i). \\ \mathcal{W}_r &: (t, q_0^A, \dots, q_{k-1}^A, q_k^A, \dots, q_{2k-1}^A, p_A^0, \dots, p_A^{k-1}) \equiv (t, q_i^A, q_j^A, p_A^i). \end{aligned}$$

Remark 2. The last coordinates are the true dynamical variables and so \mathcal{W}_r is the true phase space in the unified formalism.

Observe that $\dim \mathcal{W} = 3kn + 2$ and $\dim \mathcal{W}_r = 3kn + 1$.

Definition 3.1. A section $\psi \in \Gamma(\rho_{\mathbb{R}})$ is holonomic of type r in \mathcal{W} , $1 \leq r \leq 2k - 1$, if the section $\psi_1 = \rho_1 \circ \psi \in \Gamma(\bar{\pi}^{2k-1})$ is holonomic of type r in $J^{2k-1}\pi$.

A vector field $X \in \mathfrak{X}(\mathcal{W})$ is a semispray of type r in \mathcal{W} , $1 \leq r \leq k$, if every integral section ψ of X is holonomic of type r in \mathcal{W} .

Let $\Theta_{k-1} \in \Omega^1(T^*(J^{k-1}\pi))$ and $\Omega_{k-1} = -d\Theta_{k-1} \in \Omega^2(T^*(J^{k-1}\pi))$ be the canonical forms on $T^*(J^{k-1}\pi)$. The higher-order unified canonical forms are $\Theta = \rho_2^* \Theta_{k-1} \in \Omega^1(\mathcal{W})$ and $\Omega = \rho_2^* \Omega_{k-1} \in \Omega^2(\mathcal{W})$. Notice that $\ker \Omega = \ker \rho_{2*}$, and then $(\mathcal{W}, \Omega, \rho_{\mathbb{R}}^* \eta)$ is a precosymplectic manifold.

In natural coordinates, the above forms are given by

$$\Theta = p_A^i dq_i^A + p dt \quad , \quad \Omega = dq_i^A \wedge dp_A^i - dp \wedge dt \quad ,$$

and $\ker \Omega$ is locally given by

$$\ker \Omega = \left\langle \frac{\partial}{\partial q_k^A}, \dots, \frac{\partial}{\partial q_{2k-1}^A} \right\rangle .$$

Definition 3.2. The higher-order coupling 1-form $\hat{C} \in \Omega^1(\mathcal{W})$ is the $\rho_{\mathbb{R}}$ -semibasic form defined as follows: for every $w = (\bar{y}, \alpha_q) \in \mathcal{W}$, where $\bar{y} \in J^{2k-1}\pi$, $q = \pi_{k-1}^{2k-1}(\bar{y})$, and $\alpha_q \in T_q^*(J^{k-1}\pi)$; if $\phi \in \Gamma(\pi)$ is any representative of \bar{y} , and $u \in T_w\mathcal{W}$, then

$$\langle \hat{C}(w) | u \rangle = \langle \alpha_q | (T_w(j^{k-1}\phi \circ \rho_{\mathbb{R}}))(u) \rangle.$$

As \hat{C} is a $\rho_{\mathbb{R}}$ -semibasic form, there is a *coupling function* $\hat{C} \in C^\infty(\mathcal{W})$ such that $\hat{C} = \hat{C}\rho_{\mathbb{R}}^*\eta = \hat{C}dt$. In natural coordinates the coupling function is

$$\hat{C} = p + p_A^i q_{i+1}^A. \tag{1}$$

The dynamical information is introduced by giving a *kth-order Lagrangian density* $\mathcal{L} \in \Omega^1(J^k\pi)$, which is a $\bar{\pi}^k$ -semibasic form. So we can write $\mathcal{L} = L \cdot (\bar{\pi}^k)^*\eta$, where $L \in C^\infty(J^k\pi)$ is the *Lagrangian function*. Then we denote $\hat{\mathcal{L}} = (\pi_k^{2k-1} \circ \rho_1)^*\mathcal{L}$. As the Lagrangian density is a $\bar{\pi}^k$ -semibasic form, then $\hat{\mathcal{L}}$ is a $\rho_{\mathbb{R}}$ -semibasic 1-form, and we have that $\hat{\mathcal{L}} = \hat{L}\rho_{\mathbb{R}}^*\eta = \hat{L}dt$, where $\hat{L} = (\pi_k^{2k-1} \circ \rho_1)^*L \in C^\infty(\mathcal{W})$.

In order to have a geometric structure in \mathcal{W}_r we define the so-called *Hamiltonian submanifold*

$$\mathcal{W}_o = \left\{ w \in \mathcal{W} : \hat{\mathcal{L}}(w) = \hat{C}(w) \right\} \xrightarrow{j_o} \mathcal{W}.$$

Since \hat{C} and $\hat{\mathcal{L}}$ are both $\rho_{\mathbb{R}}$ -semibasic forms, the submanifold \mathcal{W}_o is defined by the constraint $\hat{C} - \hat{\mathcal{L}} = 0$. In natural coordinates, bearing in mind the local expression (1) of \hat{C} , the constraint function is given by $p + p_A^i q_{i+1}^A - \hat{L} = 0$.

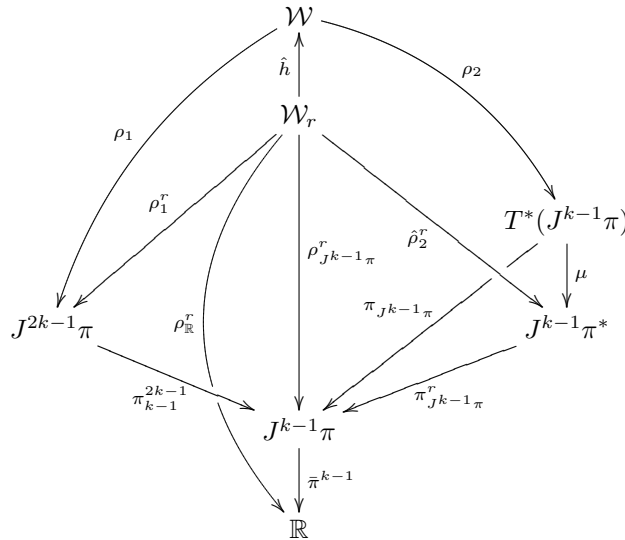
From [21] we have that:

Proposition 1. *The submanifold $\mathcal{W}_o \hookrightarrow \mathcal{W}$ is 1-codimensional, $\mu_{\mathcal{W}}$ -transverse and diffeomorphic to \mathcal{W}_r .*

As a consequence of this, if $\Upsilon : \mathcal{W}_r \rightarrow \mathcal{W}_o$ denotes this diffeomorphism, we have an induced section $\hat{h} = j_o \circ \Upsilon \in \Gamma(\mu_{\mathcal{W}})$, which is specified by giving the local *Hamiltonian function* $\hat{H} = -\hat{L} + p_A^i q_{i+1}^A \in C^\infty(\mathcal{W}_r)$; that is, we have

$$\hat{h}(t, q_i^A, q_j^A, p_A^i) = (t, q_i^A, q_j^A, -\hat{H}, p_A^i) \equiv (t, q_i^A, q_j^A, \hat{L} - p_A^i q_{i+1}^A, p_A^i).$$

The section \hat{h} is called a *Hamiltonian section* of $\mu_{\mathcal{W}}$, or a *Hamiltonian $\mu_{\mathcal{W}}$ -section*.



Next, we define the forms $\Theta_r = \hat{h}^*\Theta \in \Omega^1(\mathcal{W}_r)$ and $\Omega_r = \hat{h}^*\Omega \in \Omega^2(\mathcal{W}_r)$, whose expressions in natural coordinates are

$$\Theta_r = p_A^i dq_i^A + (\hat{L} - p_A^i q_{i+1}^A) dt \quad ; \quad \Omega_r = dq_i^A \wedge dp_A^i + d(p_A^i q_{i+1}^A - \hat{L}) \wedge dt .$$

Remark 3. The precosymplectic Hamiltonian system $(\mathcal{W}_r, \Omega_r, (\rho_{\mathbb{R}}^r)^*\eta)$ (or, what is equivalent, $(\mathcal{W}_o, \Omega_o, (\rho_{\mathbb{R}}^o)^*\eta)$, with $\Omega_o = j_o^*\Omega$) represents the higher-order dynamical system in the Lagrangian-Hamiltonian unified formalism.

4. Variational principle for the unified formalism. Next we establish the variational principle from which the dynamical equations for the unified formalism are derived. Our starting point is the precosymplectic Hamiltonian system $(\mathcal{W}_r, \Omega_r, (\rho_{\mathbb{R}}^r)^*\eta)$.

Let $\Gamma(\rho_{\mathbb{R}}^r)$ be the set of sections of $\rho_{\mathbb{R}}^r$. Consider the functional

$$\begin{aligned} \mathbf{LH}: \Gamma(\rho_{\mathbb{R}}^r) &\longrightarrow \mathbb{R} \\ \psi &\longmapsto \int_{\mathbb{R}} \psi^* \Theta_r \end{aligned} ,$$

where the convergence of the integral is assumed.

Definition 4.1 (Generalized Variational Principle). The Lagrangian-Hamiltonian variational problem for the system $(\mathcal{W}_r, \Omega_r, (\rho_{\mathbb{R}}^r)^*\eta)$ is the search for the critical (or stationary) holonomic sections of the functional **LH** with respect to the variations of ψ given by $\psi_s = \sigma_s \circ \psi$, where $\{\sigma_s\}$ is a local one-parameter group of any compact-supported $\rho_{\mathbb{R}}^r$ -vertical vector field Z in \mathcal{W}_r (in the following we denote this set by $\mathfrak{X}^{V(\rho_{\mathbb{R}}^r)}(\mathcal{W}_r)$), that is,

$$\left. \frac{d}{ds} \right|_{s=0} \int_{\mathbb{R}} \psi_s^* \Theta_r = 0 .$$

The main result of the calculus of variations in this context is the following:

Theorem 4.2. *The following assertions on a section $\psi \in \Gamma(\rho_{\mathbb{R}}^r)$ are equivalent:*

1. ψ is a solution to the Lagrangian-Hamiltonian variational problem.
2. ψ is a holonomic section solution to the equation

$$\psi^* i(Y)\Omega_r = 0, \quad \text{for every } Y \in \mathfrak{X}(\mathcal{W}_r) .$$

3. ψ is a holonomic section solution to the equation

$$i(\psi')(\Omega_r \circ \psi) = 0 ,$$

where $\psi': \mathbb{R} \rightarrow T\mathcal{W}_r$ is the canonical lifting of ψ to $T\mathcal{W}_r$.

4. ψ is an integral curve of a vector field contained in a class of $\rho_{\mathbb{R}}^r$ -transverse semisprays of type 1, $\{X\} \subset \mathfrak{X}(\mathcal{W}_r)$, satisfying the equation

$$i(X)\Omega_r = 0 . \tag{2}$$

Proof. We prove the equivalence $1 \iff 2$ following the patterns taken from [11]. For the proof of the other equivalences, see [21] (Theorem 1).

Let $Z \in \mathfrak{X}^{V(\rho_{\mathbb{R}}^r)}(\mathcal{W}_r)$ be a compact-supported vector field, and $V \subset \mathbb{R}$ an open set such that ∂V is a 0-dimensional manifold and $\rho_{\mathbb{R}}^r(\text{supp}(Z)) \subset V$. Then,

$$\begin{aligned}
 \frac{d}{ds} \Big|_{s=0} \int_{\mathbb{R}} \psi_s^* \Theta_r &= \frac{d}{ds} \Big|_{s=0} \int_V \psi_s^* \Theta_r = \frac{d}{ds} \Big|_{s=0} \int_V \psi^* \sigma_s^* \Theta_r \\
 &= \int_V \psi^* \left(\lim_{t \rightarrow 0} \frac{\sigma_s^* \Theta_r - \Theta_r}{t} \right) = \int_V \psi^* L(Z) \Theta_r \\
 &= \int_V \psi^* (i(Z) d\Theta_r + d i(Z) \Theta_r) \\
 &= \int_V \psi^* (-i(Z) \Omega_r + d i(Z) \Theta_r) \\
 &= - \int_V \psi^* i(Z) \Omega_r + \int_V d(\psi^* i(Z) \Theta_r) \\
 &= - \int_V \psi^* i(Z) \Omega_r + \int_{\partial V} \psi^* i(Z) \Theta_r = - \int_V \psi^* i(Z) \Omega_r,
 \end{aligned}$$

as a consequence of Stoke's theorem and the assumptions made on the supports of the vertical vector fields. Thus, by the fundamental theorem of the variational calculus, we conclude

$$\frac{d}{ds} \Big|_{s=0} \int_{\mathbb{R}} \psi_s^* \Theta_r = 0 \iff \psi^* i(Z) \Omega_r = 0$$

for every compact-supported $Z \in \mathfrak{X}^{V(\rho_{\mathbb{R}}^r)}(\mathcal{W}_r)$. However, since compact-supported vector fields generate locally the $C^\infty(\mathcal{W}_r)$ -module of vector fields in \mathcal{W}_r , it follows that the last equality holds for every $\rho_{\mathbb{R}}^r$ -vertical vector field Z in \mathcal{W}_r .

Now, recall that for every point $w \in \text{Im} \psi$, we have a canonical splitting of the tangent space of \mathcal{W}_r at w in a $\rho_{\mathbb{R}}^r$ -vertical subspace and a $\rho_{\mathbb{R}}^r$ -horizontal subspace, that is,

$$T_w \mathcal{W}_r = V_w(\rho_{\mathbb{R}}^r) \oplus T_w(\text{Im} \psi).$$

Thus, if $Y \in \mathfrak{X}(\mathcal{W}_r)$, then

$$Y_w = (Y_w - T_w(\psi \circ \rho_{\mathbb{R}}^r)(Y_w)) + T_w(\psi \circ \rho_{\mathbb{R}}^r)(Y_w) \equiv Y_w^V + Y_w^\psi,$$

with $Y_w^V \in V_w(\rho_{\mathbb{R}}^r)$ and $Y_w^\psi \in T_w(\text{Im} \psi)$. Therefore

$$\psi^* i(Y) \Omega_r = \psi^* i(Y^V) \Omega_r + \psi^* i(Y^\psi) \Omega_r = \psi^* i(Y^\psi) \Omega_r,$$

since $\psi^* i(Y^V) \Omega_r = 0$, by the conclusion in the above paragraph. Now, as $Y_w^\psi \in T_w(\text{Im} \psi)$ for every $w \in \text{Im} \psi$, then the vector field Y^ψ is tangent to $\text{Im} \psi$, and hence there exists a vector field $X \in \mathfrak{X}(\mathbb{R})$ such that X is ψ -related with Y^ψ ; that is, $\psi_* X = Y^\psi|_{\text{Im} \psi}$. Then $\psi^* i(Y^\psi) \Omega_r = i(X) \psi^* \Omega_r$. However, as $\dim \text{Im} \psi = \dim \mathbb{R} = 1$ and Ω_r is a 2-form, we obtain that $\psi^* i(Y^\psi) \Omega_r = 0$. Hence, we conclude that $\psi^* i(Y) \Omega_r = 0$ for every $Y \in \mathfrak{X}(\mathcal{W}_r)$.

Taking into account the reasoning of the first paragraph, the converse is obvious since the condition $\psi^* i(Y) \Omega_r = 0$, for every $Y \in \mathfrak{X}(\mathcal{W}_r)$, holds, in particular, for every $Z \in \mathfrak{X}^{V(\rho_{\mathbb{R}}^r)}(\mathcal{W}_r)$. \square

5. Analysis of the unified dynamical equations. In order to complete the Lagrangian-Hamiltonian unified formalism, it is necessary to analyze the dynamical

equations. We start by using the equations written for vector fields. Thus, equations (2), with the $\rho_{\mathbb{R}}^r$ -transverse condition, are

$$i(X)\Omega_r = 0 \quad ; \quad i(X)(\rho_{\mathbb{R}}^r)^*\eta \neq 0 .$$

It is usual to fix the $\rho_{\mathbb{R}}^r$ -transverse condition by demanding that

$$i(X)(\rho_{\mathbb{R}}^r)^*\eta = 1 . \quad (3)$$

This selects a representative in the class $\{X\}$. We will do this in the sequel.

The first important result is [21]:

Proposition 2. *The above equations are compatible only on the points of the following submanifold of \mathcal{W}_r*

$$\begin{aligned} \mathcal{W}_1 &= \left\{ w \in \mathcal{W}_r : (i(Z)d\hat{H})(w) = 0, \text{ for every } Z \in \ker \Omega_r \right\} \\ &= \left\{ w \in \mathcal{W}_r : (i(Y)\Omega_r)(w) = 0, \text{ for every } Y \in \mathfrak{X}^{V(\hat{\rho}_2^r)}(\mathcal{W}_r) \right\} . \end{aligned}$$

In natural coordinates, a generic vector field $X \in \mathfrak{X}(\mathcal{W}_r)$ is given by

$$X = f \frac{\partial}{\partial t} + f_i^A \frac{\partial}{\partial q_i^A} + F_j^A \frac{\partial}{\partial q_j^A} + G_A^i \frac{\partial}{\partial p_A^i} ,$$

and the $\rho_{\mathbb{R}}^r$ -transverse condition implies $f \neq 0$, and in particular, using (3), we get $f = 1$. Therefore, the dynamical equation (2) first gives

$$p_A^{k-1} - \frac{\partial \hat{L}}{\partial q_k^A} = 0 ,$$

which are the compatibility relations (constraints) defining locally \mathcal{W}_1 . Furthermore, for $0 \leq l \leq k-1$, $k \leq j \leq 2k-1$;

$$f_i^A = q_{i+1}^A \quad ; \quad G_A^0 = \frac{\partial \hat{L}}{\partial q_0^A} \quad ; \quad G_A^i = \frac{\partial \hat{L}}{\partial q_i^A} - p_A^{i-1} ;$$

therefore

$$X = \frac{\partial}{\partial t} + q_{i+1}^A \frac{\partial}{\partial q_i^A} + F_j^A \frac{\partial}{\partial q_j^A} + \frac{\partial \hat{L}}{\partial q_0^A} \frac{\partial}{\partial p_A^0} + \left(\frac{\partial \hat{L}}{\partial q_i^A} - p_A^{i-1} \right) \frac{\partial}{\partial p_A^i} .$$

Observe that, in a natural way, X is a semispray of type k . Nevertheless, the variational principle requires that X must be a semispray of type 1, thus

$$X = \frac{\partial}{\partial t} + \sum_{l=0}^{2k-2} q_{l+1}^A \frac{\partial}{\partial q_l^A} + F_{2k-1}^A \frac{\partial}{\partial q_{2k-1}^A} + \frac{\partial \hat{L}}{\partial q_0^A} \frac{\partial}{\partial p_A^0} + \left(\frac{\partial \hat{L}}{\partial q_i^A} - p_A^{i-1} \right) \frac{\partial}{\partial p_A^i} .$$

Next we must require X to be tangent to \mathcal{W}_1 . Thus, it is necessary to impose that $L(X)\xi|_{\mathcal{W}_1} = 0$, for every constraint function ξ defining \mathcal{W}_1 :

$$X \left(p_A^{k-1} - \frac{\partial \hat{L}}{\partial q_k^A} \right) \Big|_{\mathcal{W}_1} = 0 \iff p_A^{k-2} - \left(\frac{\partial \hat{L}}{\partial q_{k-1}^A} - d_T \left(\frac{\partial \hat{L}}{\partial q_k^A} \right) \right) = 0 \quad (\text{on } \mathcal{W}_1) .$$

(where $d_T = \frac{\partial}{\partial t} + q_{i+1}^A \frac{\partial}{\partial q_i^A}$). Repeating this procedure ($k-1$ steps), we get

$$p_A^0 - \sum_{i=0}^{k-1} (-1)^i d_T^i \left(\frac{\partial \hat{L}}{\partial q_{1+i}^A} \right) = 0 \quad (\text{on } \mathcal{W}_{k-1}) .$$

Thus we obtain a sequence of submanifolds (which can also be obtained by applying any other constraint algorithm [8, 16]),

$$\mathcal{W}_0 \leftrightarrow \mathcal{W}_1 \leftrightarrow \dots \leftrightarrow \mathcal{W}_k \equiv \mathcal{W}_{\mathcal{L}} .$$

As a consequence of the last equalities we conclude that

Proposition 3. *The submanifold $\mathcal{W}_{\mathcal{L}}$ is the graph of a map $\mathcal{F}L: J^{2k-1}\pi \rightarrow J^{k-1}\pi^*$ locally defined by*

$$\mathcal{F}L^*t = t \quad , \quad \mathcal{F}L^*q_{r-1}^A = q_{r-1}^A \quad , \quad \mathcal{F}L^*p_A^{r-1} = \sum_{i=0}^{k-r} (-1)^i d_T^i \left(\frac{\partial \hat{L}}{\partial q_{r+i}^A} \right) .$$

Definition 5.1. The map $\mathcal{F}L: J^{2k-1}\pi \rightarrow J^{k-1}\pi^*$ is the (restricted) Legendre-Ostrogradsky map.

A Lagrangian density \mathcal{L} is regular if the map $\mathcal{F}L$ is a local diffeomorphism. Otherwise, \mathcal{L} is said to be a singular Lagrangian. If $\mathcal{F}L$ is a global diffeomorphism, then \mathcal{L} is said to be hyperregular.

In natural coordinates, the regularity condition for \mathcal{L} is equivalent to

$$\det \left(\frac{\partial^2 L}{\partial q_k^B \partial q_k^A} \right) (\bar{y}) \neq 0, \text{ for every } \bar{y} \in J^k \pi . \tag{4}$$

Observe that X is not necessarily tangent to $\mathcal{W}_{\mathcal{L}}$. Thus, imposing the tangency condition to the last generation of constraints defining $\mathcal{W}_{\mathcal{L}}$, these conditions give the following equations (on $\mathcal{W}_{\mathcal{L}}$):

$$(-1)^k (F_{2k-1}^B - d_T(q_{2k-1}^B)) \frac{\partial^2 \hat{L}}{\partial q_k^B \partial q_k^A} + \sum_{i=0}^k (-1)^i d_T^i \left(\frac{\partial \hat{L}}{\partial q_i^A} \right) = 0 .$$

And, as a consequence of (4), we have:

Proposition 4. *If \mathcal{L} is regular, then there is a unique semispray of type 1, $X \in \mathfrak{X}(\mathcal{W}_r)$, tangent to $\mathcal{W}_{\mathcal{L}}$, which is a solution to the dynamical equations (on $\mathcal{W}_{\mathcal{L}}$).*

If \mathcal{L} is not regular, new constraints could appear and the algorithm continues until arriving (in the best cases) at a final constraint submanifold $\mathcal{W}_f \hookrightarrow \mathcal{W}_{\mathcal{L}}$.

If $\psi(t) = (t, q_i^A(t), q_j^A(t), p_A^i(t))$ is an integral section of X , the above equations lead to

$$\dot{q}_l^A = q_{l+1}^A \quad ; \quad \dot{q}_{2k-1}^A = F_{2k-1}^A \circ \psi \quad ; \quad \dot{p}_A^0 = \frac{\partial \hat{L}}{\partial q_0^A} \quad ; \quad \dot{p}_A^i = \frac{\partial \hat{L}}{\partial q_i^A} - p_A^{i-1} ,$$

and after some calculations we reach (on the points of $\mathcal{W}_{\mathcal{L}}$, or \mathcal{W}_f),

$$\frac{\partial L}{\partial q_0^A} \circ \psi - \frac{d}{dt} \left(\frac{\partial L}{\partial q_1^A} \circ \psi \right) + \dots + (-1)^k \frac{d^k}{dt^k} \left(\frac{\partial L}{\partial q_k^A} \circ \psi \right) = 0$$

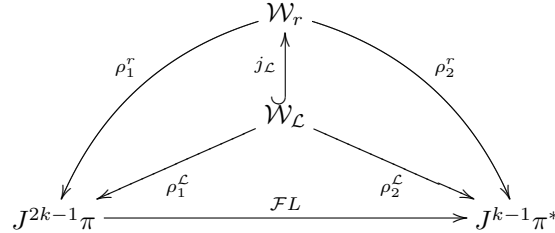
$$\dot{p}_A^0 = \frac{\partial \hat{L}}{\partial q_0^A} \quad ; \quad \dot{p}_A^i = \frac{\partial \hat{L}}{\partial q_i^A} - p_A^{i-1} ,$$

These equations compress both the higher-order Euler-Lagrange and Hamilton equations, as can be seen in the following sections.

Remark 4. It is interesting to point out that the variational principle for higher-order autonomous dynamical systems, and the corresponding dynamical equations, can be obtained as a particular case of these results when the Lagrangian function does not depend explicitly on the coordinate t .

6. Lagrangian formalism: Generalized Hamilton principle. In this section we show how to recover the Lagrangian formalism for higher-order mechanical systems. In particular, we state the classical Hamilton Variational Principle of the Lagrangian formalism for higher-order systems and study its relation with the unified variational Principle.

First, consider the diagram



As \mathcal{W}_L is the graph of the restricted Legendre-Ostrogadski map, we have that the map $\rho_1^L = \rho_1^r \circ j_L : \mathcal{W}_L \rightarrow J^{2k-1}\pi$ is a diffeomorphism. Then we can define the Poincaré-Cartan 1 and 2 forms in $J^{2k-1}\pi$ as

$$\Theta_{\mathcal{L}} = (j_L \circ (\rho_1^L)^{-1})^* \Theta_r \quad ; \quad \Omega_{\mathcal{L}} = -d\Theta_{\mathcal{L}} = (j_L \circ (\rho_1^L)^{-1})^* \Omega_r .$$

These forms can also be introduced in several equivalent ways (see, for instance, [1, 14, 22, 23]).

Remark 5. The triple $(J^{2k-1}\pi, \Omega_{\mathcal{L}}, (\bar{\pi}^{2k-1})^* \eta)$ is the higher-order non-autonomous Lagrangian system associated to $(\mathcal{W}_r, \Omega_r, (\rho_{\mathbb{R}}^r)^* \eta)$.

Now we establish the variational principle from which we can obtain the dynamical equations for the Lagrangian formalism.

Given the Lagrangian system $(J^{2k-1}\pi, \Omega_{\mathcal{L}}, (\bar{\pi}^{2k-1})^* \eta)$, let $\Gamma(\pi)$ be the set of sections of π . Consider the functional

$$\begin{aligned} \mathbf{L} : \Gamma(\pi) &\longrightarrow \mathbb{R} \\ \phi &\longmapsto \int_{\mathbb{R}} (j^{2k-1}\phi)^* \Theta_{\mathcal{L}} \quad , \end{aligned}$$

where the convergence of the integral is assumed.

Definition 6.1 (Generalized Hamilton Variational Principle). The Lagrangian variational problem (also called Hamilton variational problem) for the higher-order Lagrangian system $(J^{2k-1}\pi, \Omega_{\mathcal{L}}, (\bar{\pi}^{2k-1})^* \eta)$ is the search for the critical (or stationary) sections of the functional \mathbf{L} with respect to the variations of ϕ given by $\phi_s = \sigma_s \circ \phi$, where $\{\sigma_s\}$ is a local one-parameter group of any compact-supported $Z \in \mathfrak{X}^{V(\pi)}(E)$; that is,

$$\left. \frac{d}{ds} \right|_{s=0} \int_{\mathbb{R}} (j^{2k-1}\phi_s)^* \Theta_{\mathcal{L}} = 0 .$$

Then, as in the above section, we have:

Theorem 6.2. The following assertions on a section $\phi \in \Gamma(\pi)$ are equivalent:

1. ϕ is a solution to the Lagrangian variational problem.
2. $\psi_{\mathcal{L}} = j^{2k-1}\phi$ is a solution to the equation

$$\psi_{\mathcal{L}}^* i(Y)\Omega_{\mathcal{L}} = 0, \quad \text{for every } Y \in \mathfrak{X}(J^{2k-1}\pi) .$$

3. $\psi_{\mathcal{L}} = j^{2k-1}\phi$ is a solution to the equation

$$i(\psi'_{\mathcal{L}})(\Omega_{\mathcal{L}} \circ \psi_{\mathcal{L}}) = 0 ,$$

where $\psi'_{\mathcal{L}}: \mathbb{R} \rightarrow TJ^{2k-1}\pi$ is the canonical lifting of $\psi_{\mathcal{L}}$ to $TJ^{2k-1}\pi$.

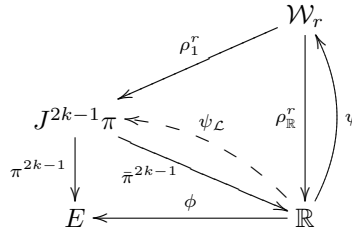
4. $\psi_{\mathcal{L}} = j^{2k-1}\phi$ is an integral curve of a vector field contained in a class of $\bar{\pi}^{2k-1}$ -transverse semisprays of type 1, $\{X_{\mathcal{L}}\} \subset \mathfrak{X}(J^{2k-1}\pi)$, satisfying

$$i(X_{\mathcal{L}})\Omega_{\mathcal{L}} = 0 .$$

Proof. The proof of the equivalence $1 \iff 2$ follows the same patterns as in Theorem 4.2. For the proof of the other equivalences, see [21] (Theorem 3). \square

Section solutions to the Hamilton variational problem are recovered from section solutions to the Lagrangian-Hamiltonian variational problem in the unified formalism. Indeed:

Theorem 6.3. *Let $\psi \in \Gamma(\rho_{\mathbb{R}}^r)$ be a holonomic section which is a solution to the Lagrangian-Hamiltonian variational problem given by the functional \mathbf{LH} . Then, the section $\psi_{\mathcal{L}} = \rho_1^r \circ \psi \in \Gamma(\bar{\pi}^{2k-1})$ is holonomic, and its projection $\phi = \pi^{2k-1} \circ \psi_{\mathcal{L}} \in \Gamma(\pi)$ is a solution to the Lagrangian variational problem given by the functional \mathbf{L} ; Conversely, from a holonomic section $\psi_{\mathcal{L}} = j^{2k-1}\phi \in \Gamma(\bar{\pi}^{2k-1})$ which is a solution to the Lagrangian variational problem, we recover a solution $\psi = (\psi_{\mathcal{L}}, \psi_{\mathcal{L}} \circ \mathcal{FL})$ to the Lagrangian-Hamiltonian variational problem.*



Proof. As $\psi \in \Gamma(\rho_{\mathbb{R}}^r)$ is holonomic, then $\psi_{\mathcal{L}} = \rho_1^r \circ \psi \in \Gamma(\bar{\pi}^{2k-1})$ is a holonomic section, by definition.

Now, ρ_1^r being a submersion, for every compact-supported vector field $X \in \mathfrak{X}^{V(\bar{\pi}^{2k-1})}(J^{2k-1}\pi)$ there exist compact-supported vector fields $Y \in \mathfrak{X}^{V(\rho_{\mathbb{R}}^r)}(\mathcal{W}_r)$ such that $\rho_{1*} Y = X$; that is, X and Y are ρ_1^r -related. In particular, this holds if X is the $(2k - 1)$ -jet lifting of a compact-supported π -vertical vector field in E ; that is, if we have $X = j^{2k-1}Z$, with $Z \in \mathfrak{X}^{V(\pi)}(E)$. We denote by $\{\sigma_s\}$ a local one-parameter group for the compact-supported vector fields $Y \in \mathfrak{X}^{V(\rho_{\mathbb{R}}^r)}(\mathcal{W}_r)$. Then,

using this particular choice of ρ_1^r -related vector fields, we have

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} \int_{\mathbb{R}} (j^{2k-1} \phi_s)^* \Theta_{\mathcal{L}} &= \frac{d}{ds} \Big|_{s=0} \int_{\mathbb{R}} (j^{2k-1} (\sigma_s \circ \phi))^* \Theta_{\mathcal{L}} \\ &= \frac{d}{ds} \Big|_{s=0} \int_{\mathbb{R}} (j^{2k-1} \phi)^* (j^{2k-1} \sigma_s)^* \Theta_{\mathcal{L}} \\ &= \int_{\mathbb{R}} \psi_{\mathcal{L}}^* L(j^{2k-1} Z) \Theta_{\mathcal{L}} = \int_{\mathbb{R}} \psi^* (\rho_1^r)^* L(j^{2k-1} Z) \Theta_{\mathcal{L}} \\ &= \int_{\mathbb{R}} \psi^* L(Y) \Theta_r = \frac{d}{ds} \Big|_{s=0} \int_{\mathbb{R}} \psi^* \sigma_s^* \Theta_r \\ &= \frac{d}{ds} \Big|_{s=0} \int_{\mathbb{R}} \psi_s^* \Theta_r = 0, \end{aligned}$$

since ψ is a critical section for the Lagrangian-Hamiltonian variational problem.

Conversely, if we have a holonomic section $\psi_{\mathcal{L}} = j^{2k-1} \phi$ which is a solution to the Lagrangian variational problem, then we can construct $\psi = (\psi_{\mathcal{L}}, \psi_{\mathcal{L}} \circ \mathcal{FL})$, which is a section $\psi: \mathbb{R} \rightarrow \mathcal{W}_{\mathcal{L}} \subset \mathcal{W}_r$ of the projection $\rho_{\mathbb{R}}^r$ (remember that, in the unified formalism, the dynamical equations have solutions only on the points of $\mathcal{W}_{\mathcal{L}}$, or in a subset of it). Then, the above reasoning also shows that if $\psi_{\mathcal{L}}$ is a solution to the Lagrangian variational problem, then ψ is a solution to the Lagrangian-Hamiltonian variational problem. \square

In natural coordinates, if ψ is given by $\psi(t) = (t, q_i^A(t), q_j^A(t), p_A^i(t))$, then $\psi_{\mathcal{L}} = (\rho_1^r \circ \psi)(t) = (t, q_i^A(t), q_j^A(t))$, and $\phi(t) = (\pi^{2k-1} \circ \psi_{\mathcal{L}})(t) = (t, q_0^A(t))$ satisfies the *kth-order Euler-Lagrange equations*

$$\frac{\partial L}{\partial q_0^A} \Big|_{j^{2k-1} \phi} - \frac{d}{dt} \frac{\partial L}{\partial q_1^A} \Big|_{j^{2k-1} \phi} + \dots + (-1)^k \frac{d^k}{dt^k} \frac{\partial L}{\partial q_k^A} \Big|_{j^{2k-1} \phi} = 0.$$

Finally, as a consequence of all the above results, we have the corresponding relation between vector field solutions to the unified dynamical equations and those which are solutions to the Lagrangian equations:

Proposition 5. *Let $X \in \mathfrak{X}(\mathcal{W}_r)$ be a vector field tangent to $\mathcal{W}_{\mathcal{L}}$ which is a solution to the equations*

$$i(X)\Omega_r = 0 \quad ; \quad i(X)(\rho_{\mathbb{R}}^r)^* \eta = 1. \tag{5}$$

Then there exists a unique semispray of type k , $X_{\mathcal{L}} \in \mathfrak{X}(J^{2k-1}\pi)$, which is a solution to the equations

$$i(X_{\mathcal{L}})\Omega_{\mathcal{L}} = 0 \quad ; \quad i(X_{\mathcal{L}})(\bar{\pi}^{2k-1})^* \eta = 1. \tag{6}$$

In addition, if \mathcal{L} is a regular Lagrangian density, then $X_{\mathcal{L}}$ is a semispray of type 1. Conversely, if $X_{\mathcal{L}} \in \mathfrak{X}(J^{2k-1}\pi)$ is a semispray of type k (resp., of type 1), which is a solution to the equations (6), then there exists a unique $X \in \mathfrak{X}(\mathcal{W}_r)$ which is a solution to the equations (5) and it is a semispray of type k in \mathcal{W}_r (resp., of type 1).

Proof. See also [21] (Theorem 2) for a detailed proof of this statement. \square

7. Hamiltonian formalism: Generalized Hamilton-Jacobi principle. In this section we state the Hamiltonian variational problem (*Hamilton-Jacobi Principle*) for higher-order systems, recovering it from the unified formalism. (See [21] for the proofs and details on the higher-order Hamiltonian formalism).

Consider the restricted Legendre-Ostrogradsky map $\mathcal{FL}: J^{2k-1}\pi \rightarrow J^{k-1}\pi^*$. First, it can be proved that the following statements are equivalent:

1. $\Omega_{\mathcal{L}}$ has maximal rank on $J^{2k-1}\pi$.
2. $\mathcal{FL}: J^{2k-1}\pi \rightarrow J^{k-1}\pi^*$ is a local diffeomorphism.
3. In natural coordinates of $J^k\pi$, $\det \left(\frac{\partial^2 L}{\partial q_k^B \partial q_k^A} \right) (\bar{y}) \neq 0$, for every $\bar{y} \in J^k\pi$.

As stated in Section 5, if these conditions are fulfilled, the Lagrangian density \mathcal{L} is said to be *regular*, and when the restricted Legendre-Ostrogradsky map is a global diffeomorphism, then \mathcal{L} is *hyperregular*.

Now, let $\tilde{\mathcal{P}} = \text{Im}(\tilde{\mathcal{FL}}) \xrightarrow{\tilde{j}} T^*(J^{k-1}\pi)$ and $\mathcal{P} = \text{Im}(\mathcal{FL}) \xrightarrow{j} J^{k-1}\pi^*$. If $\bar{\tau} = \pi_{J^{k-1}\pi}^r \circ \bar{\pi}^{k-1}: J^{k-1}\pi^* \rightarrow \mathbb{R}$ is the natural projection, we denote $\bar{\tau}_o = \bar{\tau} \circ j: \mathcal{P} \rightarrow \mathbb{R}$. A Lagrangian density $\mathcal{L} \in \Omega^1(J^k\pi)$ is said to be *almost-regular* if:

1. \mathcal{P} is a closed submanifold of $J^{k-1}\pi^*$.
2. \mathcal{FL} is a submersion onto its image.
3. For every $\bar{y} \in J^{2k-1}\pi$, the fibers $\mathcal{FL}^{-1}(\mathcal{FL}(\bar{y}))$ are connected submanifolds of $J^{2k-1}\pi$.

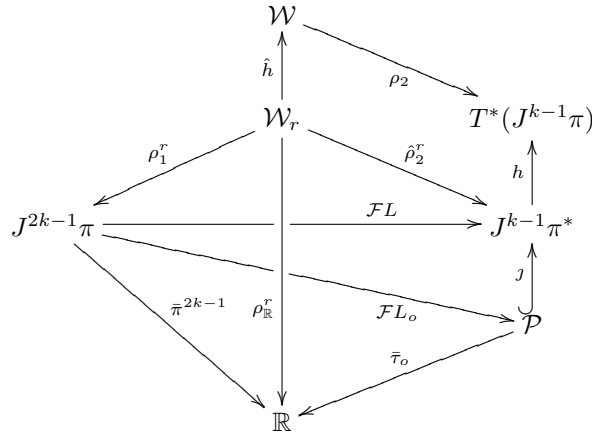
The Hamiltonian section $\hat{h} \in \Gamma(\mu_{\mathcal{W}})$ (introduced after Proposition 1) induces a Hamiltonian section $h \in \Gamma(\mu)$ defined by

$$\rho_2 \circ \hat{h} = h \circ \rho_2^r$$

Then, if Θ_{k-1} and Ω_{k-1} are the canonical 1 and 2 forms of the cotangent bundle $T^*(J^{k-1}\pi)$, we can construct the *Hamilton-Cartan forms* in $J^{k-1}\pi^*$ and \mathcal{P} by making

$$\begin{aligned} \Theta_h &= h^* \Theta_{k-1} \in \Omega^1(J^{k-1}\pi^*) & ; & \quad \Omega_h = h^* \Omega \in \Omega^2(J^{k-1}\pi^*) \\ \Theta_{\mathcal{P}} &= j^* \Theta_h \in \Omega^1(\mathcal{P}) & ; & \quad \Omega_{\mathcal{P}} = j^* \Omega_h \in \Omega^2(\mathcal{P}) . \end{aligned}$$

Observe that $\mathcal{FL}^* \Theta_h = \Theta_{\mathcal{L}}$ and $\mathcal{FL}^* \Omega_h = \Omega_{\mathcal{L}}$.



Remark 6. $(\mathcal{P}, \Omega_{\mathcal{P}}, \bar{\tau}_o^* \eta)$ is the *higher-order non-autonomous Hamiltonian system* associated with $(\mathcal{W}_r, \Omega_r, (\rho_{\mathbb{R}}^r)^* \eta)$.

In what follows, we consider that the Lagrangian density $\mathcal{L} \in \Omega^1(J^k\pi)$ is, at least, almost-regular. However, all the following results also hold for regular or hyperregular Lagrangian densities, replacing \mathcal{P} by the corresponding open subset of $J^{k-1}\pi^*$, or by $J^{k-1}\pi^*$, respectively.

First, we establish the variational principle from which we can obtain the dynamical equations for the Hamiltonian formalism, and then we show how to obtain the Hamiltonian dynamical equations.

Given the Hamiltonian system $(\mathcal{P}, \Omega_{\mathcal{P}}, \bar{\tau}_o^* \eta)$, let $\Gamma(\bar{\tau}_o)$ be the set of sections of $\bar{\tau}_o$. Consider the functional

$$\mathbf{H}: \Gamma(\bar{\tau}_o) \longrightarrow \mathbb{R}$$

$$\varphi \longmapsto \int_{\mathbb{R}} \varphi^* \Theta_{\mathcal{P}} \quad ,$$

where the convergence of the integral is assumed.

Definition 7.1 (Generalized Hamilton-Jacobi Variational Principle). The Hamiltonian or Hamilton-Jacobi variational problem for the higher-order Hamiltonian system $(\mathcal{P}, \Omega_{\mathcal{P}}, \bar{\tau}_o^* \eta)$ is the search for the critical (or stationary) sections of the functional \mathbf{H} with respect to the variations of φ given by $\varphi_s = \sigma_s \circ \varphi$, where $\{\sigma_s\}$ is a local one-parameter group of any compact-supported $Z \in \mathfrak{X}^{V(\bar{\tau}_o)}(\mathcal{P})$; that is

$$\left. \frac{d}{ds} \right|_{s=0} \int_{\mathbb{R}} \varphi_s^* \Theta_{\mathcal{P}} = 0 \tag{7}$$

Then, as in the above sections, we have:

Theorem 7.2. *The following assertions on a section $\varphi \in \Gamma(\bar{\tau}_o)$ are equivalent:*

1. φ is a solution to the Hamiltonian variational problem.
2. φ is a solution to the equation

$$\varphi^* i(Y) \Omega_{\mathcal{P}} = 0, \quad \text{for every } Y \in \mathfrak{X}(\mathcal{P}).$$

3. φ is a solution to the equation

$$i(\varphi')(\Omega_{\mathcal{P}} \circ \varphi) = 0 \quad ,$$

where $\varphi': \mathbb{R} \rightarrow T\mathcal{P}$ is the canonical lifting of φ to $T\mathcal{P}$.

4. φ is an integral curve of a vector field contained in a class of $\bar{\tau}_o$ -transverse vector fields, $\{X_h\} \subset \mathfrak{X}(\mathcal{P})$, satisfying

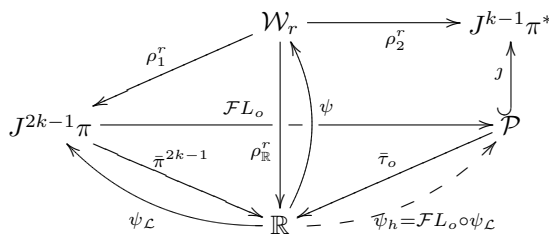
$$i(X_h) \Omega_{\mathcal{P}} = 0.$$

Proof. The proof of the equivalence $1 \iff 2$ follows the same patterns as in Theorem 4.2. For the proof of the other equivalences, see [21] (Theorem 5). \square

In addition, section solutions to the Hamilton equations are recovered from the solutions to the dynamical equations in the unified formalism. Indeed:

Theorem 7.3. *Let $\psi \in \Gamma(\rho_{\mathbb{R}}^r)$ be a critical section for the Lagrangian-Hamiltonian variational problem given by the functional \mathbf{LH} . Then, the section $\psi_h = \mathcal{F}L_o \circ \rho_1^r \circ \psi = \mathcal{F}L_o \circ \psi_{\mathcal{L}} \in \Gamma(\bar{\tau}_o)$ is a critical section for the Hamiltonian variational problem given by the functional \mathbf{H} .*

Conversely, from a section ψ_h solution to the Hamiltonian variational problem, we recover a solution ψ to the Lagrangian-Hamiltonian variational problem.



Proof. Observe that $\mathcal{F}L_o \circ \rho_1^r$ is a submersion, since it is a composition of submersions, and $(\mathcal{F}L_o \circ \rho_1^r)^* \Theta_{\mathcal{P}} = (\rho_1^r)^* (\mathcal{F}L_o^* \Theta_h) = (\rho_1^r)^* \Theta_{\mathcal{L}} = \Theta_r$. Then, for every compact-supported vector field $Z \in \mathfrak{X}^{V(\bar{\tau}_o)}(\mathcal{P})$, there exist compact-supported vector fields $Y \in \mathfrak{X}^{V(\rho_{\mathbb{R}}^r)}(\mathcal{W}_r)$ such that $(\mathcal{F}L_o \circ \rho_1^r)_* Y = Z$; that is, Z is $(\mathcal{F}L_o \circ \rho_1^r)$ -related with Y . We denote by $\{\sigma_s^r\}$ a local one-parameter group for the compact-supported vector fields $Y \in \mathfrak{X}^{V(\rho_{\mathbb{R}}^r)}(\mathcal{W}_r)$. Then, using this particular choice of $(\mathcal{F}L_o \circ \rho_1^r)$ -related vector fields, we have

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} \int_{\mathbb{R}} (\psi_h)_s^* \Theta_{\mathcal{P}} &= \frac{d}{ds} \Big|_{s=0} \int_{\mathbb{R}} (\sigma_s \circ \psi_h)^* \Theta_{\mathcal{P}} = \frac{d}{ds} \Big|_{s=0} \int_{\mathbb{R}} \psi_h^* (\sigma_s^* \Theta_{\mathcal{P}}) \\ &= \int_{\mathbb{R}} \psi_h^* L(Z) \Theta_{\mathcal{P}} = \int_{\mathbb{R}} \psi^* (\mathcal{F}L_o \circ \rho_1^r)^* L(Z) \Theta_{\mathcal{P}} \\ &= \int_{\mathbb{R}} \psi^* L(Y) \Theta_r = \frac{d}{ds} \Big|_{s=0} \int_{\mathbb{R}} \psi^* (\sigma_s^r)^* \Theta_r \\ &= \frac{d}{ds} \Big|_{s=0} \int_{\mathbb{R}} \psi_s^* \Theta_r = 0, \end{aligned}$$

since ψ is a critical section for the Lagrangian-Hamiltonian variational problem.

For the converse, following the same patterns as in the theory of singular non-autonomous first-order mechanical systems [8], it can be proved that there are holonomic sections $\psi_{\mathcal{L}}: \mathbb{R} \rightarrow J^{2k-1}\pi$ of the projection $\bar{\pi}^{2k-1}$ such that $\psi_h = \mathcal{F}L_o \circ \psi_{\mathcal{L}}$, and they are solutions to the Lagrangian dynamical equations. Then, the sections $\psi = (\psi_{\mathcal{L}}, \psi_h)$ are solutions to the Lagrangian-Hamiltonian variational problem (see the proof of Theorem 6.3). \square

Observe that the section ψ obtained from a given ψ_h is not necessarily unique.

In natural coordinates, if ψ_h is given by $\psi_h(t) = (t, q_i^A(t), p_A^i(t))$, $0 \leq i \leq k-1$, then the above equations give the classical higher-order Hamilton equations:

$$\dot{q}_i^A = \frac{\partial H}{\partial p_A^i} \Big|_{\psi_h} \quad ; \quad \dot{p}_A^i = \frac{\partial H}{\partial q_i^A} \Big|_{\psi_h} .$$

Finally, as a consequence of all the above results, we have the corresponding relation between vector field solutions to the unified dynamical equations and those which are solutions to the Hamiltonian equations:

Proposition 6. *Let $X \in \mathfrak{X}(\mathcal{W}_r)$ be a vector field tangent to $\mathcal{W}_{\mathcal{L}}$ and solution to the equations*

$$i(X)\Omega_r = 0 \quad ; \quad i(X)(\rho_{\mathbb{R}}^r)^* \eta = 1 , \tag{8}$$

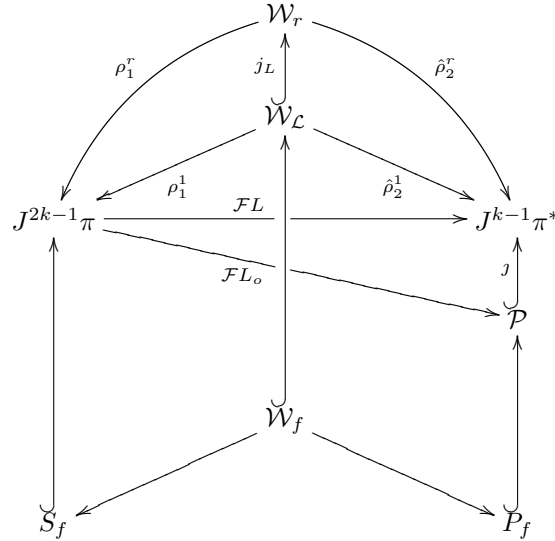
Then there exist vector fields $X_h \in \mathfrak{X}(\mathcal{P})$, which are solutions to the equations

$$i(X_h)\Omega_{\mathcal{P}} = 0 \quad , \quad i(X_h)\bar{\tau}_o^* \eta = 1 . \tag{9}$$

Conversely, if $X_h \in \mathfrak{X}(\mathcal{P})$ is a vector field which is a solution to the equations (9), then there exist vector fields $X \in \mathfrak{X}(\mathcal{W}_r)$ which are solutions to the equations (8).

Proof. : See also [21] (Theorem 4) for a detailed proof of this statement. \square

Remark 7. It is interesting to point out that, for almost-regular systems, if the unified dynamical equations have consistent solutions on a final constraint submanifold $\mathcal{W}_f \hookrightarrow \mathcal{W}_r$, then the Lagrangian and Hamiltonian equations have consistent solutions on final constraint submanifolds $S_f = \rho_1^r(\mathcal{W}_f) \hookrightarrow J^{2k-1}\pi$ and $\mathcal{P}_f = \hat{\rho}_1^r(\mathcal{W}_f) \hookrightarrow \mathcal{P}$, respectively. Then all the above results hold on the points of these submanifolds instead of \mathcal{W}_r , $J^{2k-1}\pi$, and \mathcal{P} , respectively.



8. Conclusions and further research. We have made an accurate revision of the generalization of the Lagrangian-Hamiltonian unified formalism of R. Skinner and R. Rusk to higher-order dynamical systems. We have analyzed the non-autonomous case, since the autonomous case can be obtained as a particular situation of this. This particular situation consists in using trivial bundles and removing the time-dependence (see [20]). This unified formalism constitutes a nice framework which allows us to study different kinds of problems in a simpler way. In particular, singular (constrained) systems can be analyzed more easily.

In particular, a first contribution is to obtain the complete Legendre-Ostrogradsky map as a consequence of applying the constraint algorithm to the unified dynamical equations, instead of using the total time derivative as it was done in [20, 21].

As another contribution, the classical variational principles of first-order mechanics are generalized to this framework, in order to state the dynamical equations for higher-order mechanics in several equivalent ways.

Therefore, the Lagrangian and Hamiltonian structures, equations and solutions of higher-order mechanics are recovered from those obtained in the unified formalism, which also includes the corresponding Lagrangian and Hamiltonian variational principles: the generalized Hamilton and Hamilton-Jacobi Principles respectively.

Several interesting physical examples have been studied using this formalism; for instance the Pais-Uhlenbeck oscillator and the shape of a deformed elastic cylindrical beam with fixed ends, as regular systems; the second-order relativistic particle, first as a free particle and later subjected to a potential, as singular systems [20, 21], and also underactuated control systems [6].

This generalization of the Lagrangian-Hamiltonian unified formalism to higher-order dynamical systems using a general fibre bundle E over \mathbb{R} (instead of the classical approach using trivial bundles) is a first step towards the study of higher-order classical field theory. However, replacing the base manifold \mathbb{R} with an orientable m -dimensional manifold M gives rise to new difficulties, such as defining a suitable fiber bundle to act as the phase space; obtaining the Legendre map without ambiguities, or obtaining the relation between the momenta (which is crucial in our formulation).

Nevertheless, our future aim is to obtain an unambiguous Lagrangian-Hamiltonian unified formalism for higher-order classical field theory, thus completing previous works [4, 25]. In particular, we want to recover a unique Legendre map in a canonical way from our unified formulation (at least for second-order field theory), and then both the Lagrangian and Hamiltonian formalisms. All of this will be made using jet bundles of finite order (in contrast to [25] where infinite-order jet bundles are used).

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