

GEOMETRIC HAMILTON-JACOBI THEORY FOR HIGHER-ORDER AUTONOMOUS SYSTEMS

LEONARDO COLOMBO*

MANUEL DE LEÓN†

*Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM).
C/ Nicolás Cabrera 15. Campus Cantoblanco UAM. 28049 Madrid. Spain*

PEDRO DANIEL PRIETO-MARTÍNEZ‡

NARCISO ROMÁN-ROY§

*Departamento de Matemática Aplicada IV.
Universitat Politècnica de Catalunya-Barcelona Tech.
Edificio C-3, Campus Norte UPC. C/ Jordi Girona 1. 08034 Barcelona. Spain*

September 10, 2013

Abstract

The geometric framework for the Hamilton–Jacobi theory is used to study this theory in the ambient of higher-order mechanical systems, both in the Lagrangian and Hamiltonian formalisms. Thus, we state the corresponding Hamilton–Jacobi equations in these formalisms and apply our results to analyze some particular physical examples.

Key words: Hamilton–Jacobi equation, higher-order Lagrangian and Hamiltonian systems, Symplectic geometry.

AMS s. c. (2010): 53C15, 53C80, 70H03, 70H05, 70H20, 70H50.

**e-mail:* leo.colombo@icmat.es

†*e-mail:* mdeleon@icmat.es

‡*e-mail:* peredaniel@ma4.upc.edu

§*e-mail:* nrr@ma4.upc.edu

Contents

1	Introduction	2
2	Higher-order tangent bundles and canonical structures	3
3	The Hamilton-Jacobi problem in the Lagrangian formalism	6
3.1	The generalized Lagrangian Hamilton-Jacobi problem	6
3.2	The Lagrangian Hamilton-Jacobi problem	10
3.3	Complete solutions	12
4	The Hamilton-Jacobi problem in the Hamiltonian formalism	12
4.1	The generalized Hamiltonian Hamilton-Jacobi problem	13
4.2	The Hamiltonian Hamilton-Jacobi problem	14
4.3	Complete solutions	16
4.4	Relation with the Lagrangian formulation	17
5	Examples	20
5.1	The end of a javelin	20
5.2	A (homogeneous) deformed elastic cylindrical beam with fixed ends	22

1 Introduction

As it is well known, in classical mechanics, Hamilton–Jacobi theory is a way to integrate a system of ordinary differential equations (Hamilton equations) that, through an appropriate canonical transformation [2, 18], is led to equilibrium. The equation to be satisfied by the generating function of this transformation is a partial differential equation whose solution allows us to integrate the original system. In this respect, Hamilton–Jacobi theory provides important physical examples of the deep connection between first-order partial differential equations and systems of first-order ordinary differential equations. It is also very close, from the classical side, to the Schrödinger equation of quantum mechanics since a complete solution of the Hamilton–Jacobi equation allows us to reconstruct an approximate solution of the Schrödinger equation (see for instance [14, 30]). For these reasons, Hamilton–Jacobi theory has been a matter of continuous interest and has been studied classically also in other ambients [12].

From the viewpoint of geometric mechanics, the intrinsic formulation of Hamilton–Jacobi equation is also clear [1, 28, 29]. Nevertheless, in the paper [8] a new geometric framework for the Hamilton–Jacobi theory was presented and the Hamilton–Jacobi equation was formulated both in the Lagrangian and in the Hamiltonian formalisms of autonomous and non-autonomous mechanics. A similar generalization of the Hamilton–Jacobi formalism was outlined in [20].

Later on, this new geometric framework was used to state the Hamilton–Jacobi theory in many different situations, such as non-holonomic mechanical systems [10, 17, 21, 23, 31, 32],

geometric mechanics on Lie algebroids [3] and almost-Poisson manifolds [26], singular systems [25], control theory [5, 41, 42], classical field theories [22, 24] and partial differential equations in general [40], the geometric discretization of the Hamilton–Jacobi equation [33], and others [4, 9].

The aim of this paper is to go ahead in this program, applying this geometric description of the Hamilton–Jacobi theory in the ambient of higher-order mechanical systems. These kind of systems appear in many models in theoretical and mathematical physics; for instance in the mathematical description of relativistic particles with spin, string theories, gravitation, Podolsky’s electromagnetism and others, in some problems of fluid mechanics and classical physics, and in numerical models arising from the geometric discretization of first-order dynamical systems (see [34, 35] for a long but non-exhaustive list of references). The geometric descriptions of these systems use higher-order tangent and jet bundles (see, for instance, [7, 15, 16, 19, 27, 34, 35, 36]). Up to our knowledge, the Hamilton–Jacobi equation for these kind of systems has deserved little attention (although an approach to the problem for higher-order field theories can be found in [39]). This paper wants to fill this gap.

Thus, after establishing some basic concepts and notation about higher-order tangent and jet bundles in Section 2, we state in Section 3 the generalized and the standard Hamilton–Jacobi problems for the Lagrangian formalism of higher-order systems, including a discussion about complete solutions to both problems and detailed coordinate expressions of the Lagrangian Hamilton–Jacobi equations so obtained. Section 4 is devoted to present the Hamilton–Jacobi problems and their complete solutions in the Hamiltonian formalism, as a straightforward application of the corresponding case for first-order mechanics stated in [8] and some results of the Lagrangian case. Finally, in Section 5, our model is applied to two real physical examples: the system describing the dynamics of the end of a javelin and a (homogeneous) deformed elastic cylindrical beam with fixed ends.

All the manifolds are real, second countable and C^∞ . The maps and the structures are assumed to be C^∞ . Sum over repeated indices is understood.

2 Higher-order tangent bundles and canonical structures

(See [27], [37] for details).

Let Q be a n -dimensional manifold, and $k \in \mathbb{Z}^+$. The k th order tangent bundle of Q is the $(k+1)n$ -dimensional manifold T^kQ made of the k -jets of the bundle $\pi: \mathbb{R} \times Q \rightarrow \mathbb{R}$ with fixed source point $t = 0 \in \mathbb{R}$; that is, $T^kQ = J_0^k\pi$. It is a 1-codimensional submanifold of $J^k\pi$.

We have the following natural projections (for $r \leq k$):

$$\begin{array}{ccc} \rho_r^k: T^kQ & \longrightarrow & T^rQ \\ j_0^k\phi & \longmapsto & j_0^r\phi \end{array} \quad ; \quad \begin{array}{ccc} \beta^k: T^kQ & \longrightarrow & Q \\ j_0^k\phi & \longmapsto & \phi(0) \end{array}$$

where $j_0^k\phi$ denotes a point in T^kQ ; that is, the equivalence class of a curve $\phi: I \subset \mathbb{R} \rightarrow Q$ by the k -jet equivalence relation. Notice that $\rho_0^k = \beta^k$, where T^0Q is canonically identified with Q , and $\rho_k^k = \text{Id}_{T^kQ}$. Observe also that $\rho_s^l \circ \rho_l^r = \rho_s^r$, for $0 \leq s \leq l \leq r \leq k$.

The natural projections $\rho_s^r: T^rQ \rightarrow T^sQ$ are smooth surjective submersions and, furthermore, the triple (T^rQ, ρ_s^r, T^sQ) is a smooth fiber bundle with fiber $\mathbb{R}^{(r-s)n}$ (see [37]). In particular, (T^kQ, ρ_r^k, T^rQ) is a smooth fiber bundle with fiber $\mathbb{R}^{(k-r)n}$, for $0 \leq r \leq k$; that is, T^kQ is canonically endowed with $k+1$ different affine bundle structures given by the projections

$\rho_0^k, \rho_1^k, \dots, \rho_k^k$. In the sequel, we refer to this fiber bundle structure as *the ρ_r^k -bundle structure of $T^k Q$* .

If (U, φ) is a local chart in Q , with $\varphi = (\varphi^A)$, $1 \leq A \leq n$, and $\phi: \mathbb{R} \rightarrow Q$ is a curve in Q such that $\phi(0) \in U$; by writing $\phi^A = \varphi^A \circ \phi$, the k -jet $j_0^k \phi$ is given in $(\beta^k)^{-1}(U) = T^k U$ by $(q^A, q_1^A, \dots, q_k^A)$, where

$$q^A = \phi^A(0) \quad ; \quad q_i^A = \left. \frac{d^i \phi^A}{dt^i} \right|_{t=0} .$$

with $1 \leq i \leq k$. Usually we write q_0^A instead of q^A , and so we have local coordinates $(q_0^A, q_1^A, \dots, q_k^A) = (q_i^A)$, with $0 \leq i \leq k$, in the open set $(\beta^k)^{-1}(U) \subseteq T^k Q$. Using these coordinates, the local expression of the canonical projections are

$$\rho_r^k(q_0^A, q_1^A, \dots, q_k^A) = (q_0^A, q_1^A, \dots, q_r^A) \quad ; \quad \beta^k(q_0^A, q_1^A, \dots, q_k^A) = (q_0^A) .$$

Hence, local coordinates in the open set $(\beta^k)^{-1}(U) \subseteq T^k Q$ adapted to the ρ_r^k -bundle structure are $(q_0^A, \dots, q_r^A, q_{r+1}^A, \dots, q_k^A)$, and a section $s \in \Gamma(\rho_r^k)$ is locally given in this open set by $s(q_0^A, \dots, q_r^A) = (q_0^A, \dots, q_r^A, s_{r+1}^A, \dots, s_k^A)$, where s_j^A (with $r+1 \leq j \leq k$) are local functions.

If $\phi: \mathbb{R} \rightarrow Q$ is a curve in Q , the canonical lifting of ϕ to $T^k Q$ is the curve $j^k \phi: \mathbb{R} \rightarrow T^k Q$ defined as the k -jet lifting of ϕ restricted to the submanifold $T^k Q \hookrightarrow J^k \pi$

Now, consider the bundle $\pi: \mathbb{R} \times Q \rightarrow \mathbb{R}$ and the natural projection $\pi_s^r: J^r \pi \rightarrow J^s \pi$, for $0 \leq s \leq r$. Let $\mathfrak{X}(\pi_s^r)$ be the module of vector fields along the projection π_s^r .

Definition 1 *If $X \in \mathfrak{X}(\mathbb{R})$, $\phi \in \Gamma(\pi)$ and $t_o \in \mathbb{R}$, the k th holonomic lift of X by ϕ is defined as the vector field $j^k X \in \mathfrak{X}(\pi_k^{k+1})$ whose associated derivation satisfies*

$$(d_{j^k X} f)(j^{k+1} \phi(t_o)) = d_X(f \circ j^k \phi)(t_o),$$

for every $f \in C^\infty(J^k \pi)$; where $d_{j^k X}$ and d_X are the derivations associated to $j^k X$ and X , respectively.

In local coordinates, if $X \in \mathfrak{X}(\mathbb{R})$ is given by $X = X_o \frac{\partial}{\partial t}$, then the k th holonomic lift of X is

$$j^k X = X_o \left(\frac{\partial}{\partial t} + \sum_{i=0}^k q_{i+1}^A \frac{\partial}{\partial q_i^A} \right) .$$

The (*non-autonomous*) *total time derivative* is the derivation associated to the k th holonomic lift of the coordinate vector field $\frac{\partial}{\partial t} \in \mathfrak{X}(\mathbb{R})$, which is denoted by $\frac{d}{dt} \in \mathfrak{X}(\pi_k^{k+1})$, and whose local expression is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i=0}^k q_{i+1}^A \frac{\partial}{\partial q_i^A} .$$

Using the identification $J^k \pi \cong \mathbb{R} \times T^k Q$ and denoting by $\pi_2: \mathbb{R} \times Q \rightarrow Q$ the natural projection and all the induced projections in higher-order jet bundles, we have the following diagram

$$\begin{array}{ccc} \mathbb{R} \times T^{k+1} Q & \xrightarrow{\pi_2} & T^{k+1} Q \\ \downarrow \pi_k^{k+1} & & \downarrow \rho_k^{k+1} \\ \mathbb{R} \times T^k Q & \xrightarrow{\pi_2} & T^k Q \end{array}$$

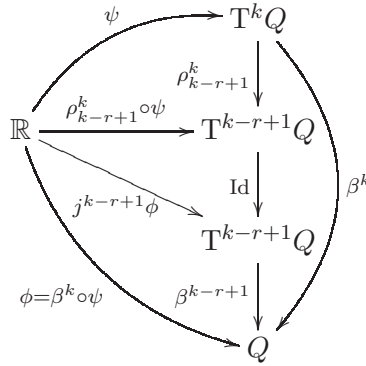
Hence, the total time derivative induces a vector field $T \in \mathfrak{X}(\rho_k^{k+1})$, which satisfies

$$T_{\text{pr}_2(j^{k+1}\phi)} = \mathbb{T}\pi_2 \circ \frac{d}{dt} \Big|_{j^{k+1}\phi},$$

and the derivation associated to the vector field T , denoted by d_T , is called *the (autonomous) total time derivative*, or *Tulczyjew's derivation*.

The total time derivative in higher-order tangent bundles can be introduced in an equivalent way without using explicitly the jet bundle structure (see [38] for details).

Definition 2 A curve $\psi: \mathbb{R} \rightarrow \mathbb{T}^k Q$ is holonomic of type r , $1 \leq r \leq k$, if $j^{k-r+1}\phi = \rho_{k-r+1}^k \circ \psi$, where $\phi = \beta^k \circ \psi: \mathbb{R} \rightarrow Q$; that is, the curve ψ is the lifting of a curve in Q up to $\mathbb{T}^{k-r+1}Q$.



In particular, a curve ψ is holonomic of type 1 if $j^k \phi = \psi$, with $\phi = \beta^k \circ \psi$. Throughout this paper, holonomic curves of type 1 are simply called holonomic.

Definition 3 A vector field $X \in \mathfrak{X}(\mathbb{T}^k Q)$ is a semispray of type r , $1 \leq r \leq k$, if every integral curve ψ of X is holonomic of type r .

The local expression of a semispray of type r is

$$X = q_1^A \frac{\partial}{\partial q_0^A} + q_2^A \frac{\partial}{\partial q_1^A} + \dots + q_{k-r+1}^A \frac{\partial}{\partial q_{k-r}^A} + F_{k-r+1}^A \frac{\partial}{\partial q_{k-r+1}^A} + \dots + F_k^A \frac{\partial}{\partial q_k^A}.$$

It is clear that every holonomic curve of type r is also holonomic of type s , for $s \geq r$. The same remark is true for semisprays.

Observe that semisprays of type 1 in $\mathbb{T}^k Q$ are the analogue to the holonomic (or SODE) vector fields in first-order mechanics. Their local expressions are

$$X = q_1^A \frac{\partial}{\partial q_0^A} + q_2^A \frac{\partial}{\partial q_1^A} + \dots + q_k^A \frac{\partial}{\partial q_{k-1}^A} + F^A \frac{\partial}{\partial q_k^A}.$$

If $X \in \mathfrak{X}(\mathbb{T}^k Q)$ is a semispray of type r , a curve $\phi: \mathbb{R} \rightarrow Q$ is said to be a *path* or *solution* of X if $j^k \phi$ is an integral curve of X ; that is, $\widetilde{j^k \phi} = X \circ j^k \phi$, where $\widetilde{j^k \phi}$ denotes the canonical lifting of $j^k \phi$ from $\mathbb{T}^k Q$ to $\mathbb{T}(\mathbb{T}^k Q)$. Then, in coordinates, ϕ verifies the following system of differential equations of order $k+1$:

$$\frac{d^{k-r+2}\phi^A}{dt^{k-r+2}} = X_{k-r+1}^A \left(\phi, \frac{d\phi}{dt}, \dots, \frac{d^k \phi}{dt^k} \right), \dots, \frac{d^{k+1}\phi^A}{dt^{k+1}} = X_k^A \left(\phi, \frac{d\phi}{dt}, \dots, \frac{d^k \phi}{dt^k} \right).$$

3 The Hamilton-Jacobi problem in the Lagrangian formalism

Let Q be the configuration space of an autonomous dynamical system of order k with n degrees of freedom (that is, Q is a n -dimensional smooth manifold), and let $\mathcal{L} \in C^\infty(\mathbb{T}^k Q)$ be the Lagrangian function for this system. From the Lagrangian function \mathcal{L} we construct the Poincaré-Cartan forms $\theta_{\mathcal{L}} \in \Omega^1(\mathbb{T}^{2k-1}Q)$ and $\omega_{\mathcal{L}} = -d\theta_{\mathcal{L}} \in \Omega^2(\mathbb{T}^{2k-1}Q)$, as well as the Lagrangian energy $E_{\mathcal{L}} \in C^\infty(\mathbb{T}^{2k-1}Q)$. The coordinate expressions of these elements are

$$\begin{aligned}\theta_{\mathcal{L}} &= \sum_{r=1}^k \sum_{i=0}^{k-r} (-1)^i d_T^i \left(\frac{\partial \mathcal{L}}{\partial q_{r+i}^A} \right) dq_{r-1}^A, \\ \omega_{\mathcal{L}} &= \sum_{r=1}^k \sum_{i=0}^{k-r} (-1)^{i+1} d_T^i d \left(\frac{\partial \mathcal{L}}{\partial q_{r+i}^A} \right) \wedge dq_{r-1}^A, \\ E_{\mathcal{L}} &= \sum_{r=1}^k q_r^A \sum_{i=0}^{k-r} (-1)^i d_T^i \left(\frac{\partial \mathcal{L}}{\partial q_{r+i}^A} \right) - \mathcal{L}.\end{aligned}$$

We assume that the Lagrangian function is regular, and then $\omega_{\mathcal{L}}$ is a symplectic form. Then the dynamical equation for this Lagrangian system, which is

$$i(X_{\mathcal{L}})\omega_{\mathcal{L}} = dE_{\mathcal{L}}, \quad (1)$$

has a unique solution $X_{\mathcal{L}} \in \mathfrak{X}(\mathbb{T}^{2k-1}Q)$ which is a semispray of type 1 in $\mathbb{T}^{2k-1}Q$. (For a detailed description of the Lagrangian and Hamiltonian formalisms of higher-order dynamical systems see, for instance, [27, 34, 35, 36]).

3.1 The generalized Lagrangian Hamilton-Jacobi problem

Following [1] and [8], we first state a general version of the Hamilton-Jacobi problem in the Lagrangian setting, which is the so-called *generalized Lagrangian Hamilton-Jacobi problem*. For first-order systems, this problem consists in finding vector fields $X \in \mathfrak{X}(Q)$ such that the liftings of any integral curve of X to $\mathbb{T}Q$ by X itself is an integral curve of the Lagrangian vector field $X_{\mathcal{L}}$. In higher-order systems we can state an analogous problem. Thus, we have the following definition:

Definition 4 *The generalized k th-order Lagrangian Hamilton-Jacobi problem consists in finding a section $s \in \Gamma(\rho_{k-1}^{2k-1})$ and a vector field $X \in \mathfrak{X}(\mathbb{T}^{k-1}Q)$ such that, if $\gamma: \mathbb{R} \rightarrow \mathbb{T}^{k-1}Q$ is an integral curve of X , then $s \circ \gamma: \mathbb{R} \rightarrow \mathbb{T}^{2k-1}Q$ is an integral curve of $X_{\mathcal{L}}$; that is,*

$$X \circ \gamma = \dot{\gamma} \implies X_{\mathcal{L}} \circ (s \circ \gamma) = \overline{s \circ \dot{\gamma}}. \quad (2)$$

Remark: Observe that, since $X_{\mathcal{L}}$ is a semispray of type 1, then every integral curve of $X_{\mathcal{L}}$ is the $(2k-1)$ -jet lifting of a curve in Q . In particular, this holds for the curve $s \circ \gamma$, that is, there exists a curve $\phi: \mathbb{R} \rightarrow Q$ such that

$$j^{2k-1}\phi = s \circ \gamma.$$

Then, composing both sides of the equality with ρ_{k-1}^{2k-1} and bearing in mind that $s \in \Gamma(\rho_{k-1}^{2k-1})$, we obtain

$$\gamma = j^{k-1}\phi,$$

that is, the curve γ is the $(k-1)$ -jet lifting of a curve in Q . This enables us to restate the problem as follows: *The generalized k th-order Lagrangian Hamilton-Jacobi problem consists in finding a vector field $X_o \in \mathfrak{X}(Q)$ such that, if $\phi: \mathbb{R} \rightarrow Q$ is an integral curve of X_o , then $j^{2k-1}\phi: \mathbb{R} \rightarrow T^{2k-1}Q$ is an integral curve of $X_{\mathcal{L}}$; that is,*

$$X_o \circ \phi = \dot{\phi} \implies X_{\mathcal{L}} \circ (j^{2k-1}\phi) = \overline{j^{2k-1}\dot{\phi}}.$$

Nevertheless, we will stick to the previous statement (Definition 4) in order to give several different characterizations of the problem.

It is clear from Definition 4 that the vector field $X \in \mathfrak{X}(T^{k-1}Q)$ cannot be chosen independently from the section $s \in \Gamma(\rho_{k-1}^{2k-1})$. In fact:

Proposition 1 *The pair $(s, X) \in \Gamma(\rho_{k-1}^{2k-1}) \times \mathfrak{X}(T^{k-1}Q)$ satisfies the condition (2) if, and only if, X and $X_{\mathcal{L}}$ are s -related; that is, $X_{\mathcal{L}} \circ s = Ts \circ X$.*

(Proof) If (s, X) satisfies the condition (2), then for every integral curve γ of X , we have

$$X_{\mathcal{L}} \circ (s \circ \gamma) = \overline{s \circ \dot{\gamma}} = Ts \circ \dot{\gamma} = Ts \circ X \circ \gamma,$$

but, as X has integral curves through every point $\bar{y} \in T^{k-1}Q$, this is equivalent to $X_{\mathcal{L}} \circ s = Ts \circ X$.

Conversely, if $X_{\mathcal{L}}$ and X are s -related and $\gamma: \mathbb{R} \rightarrow T^{k-1}Q$ is an integral curve of X , we have

$$X_{\mathcal{L}} \circ s \circ \gamma = Ts \circ X \circ \gamma = Ts \circ \dot{\gamma} = \overline{s \circ \dot{\gamma}}.$$

■

Hence, the vector field $X \in \mathfrak{X}(T^{k-1}Q)$ is related with the Lagrangian vector field $X_{\mathcal{L}}$ and with the section $s \in \Gamma(\rho_{k-1}^{2k-1})$. As a consequence of Proposition 1, composing both sides of the equality $X_{\mathcal{L}} \circ s = Ts \circ X$ with $T\rho_{k-1}^{2k-1}$ and bearing in mind that $\rho_{k-1}^{2k-1} \circ s = \text{Id}_{T^{k-1}Q}$, we have:

Corollary 1 *If (s, X) satisfies condition (2), then $X = T\rho_{k-1}^{2k-1} \circ X_{\mathcal{L}} \circ s$.*

Thus, the vector field X is completely determined by the section $s \in \Gamma(\rho_{k-1}^{2k-1})$, and it is called the *vector field associated to s* . The following diagram illustrates the situation

$$\begin{array}{ccc}
 T(T^{k-1}Q) & \xleftarrow{T\rho_{k-1}^{2k-1}} & T(T^{2k-1}Q) \\
 \uparrow X & \curvearrowright Ts & \uparrow X_{\mathcal{L}} \\
 T^{k-1}Q & \xleftarrow{\rho_{k-1}^{2k-1}} & T^{2k-1}Q \\
 & \curvearrowright s &
 \end{array}$$

Since the vector field X is completely determined by the section s , the search of a pair $(s, X) \in \Gamma(\rho_{k-1}^{2k-1}) \times \mathfrak{X}(T^{k-1}Q)$ satisfying condition (2) is equivalent to the search of a section $s \in \Gamma(\rho_{k-1}^{2k-1})$ such that the pair $(s, T\rho_{k-1}^{2k-1} \circ X_{\mathcal{L}} \circ s)$ satisfies the same condition. Thus, we can give the following definition:

Definition 5 A solution to the generalized k th-order Lagrangian Hamilton-Jacobi problem for $X_{\mathcal{L}}$ is a section $s \in \Gamma(\rho_{k-1}^{2k-1})$ such that, if $\gamma: \mathbb{R} \rightarrow \mathbb{T}^{k-1}Q$ is an integral curve of $\mathbb{T}\rho_{k-1}^{2k-1} \circ X_{\mathcal{L}} \circ s$, then $s \circ \gamma: \mathbb{R} \rightarrow \mathbb{T}^{2k-1}Q$ is an integral curve of $X_{\mathcal{L}}$, that is,

$$\mathbb{T}\rho_{k-1}^{2k-1} \circ X_{\mathcal{L}} \circ s \circ \gamma = \dot{\gamma} \implies X_{\mathcal{L}} \circ (s \circ \gamma) = \overline{s \circ \gamma}.$$

Finally, we have the following result, which gives some equivalent conditions for a section to be a solution to the generalized k th-order Lagrangian Hamilton-Jacobi problem:

Proposition 2 The following assertions on a section $s \in \Gamma(\rho_{k-1}^{2k-1})$ are equivalent.

1. The section s is a solution to the generalized k th-order Lagrangian Hamilton-Jacobi problem.
2. The submanifold $\text{Im}(s) \hookrightarrow \mathbb{T}^{2k-1}Q$ is invariant by the Euler-Lagrange vector field $X_{\mathcal{L}}$ (that is, $X_{\mathcal{L}}$ is tangent to the submanifold $s(\mathbb{T}^{k-1}Q) \hookrightarrow \mathbb{T}^{2k-1}Q$).
3. The section s satisfies the equation

$$i(X)(s^*\omega_{\mathcal{L}}) = d(s^*E_{\mathcal{L}}).$$

where $X = \mathbb{T}\rho_{k-1}^{2k-1} \circ X_{\mathcal{L}} \circ s$ is the vector field associated to s .

(Proof)

(1 \iff 2) Let s be a solution to the generalized k th-order Lagrangian Hamilton-Jacobi problem. Then by Proposition 1 the Lagrangian vector field $X_{\mathcal{L}} \in \mathfrak{X}(\mathbb{T}^{2k-1}Q)$ is s -related to the vector field $X = \mathbb{T}\rho_{k-1}^{2k-1} \circ X_{\mathcal{L}} \circ s \in \mathfrak{X}(\mathbb{T}^{k-1}Q)$ associated to s , and thus for every $\bar{y} \in \mathbb{T}^{k-1}Q$ we have

$$X_{\mathcal{L}}(s(\bar{y})) = (X_{\mathcal{L}} \circ s)(\bar{y}) = (\mathbb{T}s \circ X)(\bar{y}) = \mathbb{T}s(X(\bar{y})).$$

Hence, $X_{\mathcal{L}}(s(\bar{y})) = \mathbb{T}s(X(\bar{y}))$ and therefore $X_{\mathcal{L}}$ is tangent to the submanifold $\text{Im}(s) \hookrightarrow \mathbb{T}^{2k-1}Q$.

Conversely, if the submanifold $\text{Im}(s)$ is invariant under the flow of $X_{\mathcal{L}}$, then $X_{\mathcal{L}}(s(\bar{y})) \in \mathbb{T}_{s(\bar{y})}\text{Im}(s)$, for every $\bar{y} \in \mathbb{T}^{k-1}Q$; that is, there exists an element $u_{\bar{y}} \in \mathbb{T}_{\bar{y}}\mathbb{T}^{k-1}Q$ such that $X_{\mathcal{L}}(s(\bar{y})) = \mathbb{T}_{\bar{y}}s(u_{\bar{y}})$. If we define $X \in \mathfrak{X}(\mathbb{T}^{k-1}Q)$ as the vector field that satisfies $\mathbb{T}_{\bar{y}}s(X_{\bar{y}}) = X_{\mathcal{L}}(s(\bar{y}))$, then X is a vector field in $\mathbb{T}^{k-1}Q$, since $X = \mathbb{T}\rho_{k-1}^{2k-1} \circ X_{\mathcal{L}} \circ s$, and it is s -related with $X_{\mathcal{L}}$. Therefore, by Proposition 1, s is a solution to the generalized k th-order Lagrangian Hamilton-Jacobi problem.

(1 \iff 3) Let s be a solution to the generalized k th-order Lagrangian Hamilton-Jacobi problem. Taking the pull-back of the Lagrangian dynamical equation (1) by the section s we have

$$s^*i(X_{\mathcal{L}})\omega_{\mathcal{L}} = s^*dE_{\mathcal{L}} = d(s^*E_{\mathcal{L}}),$$

but since X and $X_{\mathcal{L}}$ are s -related by Proposition 1, we have that $s^*i(X_{\mathcal{L}})\omega_{\mathcal{L}} = i(X)s^*\omega_{\mathcal{L}}$, and hence we obtain

$$i(X)s^*\omega_{\mathcal{L}} = d(s^*E_{\mathcal{L}}).$$

Conversely, consider the following vector field along the section $s \in \Gamma(\rho_{k-1}^{2k-1})$

$$D_{\mathcal{L}} = X_{\mathcal{L}} \circ s - \mathbb{T}s \circ X: \mathbb{T}^{k-1}Q \rightarrow \mathbb{T}(\mathbb{T}^{2k-1}Q).$$

We want to prove that $D_{\mathcal{L}} = 0$ or equivalently, as $\omega_{\mathcal{L}}$ is non-degenerate, that $(\omega_{\mathcal{L}})_{s(\bar{y})}(D_{\mathcal{L}}(\bar{y}), Z_{s(\bar{y})}) = 0$, for every tangent vector $Z_{s(\bar{y})} \in T_{s(\bar{y})}T^{2k-1}Q$. Taking the pull-back of the Lagrangian dynamical equation, and using the hypothesis, we have

$$s^*(i(X_{\mathcal{L}})\omega_{\mathcal{L}}) = s^*dE_{\mathcal{L}} = d(s^*E_{\mathcal{L}}) = i(X)(s^*\omega_{\mathcal{L}}),$$

and then the form $s^*(i(X_{\mathcal{L}})\omega_{\mathcal{L}}) - i(X)(s^*\omega_{\mathcal{L}}) \in \Omega^1(T^{k-1}Q)$ vanishes. Therefore, for every $\bar{y} \in T^{k-1}Q$ and $u_{\bar{y}} \in T_{\bar{y}}T^{k-1}Q$, we have

$$\begin{aligned} 0 &= (s^*i(X_{\mathcal{L}})\omega_{\mathcal{L}} - i(X)s^*\omega_{\mathcal{L}})_{\bar{y}}(u_{\bar{y}}) \\ &= (\omega_{\mathcal{L}})_{s(\bar{y})}(X_{\mathcal{L}}(s(\bar{y})), T_{\bar{y}}s(u_{\bar{y}})) - (\omega_{\mathcal{L}})_{s(\bar{y})}(T_{\bar{y}}s(X_{\bar{y}}), T_{\bar{y}}s(u_{\bar{y}})) \\ &= (\omega_{\mathcal{L}})_{s(\bar{y})}(X_{\mathcal{L}}(s(\bar{y})) - T_{\bar{y}}s(X_{\bar{y}}), T_{\bar{y}}s(u_{\bar{y}})) \\ &= (\omega_{\mathcal{L}})_{s(\bar{y})}(D_{\mathcal{L}}(\bar{y}), T_{\bar{y}}s(u_{\bar{y}})). \end{aligned}$$

Therefore, $(\omega_{\mathcal{L}})_{s(\bar{y})}(D_{\mathcal{L}}(\bar{y}), A_{s(\bar{y})}) = 0$, for every $A_{s(\bar{y})} \in T_{s(\bar{y})}\text{Im}(s)$. Now recall that every section defines a canonical splitting of the tangent space of $T^{2k-1}Q$ at every point given by

$$T_{s(\bar{y})}T^{2k-1}Q = T_{s(\bar{y})}\text{Im}(s) \oplus V_{s(\bar{y})}(\rho_{k-1}^{2k-1}).$$

Thus, we only need to prove that $(\omega_{\mathcal{L}})_{s(\bar{y})}(D_{\mathcal{L}}(\bar{y}), B_{s(\bar{y})}) = 0$, for every vertical tangent vector $B_{s(\bar{y})} \in V_{s(\bar{y})}(\rho_{k-1}^{2k-1})$. Equivalently, as $\omega_{\mathcal{L}}$ is annihilated by the contraction of two ρ_{k-1}^{2k-1} -vertical vectors, it suffices to prove that $D_{\mathcal{L}}$ is vertical with respect to that submersion. Indeed,

$$\begin{aligned} T\rho_{k-1}^{2k-1} \circ D_{\mathcal{L}} &= T\rho_{k-1}^{2k-1} \circ (X_{\mathcal{L}} \circ s - Ts \circ X) \\ &= T\rho_{k-1}^{2k-1} \circ X_{\mathcal{L}} \circ s - T\rho_{k-1}^{2k-1} \circ Ts \circ X \\ &= T\rho_{k-1}^{2k-1} \circ X_{\mathcal{L}} \circ s - T(\rho_{k-1}^{2k-1} \circ s) \circ X \\ &= T\rho_{k-1}^{2k-1} \circ X_{\mathcal{L}} \circ s - X = 0 \end{aligned}$$

Therefore $(\omega_{\mathcal{L}})_{s(\bar{y})}(D_{\mathcal{L}}(\bar{y}), Z_{s(\bar{y})}) = 0$, for every $Z_{s(\bar{y})} \in T_{s(\bar{y})}T^{2k-1}Q$, and as $\omega_{\mathcal{L}}$ is non-degenerate, we have that $X_{\mathcal{L}}$ and X are s -related, and by Proposition 1 s is a solution to the generalized k th-order Lagrangian Hamilton-Jacobi problem. \blacksquare

Observe that if $s \in \Gamma(\rho_{k-1}^{2k-1})$ is a solution to the generalized k th-order Lagrangian Hamilton-Jacobi problem then, taking into account Corollary 1, we can conclude that the integral curves of the Lagrangian vector field $X_{\mathcal{L}}$ contained in $\text{Im}(s)$ project to $T^{k-1}Q$ by ρ_{k-1}^{2k-1} to integral curves of $T\rho_{k-1}^{2k-1} \circ X_{\mathcal{L}} \circ s$. The converse, however, is not true unless we assume further assumptions.

Remark: Notice that, except for the third item in Proposition 2, all the results stated in this Section hold for every vector field $Z \in \mathfrak{X}(T^{2k-1}Q)$, not only for the Lagrangian vector field $X_{\mathcal{L}}$. Indeed, the assumption for $X_{\mathcal{L}}$ being the Lagrangian vector field solution to the equation (1) is only needed to prove that the section $s \in \Gamma(\rho_{k-1}^{2k-1})$ and its associated vector field $X \in \mathfrak{X}(T^{k-1}Q)$ satisfy some kind of dynamical equation in $T^{k-1}Q$.

Coordinate expression: Let (q_0^A) be local coordinates in Q , and $(q_0^A, \dots, q_{2k-1}^A)$ the induced natural coordinates in $T^{2k-1}Q$. Then, local coordinates in $T^{2k-1}Q$ adapted to the ρ_{k-1}^{2k-1} -bundle structure are $(q_0^A, \dots, q_{k-1}^A; q_k^A, \dots, q_{2k-1}^A) \equiv (q_i^A; q_j^A)$, where $i = 0, \dots, k-1$ and $j = k, \dots, 2k-1$. Until the end of this section, the indices i, j will take the values above. Hence, a section $s \in \Gamma(\rho_{k-1}^{2k-1})$ is given locally by $s(q_0^A, \dots, q_{k-1}^A) = (q_0^A, \dots, q_{k-1}^A, s_k^A, \dots, s_{2k-1}^A) \equiv (q_i^A, s_j^A)$, where s_j^A are local smooth functions in $T^{k-1}Q$.

Let us check what is the local condition for a section $s \in \Gamma(\rho_{k-1}^{2k-1})$ to be a solution to the generalized k th-order Lagrangian Hamilton-Jacobi problem. By Proposition 2, this is equivalent to require the Lagrangian vector field $X_{\mathcal{L}} \in \mathfrak{X}(T^{2k-1}Q)$ to be tangent to the submanifold $\text{Im}(s) \hookrightarrow T^{2k-1}Q$. As $\text{Im}(s)$ is locally defined by the constraints $q_j^A - s_j^A = 0$, we must require $L(X_{\mathcal{L}})(q_j^A - s_j^A) \equiv X_{\mathcal{L}}(q_j^A - s_j^A) = 0$ (on $\text{Im}(s)$), for $k \leq j \leq 2k-1$, $1 \leq A \leq n$. From [27, 34] we know that the Lagrangian vector field $X_{\mathcal{L}}$ has the following local expression

$$X_{\mathcal{L}} = q_1^A \frac{\partial}{\partial q_0^A} + q_2^A \frac{\partial}{\partial q_1^A} + \dots + q_{2k-1}^A \frac{\partial}{\partial q_{2k-2}^A} + F^A \frac{\partial}{\partial q_{2k-1}^A},$$

where F^A are the functions solution to the following system of n equations

$$(-1)^k (F^B - d_T(q_{2k-1}^B)) \frac{\partial^2 \mathcal{L}}{\partial q_k^B \partial q_k^A} + \sum_{l=0}^k (-1)^l d_T^l \left(\frac{\partial \mathcal{L}}{\partial q_l^A} \right) = 0. \quad (3)$$

Hence, the condition $X_{\mathcal{L}}(q_j^A - s_j^A)|_{\text{Im}(s)} = 0$ gives the following equations

$$s_{j+1}^A - \sum_{i=0}^{k-2} q_{i+1}^B \frac{\partial s_j^A}{\partial q_i^B} - s_k^B \frac{\partial s_j^A}{\partial q_{k-1}^B} = 0 \quad ; \quad F^A|_{\text{Im}(s)} - \sum_{i=0}^{k-2} q_{i+1}^B \frac{\partial s_{2k-1}^A}{\partial q_i^B} - s_k^B \frac{\partial s_j^A}{\partial q_{k-1}^B} = 0, \quad (4)$$

This is a system of kn partial differential equations with kn unknown functions s_j^A . Thus, a section $s \in \Gamma(\rho_{k-1}^{2k-1})$ solution to the generalized k th-order Lagrangian Hamilton-Jacobi problem must satisfy the local equations (4).

3.2 The Lagrangian Hamilton-Jacobi problem

In general, to solve the generalized k th-order Lagrangian Hamilton-Jacobi problem can be a difficult task, since it amounts to find kn -dimensional submanifolds of $T^{2k-1}Q$ invariant by the Lagrangian vector field $X_{\mathcal{L}}$, or, equivalently, solutions to a large system of partial differential equations with many unknown functions. Therefore, in order to simplify the problem, it is convenient to impose some additional conditions to the section $s \in \Gamma(\rho_{k-1}^{2k-1})$, thus considering a less general problem.

Definition 6 *The k th-order Lagrangian Hamilton-Jacobi problem consists in finding sections $s \in \Gamma(\rho_{k-1}^{2k-1})$ solution to the generalized k th-order Lagrangian Hamilton-Jacobi problem satisfying that $s^* \omega_{\mathcal{L}} = 0$. Such a section is called a solution to the k th-order Lagrangian Hamilton-Jacobi problem.*

With the new assumption in Definition 6, a straightforward consequence of Proposition 2 is:

Proposition 3 *Let $s \in \Gamma(\rho_{k-1}^{2k-1})$ such that $s^* \omega_{\mathcal{L}} = 0$. The following assertions are equivalent:*

1. *The section s is a solution to the k th-order Lagrangian Hamilton-Jacobi problem.*
2. $d(s^* E_{\mathcal{L}}) = 0$.
3. *$\text{Im}(s)$ is a Lagrangian submanifold of $T^{2k-1}Q$ invariant by $X_{\mathcal{L}}$.*
4. *The integral curves of $X_{\mathcal{L}}$ with initial conditions in $\text{Im}(s)$ project onto the integral curves of $X = T\rho_{k-1}^{2k-1} \circ X_{\mathcal{L}} \circ s$.*

Coordinate expression: In coordinates we have

$$dE_{\mathcal{L}} = \frac{\partial E_{\mathcal{L}}}{\partial q_i^A} dq_i^A + \frac{\partial E_{\mathcal{L}}}{\partial q_j^A} dq_j^A \quad ; \quad (0 \leq i \leq k-1, k \leq j \leq 2k-1),$$

and taking the pull-back of this 1-form by the section $s(q_i^A) = (q_i^A, s_j^A)$, we obtain

$$s^*(dE_{\mathcal{L}}) = \left(\frac{\partial E_{\mathcal{L}}}{\partial q_i^A} + \frac{\partial E_{\mathcal{L}}}{\partial q_j^B} \frac{\partial s_j^B}{\partial q_i^A} \right) dq_i^A.$$

Hence, the condition $d(s^*E_{\mathcal{L}}) = 0$ in Proposition 3 is equivalent to the following kn partial differential equations (on $\text{Im}(s)$)

$$\frac{\partial E_{\mathcal{L}}}{\partial q_i^A} + \frac{\partial E_{\mathcal{L}}}{\partial q_j^B} \frac{\partial s_j^B}{\partial q_i^A} = 0 \quad (5)$$

Therefore, a section $s \in \Gamma(\rho_{k-1}^{2k-1})$ given locally by $s(q_i^A) = (q_i^A, s_j^A(q_i^A))$ is a solution to the k th-order Lagrangian Hamilton-Jacobi problem if, and only if, the local functions s_j^A satisfy the system of $2kn$ partial differential equations given by (4) and (5). Note that these $2kn$ partial differential equations may not be $C^\infty(U)$ -linearly independent.

In addition to the local equations for the section $s \in \Gamma(\rho_{k-1}^{2k-1})$, we can state the equations for the characteristic Hamilton-Jacobi function. These equations are a generalization to higher-order systems of the classical Lagrangian Hamilton-Jacobi equations:

$$\frac{\partial S}{\partial q^A} = \frac{\partial \mathcal{L}}{\partial v^A}(q^A, X^A) \quad (6)$$

where $S \in C^\infty(Q)$ is the characteristic function, $\mathcal{L} \in C^\infty(\text{T}Q)$ is the first-order Lagrangian, (q^A, v^A) are the natural coordinates in $\text{T}Q$ and X^A are the component functions of the vector field $X \in \mathfrak{X}(Q)$ solution to the Lagrangian Hamilton-Jacobi problem [8].

As $\omega_{\mathcal{L}} = -d\theta_{\mathcal{L}}$, it is clear that $s^*\omega_{\mathcal{L}} = 0$ if, and only if, $s^*(d\theta_{\mathcal{L}}) = d(s^*\theta_{\mathcal{L}}) = 0$; that is, $s^*\theta_{\mathcal{L}} \in \Omega^1(\text{T}^{k-1}Q)$ is a closed 1-form. Using Poincaré's Lemma, $s^*\theta_{\mathcal{L}}$ is locally exact, and thus there exists $W \in C^\infty(U)$, with $U \subseteq \text{T}^{k-1}Q$ an open set, such that $s^*\theta_{\mathcal{L}}|_U = dW$. In coordinates,

$$\theta_{\mathcal{L}} = \sum_{i=0}^{k-1} \sum_{l=0}^{k-i-1} (-1)^l d_T^l \left(\frac{\partial \mathcal{L}}{\partial q_{i+l+1}^A} \right) dq_i^A,$$

and thus the local expression of $s^*\theta_{\mathcal{L}}$ is

$$s^*\theta_{\mathcal{L}} = \sum_{i=0}^{k-1} \sum_{l=0}^{k-i-1} (-1)^l d_T^l \left(\frac{\partial \mathcal{L}}{\partial q_{i+l+1}^A} \right) \Big|_{\text{Im}(s)} dq_i^A$$

Hence, from the identity $s^*\theta_{\mathcal{L}} = dW$ we obtain

$$\frac{\partial W}{\partial q_i^A} = \sum_{l=0}^{k-i-1} (-1)^l d_T^l \left(\frac{\partial \mathcal{L}}{\partial q_{i+l+1}^A} \right) \Big|_{\text{Im}(s)}, \quad (7)$$

which is a system of kn partial differential equations for W that clearly generalizes equations (6) to higher-order systems.

3.3 Complete solutions

In the above Sections we have stated the k th-order Hamilton-Jacobi problem in the Lagrangian formalism, and a section $s \in \Gamma(\rho_{k-1}^{2k-1})$ solution to this problem gives a particular solution to the dynamical equation (1). Nevertheless, this is not a complete solution to the system, since only the integral curves of $X_{\mathcal{L}}$ with initial conditions in $\text{Im}(s)$ can be recovered from the solution to the Hamilton-Jacobi problem. Hence, in order to obtain a complete solution to the problem, we need to foliate the phase space $\mathbb{T}^{2k-1}Q$ in such a way that every leaf is the image set of a section solution to the k th-order Lagrangian Hamilton-Jacobi problem. The precise definition is:

Definition 7 *A complete solution to the k th-order Lagrangian Hamilton-Jacobi problem is a local diffeomorphism $\Phi: U \times \mathbb{T}^{k-1}Q \rightarrow \mathbb{T}^{2k-1}Q$, with $U \subseteq \mathbb{R}^{kn}$ an open set, such that for every $\lambda \in U$, the map $s_{\lambda}(\bullet) \equiv \Phi(\lambda, \bullet): \mathbb{T}^{k-1}Q \rightarrow \mathbb{T}^{2k-1}Q$ is a solution to the k th-order Lagrangian Hamilton-Jacobi problem.*

Remark: Usually, it is the set of maps $\{s_{\lambda} \mid \lambda \in U\}$ which is called a complete solution of the k th-order Lagrangian Hamilton-Jacobi problem, instead of the map Φ . Both definitions are clearly equivalent.

It follows from this last definition that a complete solution provides $\mathbb{T}^{2k-1}Q$ with a foliation transverse to the fibers, and that every leaf of this foliation has dimension kn and is invariant by the Lagrangian vector field $X_{\mathcal{L}}$.

Let Φ be a complete solution, and we consider the family of vector fields

$$\left\{ X_{\lambda} = \mathbb{T}\rho_{k-1}^{2k-1} \circ X_{\mathcal{L}} \circ s_{\lambda} \in \mathfrak{X}(\mathbb{T}^{k-1}Q) ; \lambda \in U \subseteq \mathbb{R}^{kn} \right\},$$

where $s_{\lambda} \equiv \Phi(\lambda, \bullet)$. Then, the integral curves of X_{λ} , for different $\lambda \in U$, will provide all the integral curves of the Lagrangian vector field $X_{\mathcal{L}}$. That is, if $\bar{y} \in \mathbb{T}^{2k-1}Q$, then there exists $\lambda_o \in U$ such that if $p_o = \rho_{k-1}^{2k-1}(\bar{y})$, then $s_{\lambda_o}(p_o) = \bar{y}$, and the integral curve of X_{λ_o} through p_o , lifted to $\mathbb{T}^{2k-1}Q$ by s_{λ_o} , gives the integral curve of $X_{\mathcal{L}}$ through \bar{y} .

4 The Hamilton-Jacobi problem in the Hamiltonian formalism

Let Q be a n -dimensional smooth manifold modeling the configuration space of a k th-order autonomous dynamical system with n degrees of freedom, and let $h \in C^{\infty}(\mathbb{T}^*(\mathbb{T}^{k-1}Q))$ be a Hamiltonian function containing the dynamical information for this system. Using the canonical Liouville forms of the cotangent bundle, namely $\theta_{k-1} = p_A^i dq_i^A \in \Omega^1(\mathbb{T}^*(\mathbb{T}^{k-1}Q))$ and $\omega_{k-1} = dq_i^A \wedge dp_A^i \in \Omega^2(\mathbb{T}^*(\mathbb{T}^{k-1}Q))$, where (q_i^A, p_A^i) with $1 \leq A \leq n$, $0 \leq i \leq k-1$ are canonical coordinates on $\mathbb{T}^*(\mathbb{T}^{k-1}Q)$, we can state the dynamical equation for this Hamiltonian system,

$$i(X_h)\omega_{k-1} = dh, \tag{8}$$

which has a unique solution $X_h \in \mathfrak{X}(\mathbb{T}^*(\mathbb{T}^{k-1}Q))$ due to the fact that ω_{k-1} is non-degenerate, regardless of the Hamiltonian function provided.

As the formalism is developed in the cotangent bundle $\mathbb{T}^*(\mathbb{T}^{k-1}Q)$, the statement of the Hamiltonian Hamilton-Jacobi theory for higher-order systems follows the same pattern as in the first-order case (see [8]). Next we detail the main results.

4.1 The generalized Hamiltonian Hamilton-Jacobi problem

Definition 8 *The generalized k th-order Hamiltonian Hamilton-Jacobi problem consists in finding a 1-form $\alpha \in \Omega^1(\mathbb{T}^{k-1}Q)$ and a vector field $X \in \mathfrak{X}(\mathbb{T}^{k-1}Q)$ such that, if $\gamma: \mathbb{R} \rightarrow \mathbb{T}^{k-1}Q$ is an integral curve of X , then $\alpha \circ \gamma: \mathbb{R} \rightarrow \mathbb{T}^*(\mathbb{T}^{k-1}Q)$ is an integral curve of X_h ; that is,*

$$X \circ \gamma = \dot{\gamma} \implies X_h \circ (\alpha \circ \gamma) = \overline{\alpha \circ \dot{\gamma}}. \quad (9)$$

Proposition 4 *The pair $(\alpha, X) \in \Omega^1(\mathbb{T}^{k-1}Q) \times \mathfrak{X}(\mathbb{T}^{k-1}Q)$ satisfies the condition (9) if, and only if, X and X_h are α -related, that is, $X_h \circ \alpha = \mathbb{T}\alpha \circ X$.*

(Proof) The proof is like in the Lagrangian case and is the same as the one of Prop. 5 in [8]. ■

Now, from Proposition 4, composing both sides of the equality $X_h \circ \alpha = \mathbb{T}\alpha \circ X$ with $\mathbb{T}\pi_{\mathbb{T}^{k-1}Q}$, and bearing in mind that $\alpha \in \Omega^1(\mathbb{T}^{k-1}Q) = \Gamma(\pi_{\mathbb{T}^{k-1}Q})$, we obtain the following result:

Corollary 2 *If (α, X) satisfies condition (9), then $X = \mathbb{T}\pi_{\mathbb{T}^{k-1}Q} \circ X_h \circ \alpha$.*

Hence, the vector field $X \in \mathfrak{X}(\mathbb{T}^{k-1}Q)$ is completely determined by the 1-form α , and it is called the *vector field associated to α* . The following diagram illustrates the situation

$$\begin{array}{ccc}
 \mathbb{T}(\mathbb{T}^{k-1}Q) & \xleftarrow{\mathbb{T}\pi_{\mathbb{T}^{k-1}Q}} & \mathbb{T}(\mathbb{T}^*(\mathbb{T}^{k-1}Q)) \\
 \uparrow X & \xleftarrow{\mathbb{T}\alpha} & \uparrow X_h \\
 \mathbb{T}^{k-1}Q & \xleftarrow{\pi_{\mathbb{T}^{k-1}Q}} & \mathbb{T}^*(\mathbb{T}^{k-1}Q) \\
 & \xleftarrow{\alpha} &
 \end{array}$$

Since the vector field X is completely determined by the 1-form α , the problem of finding a pair $(\alpha, X) \in \Omega^1(\mathbb{T}^{k-1}Q) \times \mathfrak{X}(\mathbb{T}^{k-1}Q)$ that satisfies the condition (9) is equivalent to the problem of finding a 1-form $\alpha \in \Omega^1(\mathbb{T}^{k-1}Q)$ satisfying the same condition with the associated vector field $\mathbb{T}\pi_{\mathbb{T}^{k-1}Q} \circ X_h \circ \alpha \in \mathfrak{X}(\mathbb{T}^{k-1}Q)$. Hence, we can define:

Definition 9 *A solution to the generalized k th-order Hamiltonian Hamilton-Jacobi problem for X_h is a 1-form $\alpha \in \Omega^1(\mathbb{T}^{k-1}Q)$ such that if $\gamma: \mathbb{R} \rightarrow \mathbb{T}^{k-1}Q$ is an integral curve of $X = \mathbb{T}\pi_{\mathbb{T}^{k-1}Q} \circ X_h \circ \alpha$, then $\alpha \circ \gamma: \mathbb{R} \rightarrow \mathbb{T}^*(\mathbb{T}^{k-1}Q)$ is an integral curve of X_h ; that is,*

$$\mathbb{T}\pi_{\mathbb{T}^{k-1}Q} \circ X_h \circ \alpha \circ \gamma = \dot{\gamma} \implies X_h \circ (\alpha \circ \gamma) = \overline{\alpha \circ \dot{\gamma}}.$$

Proposition 5 *The following conditions on a 1-form $\alpha \in \Omega^1(\mathbb{T}^{k-1}Q)$ are equivalent.*

1. *The form α is a solution to the generalized k th-order Hamiltonian Hamilton-Jacobi problem.*
2. *The submanifold $\text{Im}(\alpha) \hookrightarrow \mathbb{T}^*(\mathbb{T}^{k-1}Q)$ is invariant under the flow of the Hamiltonian vector field X_h (that is, X_h is tangent to the submanifold $\text{Im}(\alpha)$).*

3. The form α satisfies the equation

$$i(X)d\alpha = -d(\alpha^*h),$$

where $X = T\pi_{T^{k-1}Q} \circ X_h \circ \alpha$ is the vector field associated to α .

(Proof) The proof follows exactly the same pattern as in Proposition 2, taking into account that from the properties of the tautological form $\theta_{k-1} \in \Omega^1(T^*(T^{k-1}Q))$ of the cotangent bundle, that is, we have $\alpha^*\theta_{k-1} = \alpha$ for every $\alpha \in \Omega^1(T^{k-1}Q)$. Hence, taking the pull-back of the dynamical equation (8) by α we obtain

$$i(X)d\alpha = -d(\alpha^*h),$$

because we have

$$\alpha^*\omega_{k-1} = \alpha^*(-d\theta_{k-1}) = -d(\alpha^*\theta_{k-1}) = -d\alpha. \quad (10)$$

■

Coordinate expression: Let (q_0^A) be local coordinates in Q and $(q_0^A, \dots, q_{k-1}^A)$ the induced natural coordinates in $T^{k-1}Q$. Then, $(q_0^A, \dots, q_{k-1}^A, p_A^0, \dots, p_A^{k-1}) \equiv (q_i^A, p_A^i)$ are natural coordinates in $T^*(T^{k-1}Q)$, which are also the adapted coordinates to the $\pi_{T^{k-1}Q}$ -bundle structure. Hence, a 1-form $\alpha \in \Omega^1(T^{k-1}Q)$ is given locally by $\alpha(q_i^A) = (q_i^A, \alpha_A^i) = \alpha_A^i dq_i^A$, where α_A^i are local smooth functions in $T^{k-1}Q$.

If $\alpha \in \Omega^1(T^{k-1}Q)$ is a solution to the generalized k th-order Hamiltonian Hamilton-Jacobi problem, then by Proposition 5 this is equivalent to require the Hamiltonian vector field $X_h \in \mathfrak{X}(T^*(T^{k-1}Q))$ to be tangent to the submanifold $\text{Im}(\alpha) \hookrightarrow T^*(T^{k-1}Q)$. This submanifold is locally defined by the constraints $p_A^l - \alpha_A^l = 0$. Thus, we must require $L(X_h)(p_A^i - \alpha_A^i) \equiv X_h(p_A^i - \alpha_A^i) = 0$ (on $\text{Im}(\alpha)$). The Hamiltonian vector field X_h is locally given by [27, 34]

$$X_h = \frac{\partial h}{\partial p_A^i} \frac{\partial}{\partial q_i^A} - \frac{\partial h}{\partial q_i^A} \frac{\partial}{\partial p_A^i}. \quad (11)$$

Hence, the conditions $X_h(p_A^i - \alpha_A^i)|_{\text{Im}(\alpha)} = 0$ give the equations

$$-\frac{\partial h}{\partial q_i^A} - \frac{\partial h}{\partial p_B^j} \frac{\partial \alpha_A^i}{\partial q_j^B} = 0 \quad , \quad (\text{on } \text{Im}(\alpha)) . \quad (12)$$

This is a system of kn partial differential equations with kn unknown functions α_A^i which must be verified by every 1-form $\alpha \in \Omega^1(T^{k-1}Q)$ solution to the generalized k th-order Hamiltonian Hamilton-Jacobi.

4.2 The Hamiltonian Hamilton-Jacobi problem

As in the Lagrangian setting, it is convenient to consider a less general problem requiring some additional conditions to the 1-form $\alpha \in \Omega^1(T^{k-1}Q)$. Observe that from (10) the condition $\alpha^*\omega_{k-1} = 0$ is equivalent to $d\alpha = 0$; that is, α is a closed 1-form in $T^{k-1}Q$. Therefore:

Definition 10 *The k th-order Hamiltonian Hamilton-Jacobi problem consists in finding closed 1-forms $\alpha \in \Omega^1(T^{k-1}Q)$ solution to the generalized Hamiltonian Hamilton-Jacobi problem. Such a form is called a solution to the k th-order Hamiltonian Hamilton-Jacobi problem.*

A straightforward consequence of Proposition 5 is the following result:

Proposition 6 *Let $\alpha \in \Omega^1(\mathbb{T}^{k-1}Q)$ be a closed 1-form. The following assertions are equivalent:*

1. *The 1-form α is a solution to the k th-order Hamiltonian Hamilton-Jacobi problem.*
2. $d(\alpha^*h) = 0$.
3. *$\text{Im}(\alpha)$ is a Lagrangian submanifold of $\mathbb{T}^*(\mathbb{T}^{k-1}Q)$ invariant by X_h .*
4. *The integral curves of X_h with initial conditions in $\text{Im}(\alpha)$ project onto the integral curves of $X = \mathbb{T}\pi_{\mathbb{T}^{k-1}Q} \circ X_h \circ \alpha$.*

Coordinate expression: In coordinates we have

$$dh = \frac{\partial h}{\partial q_i^A} dq_i^A + \frac{\partial h}{\partial p_A^i} dp_A^i \quad ; \quad (0 \leq i \leq k-1),$$

and taking the pull-back of dh by the 1-form $\alpha = \alpha_A^i dq_i^A$, we have

$$\alpha^*(dh) = \left(\frac{\partial h}{\partial q_i^A} + \frac{\partial h}{\partial p_B^j} \frac{\partial \alpha_B^j}{\partial q_i^A} \right) dq_i^A.$$

Hence, the condition $d(\alpha^*h) = 0$ in Proposition 6 holds if, and only if, the following kn partial differential equations hold

$$\frac{\partial h}{\partial q_i^A} + \frac{\partial h}{\partial p_B^j} \frac{\partial \alpha_B^j}{\partial q_i^A} = 0 \tag{13}$$

Equivalently, we can require the 1-form $\alpha \in \Omega^1(\mathbb{T}^{k-1}Q)$ to be closed, that is, $d\alpha = 0$. Locally, this condition reads

$$\begin{aligned} \frac{\partial \alpha_A^i}{\partial q_i^B} &= 0, \quad \text{if } A \neq B, \\ \frac{\partial \alpha_A^i}{\partial q_j^B} - \frac{\partial \alpha_B^j}{\partial q_i^A} &= 0. \end{aligned} \tag{14}$$

Therefore, a 1-form $\alpha \in \Omega^1(\mathbb{T}^{k-1}Q)$ given locally by $\alpha = \alpha_A^i dq_i^A$ is a solution to the k th-Hamiltonian Hamilton-Jacobi problem if, and only if, the local functions α_A^i satisfy the $2kn$ partial differential equations given by (12) and (13), or equivalently (12) and (14). Observe that these $2kn$ partial differential equations may not be $C^\infty(U)$ -linearly independent.

In addition to the local equations for the 1-form $\alpha \in \Omega^1(\mathbb{T}^{k-1}Q)$, in this particular situation we can give the equation for the characteristic Hamilton-Jacobi function. This equation is a generalization to higher-order systems of the classical Hamiltonian Hamilton-Jacobi equation

$$h \left(q^A, \frac{\partial S}{\partial q^A} \right) = E \tag{15}$$

where $E \in \mathbb{R}$ is a constant, $S \in C^\infty(Q)$ is the characteristic function and $h \in C^\infty(\mathbb{T}^*Q)$ is the Hamiltonian function.

As $\alpha \in \Omega^1(\mathbb{T}^{k-1}Q)$ is closed, by Poincaré's Lemma there exists a function $W \in C^\infty(U)$, with $U \subseteq \mathbb{T}^{k-1}Q$ an open set, such that $\alpha = dW$. In coordinates the condition $\alpha = dW$ gives the following kn partial differential equations for W

$$\frac{\partial W}{\partial q_i^A} = \alpha_A^i. \quad (16)$$

Finally, as $\alpha^*h = h(q_i^A, \alpha_A^i) = h\left(q_i^A, \frac{\partial W}{\partial q_i^A}\right)$, the condition α^*h being locally constant gives

$$h\left(q_i^A, \frac{\partial W}{\partial q_i^A}\right) = E, \quad (17)$$

where $E \in \mathbb{R}$ is a local constant. This equation clearly generalizes the equation (15) to higher-order systems.

4.3 Complete solutions

The concept of *complete solution* is defined in an analogous way as in Section 3.3.

Definition 11 *A complete solution to the k th-order Hamiltonian Hamilton-Jacobi problem is a local diffeomorphism $\Phi: U \times \mathbb{T}^{k-1}Q \rightarrow \mathbb{T}^*(\mathbb{T}^{k-1}Q)$, where $U \subseteq \mathbb{R}^{kn}$ is an open set, such that, for every $\lambda \in U$, the map $\alpha_\lambda(\bullet) \equiv \Phi(\lambda, \bullet): \mathbb{T}^{k-1}Q \rightarrow \mathbb{T}^*(\mathbb{T}^{k-1}Q)$ is a solution to the k th-order Hamiltonian Hamilton-Jacobi problem.*

Then, the set $\{\alpha_\lambda \mid \lambda \in U\}$ is also called a complete solution to the k th-order Hamiltonian Hamilton-Jacobi problem.

It follows from the definition that a complete solution endows $\mathbb{T}^*(\mathbb{T}^{k-1}Q)$ with a foliation transverse to the fibers, and that the Hamiltonian vector field X_h is tangent to the leaves.

Let $\{\alpha_\lambda \mid \lambda \in U\}$ be a complete solution, and we consider the set of associated vector fields

$$\left\{ X_\lambda = \mathbb{T}\pi_{\mathbb{T}^{k-1}Q} \circ X_h \circ \alpha_\lambda \in \mathfrak{X}(\mathbb{T}^{k-1}Q) ; \lambda \in U \subseteq \mathbb{R}^{kn} \right\}.$$

Then, the integral curves of X_λ , for different $\lambda \in U$, will provide all the integral curves of the Hamiltonian vector field X_h . That is, if $\beta \in \mathbb{T}^*(\mathbb{T}^{k-1}Q)$, then there exists $\lambda_o \in U$ such that if $p_o = \pi_{\mathbb{T}^{k-1}Q}(\beta)$, then $\alpha_{\lambda_o}(p_o) = \beta$, and the integral curve of X_{λ_o} through p_o , lifted to $\mathbb{T}^*(\mathbb{T}^{k-1}Q)$ by α_{λ_o} , gives the integral curve of X_h through β .

Let us assume that Φ is a global diffeomorphism for simplicity. Then, given $\lambda = (\lambda_i^A) \in \mathbb{R}^{kn}$, $0 \leq i \leq k-1$, $1 \leq A \leq n$, we consider the functions $f_j^B: \mathbb{T}^*(\mathbb{T}^{k-1}Q) \rightarrow \mathbb{R}$, $0 \leq j \leq k-1$, $1 \leq B \leq n$, given by

$$f_j^B = \text{pr}_j^B \circ \text{p}_1 \circ \Phi^{-1},$$

where $\text{p}_1: \mathbb{R}^{kn} \times \mathbb{T}^{k-1}Q \rightarrow \mathbb{R}^{kn}$ is the projection onto the first factor and $\text{pr}_j^B: \mathbb{R}^{kn} \rightarrow \mathbb{R}$ is given by $\text{pr}_j^B = \text{pr}^B \circ \text{pr}_j$, where pr^B and pr_i are the natural projections

$$\begin{array}{ccc} \text{pr}_j: \mathbb{R}^{kn} & \longrightarrow & \mathbb{R}^n \\ (x_i^A) & \longmapsto & (x_j^1, \dots, x_j^n) \end{array} \quad ; \quad \begin{array}{ccc} \text{pr}^B: \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ (x^1, \dots, x^n) & \longmapsto & x^B. \end{array}$$

Therefore, $f_j^B(\alpha_\lambda(q_i^A)) = (\text{pr}_j^B \circ \text{p}_1 \circ \Phi^{-1} \circ \Phi)(\lambda_i^A, q_i^A) = \lambda_j^B$.

Proposition 7 *The functions f_j^B , $0 \leq j \leq k-1$, $1 \leq B \leq n$ are in involution.*

(Proof) Let $\beta \in \mathbb{T}^*(\mathbb{T}^{k-1}Q)$. We will show that $\{f_i^j, f_a^b\}(\beta) = 0$.

Since Φ is a complete solution, for every $\beta \in \mathbb{T}^*(\mathbb{T}^{k-1}Q)$ there exists $\lambda \in \mathbb{R}^{kn}$ such that $\alpha_\lambda(\pi_{\mathbb{T}^{k-1}Q}(\beta)) = \Phi(\lambda, \pi_{\mathbb{T}^{k-1}Q}(\beta)) = \beta$. Then we have

$$f_j^B(\beta) = (f_j^B \circ \alpha_\lambda)(\pi_{\mathbb{T}^{k-1}Q}(\beta)) = (\text{pr}_j^B \circ \text{p}_1 \circ \Phi^{-1} \circ \Phi)(\lambda, \pi_{\mathbb{T}^{k-1}Q}(\beta)) = \lambda_j^B,$$

that is, $f_j^B \circ \alpha_\lambda = (f_j^B \circ \Phi)(\lambda, \bullet): \mathbb{T}^{k-1}Q \rightarrow \mathbb{R}$ is constant for every $\lambda \in \mathbb{R}^{kn}$. Therefore, we have

$$df_j^B|_{\text{TIm}(\alpha_\lambda)} = 0.$$

Now, since Φ is a complete solution, we have that $\alpha_\lambda = \Phi(\lambda, \bullet)$ is a solution to the k th-order Hamiltonian Hamilton-Jacobi problem. Therefore, from Prop. 6, $\text{Im}(\Phi_\lambda)$ is a Lagrangian submanifold of $(\mathbb{T}^*(\mathbb{T}^{k-1}Q), \omega_{k-1})$, and then

$$(\text{TIm}(\alpha_\lambda))^\perp = \text{TIm}(\alpha_\lambda),$$

where $(\text{TIm}(\alpha_\lambda))^\perp$ denotes the ω_{k-1} -orthogonal of $\text{TIm}(\alpha_\lambda)$.

From this, the result follows from the definition of the induced Poisson bracket, which is

$$\{f_j^B, f_l^C\}(\beta) = \omega_{k-1}(X_{f_j^B}, X_{f_l^C})(\beta),$$

and the facts that ω_{k-1} is symplectic, $df_j^B \in (\text{TIm}(\alpha_\lambda))^\perp = \text{TIm}(\alpha_\lambda)$, and that there exists a unique vector field $X_{f_j^B} \in \mathfrak{X}(\mathbb{T}^*(\mathbb{T}^{k-1}Q))$ satisfying $i(X_{f_j^B})\omega_{k-1} = df_j^B$. ■

4.4 Relation with the Lagrangian formulation

Up to this point we have stated both the Lagrangian and Hamiltonian Hamilton-Jacobi problems for higher-order autonomous systems. Now, we establish a relation between the solutions of the Hamilton-Jacobi problem in both formulations. In particular, we show that there exists a bijection between the set of solutions of the (generalized) k th-order Lagrangian Hamilton-Jacobi problem and the set of solutions of the (generalized) k th-order Hamiltonian Hamilton-Jacobi problem, given by the Legendre-Ostrogradsky map.

Definition 12 *Let $(\mathbb{T}^{2k-1}Q, \mathcal{L})$ be a Lagrangian system. The Legendre-Ostrogradsky map (or generalized Legendre map) associated to \mathcal{L} is the map $\mathcal{FL}: \mathbb{T}^{2k-1}Q \rightarrow \mathbb{T}^*(\mathbb{T}^{k-1}Q)$ defined as follows: for every $u \in \mathbb{T}(\mathbb{T}^{2k-1}Q)$,*

$$\theta_{\mathcal{L}}(u) = \left\langle \mathbb{T}\rho_{k-1}^{2k-1}(u), \mathcal{FL}(\tau_{\mathbb{T}^{2k-1}Q}(u)) \right\rangle$$

This map verifies that $\pi_{\mathbb{T}^{k-1}Q} \circ \mathcal{FL} = \rho_{k-1}^{2k-1}$, that is, it is a bundle map over $\mathbb{T}^{k-1}Q$. Furthermore, we have that $\mathcal{FL}^*\theta_{k-1} = \theta_{\mathcal{L}}$ and $\mathcal{FL}^*\omega_{k-1} = \omega_{\mathcal{L}}$. Moreover, we have that $\mathcal{L} \in C^\infty(\mathbb{T}^kQ)$ is a regular Lagrangian if, and only if, $\mathcal{FL}: \mathbb{T}^{2k-1}Q \rightarrow \mathbb{T}^*(\mathbb{T}^{k-1}Q)$ is a local diffeomorphism, and \mathcal{L} is said to be *hyperregular* if \mathcal{FL} is a global diffeomorphism.

Remark: Observe that if \mathcal{L} is hyperregular, then \mathcal{FL} is a symplectomorphism and therefore the symplectic structures are in correspondence. Therefore, the induced Poisson brackets also are

in correspondence and we have the analogous result of (7) in the Lagrangian formalism, where the Poisson bracket is determined by the Poincaré-Cartan 2-form $\omega_{\mathcal{L}}$ as $\{f, g\} = \omega_{\mathcal{L}}(X_f, X_g)$.

Given a local natural chart in $\mathbb{T}^{2k-1}Q$, we can define the following local functions

$$\hat{p}_A^{r-1} = \sum_{i=0}^{k-r} (-1)^i d_T^i \left(\frac{\partial \mathcal{L}}{\partial q_{r+i}^A} \right).$$

Thus, bearing in mind the local expression of the form $\theta_{\mathcal{L}}$, we can write $\theta_{\mathcal{L}} = \sum_{r=1}^k \hat{p}_A^{r-1} dq_{r-1}^A$, and we obtain that the expression in natural coordinates of the map \mathcal{FL} is

$$\mathcal{FL}(q_i^A, q_j^A) = (q_i^A, p_A^i), \text{ with } p_A^i \circ \mathcal{FL} = \hat{p}_A^i.$$

Now we establish the relation theorem. First we need the following technical result:

Lemma 1 *Let $E_1 \xrightarrow{\pi_1} M$ and $E_2 \xrightarrow{\pi_2} M$ be two fiber bundles, $F: E_1 \rightarrow E_2$ a fiber bundle morphism, and two F -related vector fields $X_1 \in \mathfrak{X}(E_1)$ and $X_2 \in \mathfrak{X}(E_2)$. If $s_1 \in \Gamma(\pi_1)$ is a section of π_1 and we define a section of π_2 as $s_2 = F \circ s_1 \in \Gamma(\pi_2)$, then*

$$\mathbb{T}\pi_1 \circ X_1 \circ s_1 = \mathbb{T}\pi_2 \circ X_2 \circ s_2 \in \mathfrak{X}(M).$$

(Proof) As $F: E_1 \rightarrow E_2$ is a fiber bundle morphism (that is, $\pi_1 = \pi_2 \circ F$), and X_1 and X_2 are F -related (that is, $\mathbb{T}F \circ X_1 = X_2 \circ F$), we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{T}E_1 & \xrightarrow{\mathbb{T}F} & \mathbb{T}E_2 \\ X_1 \uparrow & & \uparrow X_2 \\ E_1 & \xrightarrow{F} & E_2 \\ \pi_1 \searrow & & \swarrow \pi_2 \\ & M & \end{array}$$

Then we have

$$\begin{aligned} \mathbb{T}\pi_1 \circ X_1 \circ s_1 &= \mathbb{T}(\pi_2 \circ F) \circ X_1 \circ s_1 = \mathbb{T}\pi_2 \circ \mathbb{T}F \circ X_1 \circ s_1 \\ &= \mathbb{T}\pi_2 \circ X_2 \circ F \circ s_1 = \mathbb{T}\pi_2 \circ X_2 \circ s_2. \end{aligned}$$

■

Then, the equivalence theorem is:

Theorem 1 *Let $(\mathbb{T}^{2k-1}Q, \mathcal{L})$ be a hyperregular Lagrangian system, $(\mathbb{T}^*(\mathbb{T}^{k-1}Q), \omega_{k-1}, h)$ its associated Hamiltonian system.*

1. *If $s \in \Gamma(\rho_{k-1}^{2k-1})$ is a solution to the (generalized) k th-order Lagrangian Hamilton-Jacobi problem, then the 1-form $\alpha = \mathcal{FL} \circ s \in \Omega^1(\mathbb{T}^{k-1}Q)$ is a solution to the (generalized) k th-order Hamiltonian Hamilton-Jacobi problem.*
2. *If $\alpha \in \Omega^1(\mathbb{T}^{k-1}Q)$ is a solution to the (generalized) k th-order Hamiltonian Hamilton-Jacobi problem, then the section $s = \mathcal{FL}^{-1} \circ \alpha \in \Gamma(\rho_{k-1}^{2k-1})$ is a solution to the (generalized) k th-order Lagrangian Hamilton-Jacobi problem.*

(Proof)

1. Let $X = \mathbb{T}\rho_{k-1}^{2k-1} \circ X_{\mathcal{L}} \circ s$ and $\bar{X} = \mathbb{T}\pi_{\mathbb{T}^{k-1}Q} \circ X_h \circ \alpha$ be the vector fields associated to s and $\alpha = \mathcal{FL} \circ s$, respectively. From Lemma 1 we have $X = \bar{X}$, and hence both vector fields are denoted by X .

Let s be a solution to the generalized k th-order Lagrangian Hamilton-Jacobi problem, and $\gamma: \mathbb{R} \rightarrow \mathbb{T}^{k-1}Q$ an integral curve of X . Therefore

$$\begin{aligned} X_h \circ (\alpha \circ \gamma) &= X_h \circ \mathcal{FL} \circ s \circ \gamma = \mathbb{T}\mathcal{FL} \circ X_{\mathcal{L}} \circ s \circ \gamma \\ &= \mathbb{T}\mathcal{FL} \circ \mathbb{T}s \circ X \circ \gamma = \mathbb{T}(\mathcal{FL} \circ s) \circ \dot{\gamma} \\ &= \mathbb{T}\alpha \circ \dot{\gamma} = \overline{\alpha \circ \dot{\gamma}}, \end{aligned}$$

then $\alpha \circ \gamma$ is an integral curve of X_h and thus α is a solution to the generalized k th-order Hamiltonian Hamilton-Jacobi problem.

Now, in addition, we require $s^*\omega_{\mathcal{L}} = 0$; that is, s is a solution to the k th-order Lagrangian Hamilton-Jacobi problem. Then, using (10) we have

$$d\alpha = \alpha^*\omega_{k-1} = (\mathcal{FL} \circ s)^*\omega_{k-1} = s^*(\mathcal{FL}^*\omega_{k-1}) = s^*\omega_{\mathcal{L}} = 0,$$

and hence α is a solution to the k th-order Hamiltonian Hamilton-Jacobi problem.

2. The proof is analogous to the one for the item 1, but using \mathcal{FL}^{-1} .

■

This result can be extended to complete solutions in a natural way.

Obviously, for regular but not hyperregular Lagrangian functions, all these results hold only in the open sets where \mathcal{FL} is a diffeomorphism.

Theorem 1 allows us to show that the vector field associated to a section solution to the (generalized) Hamilton-Jacobi problem is a semispray of type 1. First, we need the following technical result:

Lemma 2 *Let $X \in \mathfrak{X}(\mathbb{T}^rQ)$ be a semispray of type 1 on \mathbb{T}^rQ , and $Y \in \mathfrak{X}(\mathbb{T}^sQ)$ ($s \leq r$) which is ρ_s^r -related with X . Then Y is a semispray of type 1 on \mathbb{T}^sQ .*

(Proof) Let $\gamma: \mathbb{R} \rightarrow \mathbb{T}^rQ$ be an integral curve of X . Then, as X is a semispray of type 1, there exists a curve $\phi: \mathbb{R} \rightarrow Q$ such that $j^r\phi = \gamma$. Furthermore, as X and Y are ρ_s^r -related, the curve $\rho_s^r \circ \gamma: \mathbb{R} \rightarrow \mathbb{T}^sQ$ is an integral curve of Y . Hence, $\rho_s^r(j^r\phi) = j^s\phi$ is an integral curve of Y .

It remains to show that every integral curve of Y is the projection to \mathbb{T}^sQ via ρ_s^r of an integral curve of X , but this holds due to the fact that the vector fields are ρ_s^r -related and ρ_s^r is a surjective submersion. Therefore, Y is a semispray of type 1 in \mathbb{T}^sQ . ■

Proposition 8 *Let $(\mathbb{T}^{2k-1}Q, \mathcal{L})$ be a hyperregular Lagrangian system, and $(\mathbb{T}^*(\mathbb{T}^{k-1}Q), \omega_{k-1}, h)$ the associated Hamiltonian system. Then, if $\alpha \in \Omega^1(\mathbb{T}^{k-1}Q)$ is a solution to the k th-order Hamiltonian Hamilton-Jacobi problem, the vector field $X = \mathbb{T}\pi_{\mathbb{T}^{k-1}Q} \circ X_h \circ \alpha$ is a semispray of type 1 on $\mathbb{T}^{k-1}Q$.*

(Proof) Let $s = \mathcal{FL}^{-1} \circ \alpha \in \Gamma(\rho_{k-1}^{2k-1})$ be the section associated to α . Then, by Lemma 1, if $X = \mathbb{T}\pi_{\mathbb{T}^{k-1}Q} \circ X_h \circ \alpha$ and $\bar{X} = \mathbb{T}\rho_{k-1}^{2k-1} \circ X_{\mathcal{L}} \circ s$ are the vector fields on $\mathbb{T}^{k-1}Q$ associated to α and s respectively, then $X = \bar{X} = \mathbb{T}\rho_{k-1}^{2k-1} \circ X_{\mathcal{L}} \circ s$. Hence, as $X_{\mathcal{L}} \in \mathfrak{X}(\mathbb{T}^{2k-1}Q)$ is the Lagrangian vector field solution to the equation (1) and $\mathcal{L} \in C^\infty(\mathbb{T}^kQ)$ is a hyperregular Lagrangian function, we have that $X_{\mathcal{L}}$ is a semispray of type 1 on $\mathbb{T}^{2k-1}Q$. In particular, $X_{\mathcal{L}} \circ s$ is a semispray of type 1 along ρ_{k-1}^{2k-1} and, by Lemma 2, X is a semispray of type 1 on $\mathbb{T}^{k-1}Q$. ■

As a consequence of Proposition 8, the generalized k th-order Hamilton-Jacobi problem can be stated in the following way:

Definition 13 *The generalized k th-order Lagrangian (resp., Hamiltonian) Hamilton-Jacobi problem consists in finding a section $s \in \Gamma(\rho_{k-1}^{2k-1})$ (resp., a 1-form $\alpha \in \Omega^1(\mathbb{T}^{k-1}Q)$) such that, if $\gamma: \mathbb{R} \rightarrow Q$ satisfies that $j^{k-1}\gamma$ is an integral curve of $X = \mathbb{T}\rho_{k-1}^{2k-1} \circ X_{\mathcal{L}} \circ s$ (resp., $X = \mathbb{T}\pi_{\mathbb{T}^{k-1}Q} \circ X_h \circ \alpha$), then $s \circ j^{k-1}\gamma: \mathbb{R} \rightarrow \mathbb{T}^{2k-1}Q$ (resp., $\alpha \circ j^{k-1}\gamma: \mathbb{R} \rightarrow \mathbb{T}^*(\mathbb{T}^{k-1}Q)$) is an integral curve of $X_{\mathcal{L}}$ (resp., X_h).*

5 Examples

5.1 The end of a javelin

Consider a thrown javelin. We want to determinate the trajectory of its end. This gives rise to a 3-dimensional second-order dynamical system, which is a particular case of the problem of determining the trajectory of a particle rotating about a translating center [11]. Let $Q = \mathbb{R}^3$ be the manifold modeling the configuration space for this system with coordinates $(q_0^A, q_1^A, q_2^A) = (q_0^A)$. Using the induced coordinates in $\mathbb{T}^2\mathbb{R}^3$, the Lagrangian function for this system is

$$\mathcal{L}(q_0^A, q_1^A, q_2^A) = \frac{1}{2} \sum_{A=1}^3 ((q_1^A)^2 - (q_2^A)^2) ,$$

which is a regular Lagrangian function since the Hessian matrix of \mathcal{L} with respect to the second-order velocities is

$$\left(\frac{\partial^2 \mathcal{L}}{\partial q_2^B \partial q_2^A} \right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

The Poincaré-Cartan forms $\theta_{\mathcal{L}}$ and $\omega_{\mathcal{L}}$, and the Lagrangian energy are locally given by

$$\begin{aligned} \theta_{\mathcal{L}} &= \sum_{A=1}^3 ((q_1^A + q_3^A) dq_0^A - q_2^A dq_1^A) \quad ; \quad \omega_{\mathcal{L}} = \sum_{A=1}^3 (dq_0^A \wedge dq_1^A + dq_0^A \wedge dq_3^A - dq_1^A \wedge dq_2^A) \\ E_{\mathcal{L}} &= \frac{1}{2} \sum_{A=1}^3 ((q_1^A)^2 + 2q_1^A q_3^A - (q_2^A)^2) . \end{aligned}$$

Thus, the semispray of type 1, $X_{\mathcal{L}} \in \mathfrak{X}(\mathbb{T}^3\mathbb{R}^3)$ solution to the dynamical equation (1) is

$$X_{\mathcal{L}} = q_1^A \frac{\partial}{\partial q_0^A} + q_2^A \frac{\partial}{\partial q_1^A} + q_3^A \frac{\partial}{\partial q_2^A} - q_2^A \frac{\partial}{\partial q_3^A} .$$

Consider the projection $\rho_1^3: \mathbb{T}^3\mathbb{R}^3 \rightarrow \mathbb{T}\mathbb{R}^3$. From Proposition 2 we know that the generalized second-order Lagrangian Hamilton-Jacobi problem consists in finding sections $s \in \Gamma(\rho_1^3)$ such

that the Lagrangian vector field $X_{\mathcal{L}}$ is tangent to the submanifold $\text{Im}(s) \hookrightarrow \mathbb{T}^3\mathbb{R}^3$. Suppose that the section s is given locally by $s(q_0^A, q_1^A) = (q_0^A, q_1^A, s_2^A, s_3^A)$. As the submanifold $\text{Im}(s)$ is defined locally by the constraint functions $q_2^A - s_2^A$ and $q_3^A - s_3^A$, then the tangency condition gives the following system of 6 partial differential equations for the component functions of the section

$$s_3^A - q_1^B \frac{\partial s_2^A}{\partial q_0^B} - s_2^B \frac{\partial s_2^A}{\partial q_1^B} = 0 \quad ; \quad s_2^A + q_1^B \frac{\partial s_3^A}{\partial q_0^B} + s_2^B \frac{\partial s_3^A}{\partial q_1^B} = 0.$$

In order to obtain the equations of the second-order Lagrangian Hamilton-Jacobi problem, we require in addition the section $s \in \Gamma(\rho_1^3)$ to satisfy the condition $d(s^*E_{\mathcal{L}}) = 0$, or, equivalently, $s^*\omega_{\mathcal{L}} = 0$. From the local expression of the Cartan 2-form $\omega_{\mathcal{L}} \in \Omega^2(\mathbb{T}^3\mathbb{R}^3)$ given above, taking the pull-back by the section $s(q_0^A, q_1^A) = (q_0^A, q_1^A, s_2^A, s_3^A)$ we obtain

$$s^*\omega_{\mathcal{L}} = \sum_{A=1}^3 \left[dq_0^A \wedge dq_1^A + \frac{\partial s_3^A}{\partial q_0^B} dq_0^A \wedge dq_0^B + \left(\frac{\partial s_3^A}{\partial q_1^B} + \frac{\partial s_2^B}{\partial q_0^A} \right) dq_0^A \wedge dq_1^B - \frac{\partial s_2^A}{\partial q_1^B} dq_1^A \wedge dq_1^B \right].$$

Hence, the condition $s^*\omega_{\mathcal{L}} = 0$ gives the following partial differential equations

$$\frac{\partial s_3^A}{\partial q_0^B} = \frac{\partial s_2^A}{\partial q_1^B} = \frac{\partial s_3^A}{\partial q_1^B} + \frac{\partial s_2^B}{\partial q_0^A} = 0, \quad \text{if } A \neq B, \quad \frac{\partial s_3^A}{\partial q_1^A} + \frac{\partial s_2^A}{\partial q_0^A} + 1 = 0.$$

Hence, the section $s \in \Gamma(\rho_1^3)$ is a solution to the second-order Lagrangian Hamilton-Jacobi problem if the following system of partial differential equations hold

$$\begin{aligned} s_3^A &= q_1^B \frac{\partial s_2^A}{\partial q_0^B} + s_2^A \frac{\partial s_2^A}{\partial q_1^A} \quad , \quad q_1^A \frac{\partial s_3^A}{\partial q_0^A} + s_2^B \frac{\partial s_3^A}{\partial q_1^B} + s_2^A = 0, \\ \frac{\partial s_3^A}{\partial q_1^B} + \frac{\partial s_2^B}{\partial q_0^A} &= 0 \quad , \quad \frac{\partial s_3^A}{\partial q_1^A} + \frac{\partial s_2^A}{\partial q_0^A} + 1 = 0. \end{aligned}$$

Finally, we compute the equations for the generating function W . The pull-back of the Cartan 1-form $\theta_{\mathcal{L}}$ by the section s gives in coordinates

$$s^*\theta_{\mathcal{L}} = \sum_{A=1}^3 ((q_1^A + s_3^A)dq_0^A - s_2^A dq_1^A)$$

Hence, requiring $s^*\theta_{\mathcal{L}} = dW$ for a local function W defined in TQ we obtain

$$\frac{\partial W}{\partial q_0^A} = q_1^A + s_3^A \quad ; \quad \frac{\partial W}{\partial q_1^A} = -s_2^A,$$

and thus from $d(s^*E_{\mathcal{L}}) = 0$, we have $s^*E_{\mathcal{L}} = \text{const.}$, that is,

$$\sum_{A=1}^3 \left(q_1^A \frac{\partial W}{\partial q_0^A} - \frac{1}{2} \left((q_1^A)^2 + \left(\frac{\partial W}{\partial q_1^A} \right)^2 \right) \right) = \text{const.}$$

Remark: This equation cannot be stated in the general case, since we need to clear the higher-order velocities from the previous equations. This calculation is easy for this particular example, but it depends on the Lagrangian function provided in the general case.

Now, to establish the Hamiltonian formalism for the Hamilton-Jacobi problem, we consider natural coordinates $(q_0^A, q_1^A, p_A^0, p_A^1)$ on the cotangent bundle $\text{T}^*(\mathbb{T}\mathbb{R}^3)$, then the Legendre-Ostrogradsky map $\mathcal{FL}: \mathbb{T}^3\mathbb{R}^3 \rightarrow \text{T}^*(\mathbb{T}\mathbb{R}^3)$ associated to the Lagrangian function \mathcal{L} is

$$\mathcal{FL}^*q_0^A = q_0^A \quad ; \quad \mathcal{FL}^*q_1^A = q_1^A \quad ; \quad \mathcal{FL}^*p_A^0 = q_1^A + q_3^A \quad ; \quad \mathcal{FL}^*p_A^1 = -q_2^A,$$

and the inverse map $\mathcal{FL}^{-1}: \mathbb{T}^*(\mathbb{T}\mathbb{R}^3) \rightarrow \mathbb{T}^3\mathbb{R}^3$ is given by

$$(\mathcal{FL}^{-1})^*q_0^A = q_0^A \quad ; \quad (\mathcal{FL}^{-1})^*q_1^A = q_1^A \quad ; \quad (\mathcal{FL}^{-1})^*q_2^A = -p_A^1 \quad ; \quad (\mathcal{FL}^{-1})^*q_3^A = p_A^0 - q_1^A.$$

From these coordinate expressions it is clear that the Legendre-Ostrogradsky map is a global diffeomorphism, that is, \mathcal{L} is a hyperregular Lagrangian function.

The Hamiltonian function $h \in C^\infty(\mathbb{T}^*(\mathbb{T}\mathbb{R}^3))$ is

$$h = (\mathcal{FL}^{-1})^*E_{\mathcal{L}} = \sum_{A=1}^3 \left[p_A^0 q_1^A - \frac{1}{2} ((q_1^A)^2 + (p_A^1)^2) \right].$$

From this Hamiltonian, applying the procedure given in Section (4), one can obtain the Hamilton-Jacobi equation for this problem which coincides with the Hamilton-Jacobi equation given previously in the Lagrangian problem.

A particular solution of this Hamilton-Jacobi equation in dimension 1 has been obtained in [11]. This particular solution is

$$W(q_0, q_1) = \sqrt{2} \int dq_1 \sqrt{-\frac{1}{2}q_1^2 + c_2q_1 - c_1 + c_2q_0} \quad , \quad (c_1, c_2 \in \mathbb{R}) .$$

5.2 A (homogeneous) deformed elastic cylindrical beam with fixed ends

Consider a deformed elastic cylindrical beam with both ends fixed. The problem is to determine its shape; that is, the width of every section transversal to the axis. This gives rise to a 1-dimensional second-order dynamical system, which is autonomous if we require the beam to be homogeneous [6, 13, 35]. Let Q be the 1-dimensional smooth manifold modeling the configuration space of the system with local coordinate (q_0) . Then, in the natural coordinates of \mathbb{T}^2Q , the Lagrangian function for this system is

$$\mathcal{L}(q_0, q_1, q_2) = \frac{1}{2}\mu q_2^2 + \rho q_0 ,$$

where $\mu, \rho \in \mathbb{R}$ are constants, and $\mu \neq 0$. This is a regular Lagrangian function because the Hessian matrix $\left(\frac{\partial^2 \mathcal{L}}{\partial q_2 \partial q_2} \right) = \mu$ has maximum rank equal to 1 when $\mu \neq 0$.

The local expressions for the Poincaré-Cartan forms $\theta_{\mathcal{L}} \in \Omega^1(\mathbb{T}^3Q)$ and $\omega_{\mathcal{L}} \in \Omega^2(\mathbb{T}^3Q)$, and the Lagrangian energy $E_{\mathcal{L}} \in C^\infty(\mathbb{T}^3Q)$ are

$$\theta_{\mathcal{L}} = \mu(-q_3 dq_0 + \mu q_2 dq_1) \quad ; \quad \omega_{\mathcal{L}} = \mu(-dq_0 \wedge dq_3 + dq_1 \wedge dq_2) \quad ; \quad E_{\mathcal{L}} = -\rho q_0 + \frac{1}{2}\mu q_2^2 - \mu q_1 q_3 .$$

Thus, the semispray of type 1 $X_{\mathcal{L}} \in \mathfrak{X}(\mathbb{T}^3Q)$ solution to the dynamical equation (1) is given locally by

$$X_{\mathcal{L}} = q_1 \frac{\partial}{\partial q_0} + q_2 \frac{\partial}{\partial q_1} + q_3 \frac{\partial}{\partial q_2} - \frac{\rho}{\mu} \frac{\partial}{\partial q_3} .$$

Observe that the Euler-Lagrange equation for this 1-dimensional system is

$$\frac{d^4}{dt^4} \gamma = -\frac{\rho}{\mu} ,$$

where $\gamma: \mathbb{R} \rightarrow Q$ is a curve. Therefore, it is straightforward to obtain the general solution, which is a 4th order polynomial given by

$$\gamma(t) = -\frac{\rho}{\mu}t^4 + c_3t^3 + c_2t^2 + c_1t + c_0$$

where $c_0, c_1, c_2, c_3 \in \mathbb{R}$ are constants depending on the initial conditions given.

Now, we state the equations of the Lagrangian Hamilton-Jacobi problem for this system. Consider the projection $\rho_1^3: T^3Q \rightarrow TQ$. By Proposition 2, the generalized second-order Lagrangian Hamilton-Jacobi problem consists in finding sections $s \in \Gamma(\rho_1^3)$, given locally by $s(q_0, q_1) = (q_0, q_1, s_2, s_3)$, such that the submanifold $\text{Im}(s) \hookrightarrow T^3Q$ is invariant by the Lagrangian vector field $X_{\mathcal{L}} \in \mathfrak{X}(T^3Q)$. Since the constraints defining locally $\text{Im}(s)$ are $q_2 - s_2 = 0$, $q_3 - s_3 = 0$, then the equations for the section s are

$$s_3 - q_1 \frac{\partial s_2}{\partial q_0} - s_2 \frac{\partial s_2}{\partial q_1} = 0 \quad ; \quad -\frac{\rho}{\mu} - q_1 \frac{\partial s_3}{\partial q_0} - s_2 \frac{\partial s_3}{\partial q_1} = 0.$$

For the second-order Lagrangian Hamilton-Jacobi problem, we must require, in addition, that the section $s \in \Gamma(\rho_1^3)$ satisfies $d(s^*E_{\mathcal{L}}) = s^*dE_{\mathcal{L}} = 0$. From the local expression of the Lagrangian energy $E_{\mathcal{L}} \in C^\infty(T^3Q)$ given above, we have

$$dE_{\mathcal{L}} = -\rho dq_0 - \mu q_3 dq_1 + \mu q_2 dq_2 - \mu q_1 dq_3.$$

Thus, taking the pull-back of $dE_{\mathcal{L}}$ by the section $s(q_0, q_1) = (q_0, q_1, s_2, s_3)$, we obtain

$$s^*dE_{\mathcal{L}} = \mu \left(-\frac{\rho}{\mu} + s_2 \frac{\partial s_2}{\partial q_0} - q_1 \frac{\partial s_2}{\partial q_1} \right) dq_0 + \mu \left(-s_3 + s_2 \frac{\partial s_2}{\partial q_1} - q_1 \frac{\partial s_3}{\partial q_1} \right) dq_1.$$

Hence, the section $s \in \Gamma(\rho_1^3)$ is a solution to the second-order Lagrangian Hamilton-Jacobi problem if its component functions satisfy the following system of partial differential equations:

$$\begin{aligned} s_3 - q_1 \frac{\partial s_2}{\partial q_0} - s_2 \frac{\partial s_2}{\partial q_1} = 0 \quad ; \quad -\frac{\rho}{\mu} - q_1 \frac{\partial s_3}{\partial q_0} - s_2 \frac{\partial s_3}{\partial q_1} = 0 \\ -\frac{\rho}{\mu} - q_1 \frac{\partial s_3}{\partial q_0} + s_2 \frac{\partial s_2}{\partial q_0} = 0 \quad ; \quad -s_3 - q_1 \frac{\partial s_3}{\partial q_1} + s_2 \frac{\partial s_2}{\partial q_1} = 0 \end{aligned}$$

These 4 partial differential equations are not linearly independent. In particular, the equations obtained requiring $d(s^*E_{\mathcal{L}}) = 0$ can be reduced to a single one by computing the pull-back of the Poincaré-Cartan 2-form by the section s ,

$$s^*\omega_{\mathcal{L}} = -\mu \left(\frac{\partial s_3}{\partial q_1} + \frac{\partial s_2}{\partial q_0} \right) dq_0 \wedge dq_1.$$

Therefore, requiring $s^*\omega_{\mathcal{L}} = 0$ instead of the equivalent condition $d(s^*E_{\mathcal{L}}) = 0$, we have that s is a solution to the second-order Lagrangian Hamilton-Jacobi problem if its component functions satisfy the following equivalent system of partial differential equations

$$s_3 - q_1 \frac{\partial s_2}{\partial q_0} - s_2 \frac{\partial s_2}{\partial q_1} = 0 \quad ; \quad -\frac{\rho}{\mu} - q_1 \frac{\partial s_3}{\partial q_0} - s_2 \frac{\partial s_3}{\partial q_1} = 0 \quad ; \quad \frac{\partial s_3}{\partial q_1} + \frac{\partial s_2}{\partial q_0} = 0,$$

where these 3 equations are now linearly independent.

Finally, we compute the equations for the generating function W . The pull-back of $\theta_{\mathcal{L}}$ by s gives, in coordinates:

$$s^*\theta_{\mathcal{L}} = -\mu s_3 dq_0 + \mu s_2 dq_1.$$

Thus, requiring $s^*\theta_{\mathcal{L}} = dW$, for a local function W in TQ , we obtain

$$\frac{\partial W}{\partial q_0} = -\mu s_3 \quad ; \quad \frac{\partial W}{\partial q_1} = \mu s_2 .$$

and thus from $d(s^*E_{\mathcal{L}}) = 0$, we have $s^*E_{\mathcal{L}} = \text{const.}$, that is,

$$-\rho q_0 + \frac{1}{2\mu} \left(\frac{\partial W}{\partial q_1} \right)^2 + q_1 \frac{\partial W}{\partial q_0} = \text{const.} ,$$

which is a the Lagrangian Hamilton-Jacobi equation for this problem.

Remark: Observe that, in this particular example, the Hamilton-Jacobi equation is clearly more difficult to solve than the Euler-Lagrange equation. Therefore, this example shows that it is important to be careful when applying the Hamilton-Jacobi theory to a system, since the Hamilton-Jacobi equations obtained can be harder to solve than the usual Euler-Lagrange (or Hamilton's) equations of the system. Nevertheless, observe that a solution of the system can be obtained from a solution $\gamma: \mathbb{R} \rightarrow Q$ of the Euler-Lagrange equations as (see [39])

$$W(q_0, q_1) = \int_{t_0}^{t_1} \mathcal{L}(j^2\gamma(t)) dt .$$

Now, to establish the Hamiltonian formalism for the Hamilton-Jacobi problem, we consider natural coordinates on T^*TQ and in these coordinates the Legendre-Ostrogradsky map $\mathcal{FL}: T^3Q \rightarrow T^*TQ$ associated to the Lagrangian function \mathcal{L} is locally given by

$$\mathcal{FL}^*q_0 = q_0 \quad ; \quad \mathcal{FL}^*q_1 = q_1 \quad ; \quad \mathcal{FL}^*p^0 = -\mu q_3 \quad ; \quad \mathcal{FL}^*p^1 = \mu q_2 .$$

Moreover, the inverse map $\mathcal{FL}^{-1}: T^*TQ \rightarrow T^3Q$ is

$$(\mathcal{FL}^{-1})^*q_0 = q_0 \quad ; \quad (\mathcal{FL}^{-1})^*q_1 = q_1 \quad ; \quad (\mathcal{FL}^{-1})^*q_2 = \frac{p^1}{\mu} \quad ; \quad (\mathcal{FL}^{-1})^*q_3 = -\frac{p^0}{\mu} .$$

From these coordinate expressions it is clear that \mathcal{L} is a hyperregular Lagrangian function, since the Legendre-Ostrogradsky map is a global diffeomorphism.

The Hamiltonian function $h \in C^\infty(T^*TQ)$ is

$$h = (\mathcal{FL}^{-1})^*E_{\mathcal{L}} = -\rho q_0 + \frac{(p^1)^2}{2\mu} + q_1 p^0 .$$

From this Hamiltonian applying the procedure given in Section (4) one can obtain the Hamilton-Jacobi equations for this problem which coincides with the Hamilton-Jacobi equations given previously by the Lagrangian problem, which is

$$-\rho q_0 + \frac{1}{2\mu} \left(\frac{\partial W}{\partial q_1} \right)^2 + q_1 \frac{\partial W}{\partial q_0} = \text{const.}$$

Acknowledgments

We acknowledge the financial support of the *Ministerio de Ciencia e Innovación* (Spain), projects MTM 2010-21186-C02-01, MTM2011-22585 and MTM2011-15725-E; AGAUR, project 2009 SGR:1338.; IRSES-project ‘‘Geomech-246981’’; and ICMAT Severo Ochoa project SEV-2011-0087. P.D. Prieto-Martínez wants to thank the UPC for a Ph.D grant, and L.Colombo wants to thank CSIC for a JAE-Pre grant. We wish to thanks Profs. D. Martín de Diego and M.C. Muñoz-Lecanda for fruitful discussions. We thank Mr. Jeff Palmer for his assistance in preparing the English version of the manuscript.

References

- [1] R. Abraham, J.E. Marsden, *Foundations of Mechanics*, 2nd edition, Addison–Wesley, 1978.
- [2] V.I. Arnol'd, *Mathematical methods of classical mechanics*, 2nd edition, Springer, 1989.
- [3] P. Balseiro, J.C. Marrero, D. Martín de Diego, E. Padrón, “A unified framework for mechanics: Hamilton–Jacobi equation and applications”, *Nonlinearity* **23**(8) (2010) 1887–1918.
- [4] M. Barbero-Liñán, M. de León, D. Martín de Diego, “Lagrangian submanifolds and Hamilton–Jacobi equation”, *Monatshefte fr Mathematik* **171** Issue 3-4, 269–290 (2013).
- [5] M. Barbero-Liñán, M. de León, D. Martín de Diego, J.C. Marrero, M.C Muñoz-Lecanda, “Kinematic reduction and the Hamilton–Jacobi equation” *J. Geom. Mech.* **4**(3) (2012) 207–237.
- [6] D.J. Benson, *Music: A Mathematical Offering*, Cambridge University Press, 2006.
- [7] F. Cantrijn, M. Crampin, W. Sarlet, “Higher-order differential equations and higher-order Lagrangian mechanics”, *Math. Proc. Cambridge Philos. Soc.* **99**(3) (1986) 565–587.
- [8] J.F. Cariñena, X. Gràcia, G. Marmo, E. Martínez, M.C. Muñoz–Lecanda, N. Román–Roy, “Geometric Hamilton–Jacobi theory”, *Int. J. Geometric Methods Mod. Phys.* **3** (2006) 1417–1458.
- [9] J.F. Cariñena, X. Gràcia, G. Marmo, E. Martínez, M.C. Muñoz–Lecanda, N. Román–Roy, “Hamilton–Jacobi theory and the evolution operator”, pp. 177–186 in *Mathematical physics and field theory. Julio Abad, in memoriam*, edited by M. Asorey, J.V. García Esteve, M.F. Rañada and J. Sesma; Prensas Universitarias de Zaragoza, 2009.
- [10] J.F. Cariñena, X. Gràcia, G. Marmo, E. Martínez, M.C. Muñoz–Lecanda, N. Román–Roy, “Non-holonomic Geometric Hamilton–Jacobi theory for nonholonomic dynamical systems”, *Int. J. Geometric Methods Mod. Phys.* **7**(3) (2010) 431–454.
- [11] G.C. Constantelos, “On the Hamilton–Jacobi Theory with Derivatives of Higher Order”, *Nuovo Cimento B (11)* **84**(1) (1984) 91–101.
- [12] D. Dominici, J. Gomis, G. Longhi, J.M. Pons, “Hamilton-jacobi Theory For Constrained Systems”, *J. Math. Phys.* **25** (1984) 2439–2460.
- [13] L. Elsgoltz, *Differential equations and the calculus of variations* (3rd ed.), Mir Publishers, 1983.
- [14] G. Esposito, G. Marmo, G. Sudarshan, *From Classical to Quantum mechanics*, Cambridge University Press, 2004.
- [15] X. Gràcia, J.M. Pons, and N. Román-Roy, “Higher-order Lagrangian systems: Geometric structures, dynamics and constraints”, *J. Math. Phys.* **32**(10) (1991) 2744–2763.
- [16] X. Gràcia, J.M. Pons, and N. Román-Roy, “Higher-order conditions for singular Lagrangian systems”, *J. Phys. A: Math. Gen.* **25** (1992) 1981–2004.
- [17] D. Iglesias-Ponte, M. de León, D. Martín de Diego, “Towards a Hamilton–Jacobi theory for non-holonomic mechanical systems”, *J. Phys. A: Math. Theor.* **41** (2008) 015205, 14 pp.
- [18] J.V. José, E.J. Saletan, *Classical dynamics: a contemporary approach*, Cambridge Univ. Press, 1998.
- [19] O. Krupkova, “Higher-order mechanical systems with constraints”, *J. Math. Phys.* **41**(8) (2000) 5304–5324.
- [20] O. Krupková, A. Vondra, “On some integration methods for connections on fibered manifolds”, pp 89–101 in *Differential Geometry and its Applications, Proc. Conf.* (Opava, 1992). Silesian University, Opava, 1993.

- [21] M. Leok, T. Ohsawa, D. Sosa, “Hamilton-Jacobi theory for degenerate Lagrangian systems with holonomic and nonholonomic constraints”, *J. Math. Phys.* **53**(7) (2012) 072905, 29 pp.
- [22] M. de León, J.C. Marrero, D. Martín de Diego, A geometric Hamilton-Jacobi theory for classical field theories. Variations, geometry and physics, pp. 129–140 in *Nova Sci. Publ.*, New York, 2009.
- [23] M. de León, J.C. Marrero, D. Martín de Diego, “Linear almost Poisson structures and Hamilton–Jacobi equation. Applications to nonholonomic mechanics”, *J. Geom. Mech.* **2**(2) (2010) 159–198.
- [24] M. de León, D. Martín de Diego, J.C. Marrero, M. Salgado, S. Vilariño, “Hamilton-Jacobi theory in k -symplectic field theories”, *Int. J. Geom. Methods Mod. Phys.* **7**(8) (2010) 1491–1507.
- [25] M. de León, D. Martín de Diego, M. Vaquero, “A Hamilton-Jacobi theory for singular Lagrangian systems in the Skinner and Rusk setting”, *Int. J. Geom. Methods Mod. Phys.* **9**(8) (2012) 1250074, 24 pp.
- [26] M. de León, D. Martín de Diego, M. Vaquero, “A Universal Hamilton-Jacobi Theory”, arXiv:1209.5351 [math-ph] (2012).
- [27] M. de León, P.R. Rodrigues, *Generalized classical mechanics and field theory*, North-Holland Math. Studies **112**, Elsevier Science Publishers B.V., Amsterdam 1985.
- [28] P. Libermann, C.M. Marle, *Symplectic geometry and analytical dynamics*, D. Reidel Publishing Company, Dordrecht, 1987.
- [29] G. Marmo, G. Morandi, N. Mukunda, “A geometrical approach to the Hamilton–Jacobi form of dynamics and its generalizations”, *Riv. Nuovo Cimento* **13** (1990) 1–74.
- [30] G. Marmo, G. Morandi, N. Mukunda, “The Hamilton-Jacobi theory and the analogy between classical and quantum mechanics”, *J. Geom. Mech.* **1**(3) (2009) 317–355.
- [31] T. Ohsawa, A.M. Bloch, M. Leok, “Nonholonomic Hamilton-Jacobi theory via Chaplygin Hamiltonization” *J. Geom. Phys.* **61**(8) (2011) 1263–1291.
- [32] T. Ohsawa, A.M. Bloch, “Nonholonomic Hamilton–Jacobi equation and integrability”, *J. Geom. Mech.* **1**(4) (2011) 461–481.
- [33] T. Ohsawa, A.M. Bloch, M. Leok, “Discrete Hamilton-Jacobi theory”, *SIAM J. Control Optim.* **49**(4) (2011) 1829–1856.
- [34] P.D. Prieto-Martínez, N. Román-Roy, “Lagrangian-Hamiltonian unified formalism for autonomous higher-order dynamical systems”, *J. Phys. A: Math. Teor.* **44**(38) (2011) 385203.
- [35] P.D. Prieto-Martínez, N. Román-Roy, “Unified formalism for higher-order non-autonomous dynamical systems”, *J. Math. Phys.* **53**(3) (2012) 032901.
- [36] P.D. Prieto-Martínez, N. Román-Roy, “Higher-order Mechanics: Variational Principles and other topics”, arXiv:1201.4102 [math-ph], *J. Geom. Mech.* (2013) (to appear).
- [37] D.J. Saunders, *The geometry of jet bundles*, London Mathematical Society, Lecture notes series **142**, Cambridge University Press, Cambridge, New York 1989.
- [38] W.M. Tulczyjew, “Sur la différentielle de Lagrange” *C. R. Acad. Sci. Paris, Série A* **280** (1975), 1295–1298
- [39] L. Vitagliano, “The Hamilton–Jacobi formalism for higher-order field theory”, *Int. J. Geom. Methods Mod. Phys.* **07** (2010) 1413–1436.
- [40] L. Vitagliano, “Hamilton–Jacobi diffieties”, *J. Geom. Phys.* **61** (2011) 1932–1949.
- [41] H. Wang, “Hamilton–Jacobi Theorems for Regular Controlled Hamiltonian System and Its Reductions”, arXiv:1305.3457 [math.SG] (2013).
- [42] H. Wang, “Symmetric Reduction and Hamilton–Jacobi Equation of Rigid Spacecraft with a Rotor”, arXiv:1307.1606 [math.DS] (2013).