ON AUTOMORPHISM-FIXED SUBGROUPS OF A FREE GROUP

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Abstract

Let F be a finitely generated free group, and let n denote its rank. A subgroup H of F is said to be *automorphism-fixed*, or *auto-fixed* for short, if there exists a set S of automorphisms of F such that H is precisely the set of elements fixed by every element of S; similarly, H is 1-*auto-fixed* if there exists a single automorphism of F whose set of fixed elements is precisely H. We show that each auto-fixed subgroup of F is a free factor of a 1-auto-fixed subgroup of F. We show also that if (and only if) $n \geq 3$, then there exist free factors of 1-auto-fixed subgroups of F which are not auto-fixed subgroups of F. A 1-auto-fixed subgroup H of F has rank at most n, by the Bestvina-Handel Theorem, and if H has rank exactly n, then H is said to be a *maximum-rank* 1-auto-fixed subgroup of F, and similarly for auto-fixed subgroups. Hence a maximum-rank auto-fixed subgroup of F is a (maximum-rank) 1-auto-fixed subgroup of F. We further prove that if H is a maximum-rank 1-auto-fixed subgroup of F, then the group of automorphisms of F which fix every element of H is free abelian of rank at most $n-1$. All of our results apply also to endomorphisms.

1 Introduction

Recall that the *rank* of a group is the minimum of the set of the cardinals of the generating sets of the group.

Throughout, let n be a positive integer and F_n a free group of rank n.

Let $\text{End}(F_n)$ denote the endomorphism monoid of F_n, and $\text{Aut}(F_n)$ the automorphism group of F_n, so $\text{Aut}(F_n)$ is the group of units of $\text{End}(F_n)$. Let $\text{Inj}(F_n)$ denote the set of injective (or monic) endomorphisms of F_n, a submonoid of $\text{End}(F_n)$ containing $\text{Aut}(F_n)$.

Throughout, we let elements of $\text{End}(F_n)$ act on the right on F_n, so $x \mapsto (x)\phi$.

For any $S \subseteq \text{End}(F_n)$, let $\text{Fix}(S)$ denote the set consisting of the elements of F_n which are fixed by every element of S (read $\text{Fix}(S) = F_n$ for the case where S is empty). Then $\text{Fix}(S)$ is a subgroup of F_n, called the *S-fixed* subgroup of F_n, or *the subgroup of F_n fixed by S*.

A subgroup H of F_n is called an *endo-fixed* subgroup of F_n if $H = \text{Fix}(S)$ for some subset S of $\text{End}(F_n)$. If S can be chosen to lie in $\text{Inj}(F_n)$ (resp. $\text{Aut}(F_n)$) we further say that H is a *mono-fixed* (resp. *auto-fixed*) subgroup of F_n.

A subgroup H of F_n is called a 1-*endo-fixed* subgroup of F_n if $H = \text{Fix}(\phi)$ for some $\phi \in \text{End}(F_n)$ (here, and throughout, to simplify notation we write $\text{Fix}(\phi)$ rather than $\text{Fix}(\{\phi\})$). If ϕ can be chosen to lie in $\text{Inj}(F_n)$ (resp. $\text{Aut}(F_n)$), we further say that H is a 1-*mono-fixed* (resp. 1-*auto-fixed*) subgroup of F_n. For example, any maximal cyclic subgroup of F_n is 1-auto-fixed, since it is the subgroup fixed by a suitable inner automorphism. Notice that an auto-fixed subgroup is an intersection of 1-auto-fixed subgroups, and vice-versa.

The most important results about 1-auto-fixed subgroups of F_n were obtained by M. Bestvina and M. Handel, in [BH], where they showed that every 1-auto-fixed subgroup of F_n has rank at most n, which had previously been conjectured by G. P. Scott. In fact, Bestvina-Handel proved, but did not state, that any 1-mono-fixed subgroup of F_n has rank at most n, see [DV]. Imrich-Turner [IT], using the result of [BH], showed that any 1-endo-fixed subgroup of F_n has rank at most n. Dicks-Ventura [DV], using the techniques of [BH], showed that any mono-fixed subgroup of F_n has rank at most n, and G. M. Bergman [B], using the result of [DV], showed that any endo-fixed subgroup of F_n has rank at most n. This brief history is appropriate for our purposes, but is far from complete; for example, it does not mention the ground-breaking work of S. M. Gersten, who showed that 1-auto-fixed subgroups are finitely generated.

A 1-auto-fixed subgroup of F_n which has rank n is said to be a *maximum-rank* 1-auto-fixed subgroup of F_n, and similarly for the other five subgroup qualifiers defined above.

The work of Bestvina-Handel has been extended in many other directions; see, for example, [Tu], [CT], [V].

This paper continues the line of investigating auto-fixed subgroups of F_n, addressing the question of whether the following holds.

Conjecture 1.1 *Every auto-fixed subgroup of F_n is a 1-auto-fixed subgroup.*

The case where $n \leq 2$ was proved in Theorem 3.9 of [V]. We obtain partial results which we believe constitute a useful step towards proving Conjecture 1.1 in general. Recall that a *free factor* of a group is a member of a free-product decomposition of the group. Our main result, Theorem 3.3, is that, for any submonoid M of $\mathrm{End}(F_n)$, $\mathrm{Fix}(M)$ is a free factor of $\mathrm{Fix}(\phi)$ for some $\phi \in M$. Thus we use results of [BH], [DV] and [B] to recover the fact that endo-fixed subgroups have rank at most n. Observe that Theorem 3.3 proves the case of Conjecture 1.1 where the auto-fixed subgroup has maximum rank.

In the case where $n \leq 2$, it is a simple matter to show that a free factor of a 1-auto-fixed subgroup of F_n is 1-auto-fixed, so Theorem 3.3 can be used to deduce this previously known case of Conjecture 1.1. However, the same approach fails for larger n, since Proposition 5.4 shows that, for $n \geq 3$, there exist free factors of 1-auto-fixed subgroups of F_n which are not endo-fixed.

Section 5 considers the Galois correspondence between subgroups of F_n and subgroups of $\mathrm{Aut}(F_n)$. We see that if H is a maximum-rank 1-auto-fixed subgroup of F_n, then the corresponding set of automorphism of F_n which fix every element of H is a free abelian subgroup of $\mathrm{Aut}(F_n)$ of rank at most $n-1$.

2 Background

In this section we collect together the results we shall use in the proof of our main result.

The following is well known, and can be viewed as a special case of the Kurosh Subgroup Theorem.

Lemma 2.1 *If A, B, C are subgroups of a group G, and A is a free factor of B, then $A \cap C$ is a free factor of $B \cap C$.*

Proof. Here A is a free factor of B, say $B = A * D$. By Bass-Serre Theory (see pp. 1–55 of [Se], or pp. 1–35 of [DD1]), B acts on a tree T with trivial edge stabilizers, having A as a vertex stabilizer. Hence $B \cap C$ acts on T with trivial edge stabilizers, having $A \cap (B \cap C) = A \cap C$ as a vertex stabilizer. By Bass-Serre Theory again, $A \cap C$ is a free factor of $B \cap C$. □

In the case where G is a free group, which is the only case of interest to us, we remark that it is straightforward to use Stallings' graph pullback techniques [St] to obtain an alternative proof.

We now turn to the free group setting, and recall two classical results of M. Takahasi, and one of A. G. Howson.

Theorem 2.2 (Takahasi) *If H is a finitely generated subgroup of F_n, then there exists a finite set $\mathcal{O}$ of finitely generated subgroups of F_n which contain H, such that each subgroup of F_n which contains H has a free factor which belongs to $\mathcal{O}$.*

Proof. See Theorem 2 of [Ta]. A graph-theoretic proof can be found in [V]. □

Theorem 2.3 (Takahasi) *If $(H_m \mid m \geq 1)$ is a countable descending chain of subgroups of a free group, and some positive integer bounds the rank of H_m for all m, then the intersection $\bigcap_{m\geq 1} H_m$ is a free factor of H_m for all but finitely many m.*

Proof. See [Ta], or Exercises 33–36 of Section 2.4 of [MKS]. A graph-theoretic proof is contained in the proof of Theorem I.4.11 of [DV]. □

Theorem 2.4 (Howson) *If A and B are finitely generated subgroups of a free group then $A \cap B$ is finitely generated.*

Proof. See [H]. Gersten's very short graph-theoretic proof is given in Section 7.7 of [St]. □

For the final topic of this review, we consider endomorphisms of F_n. The deepest result we shall use is the following.

Theorem 2.5 (Bestvina-Handel-Imrich-Turner) *Every 1-endo-fixed subgroup of F_n has rank at most n.*

Proof. This was proved by W. Imrich and E. C. Turner [IT] using the 1-auto-fixed case proved by M. Bestvina and M. Handel [BH]. Essentially, it suffices to consider the action of ϕ on the image of a sufficiently high power of ϕ, since the rank of such images eventually stabilizes. □

We now recall a forerunner of the above.

Theorem 2.6 (Dyer-Scott [DS]) *If an automorphism* ϕ *of* F_n *has finite order, then its fixed subgroup* $\mathrm{Fix}(\phi)$ *is a free factor of* F_n*.* □

These results have consequences which are known to experts, but do not seem to have standard references, so we recall the (elementary) proofs.

Corollary 2.7 *If* $\phi \in \mathrm{End}(F_n)$ *and* m *is a positive integer, then* $\mathrm{Fix}(\phi)$ *is a free factor of* $\mathrm{Fix}(\phi^m)$*.*

Proof. By Theorem 2.5, $\mathrm{Fix}(\phi^m)$ is free of finite rank, and ϕ acts on it as an automorphism of finite order, so the result follows from Theorem 2.6. □

Corollary 2.8 *If* $\phi \in \mathrm{End}(F_n)$ *then* $\{\mathrm{Fix}(\phi^m) \mid m \geq 1\}$ *has a maximum element under inclusion.*

Proof. For each positive integer m, let us write $H_m = \mathrm{Fix}(\phi^m)$. By Theorem 2.5, each H_m is free of rank at most n. Thus we can choose m such that H_m has maximum possible rank. By Corollary 2.7, if r is a positive integer, then H_m and H_r are free factors of H_{mr}. By the maximality of the rank of H_m, we see that $H_m = H_{mr}$, so H_r is a free factor of H_m. Thus H_m is a maximum element. □

We remark that this maximum element consists of all the finite orbits of ϕ, and is sometimes called the *periodic* set of ϕ.

We conclude with a recent result. Recall that endomorphisms act on the right.

Theorem 2.9 (Bergman, [B]) *If* M *is a submonoid of* $\mathrm{End}(F_n)$*, then there exists* $\psi \in M$ *such that* $\mathrm{Fix}(M)$ *is a free factor of* $\mathrm{Fix}(M\psi)$*, and the subsemigroup* $M\psi$ *of* M *viewed as a subsemigroup of* $\mathrm{End}((F_n)\psi)$ *lies in* $\mathrm{Inj}((F_n)\psi)$*.*

Proof. ([B], p. 1540) Take $\psi \in M$ minimizing the rank of $(F_n)\psi$. It is clear that $M\psi$ acts injectively on $(F_n)\psi$ so, $M\psi$ can be viewed as a subsemigroup of $\mathrm{Inj}((F_n)\psi)$. Consider now $H = (\mathrm{Fix}(M\psi))\psi^{-1} \leq F_n$. For every $\phi \in M$, its restriction to $\mathrm{Fix}(M\psi)$ is a section of the surjective homomorphism $\psi\colon H \to \mathrm{Fix}(M\psi)$, since $\phi\psi \in M\psi$. By Corollary 12 of [B], the equalizer of this family of sections is a free factor of $\mathrm{Fix}(M\psi)$. But M contains the identity so, the previous equalizer is precisely $\mathrm{Fix}(M)$. Thus, $\mathrm{Fix}(M)$ is a free factor of $\mathrm{Fix}(M\psi)$.

For an alternative argument, see Remark 5.7 in [DD2]. □

We remark that, since $M\psi$ acts on both F_n and $(F_n)\psi$, there is an apparent ambiguity about the interpretation of $\mathrm{Fix}(M\psi)$, but this causes no problem since the two interpretations give the same subgroup.

3 Fixed subgroups

In this section, we prove that given a subset $S \subseteq \mathrm{End}(F_n)$ there exists ϕ in the submonoid of $\mathrm{End}(F_n)$ generated by S such that $\mathrm{Fix}(S)$ is a free factor of $\mathrm{Fix}(\phi)$. In particular, any auto-fixed (resp. mono-fixed, endo-fixed) subgroup of F_n is a free factor of a 1-auto-fixed (resp. 1-mono-fixed, 1-endo-fixed) subgroup of F_n.

Let us consider first the injective case with only two morphisms, then the general injective case and finally the general case. Recall that endomorphisms act on the right.

Lemma 3.1 *If $\phi, \psi \in \mathrm{End}(F_n)$ and ψ is injective, then there exists a positive integer t such that $\mathrm{Fix}(\phi, \psi)$ is a free factor of $\mathrm{Fix}(\phi\psi^t)$.*

Proof. By Corollary 2.8, there exists a positive integer m such that $\mathrm{Fix}(\psi^{mr}) = \mathrm{Fix}(\psi^m)$ for every positive integer r. Let $\eta = \psi^m$, so $\mathrm{Fix}(\eta^r) = \mathrm{Fix}(\eta)$ for every positive integer r, and η is injective.

By Theorem 2.5 and Theorem 2.4, $\mathrm{Fix}(\phi, \eta)$ is finitely generated. By Theorem 2.2, there exists a finite set $\mathcal{O}$ of finitely generated subgroups of F_n which contain $\mathrm{Fix}(\phi, \eta)$, such that every subgroup of F_n which contains $\mathrm{Fix}(\phi, \eta)$ has some element of $\mathcal{O}$ as a free factor.

Let r be a positive integer. Then $\mathrm{Fix}(\phi\eta^r)$ is a subgroup of F_n which contains $\mathrm{Fix}(\phi, \eta)$, so there exists some $M_r \in \mathcal{O}$ such that M_r is a free factor of $\mathrm{Fix}(\phi\eta^r)$.

For any positive integer $s > r$,

$$\mathrm{Fix}(\phi, \eta) \leq M_r \cap M_s \leq \mathrm{Fix}(\phi\eta^r, \phi\eta^s).$$

But if $x \in \mathrm{Fix}(\phi\eta^r, \phi\eta^s)$ then $(x)\eta^{s-r} = ((x)\phi\eta^r)\eta^{s-r} = (x)\phi\eta^s = x$, so $(x)\eta = x$. Hence $(x)\phi\eta^r = x = (x)\eta^r$, but η^r is injective, so $(x)\phi = x$. Thus $x \in \mathrm{Fix}(\phi, \eta)$. This shows that $\mathrm{Fix}(\phi\eta^r, \phi\eta^s) \leq \mathrm{Fix}(\phi, \eta)$, so $\mathrm{Fix}(\phi, \eta) = M_r \cap M_s$ for all positive distinct integers r, s.

Since $\mathcal{O}$ is finite, there exist positive integers $s > r$ such that $M_r = M_s$, and hence $M_r = M_r \cap M_s = \mathrm{Fix}(\phi, \eta)$, so $\mathrm{Fix}(\phi, \eta)$ is a free factor of $\mathrm{Fix}(\phi\eta^r) = \mathrm{Fix}(\phi\psi^{mr})$.

By Corollary 2.7, $\mathrm{Fix}(\psi)$ is a free factor of $\mathrm{Fix}(\eta)$. And by Lemma 2.1, $\mathrm{Fix}(\phi) \cap \mathrm{Fix}(\psi) = \mathrm{Fix}(\phi, \psi)$ is a free factor of $\mathrm{Fix}(\phi) \cap \mathrm{Fix}(\eta) = \mathrm{Fix}(\phi, \eta)$, which we have seen is a free factor of $\mathrm{Fix}(\phi\psi^{mr})$. Hence $\mathrm{Fix}(\phi, \psi)$ is a free factor of $\mathrm{Fix}(\phi\psi^{mr})$. □

We can now prove the injective case of the main result.

Lemma 3.2 *Let S be a nonempty subset of $\mathrm{Inj}(F_n)$, and $M(S)$ the subsemigroup of $\mathrm{Inj}(F_n)$ generated by S. Then there exists $\phi \in M(S)$ such that $\mathrm{Fix}(S)$ is a free factor of $\mathrm{Fix}(\phi)$.*

Proof. Let κ denote the cardinal of S, so $1 \leq \kappa \leq \aleph_0$.

The case $\kappa = 1$ is clearly valid.

Suppose that $2 \leq \kappa < \aleph_0$, and that the result is true for smaller sets. Let $\psi \in S$. By the induction hypothesis, there exists $\phi \in M(S - \{\psi\})$ such that $\mathrm{Fix}(S - \{\psi\})$ is a free factor of $\mathrm{Fix}(\phi)$. By Lemma 2.1,

$$\mathrm{Fix}(S - \{\psi\}) \cap \mathrm{Fix}(\psi) = \mathrm{Fix}(S)$$

is a free factor of $\mathrm{Fix}(\phi) \cap \mathrm{Fix}(\psi) = \mathrm{Fix}(\phi, \psi)$. And, by Lemma 3.1, there exists a positive integer t such that $\mathrm{Fix}(\phi, \psi)$ is a free factor of $\mathrm{Fix}(\phi\psi^t)$. Since $\phi\psi^t \in M(S)$, we see that, by induction, the result holds for finite sets.

It remains to consider the case where $\kappa = \aleph_0$, so S is the union of a countable ascending chain of finite nonempty subsets $(S_m \mid m \geq 1)$. Then $(\mathrm{Fix}(S_m) \mid m \geq 1)$ is a countable descending chain of subgroups whose intersection is $\mathrm{Fix}(S)$. By the preceding paragraph, for each $m \geq 1$, there exists $\phi_m \in M(S_m)$ such that $\mathrm{Fix}(S_m)$ is a free factor of $\mathrm{Fix}(\phi_m)$. By Theorem 2.5, each $\mathrm{Fix}(S_m)$ has rank at most n, so, by Theorem 2.3, there exists a positive integer m such that $\mathrm{Fix}(S)$ is a free factor of $\mathrm{Fix}(S_m)$, so is a free factor of $\mathrm{Fix}(\phi_m)$. □

We can now obtain our main result.

Theorem 3.3 *Let n be a positive integer, F_n a free group of rank n, S a subset of $\mathrm{End}(F_n)$, and M the submonoid of $\mathrm{End}(F_n)$ generated by S. Then there exists $\phi \in M$ such that $\mathrm{Fix}(S)$ is a free factor of $\mathrm{Fix}(\phi)$.*

Proof. By Theorem 2.9, there exists $\psi \in M$ such that $M\psi$ can be viewed as a subsemigroup of $\mathrm{Inj}((F_n)\psi)$, and such that $\mathrm{Fix}(M)$ is a free factor of $\mathrm{Fix}(M\psi)$. By Lemma 3.2 applied to the nonempty subset (and subsemigroup) $M\psi$ of $\mathrm{Inj}((F_n)\psi)$, there exists $\phi \in M\psi \subseteq M \subseteq \mathrm{End}(F_n)$ such that $\mathrm{Fix}(M\psi)$ is a free factor of $\mathrm{Fix}(\phi)$ (recall the two coinciding interpretations of the term "Fix" in this context). Hence $\mathrm{Fix}(S) = \mathrm{Fix}(M)$ is a free factor of $\mathrm{Fix}(\phi)$. $\square$

It is not known in general if the set of 1-endo-fixed subgroups is closed under arbitrary (or even finite) intersections; this is precisely the fact conjectured in Conjecture 1.1. However, the subset of those subgroups $H = \mathrm{Fix}(\phi)$ with $\phi^2 = \phi$ is closed under arbitrary intersections (see Lemma 18 in [B]).

In light of Theorem 3.3, it is natural to ask whether a free factor of a 1-auto-fixed subgroup of F_n is necessarily auto-fixed. It is straightforward to prove this for $n \leq 2$, but, in Proposition 5.4, we will see that there are counter-examples for all $n \geq 3$.

We record the following consequences of Theorem 3.3.

Corollary 3.4 *Each auto-fixed subgroup of* F_n *is a free factor of some* 1-*auto-fixed subgroup.*

Each mono-fixed subgroup of F_n *is a free factor of some* 1-*mono-fixed subgroup.*

Each endo-fixed subgroup of F_n *is a free factor of some* 1-*endo-fixed subgroup.* $\square$

For completeness we mention the following.

Corollary 3.5 (Dicks-Ventura-Bergman) *Every endo-fixed subgroup of* F_n *has rank at most* n. $\square$

This result was originally obtained by G. M. Bergman in [B] using Theorem 2.5, results of [DV] and Theorem 2.9; here we have used the first and third, but completely bypassed the second.

4 Maximum-rank fixed subgroups

We have introduced six types of fixed subgroups of F_n, and, for each type, the maximum possible rank is n. In this section we consider the case where this maximum rank is achieved.

We begin by observing two consequences of Theorem 3.3 for the maximum-rank case.

Corollary 4.1 *If* S *is a subset of* $\mathrm{End}(F_n)$ *such that* $\mathrm{Fix}(S)$ *has rank* n, *then the submonoid of* $\mathrm{End}(F_n)$ *generated by* S *contains some element* ϕ *such that* $\mathrm{Fix}(S) = \mathrm{Fix}(\phi)$. $\square$

Corollary 4.2 *Every maximum-rank auto-fixed subgroup of F_n is a maximum-rank 1-auto-fixed subgroup.*

Every maximum-rank mono-fixed subgroup of F_n is a maximum-rank 1-mono-fixed subgroup.

Every maximum-rank endo-fixed subgroup of F_n is a maximum-rank 1-endo-fixed subgroup. □

Notice the first part is a special case of Conjecture 1.1.

Now recall the important work of Collins and Turner in this area.

Theorem 4.3 (Turner [Tu]) *If $\phi \in \mathrm{End}(F_n)$ and $\mathrm{Fix}(\phi)$ has rank n, then $\phi \in \mathrm{Aut}(F_n)$.* □

Let F_n^{ab} denote the abelianization of F_n, a free abelian group of rank n. For elements a, b of F_n, we write $[a,b] = a^{-1}b^{-1}ab$.

Theorem 4.4 (Collins-Turner [CT]) *Let H be a subgroup of F_n, and let m denote the rank of the (free abelian) image of H in F_n^{ab}. Then H is a maximum-rank 1-auto-fixed subgroup of F_n if and only if there exists a basis $(x_i \mid 1 \le i \le n)$ of F_n, such that, setting $F_l = \langle x_i \mid 1 \le i \le l\rangle$ for $0 \le l \le n$, there exists a basis $(y_i \mid 1 \le i \le n)$ of H, such that, for $1 \le j \le m$, $y_j = x_j$, and, for $m+1 \le k \le n$, $y_k = [w_k, x_k]$ for some $w_k \in H \cap F_{k-1}$ such that w_k is not a proper power of any element of F_n (so, in particular, $w_k \ne 1$).*

In this event, $(y_i \mid 1 \le i \le l)$ is a basis of $H \cap F_l$, for $0 \le l \le n$. □

We now combine all the above results to obtain a generalization of Theorem 3.9 of [V], which dealt with the case $n = 2$.

Theorem 4.5 *Let n be a positive integer, F_n a free group of rank n, and H a subgroup of F_n of rank n. Let m denote the rank of the (free abelian) image of H in F_n^{ab}. Then the following are equivalent:*

(a) *H is a 1-auto-fixed subgroup of F_n.*

(b) *H is a 1-mono-fixed subgroup of F_n.*

(c) *H is a 1-endo-fixed subgroup of F_n.*

(d) *H is an auto-fixed subgroup of F_n.*

(e) *H is a mono-fixed subgroup of F_n.*

(f) *H is an endo-fixed subgroup of F_n.*

(g) *There exist a basis* $(x_i \mid 1 \le i \le n)$ *of* F_n*, and a basis* $(y_i \mid 1 \le i \le n)$ *of* H*, such that, if* $1 \le j \le m$*, then* $y_j = x_j$*, and, if* $m+1 \le k \le n$*, then* $y_k = [w_k, x_k]$ *for some* $w_k \in H \cap F_{k-1}$ *such that* w_k *is not a proper power.*

Proof. Corollary 4.2 shows that (a) is equivalent to (d), that (b) is equivalent to (e) and that (c) is equivalent (f). Theorem 4.3 shows that (a), (b) and (c) are equivalent, while Theorem 4.4 shows that (a) and (g) are equivalent. □

5 Galois groups

For any subgroup H of F_n, let us write $\mathrm{Aut}_H(F_n)$ for the set of elements of $\mathrm{Aut}(F_n)$ which fix every element of H; this is sometimes called the *pointwise-stabilizer* of H.

In the usual way, we have a Galois correspondence between subsets of $\mathrm{Aut}(F_n)$ and subgroups of F_n, given by $S \mapsto \mathrm{Fix}(S)$ in one direction, and $H \mapsto \mathrm{Aut}_H(F_n)$ in the other direction. This gives rise to the standard bijection between the corresponding closed subsets on both sides. Thus the closed subgroups of F_n are the auto-fixed subgroups of F_n, while the closed subsets of $\mathrm{Aut}(F_n)$ are the pointwise-stabilizers of subgroups of F_n.

Thus for any subgroup H of F_n, we define the *auto-fixed closure* of H in F_n to be $\mathrm{Fix}(\mathrm{Aut}_H(F_n))$, that is, the intersection of all those 1-auto-fixed subgroups of F_n containing H.

Unlike the situation for finite field extensions, we get a different Galois correspondence if we consider endomorphisms of F_n. We shall deal mostly with the maximal rank case, where no difference arises. We define $\mathrm{End}_H(F_n)$ in the natural way.

The purpose of this section is to calculate some interesting special cases. We begin by describing those closed subgroups of $\mathrm{Aut}(F_n)$ which correspond to maximum-rank auto-fixed subgroups.

Proposition 5.1 *Let* H *be a maximum-rank auto-fixed subgroup of* F_n*, and let* m *denote the rank of the (free abelian) image of* H *in* F_n^{ab}*. Then* $\mathrm{End}_H(F_n) = \mathrm{Aut}_H(F_n)$ *is a free abelian subgroup of* $\mathrm{Aut}(F_n)$ *of rank* $n-m$*.*

Proof. By Theorem 4.4, there is a basis $(x_i \mid 1 \le i \le n)$ of F_n, and a basis $(y_i \mid 1 \le i \le n)$ of H, such that, for $1 \le j \le m$, $y_j = x_j$, and, for $m+1 \le k \le n$, $y_k = [w_k, x_k]$ for some $w_k \in H \cap F_{k-1}$ not being a proper power.

Let l be an integer, $m+1 \leq l \leq n$. There exists a unique endomorphism ϕ_l of F_n such that, for $1 \leq i \leq n$, $(x_i)\phi_l = x_i$ if $i \neq l$, and $(x_l)\phi_l = w_l x_l$. It is clear that ϕ_l is an automorphism, and fixes w_l, and a straightforward induction argument shows that ϕ_l fixes all the y_i and all the w_i, so $\phi_l \in \mathrm{Aut}_H(F_n)$.

It is easy to see that the ϕ_l $(m+1 \leq l \leq n)$ commute with each other.

Now consider any $\phi \in \mathrm{End}_H(F_n)$.

For $1 \leq j \leq m$, we see $(x_j)\phi = (y_j)\phi = y_j = x_j$.

We claim that, if $m+1 \leq k \leq n$, then there exists a unique integer r_k such that $(x_k)\phi = w_k^{r_k} x_k$. To see this let $X_k = (x_k)\phi$, and notice that ϕ simultaneously fixes w_k and y_k, since they both lie in H. So,

$$w_k^{-1} x_k^{-1} w_k x_k = y_k = (y_k)\phi = ([w_k, x_k])\phi = [w_k, X_k] = w_k^{-1} X_k^{-1} w_k X_k.$$

Hence $X_k x_k^{-1}$ commutes with w_k, and so $X_k x_k^{-1} = w_k^{r_k}$ for a unique integer r_k, since w_k is not a proper power. This proves the claim.

Thus, we have proved that $\phi = \phi_{m+1}^{r_{m+1}} \cdots \phi_n^{r_n}$ for unique integers r_k. The result now follows. □

Corollary 5.2 *Suppose S is a subset of $\mathrm{End}(F_n)$ such that $\mathrm{Fix}(S)$ has rank n, and let m denote the rank of the (free abelian) image of $\mathrm{Fix}(S)$ in F_n^{ab}. Then S lies in a free abelian subgroup of $\mathrm{Aut}(F_n)$ of rank $n-m$ (and generates a free abelian subgroup of $\mathrm{Aut}(F_n)$ of rank at most $n-m$).* □

Ignoring the trivial case $n = m$, we see that S can be chosen to consist of a single element, in which case the rank of the (free abelian) subgroup it generates is 1.

Lemma 5.3 *Let a, b, c be distinct elements of some basis $\mathcal{B}$ of F_n. If A, B, C are elements of F_n such that $[A,B] = [a,b]$ and $[A,C] = [a,c]$, then there exist integers r, s such that $A = a$, $B = a^r b$, and $C = a^s c$.*

Proof. Let A^{red} be the cyclic reduction of A and write $A = \alpha A^{\mathrm{red}} \alpha^{-1}$ for some $\alpha \in F_n$. We have that $[a,b] = [A,B]$ belongs to the normal closure of A so, by Proposition II.5.1 in [LS], $A^{\mathrm{red}} \in \langle a, b\rangle$. In the same way, $A^{\mathrm{red}} \in \langle a, c\rangle$. Thus, A^{red} is a power of a. But killing A^{red}, $a^{-1}b^{-1}ab = A^{-1}B^{-1}AB$ becomes trivial, so $A^{\mathrm{red}} = a^{\epsilon}$ and $A = \alpha a^{\epsilon} \alpha^{-1}$ for $\epsilon = \pm 1$.

Write $B = \alpha B' \alpha^{-1}$ and $C = \alpha C' \alpha^{-1}$. Note that $[a^{\epsilon}, B'] = \alpha^{-1}[a,b]\alpha$, so

$$\langle a, B'^{-1} a B' \rangle \;=\; \langle a, a^{-\epsilon} B'^{-1} a^{\epsilon} B' \rangle \;=\; \langle a, \alpha^{-1} a^{-1} b^{-1} a b \alpha \rangle.$$

Figure 1: Three $\mathcal{B}$-labelled graphs for the subgroup H of F_n.

The $\mathcal{B}$-labelled graphs depicted in figures 1(a) and 1(b) represent this subgroup of F_n, say H, with basepoints in u_1 and v_1, respectively (see [V] for notation). These two $\mathcal{B}$-labelled graphs are locally injective everywhere except, possibly, in vertices u_1 and u_2 and v_1 and v_2, respectively. And, after folding, they both give the same $\mathcal{B}$-labelled graph immersion. But $r(H) = 2$, so $B' \notin \langle a \rangle$ and hence the path α in figure 1(b) must be completely folded. Thus, $\alpha \in \langle a, b \rangle$ and the $\mathcal{B}$-labelled graph immersion for H is that depicted in figure 1(c), with either w_1 or w_2 as a basepoint. We deduce that $B' = a^r b^\delta a^p$ for $\delta = \pm 1$ and some integers r, p.

An analogous argument shows that $\alpha \in \langle a, c \rangle$ and that $C' = a^s c^\nu a^q$ for $\nu = \pm 1$ and some integers s, q.

So, $\alpha \in \langle a, b \rangle \cap \langle a, c \rangle = \langle a \rangle$ and we may assume $\alpha = 1$. Hence, $A = a^\epsilon$, $B = B' = a^r b^\delta a^p$ and $C = C' = a^s c^\nu a^q$. Now, writting the equations $[A, B] = [a, b]$ and $[A, C] = [a, c]$, we deduce that $p = 0$ and $\delta = \epsilon = 1$ and that $q = 0$ and $\nu = \epsilon = 1$, respectively. So, $A = a$, $B = a^r b$, and $C = a^s c$. $\square$

Proposition 5.4 *Let (n be at least three and) a, b, c be distinct elements of some basis X of F_n, let $H = \langle X - \{a, b, c\} \cup \{[a, b], [a, c]\} \rangle$, and let*

$$K = \langle X - \{b, c\} \cup \{[a, b], [a, c]\} \rangle = H * \langle a \rangle.$$

Then the endo-fixed closure of H is K, and H is a proper free factor of K, and K is a maximum-rank 1-auto-fixed subgroup of F_n.

Proof. It follows from Lemma 5.3 that any endomorphsim of F_n which fixes H also fixes a, so K lies in the endo-fixed closure of H. Since

$$X - \{b, c\} \cup \{[a, b], [a, c]\}$$

is a basis for K of the form given in Theorem 4.4, we see that K is a maximum-rank 1-auto-fixed subgroup of F_n. □

Thus, for $n \geq 3$, Proposition 5.4 provides examples of free factors of 1-auto-fixed subgroups of F_n which are not auto-fixed subgroups, in fact, not even endo-fixed.

It is clear that the set of all auto-fixed subgroups of F_n is closed under arbitrary intersections. By Proposition 5.4, this set is not closed under taking free factors, if $n \geq 3$. It is obvious that every 1-auto fixed subgroup of F_n is an auto-fixed subgroup of F_n. The converse of this implication is precisely Conjecture 1.1.

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