Discrete Applied Mathematics 170 (2014) 7–20

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Partnership formation and multinomial values[☆]

José Miguel Giménez [a], Maria Dolors Llongueras [b], María Albina Puente [a,*]

[a] *Department of Applied Mathematics III and Engineering School of Manresa, Technical University of Catalonia, Spain*

[b] *Department of Applied Mathematics II and Industrial and Aeronautical Engineering School of Terrassa, Technical University of Catalonia, Spain*

ARTICLE INFO

Article history:
Received 17 April 2013
Received in revised form 22 January 2014
Accepted 26 January 2014
Available online 18 February 2014

Keywords:
Cooperative game
Coalition formation
Partnership
Multinomial values

ABSTRACT

We use multinomial values to study the effects of the partnership formation in cooperative games, comparing the joint effect on the involved players with the alternative alliance formation. The simple game case is especially considered and the application to the Catalonia Parliament (Legislature 2003–2007) is also studied.

1. Introduction

The notion of coalition of partners or *partnership* – as it will be called here – was introduced in [16]. In [3] the significance and scope of this concept were emphasized, first in cooperative games and later on for simple games, and the natural way to *impose* partnerships in a given game by means of commitments between players was also suggested. In the present paper we focus on a subfamily of probabilistic values called *multinomial (probabilistic) values*. These values were introduced in reliability by Puente [20] (see also [15]) with the name of "multibinary probabilistic values". They were independently defined by Carreras [4], for simple games only – i.e. as power indices – in a work on decisiveness where they were called "Banzhaf α-indices". Recently, Carreras and Puente [11] have given two characterizations of the multinomial values within the class of probabilistic values: one for each value and another for the whole family.

For more than a decade, our research group has been studying *semivalues*, a subfamily of probabilistic values introduced by Dubey et al. [14], characterized by anonymity, and including the Shapley value as the only *efficient* member. In the analysis of certain cooperative problems we have successfully used *binomial semivalues* [20] that include the Banzhaf value introduced by Owen [17].[1] From this experience, we feel that multinomial values (n parameters, n being the number of players) offer a deal of flexibility clearly greater than binomial semivalues (one parameter) and hence many more possibilities to introduce additional information when evaluating a game.

The aim of this paper is the application of *multinomial values* to study the effects of the partnership formation. Our first goal is to investigate how these values are modified if several players agree to form a partnership and generalize the previous

☆ Research supported by Grant SGR 2009-01029 of the Catalonia Government (*Generalitat de Catalunya*) and Grant MTM 2012-34426 of the Science and Innovation Spanish Ministry.

* Corresponding author. Tel.: +34 938777249.
E-mail addresses: jose.miguel.gimenez@upc.edu (J.M. Giménez), m.dolors.llongueras@upc.edu (M.D. Llongueras), m.albina.puente@upc.edu (M.A. Puente).

1 [1,6,9,10] and [12] are samples of our work in this line.

http://dx.doi.org/10.1016/j.dam.2014.01.018
0166-218X/

results found by using binomial semivalues in [10]. Our second goal is the study of a real life political instance: the Catalonia Parliament during the Legislature 2003–2007.

The organization of the paper is then as follows. In Section 2, a minimum of preliminaries is provided. In Section 3, general statements for cooperative games are first given and concern the variation of the multinomial values, when a partnership is formed, and refer to (a) inner players and (b) outside players; next, a comparison is established between the multinomial values of the coalition as (i) a partnership and (ii) an alliance. In Section 4, we analyze partnerships in simple games: in this case, we determine the maximum and minimum values of the differences found for any multinomial value in the three cases mentioned above and supply games where these extreme values are attained. Proofs of the statements in Sections 3 and 4 will be found in Appendices A and B, respectively. Section 5 contains the analysis of the Catalonia Parliament if a partnership is formed. Finally, Section 6 states some conclusions.

2. Preliminaries

Let $N = \{1, 2, \ldots, n\}$ denote a finite set of *players*. A *cooperative game* in N is a function $v : 2^N \to \mathbb{R}$, which assigns a real number $v(S)$ to each *coalition* $S \subseteq N$ and satisfies $v(\emptyset) = 0$. A game v is *monotonic* if $v(S) \leq v(T)$ whenever $S \subset T \subseteq N$. Player $i \in N$ is a *dummy* in v if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N\backslash\{i\}$, and *null* in v if, moreover, $v(\{i\}) = 0$. Players $i, j \in N$ are *symmetric* in v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N\backslash\{i, j\}$. For example, if $\emptyset \neq S \subseteq N$, the *unanimity game* u_S is defined by $u_S(T) = 1$ if $S \subseteq T$ and $u_S(T) = 0$ otherwise. In this monotonic game, every $j \notin S$ is a null player and all members of S are symmetric players. The vector space of all games in N will be denoted as $\mathcal{G}_N$. Finally, every permutation θ of N induces a linear automorphism of $\mathcal{G}_N$ given by $(\theta v)(S) = v(\theta^{-1}S)$ for all $S \subseteq N$ and all v.

2.1. Probabilistic values

Following Weber's [23] axiomatic definition, $\phi : \mathcal{G}_N \to \mathbb{R}^N$ is a (group) *probabilistic value* iff it satisfies the following properties:

(i) *linearity*: $\phi[v + v'] = \phi[v] + \phi[v']$ and $\phi[\lambda v] = \lambda\phi[v]$ for all $v, v' \in \mathcal{G}_N$ and $\lambda \in \mathbb{R}$;
(ii) *positivity*[2]: if v is monotonic, then $\phi[v] \geq 0$;
(iii) *dummy player property*: if $i \in N$ is a dummy in game v, then $\phi_i[v] = v(\{i\})$.

There is an interesting characterization of the probabilistic values, also in [23]: (a) given a set of $n2^{n-1}$ *weighting coefficients* $\{p_S^i : i \in N,\ S \subseteq N\backslash\{i\}\}$ such that $\sum_{S \subseteq N\backslash\{i\}} p_S^i = 1$ for each $i \in N$ and $p_S^i \geq 0$ for all $i \in N$ and $S \subseteq N\backslash\{i\}$, the expression

$$\phi_i[v] = \sum_{S \subseteq N\backslash\{i\}} p_S^i[v(S \cup \{i\}) - v(S)] \quad \text{for all } i \in N \text{ and } v \in \mathcal{G}_N \tag{1}$$

defines a probabilistic value ϕ on N; (b) conversely, every probabilistic value can be obtained in this way; (c) the correspondence given by $\{p_S^i : i \in N,\ S \subseteq N\backslash\{i\}\} \mapsto \phi$ is one-to-one.

Thus, the payoff that a probabilistic value allocates to every player in any game is a weighted sum of his marginal contributions in the game. We quote from [23]:

> "Let player i view his participation in a game v as consisting merely of joining some coalition S and then receiving as a reward his marginal contribution to the coalition. If p_S^i is the probability that he joins coalition S, then $\phi_i[v]$ is his expected payoff from the game".

Among the probabilistic values, *semivalues*, introduced by Dubey et al. [14], are characterized by the *anonymity* property: $\phi_{\theta i}[\theta v] = \phi_i[v]$ for all $i \in N$, $v \in \mathcal{G}_N$ and θ, permutation on N. Alternatively, this is equivalent to saying that, if $n = |N|$, there is a vector $\{p_s\}_{s=0}^{n-1}$ such that $p_S^i = p_s$ for all $i \in N$ and all $S \subseteq N\backslash\{i\}$, where $s = |S|$, so that all coalitions of a given size share a common weight that applies to all (external) players, and hence Eq. (1) reduces to

$$\phi_i[v] = \sum_{S \subseteq N\backslash\{i\}} p_s[v(S \cup \{i\}) - v(S)] \quad \text{for all } i \in N \text{ and } v \in \mathcal{G}_N.$$

The weighting coefficients $\{p_s\}_{s=0}^{n-1}$ of any semivalue ϕ satisfy therefore two characteristic conditions:

$$\text{each } p_s \geq 0 \quad \text{and} \quad \sum_{s=0}^{n-1} \binom{n-1}{s} p_s = 1.$$

Well-known examples of semivalues are the *Shapley value* φ [21], for which $p_s = 1/n\binom{n-1}{s}$, and the *Banzhaf value* β [17], for which $p_s = 2^{1-n}$. The Shapley value φ is the only *efficient* semivalue, in the sense that $\sum_{i \in N} \varphi_i[v] = v(N)$ for every $v \in \mathcal{G}_N$. Note that these two classical values are defined for each N.

---

[2] In [23] this property is called *monotonicity*, but we prefer to call to it *positivity* as in [14].

Finally, the *multilinear extension* (Owen [18]) of a game $v \in \mathcal{G}_N$ is the real-valued function defined on $\mathbb{R}^n$ by

$$f(x_1, x_2, \dots, x_n) = \sum_{S \subseteq N} \prod_{i \in S} x_i \prod_{j \in N \setminus S} (1 - x_j) v(S). \tag{2}$$

As is well known, both the Shapley and Banzhaf values of any game v can be obtained from its multilinear extension. Indeed, $\varphi[v]$ can be calculated by integrating the partial derivatives of the multilinear extension of the game along the main diagonal $x_1 = x_2 = \cdots = x_n$ of the cube $[0, 1]^n$ [18], while the partial derivatives of that multilinear extension evaluated at point $(1/2, 1/2, \dots, 1/2)$ give $\beta[v]$ [17].

2.2. Multinomial (probabilistic) values

The multinomial (probabilistic) values were introduced by Puente [20] (see also Freixas and Puente [15] and Carreras and Puente [12,13]) as follows.

Definition 2.1. Set $N = \{1, 2, \dots, n\}$ and let $\mathbf{p} \in [0, 1]^n$, that is, $\mathbf{p} = (p_1, p_2, \dots, p_n)$ with $0 \le p_i \le 1$ for $i = 1, 2, \dots, n$, be given. Then the coefficients

$$p^i_S = \prod_{j \in S} p_j \prod_{\substack{k \in N \setminus S \\ k \neq i}} (1 - p_k) \quad \text{for all } i \in N \text{ and } S \subseteq N \setminus \{i\} \tag{3}$$

(where the empty product, arising if $S = \emptyset$ or $S = N \setminus \{i\}$, is taken to be 1) define a probabilistic value on $\mathcal{G}_N$ that will be called the $\mathbf{p}$-*multinomial value* and denoted as $\lambda^{\mathbf{p}}$. Thus,

$$\lambda^{\mathbf{p}}_i[v] = \sum_{N \supseteq S \ni i} \prod_{j \in S \setminus \{i\}} p_j \prod_{k \in N \setminus S} (1 - p_k)[v(S) - v(S \setminus \{i\})] \quad \text{for all } i \in N \text{ and } v \in \mathcal{G}_N.$$

Although agreeing with Weber's [23] interpretation of the weighting coefficient p^i_S as the probability that player i joins coalition S, we feel that the collection $\{p^i_S : i \in N,\ S \subseteq N \setminus \{i\}\}$ defining a probabilistic value contains, in general, too many parameters to be precisely determined and handled for discussion in most cases in practice. On the contrary, the weighting coefficients $\{p_s\}_{s=0}^{n-1}$ defining a semivalue [14], and particularly parameter q in the binomial case [20,2], may well lack the necessary flexibility to deal with players' behavior since, precisely, they are not able, because of their anonymity, to discriminate among players. Thus, using *profiles* $\mathbf{p} = (p_1, p_2, \dots, p_n)$, which define multinomial values, appears as an intermediate and reasonably balanced possibility between both extreme cases, since these vectors provide just *one parameter per player*. Therefore we will attach to p_i the meaning of *generical tendency of player i to form coalitions*, and will assume that p_i and p_j are independent of each other whenever $i \neq j$. Thus we will say that $\mathbf{p}$ is a *(tendency) profile* on N. According to Eq. (3), this implies that coefficient p^i_S, the probability of i to join S, will depend on the positive tendencies of the members of S to form coalitions and also on the negative tendencies in this sense of the outside players, i.e. the members of $N \setminus (S \cup \{i\})$.

Remark 2.2. (a) For example, for $n = 2$ we have $\mathbf{p} = (p_1, p_2)$ and, if $i \neq j$,

$$\lambda^{\mathbf{p}}_i[v] = (1 - p_j)[v(\{i\}) - v(\emptyset)] + p_j[v(\{i, j\}) - v(\{j\})].$$

Thus, the payoff allocated by $\lambda^{\mathbf{p}}$ to player i does not depend on p_i but only on p_j. If player j is not greatly interested in cooperating, and hence p_j is small, player i mainly receives his individual utility whereas, otherwise, if player j is interested in cooperating, and hence p_j is great, player i mainly receives his marginal contribution to the grand coalition.

(b) It is easy to check that the action of $\lambda^{\mathbf{p}}$ on an unanimity game u_T is given by:

$$\lambda^{\mathbf{p}}_i[u_T] = \prod_{\substack{j \in T \\ j \neq i}} p_j \quad \text{if } i \in T \quad \text{and} \quad \lambda^{\mathbf{p}}_i[u_T] = 0 \quad \text{otherwise}. \tag{4}$$

(c) Whenever, in particular, $p_1 = p_2 = \cdots = p_n = q$ for some $q \in [0, 1]$, coefficients p^i_S reduce, for all $i \in N$, to

$$p^i_S = p_s = q^s (1 - q)^{n-s-1} \quad \text{for } s = 0, 1, \dots, n-1,$$

where $s = |S|$ and $0^0 = 1$ by convention in cases $q = 0$ and $q = 1$. These coefficients $\{p_s\}_{s=0}^{n-1}$ define the *q-binomial semivalue* ψ^q introduced by Puente [20] (see also [2]) and, obviously, $\lambda^{\mathbf{p}} = \psi^q$. Of course, $\psi^{1/2} = \beta$, the Banzhaf value, and ψ^0 and ψ^1 are, respectively, the *dictatorial* and the *marginal* values, introduced in [19] and defined by $\psi^0_i[v] = v(\{i\})$ and $\psi^1_i[v] = v(N) - v(N \setminus \{i\})$.

(d) As is shown in [20], the multilinear extension procedure extends well to all binomial semivalues. In [20,15], the method is also extended to any multinomial value: if $\lambda^{\mathbf{p}}$ is such a value and f is the multilinear extension of game $v \in \mathcal{G}_N$ then

$$\lambda^{\mathbf{p}}_i[v] = \frac{\partial f}{\partial x_i}(p_1, p_2, \dots, p_n) \quad \text{for all } i \in N. \tag{5}$$

2.3. Partnerships

Definition 2.3 (*[16]*)**.** A coalition $S \subseteq N$ is a *partnership* in a game v iff $v(R \cup T) = v(R)$ for all $T \subset S$ and all $R \subseteq N \backslash S$.

A partnership is a coalition that possesses a trivial internal structure and behaves in some sense like an individual member, since all of its strict subcoalitions are powerless (if we set $R = \emptyset$ then $v(T) = 0$ for all $T \subset S$). The formation of a partnership suggests for the involved players a way to obtain strategic advantages. The partnership condition demands more than an agreement to coordinate strategies, since it also forbids any deal or bargain between strict subsets and external groups of players and restricts, then, the set of available worths. The case where only one coalition S (with $|S| \geq 2$) turns into a partnership will be the only considered throughout this paper. In this case, the partnership formation is stated as follows.

Definition 2.4 (*[3]*)**.** Let v be a game in N and $\emptyset \neq S \subseteq N$. The *partnership game* v^S in N (where S clearly becomes a partnership) is defined by

$$v^S(T) = \begin{cases} v(T) & \text{if} \quad S \subseteq T, \\ v(T \backslash S) & \text{if} \quad S \not\subseteq T. \end{cases}$$

It is easy to see that any two players i, j of a partnership S in a game v are symmetric in v^S.

Example 2.5. Assume that three nearby towns wish to get some supply from a distribution center. By, say, geographic reasons, the contract for a single town amounts to 1000 units of money but a reduced cost of 1860 units is offered by the supplier to towns 1 and 2 for a joint contract and, similarly, a cost of 1900 units is offered to towns 1 and 3. Finally, the supplier tenders a contract involving all towns at a price of 2790 units.

The game v, defined in $N = \{1, 2, 3\}$, that assigns to each coalition the cost savings derived from a joint contract of its members with the supplier is given by $v(\{i\}) = 0$ for all $i \in N$ and

$$v(\{1, 2\}) = 140, \qquad v(\{1, 3\}) = 100, \qquad v(\{2, 3\}) = 0 \quad \text{and} \quad v(N) = 210.$$

If towns 2 and 3 agree to reject the possibility of a contract joining either to town 1 – it does not mean that they form a coalition, which would give no profit to them – then this implies that coalition $S = \{2, 3\}$ turns into a partnership, and a new game v^S arises. As $v^S = 210u_N$, the unanimity game on N, towns 2 and 3 are now better placed to bargain with town 1.

3. Partnership formation

The multinomial values will be used here to measure the internal and external effects of the partnership formation. Jointly with the $\mathbf{p}$-multinomial value of each player $\lambda_i^{\mathbf{p}}[v]$, it will be useful to consider the *(additive)* $\mathbf{p}$*-multinomial value* of a nonempty coalition $S \subseteq N$ in a game v, defined in a natural way by

$$\lambda_S^{\mathbf{p}}[v] = \sum_{i \in S} \lambda_i^{\mathbf{p}}[v].$$

3.1. Partnership game vs. original game

In particular, if $v = v^S$, the partnership game introduced in Definition 2.4, the *partnership* $\mathbf{p}$*-multinomial value of* S is $\lambda_S^{\mathbf{p}}[v^S]$.

For all $v \in \mathcal{G}_N$ and $S \subseteq N$ ($|S| \geq 2$), we will consider the insider and outsider increments

$$\Delta_i^S \lambda^{\mathbf{p}}[v] = \lambda_i^{\mathbf{p}}[v^S] - \lambda_i^{\mathbf{p}}[v] \quad \text{for each } i \in S,$$
$$\Delta_j^S \lambda^{\mathbf{p}}[v] = \lambda_j^{\mathbf{p}}[v] - \lambda_j^{\mathbf{p}}[v^S] \quad \text{for each } j \in N \backslash S$$

and the additive increment

$$\Delta_S \lambda^{\mathbf{p}}[v] = \lambda_S^{\mathbf{p}}[v^S] - \lambda_S^{\mathbf{p}}[v] = \sum_{i \in S} \Delta_i^S \lambda^{\mathbf{p}}[v].$$

As for the convenience to form partnership S, it seems important, in principle, that $\Delta_i^S \lambda^{\mathbf{p}}[v] > 0$ is satisfied for every $i \in S$. However, if at least $\Delta_S \lambda^{\mathbf{p}}[v] > 0$ holds, some kind of utility redistribution within S might be expected which satisfies all its players. Then we can consider that the partnership formed by coalition $S \subseteq N$ in v, which gives rise to v^S, is *positive* if $\Delta_S \lambda^{\mathbf{p}}[v] > 0$, *negative* if $\Delta_S \lambda^{\mathbf{p}}[v] < 0$, and *null* if $\Delta_S \lambda^{\mathbf{p}}[v] = 0$. The next statement provides expressions for the above increments. Note that, a priori, players $j \in N \backslash S$ are expected to be damaged by the partnership formation and hence their increment is defined in a way opposed to the definition for players $i \in S$. To ease the notation, if $i \in T$ we will use T_i for $T \backslash \{i\}$.

Table 1
Shapley, Banzhaf and multinomial values in v and v^S, $S = \{1, 2\}$.

i	$\varphi_i[v]$ / $\varphi_i[v^S]$	$\beta_i[v]$ / $\beta_i[v^S]$	$\lambda_i^{\mathbf{p}}[v]$ / $\lambda_i^{\mathbf{p}}[v^S]$
1	23.50	12.75	$1+p_4+(1-p_4)[p_3+(1-p_3)p_2]+87p_2p_3p_4$
	24.17	13.25	$p_2[3+(1+p_3)(1+p_4)+85p_3p_4]$
2	24.50	13.75	$2+p_4+(1-p_4)[p_3+(1-p_3)p_1]+87p_1p_3p_4$
	24.17	13.25	$p_1[3+(1+p_3)(1+p_4)+85p_3p_4]$
3	25.50	14.75	$3+p_4+(1-p_4)[p_2+(1-p_2)p_1]+87p_1p_2p_4$
	25.33	14.50	$3+p_1p_2(1+86p_4)+p_4$
4	26.50	15.75	$4+p_3+(1-p_3)[p_2+(1-p_2)p_1]+87p_1p_2p_3$
	26.33	15.50	$4+p_1p_2(1+86p_3)+p_3$

Proposition 3.1. *Let v be a game in N and $\mathbf{p} \in [0, 1]^n$ be a profile on N. Then, for any $S \subseteq N$ with $|S| \geq 2$, and being $p_S^i = \prod_{j\in S} p_j \prod_{\substack{k\in N\setminus S\\ k\neq i}} (1-p_k)$ for all $i \in N$ and $S \subseteq N\setminus\{i\}$*

(a) $\Delta_i^S\lambda^{\mathbf{p}}[v] = \sum_{T\supseteq S} p_{T_i}^i[v(T_i) - v(T\setminus S)] - \sum_{\substack{T\ni i\\ S\not\subseteq T}} p_{T_i}^i[v(T) - v(T_i)]$ *for all $i \in S$.*

(b) $\Delta_S\lambda^{\mathbf{p}}[v] = \sum_{i\in S}\left\{\sum_{T\supseteq S} p_{T_i}^i[v(T_i) - v(T\setminus S)] - \sum_{\substack{T\ni i\\ S\not\subseteq T}} p_{T_i}^i[v(T) - v(T_i)]\right\}.$

(c) $\Delta_j^S\lambda^{\mathbf{p}}[v] = \sum_{\substack{T\ni j\\ S\not\subseteq T}} p_{T_j}^j\Big[v(T) - v(T_j) - v(T\setminus S) + v(T_j\setminus S)\Big]$ *for all $j \in N\setminus S$.*

Corollary 3.2. *Let v be a monotonic game in N and $\{i, j\} = S \subseteq N$. Then:*

(a) $\Delta_S\lambda^{\mathbf{p}}[v] \geq 2(p_i + p_j - 1)v(N)$ *if* $p_i + p_j < 1$ *and*
$\Delta_S\lambda^{\mathbf{p}}[v] \leq 2(p_i + p_j - 1)v(N)$ *if* $p_i + p_j > 1$.

(b) $\Delta_S\lambda^{\mathbf{p}}[v] \leq 0$ *if* $0 \leq p_i + p_j \leq 1$.

(c) $\Delta_S\lambda^{\mathbf{p}}[v] \geq 0$ *if* $p_i + p_j \geq 1$.

Remark 3.3. Corollary 3.2 states that, in monotonic games, the two-player partnership formation can be positive if, in particular, there is at least one player with a rather great tendency to form coalitions ($p_i \geq 1/2$ or $p_j \geq 1/2$). It also gives bounds for the increase or decrease of the additive value of the coalition, once it becomes a partnership, in terms of $p_i + p_j$ and $v(N)$.

Example 3.4. Let v be the monotonic game in $N = \{1, 2, 3, 4\}$ defined by

$$v(\{i\}) = i, \qquad v(\{i, j\}) = 1 + i + j, \qquad v(\{i, j, k\}) = 2 + i + j + k \quad \text{and} \quad v(N) = 100.$$

Assume that coalition $S = \{1, 2\}$ turns into a partnership and hence a new game v^S arises. We study whether the constitution of the partnership is interesting for its members by using the Shapley and Banzhaf values and comparing these results with the multinomial values (see Table 1).

(a) According to the Shapley and Banzhaf values, player 1 gets profit from the partnership formation, whereas player 2 does not. On the other hand, players 3 and 4 are damaged by the partnership formation.
(b) According to the multinomial values, we obtain:

$$\lambda_1^{\mathbf{p}}[v^S] - \lambda_1^{\mathbf{p}}[v] = (2p_2 - 1)[1 + p_3 + p_4(1 - p_3)] + p_2$$
$$\lambda_2^{\mathbf{p}}[v^S] - \lambda_2^{\mathbf{p}}[v] = p_1[1 + p_3 + p_4(1 - p_3)] - (1 - p_1)[2 + p_3 + p_4(1 - p_3)]$$
$$\lambda_3^{\mathbf{p}}[v] - \lambda_3^{\mathbf{p}}[v^S] = (1 - p_4)(p_1 + p_2 - 2p_1p_2)$$
$$\lambda_4^{\mathbf{p}}[v] - \lambda_4^{\mathbf{p}}[v^S] = (1 - p_3)(p_1 + p_2 - 2p_1p_2).$$

This shows that for all $\mathbf{p} \in (0, 1)^4$, players 3 and 4 are damaged by the partnership formation. If $p_2 > 1/2$ player 1 gets profit from the partnership formation whereas the result over player 2 depends on the profile $\mathbf{p}$.

Example 3.5. Let v be the monotonic symmetric game in $N = \{1, 2, 3, 4\}$ defined by $v(S) = s(s - 1)/2$ for all $S \subseteq N$ and let $\mathbf{p} = (2/3, 3/5, 1/3, 1/4)$ be the tendency profile.

We will assume that three different coalitions turn into a partnership and hence a new game v^S arises. For each case we will study whether the constitution of the partnership is interesting for its members.

First we consider that players 1 and 2 agree to form a partnership, that is, $S = \{1, 2\}$. In this case $\Delta_1\lambda^{\mathbf{p}}[v] = 7/60$ and $\Delta_2\lambda^{\mathbf{p}}[v] = 7/36$. In accordance with Corollary 3.2, $\Delta_S\lambda^{\mathbf{p}}[v] = 14/45 \geq 0$ because $p_1 + p_2 \geq 1$.

However, if coalition $S = \{3, 4\}$ turns into a partnership, we obtain $\Delta_3\lambda^{\mathbf{p}}[v] = -19/30$ and $\Delta_4\lambda^{\mathbf{p}}[v] = -19/45$. In accordance with Corollary 3.2, we have $\Delta_S\lambda^{\mathbf{p}}[v] = -19/18 \leq 0$ because $p_3 + p_4 \leq 1$.

Finally, for $S = \{1, 3\}$, the partnership formation is globally null in the neutral case, in fact player 3 gains some amount $(17/60)$, which player 1 loses.

3.2. Partnership vs. alliance

A *coalition structure* or *system of unions* in N is a partition $B = \{B_1, B_2, \ldots, B_m\}$ of this set. A *game with a coalition structure* is a pair $[v; B]$ where $v \in \mathcal{G}_N$ and B is a coalition structure in N. The *quotient game* v_B is the game played by the unions, or, rather, by the set $M = \{1, 2, \ldots, m\}$ of their representatives, as follows:

$$v_B(R) = v\left(\bigcup_{r\in R} B_r\right) \quad \text{for all } R \subseteq M.$$

Here we shall only deal with coalition structures where just one nonempty coalition $S \subseteq N$ (with $|S| \geq 2$) forms. In this case, it will be useful to denote as $[S]$ such a coalition structure. Given $\emptyset \neq S \subseteq N$, we will denote by $\lambda_0^{\overline{\mathbf{p}}}[v_{[S]}]$ the $\overline{\mathbf{p}}$-multinomial value of the representative of S in the quotient game $v_{[S]}$ played in $M = \{0\} \cup N\backslash S$ (a non-standard notation that seems, however, more suitable in this special case, where 0 represents S and each $j \in N\backslash S$ represents himself). In this case $\overline{\mathbf{p}}$ is the tendency profile induced by $\mathbf{p}$ in M with $\overline{p}_i = p_i$ for all $i \in N\backslash S$ and, among the infinite many possibilities to define $\overline{p}_0$ in terms of $\mathbf{p}$, let us suggest a few ones:

(a) $\overline{p}_0 = \min_{i\in S}\{p_i\}$
(b) $\overline{p}_0 = p_i$ for some $i \in S$ arbitrarily chosen
(c) $\overline{p}_0 = \frac{1}{s}\sum_{i\in S} p_i$, where $s = |S|$
(d) $\overline{p}_0 = \max_{i\in S}\{p_i\}$.

We will not discuss here which is the best option (if any). The theory developed in this paper will be of application provided that $\overline{\mathbf{p}}$ is a profile induced by $\mathbf{p}$, no matter by which option.

We will compare $\lambda_0^{\overline{\mathbf{p}}}[v_{[S]}]$ with $\lambda_S^{\mathbf{p}}[v^S]$, by introducing

$$\Delta_S\lambda^{\mathbf{p}}[v^S, v_{[S]}] = \lambda_S^{\mathbf{p}}[v^S] - \lambda_0^{\overline{\mathbf{p}}}[v_{[S]}].$$

Given a set of players S, the following statement will be useful to determine the cases where the formation of a partnership is more interesting than the effective constitution of a coalition in the very sense of the term (that is why we use the term "alliance" to distinguish it from "coalition", commonly understood as a mere synonymous of "subset of N").

Proposition 3.6. *Let v be a game in N. Then, for all $S \subseteq N$ with $|S| \geq 2$,*

$$\Delta_S\lambda^{\mathbf{p}}[v^S, v_{[S]}] = \begin{cases} \left[\displaystyle\sum_{i\in S}\prod_{r\in S_i} p_r - 1\right]\displaystyle\sum_{T\supseteq S}\prod_{k\in T\backslash S} p_k \prod_{l\in N\backslash T}(1-p_l)[v(T) - v(T\backslash S)] & \text{if } 0 < p_i \leq 1, \\ -v(S) & \text{if } p_i = 0, \end{cases}$$

for all $i \in N$.

Corollary 3.7. *Let v be a monotonic game in N and $S \subseteq N$ be such that $|S| \geq 2$. Then:*

(a) *If $p_i = 0$ for all $i \in N$ then $\Delta_S\lambda^{\mathbf{0}}[v^S, v_{[S]}] < 0$ unless $v(S) = 0$, in which case the difference vanishes.*
(b) *If $p_i = 1$ for all $i \in N$ then $\Delta_S\lambda^{\mathbf{1}}[v^S, v_{[S]}] > 0$ unless $v(N\backslash S) = v(N)$, in which case the difference vanishes.*
(c) *If $0 < p_i < 1$ for all $i \in N$ then $\Delta_S\lambda^{\mathbf{p}}[v^S, v_{[S]}] > 0, = 0$ or < 0 according to either $\sum_{i\in S}\prod_{r\in S_i} p_r > 1$, $\sum_{i\in S}\prod_{r\in S_i} p_r = 1$ or $\sum_{i\in S}\prod_{r\in S_i} p_r < 1$.*
(d) *If $0 < p_i < 1$, $q_M = \max_{r\in S} p_r$ and $q_m = \min_{r\in S} p_r$ then*

$$\Delta_S\lambda^{\mathbf{p}}[v^S, v_{[S]}] < (sq_M^{s-1} - 1)v(N) \quad \text{if } \sum_{i\in S}\prod_{r\in S_i} p_r > 1$$

$$\Delta_S\lambda^{\mathbf{p}}[v^S, v_{[S]}] > (sq_m^{s-1} - 1)v(N) \quad \text{if } \sum_{i\in S}\prod_{r\in S_i} p_r < 1.$$

Remark 3.8. Corollary 3.7 establishes that, in monotonic games, the partnership formation is more advantageous than the alliance formation whenever for all $i \in S$, p_i is high enough. For example, if $S = \{1, 2\}$ then the difference $\Delta_S\lambda^{\mathbf{p}}[v^S, v_{[S]}]$ is positive iff $p_1 + p_2 > 1$, in particular, if $p_1, p_2 > 1/2$; and it vanishes iff $p_1 + p_2 = 1$. If $S = \{1, 2, 3\}$ then the difference is positive if $p_i > 1/\sqrt{3}$ for all $i \in S$, and so on. In general, increasing the number of involved players requires that the tendency to forming coalitions of all of them increases in order to get more profit from forming a partnership than an alliance. In 3.7(d), bounds for the difference are provided in terms of s, q_M, q_m and $v(N)$.

4. Partnerships in simple games

Simple games form an interesting class of cooperative games, not only as a test bed for many cooperative concepts but also for the variety of their interpretations, in political science and other fields. In particular, they have been often applied to describe and analyze collective decision-making mechanisms, and the notion of voting power has been closely attached to them. We will specialize here on this class of games.

A game v in N is *simple* if it is monotonic, $v(T) = 0$ or 1 for every $T \subseteq N$, and $v(N) = 1$. For example, the unanimity game u_S, for any nonempty $S \subseteq N$, is simple. A coalition $T \subseteq N$ is *winning in* v if $v(T) = 1$ (otherwise it is called *losing*), and $W = W(v)$ denotes the set of winning coalitions in v. Due to monotonicity, the set W^m of all *minimal* winning coalitions determines W and hence the game. A simple game v is a *weighted majority game* if there are nonnegative *weights* $w_1, w_2, \ldots, w_n$ allocated to the players and a positive *quota* q such that

$$v(S) = 1 \quad \text{iff} \sum_{i \in S} w_i \geq q.$$

We then write $v = [q; w_1, w_2, \ldots, w_n]$. (For additional details on simple games we refer the reader to e.g. [8,22,7] or [5].)

We denote as $\mathcal{SG}_N$ the set of all simple games in N. Endowed with the standard composition laws, given by $(v \vee v')(S) = \max\{v(S), v'(S)\}$ and $(v \wedge v')(S) = \min\{v(S), v'(S)\}$ for all $S \subseteq N$, it becomes a distributive lattice. This lattice remains invariant under the partnership formation, i.e. if $v \in \mathcal{SG}_N$ and $\emptyset \neq S \subseteq N$ then $v^S \in \mathcal{SG}_N$.

A *power index* on $\mathcal{SG}_N$ is a map $f : \mathcal{SG}_N \to \mathbb{R}^N$. Following Remark 2.3(c) in [4], an alternative interpretation in simple games of the profile that defines a multinomial value is as follows. There is a status quo Q and a proposal P to modify it. The action of the parliamentary members reduces voting for or against P. Then *each p_i can be viewed as the probability that player i votes for P*. Since the result of a vote is essentially equivalent to forming a coalition (the coalition of players that vote for P), this interpretation of p_i agrees with that of "tendency to form a coalition" that we are using in this paper.

In this section we determine the maximum and minimum values of the increments $\Delta_i^S \lambda^{\mathbf{p}}[v]$ for $i \in S$, $\Delta_j^S \lambda^{\mathbf{p}}[v]$ for $j \in N \backslash S$, and $\Delta_S \lambda^{\mathbf{p}}[v^S, v_{[S]}]$, when v ranges $\mathcal{SG}_N$, as a generalization of the results obtained in [10].

Theorem 4.1. *Let $\mathcal{SG}_N$ be the set of all simple games in N, $S \subseteq N$ with $|S| \geq 2$ and $i \in S$. Then, for any $\mathbf{p} \in [0, 1]^n$ and each such S:*

(a) *The maximum value of $\Delta_i^S \lambda^{\mathbf{p}}[v]$ is $\prod_{k \in S_i} p_k$, and it is attained for $v = u_{S \backslash \{i\}}$.*
(b) *The minimum value of $\Delta_i^S \lambda^{\mathbf{p}}[v]$ is $\prod_{k \in S_i} p_k - 1$, and it is attained for $v = u_{\{i\}}$.*

Remark 4.2. (a) Theorem 4.1(a) states that, for any $\mathbf{p} \in [0, 1]^n$, the best profit is obtained by a null player in an *oligarchy* (unanimity game) when this player is accepted to form a partnership with all the oligarchic members. Note that, therefore, $v = u_{S \backslash \{i\}}$ and $v^S = u_S$, so that $\lambda_i^{\mathbf{p}}[v] = 0$, $\lambda_i^{\mathbf{p}}[v^S] = \prod_{k \in S_i} p_k$ and hence $\Delta_i^S \lambda^{\mathbf{p}}[v] = \prod_{k \in S_i} p_k$ (an alternative way to obtain the maximum value). Of course, the "best of the best" arises in case $s = 2$, where the oligarchy reduces to a *dictatorship* (unanimity game $u_{\{j\}}$) and a null player $i \neq j$ joins j and forms partnership $S = \{i, j\}$, with a profit p_j for him. The "worst of the best" occurs when $S = N$, that is, when the only null player in an oligarchy is admitted to form a partnership with all the other players.

(b) Analogously, 4.1(b) says that the worst result is precisely obtained by a dictator who forms a partnership with one or more null players. In this case, $v = u_{\{i\}}$ and $v^S = u_S$, and thus $\lambda_i^{\mathbf{p}}[v] = 1$, $\lambda_i^{\mathbf{p}}[v^S] = \prod_{k \in S_i} p_k$ and $\Delta_i^S \lambda^{\mathbf{p}}[v] = \prod_{k \in S_i} p_k - 1$ (alternative calculus). The "worst of the worst" is found for $S = N$, that is, when all null players join the dictator, who loses $1 - \prod_{k \in N_i} p_k$, and the "best of the worst" whenever $s = 2$, only one null player joins him, and the dictator's loss is $1 - p_j$.

Theorem 4.3. *Let $\mathcal{SG}_N$ be the set of all simple games in N, $S \subseteq N$ with $s = |S|$ such that $2 \leq s < n$, and $j \in N \backslash S$. Then, for any $\mathbf{p} \in [0, 1]^n$ and each such S:*

(a) *The maximum value of $\Delta_j^S \lambda^{\mathbf{p}}[v]$ is $1 - \prod_{k \in S} p_k - \prod_{l \in S}(1 - p_l)$, and it is attained for game v defined by $W = \{T \subseteq N : j \in T,\ S \cap T \neq \emptyset\}$.*
(b) *The minimum value of $\Delta_j^S \lambda^{\mathbf{p}}[v]$ is $\prod_{l \in S}(1 - p_l) + \prod_{k \in S} p_k - 1$, and it is attained for game v defined by $W = \{T \subseteq N : (S \cup \{j\}) \cap T \neq \emptyset\}$.*

Remark 4.4. Theorem 4.3 is a bit more complicated than Theorem 4.1 but also interesting, especially due to the games concerned with the extreme values. Let us define, when $\emptyset \neq R \subseteq N$, the *intersection game* σ_R by $\sigma_R(T) = 1$ if $T \cap R \neq \emptyset$ or else $\sigma_R(T) = 0$. Each σ_R is a simple game. In fact, $\sigma_{\{i\}} = u_{\{i\}}$, a dictatorship, whereas for $|R| \geq 2$ we have $\sigma_R = \bigvee_{i \in R} \sigma_{\{i\}} = \bigvee_{i \in R} u_{\{i\}}$, a *purely individualistic game* because each $i \in R$ is a *winner* ($\{i\}$ is winning but i is not a dictator) and all $k \notin R$ are null players. Obviously, $\sigma_R = [q; w_1, w_2, \ldots, w_n]$ where $w_i = 1$ if $i \in R$, $w_k = 0$ if $k \notin R$, and $q = 1$.

(a) The game v where $\Delta_j^S \lambda^{\mathbf{p}}[v]$ reaches its maximum value can also be described by $W^m = \{\{i, j\} : i \in S\}$, so that j is the only *veto player* (i.e. belongs to every winning coalition but is not a dictator), S is a *blocking* coalition (i.e. losing but powerful enough to prevent $N\backslash S$ to win) and the remaining players $k \notin S \cup \{j\}$ are null. Of course, $v = [q; w_1, w_2, \dots, w_n]$ where $w_i = 1$ for each $i \in S$, $w_j = s$, $w_k = 0$ for any $k \notin S \cup \{j\}$, and $q = s + 1$. Besides,

$$v = u_{\{j\}} \wedge \sigma_S = u_{\{j\}} \wedge \bigvee_{i\in S} u_{\{i\}} = \bigvee_{i\in S}(u_{\{j\}} \wedge u_{\{i\}}) = \bigvee_{i\in S} \sigma_{\{i,j\}}.$$

As $v^S = u_{S\cup\{j\}}$, it is not difficult to find that $\lambda_j^{\mathbf{p}}[v] = 1 - \prod_{l\in S}(1 - p_l)$ and $\lambda_j^{\mathbf{p}}[v^S] = \prod_{k\in S} p_k$ whence $\Delta_j^S\lambda^{\mathbf{p}}[v] = 1 - \prod_{l\in S}(1 - p_l) - \prod_{k\in S} p_k$ (alternative but not easier computation of the maximum).

(b) The game v where $\Delta_j^S\lambda^{\mathbf{p}}[v]$ reaches its minimum value can also be described by $W^m = \{\{k\} : k \in S \cup \{j\}\}$. It is an intersection game, since $v = \sigma_{S\cup\{j\}}$, and the first paragraph of this remark applies to it. In particular, $v = \bigvee_{k\in S\cup\{j\}} u_{\{k\}}$. As $v^S = u_S \vee u_{\{j\}}$, here we have

$$\lambda_j^{\mathbf{p}}[v] = \prod_{l\in S}(1 - p_l),$$
$$\lambda_j^{\mathbf{p}}[v^S] = 1 - \prod_{k\in S} p_k,$$
$$\Delta_j^S\lambda^{\mathbf{p}}[v] = \prod_{l\in S}(1 - p_l) + \prod_{k\in S} p_k - 1$$

(again, an alternative but not easier computation of the minimum).

Theorem 4.5. *Let $\mathcal{SG}_N$ be the set of all simple games in N and $S \subseteq N$ with $|S| \geq 2$. Then, for each such S:*

(a) *If $\sum_{i\in S}\prod_{r\in S_i} p_r > 1$ then the maximum value of $\Delta_S\lambda^{\mathbf{p}}[v^S, v_{[S]}]$ is $\sum_{i\in S}\prod_{r\in S_i} p_r - 1 > 0$, and it is attained for $v = u_S$; the minimum value is 0 and it is attained in any game v where all members of S are null players.*
(b) *If $\sum_{i\in S}\prod_{r\in S_i} p_r < 1$ then the minimum value of $\Delta_S\lambda^{\mathbf{p}}[v^S, v_{[S]}]$ is $\sum_{i\in S}\prod_{r\in S_i} p_r - 1 < 0$, and it is attained for $v = u_S$; the maximum value is 0 and it is attained in any game v where all members of S are null players.*

Remark 4.6. (a) The extreme values of $\Delta_S\lambda^{\mathbf{p}}[v^S, v_{[S]}]$ are quite obvious in three cases that have been omitted in Theorem 4.5. Indeed, from Corollary 3.7 it follows that:

- For $p_i = 0$ for all $i \in N$, the maximum is 0 and it is attained in all games where $S \notin W$, whereas the minimum is -1 and it is attained in all games where $S \in W$.
- Instead, for $p_i = 1$ for all $i \in N$, the maximum is $s - 1$ and it is attained in all games where $N\backslash S \notin W$, whereas the minimum is 0 and it is attained in all games where $N\backslash S \in W$.
- Finally, if $\sum_{i\in S}\prod_{r\in S_i} p_r = 1$ then $\Delta_S\lambda^{\mathbf{p}}[v^S, v_{[S]}] = 0$ for all $S \subseteq N$ with $s = |S| \geq 2$ and all v.

(b) For the remaining values of p_i, Theorem 4.5 points out that the greatest difference (in both senses) between the result of forming a partnership and that of forming an alliance is found in the oligarchy of the involved coalition. Note that if $v = u_S$ then $v^S = u_S$ and $v_{[S]} = u_{\{0\}}$, so that $\lambda_S^{\mathbf{p}}[v^S] = \sum_{i\in S}\prod_{r\in S_i} p_r$, $\lambda_0^{\overline{\mathbf{p}}}[v_{[S]}] = 1$ and therefore $\Delta_S\lambda^{\mathbf{p}}[v^S, v_{[S]}] = \sum_{i\in S}\prod_{r\in S_i} p_r - 1$ (an alternative way to derive the extreme no-null value in both cases).

5. The Catalonia Parliament, Legislature 2003–2007

In this section, we shall apply multinomial values to the analysis of a political structure.

5.1. The political framework

We consider here the Catalonia Parliament in Legislature 2003–2006, prematurely finished.[3]

Let us briefly describe the political positions of the agents in this game:

1. CiU (*Convergència i Unió*), Catalan nationalist middle-of-the-road coalition of two federated parties.
2. PSC (*Partit dels Socialistes de Catalunya*), moderate left-wing socialist party, federated to the *Partido Socialista Obrero Español*.
3. ERC (*Esquerra Republicana de Catalunya*), radical Catalan nationalist left-wing party.
4. PPC (*Partit Popular de Catalunya*), conservative party, Catalan delegation of the *Partido Popular*.
5. ICV (*Iniciativa per Catalunya-Verds*), coalition of Catalan eurocommunist parties, federated to *Izquierda Unida*, and ecologist groups (*"Verds"*).

In Catalonia, politics is based on two main axes: the classical left-to-right axis and an orthogonal axis going from Spanish centralism to Catalanism (Catalan nationalism) (see Fig. 1). In 2003, Esquerra Republicana de Catalunya (ERC), a radical

[3] The analysis remains valid for Legislature 2006–2010: in spite of the modification of the seat distribution issued from the elections held in November 1, 2006, the strategic possibilities are exactly the same.

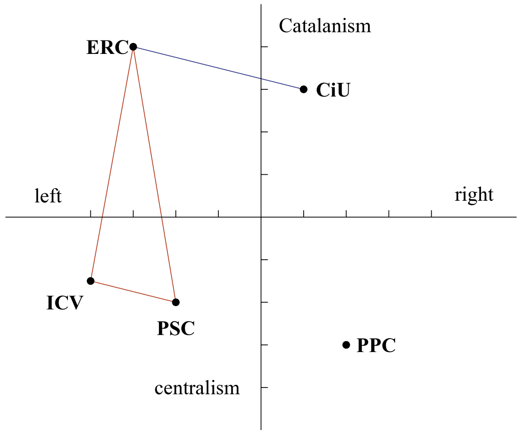

Fig. 1. Position of parties in a two-dimensional ideological space.

nationalist and left-wing party, was faced with the dilemma of choosing between either a Catalanist majority coalition with Convergència i Unió (CiU) or a left-wing majority coalition with the Partitp dels Socialistes de Catalunya (PSC) and Iniciativa per Catalunya-Verds (ICV). Thus, the role of ERC in this scenario would be crucial. Nevertheless, a previous partnership formation concerning PSC and ICV would have been only natural by ideological reasons.

5.2. Initial evaluation

Under the standard absolute majority rule, and assuming voting discipline within parties, the structure of this parliamentary body can be represented by the weighted majority game

$$v \equiv [68; 46, 42, 23, 15, 9].$$

Therefore, the strategic situation is given by

$$W^m(v) = \{\{1, 2\}, \{1, 3\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}\},$$

so that players 2 and 3 on one hand, and 4 and 5 on the other, are symmetric in v.

The introduction of tendency profiles will break these symmetries. Nevertheless, a structural symmetry still exists, between $\lambda_2^{\mathbf{p}}[v]$ and $\lambda_3^{\mathbf{p}}[v]$ on one hand and $\lambda_4^{\mathbf{p}}[v]$ and $\lambda_5^{\mathbf{p}}[v]$ on the other, since e.g. $\lambda_3^{\mathbf{p}}[v]$ follows from $\lambda_2^{\mathbf{p}}[v]$ by replacing p_3 with p_2. This is due to the symmetrical positions of each pair of players in the game, which translates to the multilinear extension.

The calculation of $\lambda^{\mathbf{p}}[v]$ derives for each player:

$$\lambda_1^{\mathbf{p}}[v] = p_2 + p_3 - p_2p_3(1 + p_4 + p_5 - p_4p_5) + p_4p_5(1 - p_2 - p_3 + p_2p_3),$$
$$\lambda_2^{\mathbf{p}}[v] = p_1 + p_3(p_4 + p_5) - p_1p_3(1 + p_4 + p_5 - p_4p_5) - p_4p_5(p_1 + p_3 - p_1p_3),$$
$$\lambda_3^{\mathbf{p}}[v] = p_1 + p_2(p_4 + p_5) - p_1p_2(1 + p_4 + p_5 - p_4p_5) - p_4p_5(p_1 + p_2 - p_1p_2),$$
$$\lambda_4^{\mathbf{p}}[v] = p_2p_3(1 - p_1 - p_5 + p_1p_5) + p_1p_5(1 - p_2 - p_3 + p_2p_3),$$
$$\lambda_5^{\mathbf{p}}[v] = p_2p_3(1 - p_1 - p_4 + p_1p_4) + p_1p_4(1 - p_2 - p_3 + p_2p_3).$$

5.3. To be or not to be (in a previous partnership)

We wish to investigate here the effects of PSC and ICV turning into a partnership by applying the multinomial values $\lambda^{\mathbf{p}}$ to games v and v^S, where $S = \{2, 5\}$. Following [3], we get v^S from v by inserting, in each minimal winning coalition $T \in W^m$ containing some member of S, the remaining members of S (if any), and removing, finally, the members of the new family that are not minimal. This gives rise to

$$(W^S)^m = \{\{1, 2, 5\}, \{1, 3\}, \{2, 3, 5\}\}.$$

All multinomial values applied in this section have been computed using derivations of the multilinear extension technique (5) that can be found in [15]. The results of applying multinomial values to the two games mentioned before are given in Table 2.

Now, we look at Table 2, where the effects of $S = \{2, 5\}$ turning into a partnership are described. By comparing $\lambda_i^{\mathbf{p}}[v]$ and $\lambda_i^{\mathbf{p}}[v^S]$ we obtain:

- PSC and ICV become symmetric players in the partnership game v^S. The insider increments are:

$$\Delta_2^S\lambda^{\mathbf{p}}[v] = -p_1(1 - p_3)(1 - p_5) - p_4p_3(1 - p_1) - p_4p_5(2p_1p_3 - p_1 - p_3),$$
$$\Delta_5^S\lambda^{\mathbf{p}}[v] = p_1p_2(1 - p_3) + p_4p_2p_3 + p_4p_1(p_2 + p_3 - 2p_2p_3 - 1).$$

Table 2
Multinomial values in v and v^S, $S = \{2, 5\}$.

1. CiU	$\lambda_1^{\mathbf{p}}[v] = p_2 + p_3 - p_2p_3(1 + p_4 + p_5 - p_4p_5) + p_4p_5(1 - p_2 - p_3 + p_2p_3)$
	$\lambda_1^{\mathbf{p}}[v^S] = p_3 + p_2p_5(1 - 2p_3)$
2. PSC	$\lambda_2^{\mathbf{p}}[v] = p_1 + p_3(p_4 + p_5) - p_1p_3(1 + p_4 + p_5 - p_4p_5) - p_4p_5(p_1 + p_3 - p_1p_3)$
	$\lambda_2^{\mathbf{p}}[v^S] = p_5(p_1 + p_3 - 2p_1p_3)$
3. ERC	$\lambda_3^{\mathbf{p}}[v] = p_1 + p_2(p_4 + p_5) - p_1p_2(1 + p_4 + p_5 - p_4p_5) - p_4p_5(p_1 + p_2 - p_1p_2)$
	$\lambda_3^{\mathbf{p}}[v^S] = p_1 + p_2p_5(1 - 2p_1)$
4. PPC	$\lambda_4^{\mathbf{p}}[v] = p_2p_3(1 - p_1 - p_5 + p_1p_5) + p_1p_5(1 - p_2 - p_3 + p_2p_3)$
	$\lambda_4^{\mathbf{p}}[v^S] = 0$
5. ICV	$\lambda_5^{\mathbf{p}}[v] = p_2p_3(1 - p_1 - p_4 + p_1p_4) + p_1p_4(1 - p_2 - p_3 + p_2p_3)$
	$\lambda_5^{\mathbf{p}}[v^S] = p_2(p_1 + p_3 - 2p_1p_3)$

It is difficult, in principle, to say anything of interest about the expressions for $\Delta_i^S\lambda^{\mathbf{p}}[v]$. Fortunately, a simplification can be reasonably achieved. Indeed, the almost isolated political position of PPC (party 4) with regard to the remaining parties strongly suggests taking $p_4 \approx 0$, which will make things easier:

$$\Delta_2^S\lambda^{\mathbf{p}}[v] \approx -p_1(1 - p_3)(1 - p_5),$$
$$\Delta_5^S\lambda^{\mathbf{p}}[v] \approx p_1p_2(1 - p_3).$$

In this case the insider increments become negative and positive respectively for all $\mathbf{p} \in (0, 1)^5$ with $p_4 \approx 0$. That is, ICV gets profit from the partnership formation whereas PSC is always damaged by it. In accordance with Corollary 3.2, the additive increment $\Delta_S\lambda^{\mathbf{p}}[v]$ is nonnegative if and only if $p_2 + p_5 \geq 1$.

- CiU and ERC become symmetric players once the partnership is formed (game v^S). The outsider increments are:

$$\Delta_1^S\lambda^{\mathbf{p}}[v] = p_2(1 - p_3)(1 - p_5) - p_4p_2p_3 + p_4p_5(1 - p_2 - p_3 + 2p_2p_3),$$
$$\Delta_3^S\lambda^{\mathbf{p}}[v] = -p_1p_2(1 - p_5) + p_4p_2(1 - p_1) + p_4p_5(2p_1p_2 - p_1 - p_2).$$

The above assumption $p_4 \approx 0$ will reduce to:

$$\Delta_1^S\lambda^{\mathbf{p}}[v] \approx p_2(1 - p_3)(1 - p_5),$$
$$\Delta_3^S\lambda^{\mathbf{p}}[v] \approx -p_1p_2(1 - p_5).$$

This shows that for all $\mathbf{p} \in (0, 1)^5$ with $p_4 \approx 0$, CiU is damaged by the partnership formation whereas ERC gets profit from it.

- PPC is fully damaged by the partnership formation since it becomes a null player in v^S.

5.4. *Partnership vs. alliance*

Assume that players 2 and 5 form an alliance $S = \{2, 5\}$. We study if the formation of the previous partnership is more interesting than the effective constitution of this alliance.

The quotient game $v_{[S]}$ played in $M = \{0\} \cup N\backslash S$, where 0 represents S and each $j \in N\backslash S$ represents himself, is

$$v_{[S]} = [68; 51, 46, 23, 15, 9].$$

The set of winning coalitions is

$$W^m\left(v_{[S]}\right) = \{\{0, 1\}, \{0, 3\}, \{1, 3\}\}.$$

We denote $\lambda_0^{\overline{\mathbf{p}}}[v_{[S]}]$ the $\overline{\mathbf{p}}$-multinomial value of S in the quotient game $v_{[S]}$ and $\lambda_S^{\mathbf{p}}[v^S]$ the partnership $\mathbf{p}$-multinomial value of S. Notice that profile $\overline{\mathbf{p}}$ does not appear in $\lambda_0^{\overline{\mathbf{p}}}[v_{[S]}]$ because the payoffs to the members of a union depend only on the tendencies of the remaining unions, that, in this case, reduce to a singleton $\{j\}$ and $\overline{p}_j = p_j$, for $j \in N\backslash S$. We obtain

$$\lambda_0^{\overline{\mathbf{p}}}[v_{[S]}] = p_1 + p_3 - 2p_1p_3$$
$$\lambda_S^{\mathbf{p}}[v^S] = (p_1 + p_3 - 2p_1p_3)(p_2 + p_5)$$
$$\lambda_S^{\mathbf{p}}[v^S] - \lambda_0^{\overline{\mathbf{p}}}[v_{[S]}] = (p_1 + p_3 - 2p_1p_3)(p_2 + p_5 - 1).$$

According to Corollary 3.7 the partnership formation is more advantageous than the alliance formation if $p_2 + p_5 > 1$.

6. Conclusions

We have used multinomial values to measure the effects of the partnership formation in a game as a generalization of the study done by using binomial semivalues in [10], whose monoparametric condition implies a limited capability of analysis of such situations. These values form an n-parametric family depending on $\mathbf{p} \in [0, 1]^n$ on which they offer a new view. Profile

p has been given a probabilistic interpretation and supplies information not included in the characteristic function of the game. By using multinomial values we get a much more precise approach to the influence of players' different personalities on the partnership formation problem. The reader is referred to the example of Catalonia Parliament (Section 5) for a detailed analysis. This influence, and hence the increase of strategic options for the different parties, cannot be discovered by merely using the traditional and more rigid values: it arises from the possibility to attach a parameter to each player, which is just the characteristic of the multinomial values. The fact that they are based on tendency profiles provide new tools to assume a wide variety of situations from players' personality when playing a given game.

The partnership formation involve any group of players in any game and it gives rise to a new game. We have compared the effects of the partnership formation, with regard to the original game, in the following situations (a) for the members of the partnership, the *insider increment*, (b) for the remaining players, the *outsider increment*, and (c) a comparison for all the partnership members, the *additive value*, in the partnership game and its value as a player in the quotient game that arises from the alliance formation is also useful to decide which strategy is better. Moreover, in the case of simple games we also determine the maximum and minimum values for the differences found in cases (a)–(c), and provide games where these extreme values are attained.

Future work might concern (a) the study of mixed situations between coalition structures and partnerships formations by extending the **p**-multinomial values to games with a coalition structure (the coalitional **p**-multinomial value [13]) in a similar way than the symmetric binomial coalitional semivalue [1,10,12] extends the binomial semivalue in these games (we are already working on it). They apply to games with a coalition structure by combining the Shapley value and the multinomial values. Here we first apply the $\overline{\mathbf{p}}$-multibinary probabilistic value $\lambda^{\overline{\mathbf{p}}}$ in the quotient game to get a payoff for each union; next, we use within each union the Shapley value, to share the payoff efficiently by applying it to a *reduced game* played in that union. (b) The study of two or more coalitions forming partnerships.

Acknowledgments

The authors wish to thank the anonymous reviewers for their interesting comments and helpful suggestions, most of which have been incorporated to the text.

Appendix A. Proofs of Section 3

Proof of Proposition 3.1. (a) Let $i \in S$. From Definitions 2.1 and 2.4,

$$\lambda_i^{\mathbf{p}}[v^S] = \sum_{T \ni i} p_{T_i}^i [v^S(T) - v^S(T_i)].$$

Therefore

$$\begin{aligned}\lambda_i^{\mathbf{p}}[v^S] &= \sum_{T \supseteq S} p_{T_i}^i [v^S(T) - v^S(T_i)] + \sum_{\substack{T \ni i \\ S \not\subseteq T}} p_{T_i}^i [v^S(T) - v^S(T_i)] \\ &= \sum_{T \supseteq S} p_{T_i}^i [v(T) - v(T_i \backslash S)] + \sum_{\substack{T \ni i \\ S \not\subseteq T}} p_{T_i} [v(T \backslash S) - v(T_i \backslash S)] = \sum_{T \supseteq S} p_{T_i}^i [v(T) - v(T \backslash S)] \end{aligned} \tag{6}$$

because $i \in S$ and $T_i \backslash S = T \backslash S$. Then

$$\begin{aligned}\lambda_i^{\mathbf{p}}[v^S] - \lambda_i^{\mathbf{p}}[v] &= \sum_{T \supseteq S} p_{T_i}^i [v(T) - v(T \backslash S)] - \sum_{T \ni i} p_{T_i}^i [v(T) - v(T_i)] \\ &= \sum_{T \supseteq S} p_{T_i}^i [v(T) - v(T \backslash S)] - \sum_{T \supseteq S} p_{T_i}^i [v(T) - v(T_i)] - \sum_{\substack{T \ni i \\ S \not\subseteq T}} p_{T_i}^i [v(T) - v(T_i)].\end{aligned}$$

Hence

$$\lambda_i^{\mathbf{p}}[v^S] - \lambda_i^{\mathbf{p}}[v] = \sum_{T \supseteq S} p_{T_i}^i [v(T_i) - v(T \backslash S)] - \sum_{\substack{T \ni i \\ S \not\subseteq T}} p_{T_i}^i [v(T) - v(T_i)].$$

(b) It readily follows from (a).

(c) Let $j \in N \backslash S$. Then

$$\lambda_j^{\mathbf{p}}[v] - \lambda_j^{\mathbf{p}}[v^S] = \sum_{T \ni j} p_{T_j}^j [v(T) - v(T_j)] - \sum_{T \ni j} p_{T_j}^j [v^S(T) - v^S(T_j)].$$

Writing v^S in terms of v and splitting again both sums according to either $S \subseteq T$ or $S \nsubseteq T$,

$$\begin{aligned}\lambda_j^{\mathbf{p}}[v] - \lambda_j^{\mathbf{p}}[v^S] = &\sum_{\substack{T\ni j\\ S\subseteq T}} p^j_{T_j}[v(T) - v(T_j)] + \sum_{\substack{T\ni j\\ S\nsubseteq T}} p^j_{T_j}[v(T) - v(T_j)] \\ &- \sum_{\substack{T\ni j\\ S\subseteq T}} p^j_{T_j}[v(T) - v(T_j)] - \sum_{\substack{T\ni j\\ S\nsubseteq T}} p^j_{T_j}[v(T\backslash S) - v(T_j\backslash S)].\end{aligned}$$

Therefore

$$\lambda_j^{\mathbf{p}}[v] - \lambda_j^{\mathbf{p}}[v^S] = \sum_{\substack{T\ni j\\ S\nsubseteq T}} p^j_{T_j}[v(T) - v(T_j) - v(T\backslash S) + v(T_j\backslash S)]. \quad \square$$

Proof of Corollary 3.2. (a) $|S| = 2$ and hence, for each $i \in S$,

$$\{T_i : i \in T, S \nsubseteq T\} = \{T\backslash S : S \subseteq T\}.$$

Indeed,

$$\Delta_S\lambda^{\mathbf{p}}[v] \quad = \quad \sum_{i\in S}\left\{\sum_{T\supseteq S} p^i_{T_i}[v(T_i) - v(T\backslash S)] - \sum_{\substack{T\ni i\\ S\nsubseteq T}} p^i_{T_i}[v(T) - v(T_i)]\right\}.$$

Besides, if $S = \{i, j\}$ then

$$\{T_i : S \subseteq T\} = \{T : j \in T, S \nsubseteq T\} \quad \text{and} \quad \{T_j : S \subseteq T\} = \{T : i \in T, S \nsubseteq T\}.$$

Thus

$$\begin{aligned}\Delta_S\lambda^{\mathbf{p}}[v] &= \sum_{i\in S}\sum_{T\supseteq S}(p^i_{T_i} - p^{S\backslash\{i\}}_{T\backslash S})[v(T_i) - v(T\backslash S)] \\ &= (p_i + p_j - 1)\sum_{T\supseteq S}\prod_{l\in T_{ij}} p_l \cdot \prod_{l\in N\backslash T}(1 - p_l)[v(T_i) + v(T_j) - 2v(T\backslash S)], \end{aligned} \tag{7}$$

The inequalities follow from formula (7) and the fact that

$$\sum_{T\supseteq S}\prod_{k\in T_{ij}} p_k \cdot \prod_{l\in N\backslash T}(1 - p_l) = 1.$$

(b) and (c) follow from formula (7) which evidences the sign of $\Delta_S\lambda^{\mathbf{p}}[v]$ as every p_i ranges [0, 1], provided that v is monotonic. $\square$

Proof of Proposition 3.6. By applying Definition 2.1 to the quotient game $v_{[S]}$ we obtain

$$\lambda_0^{\bar{\mathbf{p}}}[v_{[S]}] = \sum_{\substack{R\subseteq M\\ 0\in R}} p^0_R[v_{[S]}(R) - v_{[S]}(R\backslash\{0\})] = \sum_{\substack{T\subseteq N\\ S\subseteq T}}\prod_{k\in T\backslash S} p_k \cdot \prod_{l\in N\backslash T}(1 - p_l)[v(T) - v(T\backslash S)].$$

Taking into account the definition of quotient game and using formula (6) from the proof of Proposition 3.1(a),

$$\begin{aligned}\Delta_S\lambda^{\mathbf{p}}[v^S, v_{[S]}] &= \sum_{i\in S}\sum_{T\supseteq S} p^i_{T_i}[v(T) - v(T\backslash S)] - \sum_{T\supseteq S}\prod_{k\in T\backslash S} p_k \cdot \prod_{l\in N\backslash T}(1 - p_l)[v(T) - v(T\backslash S)] \\ &= \sum_{T\supseteq S}\left\{\sum_{i\in S} p^i_{T_i} - \prod_{k\in T\backslash S} p_k \cdot \prod_{l\in N\backslash T}(1 - p_l)\right\}\Big[v(T) - v(T\backslash S)\Big]\end{aligned}$$

if $p_i > 0$ for all $i \in N$. Instead, in case $p_i = 0$, for all $i \in N$, we get

$$\Delta_S\lambda^{\mathbf{p}}[v^S, v_{[S]}] = -v(S). \quad \square$$

Proof of Corollary 3.7. Items (a) and (b) are obvious because the expression of $\Delta_S\lambda^{\mathbf{p}}[v^S, v_{[S]}]$ obtained in Proposition 3.6 is $-v(S)$ if $p_i = 0$ for all $i \in N$ and $(s-1)[v(N) - v(N\backslash S)]$ if $p_i = 1$ for all $i \in N$. As for (c), the monotonicity of v reduces the sign question to factor $\sum_{i\in S}\prod_{r\in S_i} p_r - 1$. Finally, if $q_M = \max_{r\in S} p_r$ and $q_m = \min_{r\in S} p_r$, (d) follows from the inequalities:

$$q_m^{s-1} - 1 \le \prod_{r\in S_i} p_r - 1 \le q_M^{s-1} - 1, \quad \text{and} \quad v(S\cup R) - v(R) \le v(S\cup R) \le v(N)$$

and the relation

$$\sum_{T\supseteq S}\prod_{k\in T\setminus S}p_k\cdot\prod_{l\in N\setminus T}(1-p_l)=1.\quad\square$$

Appendix B. Proofs of Section 4

Proof of Theorem 4.1. From Proposition 3.1(a) it is easy to see that, if v is simple and $i\in S$, then

$$\Delta_i^S\lambda^{\mathbf{p}}[v]=\lambda_i^{\mathbf{p}}[v^S]-\lambda_i^{\mathbf{p}}[v]=\sum_{T\in W_1}p^i_{T_i}-\sum_{T\in W_2}p^i_{T_i},$$

where

$$W_1=\{T\in W:S\subseteq T,\ T_i\in W,\ T\setminus S\notin W\}\quad\text{and}\quad W_2=\{T\in W:i\in T,\ S\not\subseteq T,\ T_i\notin W\}.$$

From this we have, for any $v\in\mathcal{SG}_N$,

$$-\sum_{T\in W_2}p^i_{T_i}\le\Delta_i^S\lambda^{\mathbf{p}}[v]\le\sum_{T\in W_1}p^i_{T_i}.$$

(a) The maximum will be reached when W_1 is formed by the maximum number of coalitions, i.e. $W_1=\{T\subseteq N:S\subseteq T\}$, and $W_2=\emptyset$. To this end, let us take $v=u_{S\setminus\{i\}}$. Then $W=\{T\subseteq N:S\setminus\{i\}\subseteq T\}$ and, indeed, $W_1=\{T\subseteq N:S\subseteq T\}$ and $W_2=\emptyset$ as desired. Hence

$$\max_{v\in\mathcal{SG}_N}\Delta_i^S\lambda^{\mathbf{p}}[v]=\sum_{S\subseteq T\subseteq N}p^i_{T_i}=\prod_{k\in S_i}p_k.$$

(b) In this case, the minimum will be reached when $W_1=\emptyset$ and W_2 is formed by the maximum number of coalitions, i.e. $W_2=\{T\subseteq N:i\in T,\ S\not\subseteq T\}$. Now let us take $v=u_{\{i\}}$. Then $W=\{T\subseteq N:i\in T\}$ and, in effect, $W_1=\emptyset$ and W_2 is as desired. Here

$$\min_{v\in\mathcal{SG}_N}\Delta_i^S\lambda^{\mathbf{p}}[v]=-\sum_{T\in W_2}p^i_{T_i}=\prod_{k\in S_i}p_k-1.\quad\square$$

Proof of Theorem 4.3. From Proposition 3.1(c) it follows that, if v is simple and $j\in N\setminus S$, then

$$\Delta_j^S\lambda^{\mathbf{p}}[v]=\lambda_j^{\mathbf{p}}[v]-\lambda_j^{\mathbf{p}}[v^S]=\sum_{\substack{T\in W:\,j\in T,\\ S\not\subseteq T,\,S\cap T\neq\emptyset}}p^i_{T_i}=\sum_{T\in W_3}p^i_{T_i}-\sum_{T\in W_4}p^i_{T_i},$$

where

$$W_3=\{T\in W:j\in T,\ S\not\subseteq T,\ S\cap T\neq\emptyset,\ T_j\notin W,\ T\setminus S\notin W\}\quad\text{and}$$
$$W_4=\{T\in W:j\in T,\ S\not\subseteq T,\ S\cap T\neq\emptyset,\ T_j\in W,\ T\setminus S\in W,\ T_j\setminus S\notin W\},$$

because, if $j\in T$ and $S\not\subseteq T$ but $T\notin W$ or $S\cap T=\emptyset$, then the term of T in the original expression of $\Delta_j^S\lambda^{\mathbf{p}}[v]$ vanishes. Then, for any $v\in\mathcal{SG}_N$,

$$-\sum_{T\in W_4}p^i_{T_i}\le\Delta_j^S\lambda^{\mathbf{p}}[v]\le\sum_{T\in W_3}p^i_{T_i}.$$

(a) The maximum will be reached when W_3 is formed by the maximum number of coalitions, i.e. $W_3=\{T\subseteq N:j\in T,\ S\not\subseteq T,\ S\cap T\neq\emptyset\}$, and $W_4=\emptyset$. To this end, let us take v defined by $W=\{T\subseteq N:j\in T,\ S\cap T\neq\emptyset\}$. It is not difficult to check that W_3 is as desired and $W_4=\emptyset$. Hence

$$\max_{v\in\mathcal{SG}_N}\Delta_j^S\lambda^{\mathbf{p}}[v]=\sum_{R\subseteq N\setminus\{S\cup j\}}p^j_R\cdot\sum_{\emptyset\neq R\subset S}p^j_R=1-p^j_\emptyset-p^j_S=1-\prod_{k\in S}(1-p_k)-\prod_{k\in S}p_k.$$

(b) In this case, the minimum will be reached when $W_3=\emptyset$ and W_4 is formed by the maximum number of coalitions, i.e. $W_4=\{T\subseteq N:j\in T,\ S\not\subseteq T,\ S\cap T\neq\emptyset\}$. Now let us take v defined by $W=\{T\subseteq N:(S\cup\{j\})\cap T\neq\emptyset\}$. Again, one easily verifies that $W_3=\emptyset$ and W_4 is as desired. Here

$$\min_{v\in\mathcal{SG}_N}\Delta_j^S\lambda^{\mathbf{p}}=-\sum_{R\subseteq N\setminus\{S\cup j\}}p^j_R\cdot\sum_{\emptyset\neq R\subset S}p^j_R=p^j_\emptyset+p^j_S-1=\prod_{k\in S}(1-p_k)+\prod_{k\in S}p_k-1.\quad\square$$

Proof of Theorem 4.5. If $p_i\in(0,1)$ for all $i\in N$ and v is simple then, from Proposition 3.6,

$$\Delta_S\lambda^{\mathbf{p}}[v^S,v_{[S]}]=\left[\sum_{i\in S}\prod_{r\in S_i}p_r-1\right]\sum_{T\supseteq S}\prod_{k\in T\setminus S}p_k\prod_{l\in N\setminus T}(1-p_l)[v(T)-v(T\setminus S)].$$

(a) According to Corollary 3.7(c), if $\sum_{i\in S}\prod_{r\in S_i} p_r > 1$ this expression is nonnegative, and will attain its maximum value whenever all $T \supseteq S$ satisfy $T \in W$ and $T\backslash S \notin W$, that is, in the unanimity game $v = u_S$. This value is

$$\Delta_S \lambda^{\mathbf{p}}[v^S, v_{[S]}] = \sum_{i\in S}\prod_{r\in S_i} p_r - 1 > 0.$$

As $\sum_{i\in S}\prod_{r\in S_i} p_r > 1$, the only way the expression of $\Delta_S \lambda^{\mathbf{p}}[v^S, v_{[S]}]$ vanishes is by forcing the sum to be empty. This happens in any game where, for all $T \in W$ such that $S \subseteq T$, we have $T\backslash S \in W$, and this means that all members of S are null players in this game.

(b) The proof follows the same guidelines as in (a). □

References

[1] J.M. Alonso, F. Carreras, M.A. Puente, Axiomatic characterizations of the symmetric coalitional binomial semivalues, Discrete Appl. Math. 155 (2007) 2282–2293.
[2] R. Amer, J.M. Giménez, Modification of semivalues for games with coalition structures, Theory and Decision 54 (2003) 185–205.
[3] F. Carreras, On the existence and formation of partnerships in a game, Games Econom. Behav. 12 (1996) 54–67.
[4] F. Carreras, α–decisiveness in simple games, Theory and Decision 56 (2004) 77–91. Also in: Essays on Cooperative Games, G. Gambarelli, (Ed.), Kluwer Academic Publishers, 77–91.
[5] F. Carreras, Elementary theory of simple games. Working paper MA2-IT-01-00001, Department of Applied Mathematics II, Technical University of Catalonia. Spain, 2001.
[6] F. Carreras, J. Freixas, Semivalue versatility and applications, Ann. Oper. Res. 109 (2002) 343–358.
[7] F. Carreras, J. Freixas, Complete simple games, Math. Social Sci. 32 (1996) 139–155.
[8] F. Carreras, J. Freixas, M.A Puente, A note on decisive symmetric games, Decision Support Systems 51 (2011) 424–433.
[9] F. Carreras, M.D. Llongueras, A. Magaña, On the convenience to form coalitions or partnerships in simple games, Ann. Oper. Res. 137 (2005) 67–89.
[10] F. Carreras, M.D Llongueras, M.A. Puente, Partnership formation and binomial semivalues, European J. Oper. Res. 192 (2009) 487–499.
[11] F. Carreras, M.A. Puente, Cooperation tendencies: values defined by one parameter per player. ICEIS 2013, Proceedings of the 15th International Conference on Enterprise Information Systems, 397–404, 2012.
[12] F. Carreras, M.A. Puente, Symmetric coalitional binomial semivalues, Group Decision and Negotiation 21 (2012) 637–662.
[13] F. Carreras, M.A. Puente, Coalitional multinomial probabilistic values, European J. Oper. Res. (2013) forthcoming.
[14] P. Dubey, A. Neyman, R.J. Weber, Value theory without efficiency, Math. Oper. Res. 6 (1981) 122–128.
[15] J. Freixas, M.A. Puente, Reliability importance measures of the components in a system based on semivalues and probabilistic values, Ann. Oper. Res. 109 (2002) 331–342.
[16] E. Kalai, D. Samet, On weighted Shapley values, Internat. J. Game Theory 16 (1987) 205–222.
[17] G. Owen, Multilinear extensions and the Banzhaf value, Naval Research Logistics Quarterly 22 (1975) 741–750.
[18] G. Owen, Multilinear extension of games, Manag. Sci. 18 (1972) 64–79.
[19] G. Owen, A characterization of the Banzhaf–Coleman index, SIAM J. Appl. Math. 35 (1978) 315–327.
[20] M.A. Puente, Contributions to the representability of simple games and to the calculus of solutions for this class of games (in Spanish). Ph.D. Thesis. Department of Applied Mathematics III, Technical University of Catalonia, Spain. 2000.
[21] L.S. Shapley, A value for n-person games, in: H.W. Kuhn, A.W. Tucker (Eds.), Contributions to the Theory of Games II, Princeton University Press, 1953, pp. 307–317.
[22] A.D. Taylor, W.S. Zwicker, Simple Games: Desirability Relations, Trading, and Pseudoweightings, Princeton University Press, 1999.
[23] R.J. Weber, Probabilistic values for games, in: A.E. Roth (Ed.), Cambridge University Press, 1988, pp. 101–119.