

Geometric Biplane Graphs II: Graph Augmentation*

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Abstract

We study biplane graphs **drawn on** a finite point set S in the plane in general position. This is the family of geometric graphs whose vertex set is S and can be decomposed into two plane graphs. It is shown that every sufficiently large set S admits a 5-connected biplane graph; and there are arbitrarily large point sets that do not admit any 6-connected biplane graph. Every plane graph other than the wheel can be augmented into a 4-connected biplane graph; and there are arbitrarily large plane graphs that cannot be augmented to a 5-connected biplane graph. In a companion paper we study extremal properties of biplane graphs such as the maximal number of edges and the largest minimum degree of biplane graphs over n -element point sets.

1 Introduction

In a *geometric graph* $G = (V, E)$, the vertices are distinct points in the plane in general position, and the edges are straight line segments between pairs of vertices. A *plane graph* is a geometric graph in which no two edges cross. Every (abstract) graph has a realization as a geometric graph (by simply mapping the vertices into distinct points in the plane, no three of which are collinear), and every planar graph can be realized as a plane graph by Fáry's theorem.

We consider a generalization of plane graphs. A geometric graph $G = (V, E)$ is k -*plane* for some $k \in \mathbb{N}$ if it admits a partition of its edges $E = E_1 \dot{\cup} \dots \dot{\cup} E_k$ such that $G_1 = (V, E_1), \dots, G_k = (V, E_k)$ are each plane graphs. Let S be a point set in the plane in general position, that is, no three points in S are collinear. Denote by $\mathcal{G}_k(S)$ the family of k -plane graphs with vertex set S . With this terminology, $\mathcal{G}_1(S)$ is the family of plane graphs with vertex set S , and $\mathcal{G}_2(S)$ is the family of 2-plane graphs (also known as *biplane graphs*) with vertex set S .

In this and a companion paper [13], we study $\mathcal{G}_2(S)$ and contrast combinatorial properties of plane graphs $\mathcal{G}_1(S)$ and biplane graphs $\mathcal{G}_2(S)$ for point sets S in general position in the plane. If only plane graphs **on** S are considered, there are limitations on achieving some desirable properties, such as high connectivity, as it is known that every plane graph H has a vertex with degree at most 5, hence $\kappa(H) \leq \lambda(H) \leq \delta(H) \leq 5$ (we use standard graph theory notation as in [7]). It is natural to expect

*A. G., F. H., M. K., R.I. S. and J. T. were partially supported by ESF EUROCORES programme EuroGIGA, CRP ComPoSe: grant EUI-EURC-2011-4306, and by project MINECO MTM2012-30951. F. H., and R.I. S. were also supported by project Gen. Cat. DGR 2009SGR1040. M. K. was supported by the Secretary for Universities and Research of the Ministry of Economy and Knowledge of the Government of Catalonia and the European Union. I. M. was supported by FEDER funds through COMPETE–Operational Programme Factors of Competitiveness, CIDMA and FCT within project PEst-C/MAT/UI4106/2011 with COMPETE number FCOMP-01-0124-FEDER-022690. M. S. was supported by ESF EuroGIGA project ComPoSe as F.R.S.-FNRS - EUROGIGA NR 13604. R.I. S. was also partially funded by the FP7 Marie Curie Actions Individual Fellowship PIEF-GA-2009-251235 and by FCT through grant SFRH/BPD/88455/2012. C. T. was supported in part by NSERC (RGPIN 35586) and NSF (CCF-0830734). A. G. and J. T. were also supported by project E58-DGA.

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that significantly better values can be obtained if the larger family $\mathcal{G}_2(S)$ is used. This is precisely the topic we explore in this paper, mostly focusing on graph connectivity.

Related concepts. Note that the above generalization of plane graphs is reminiscent to, although more restrictive, than the notion of thickness, geometric thickness, and book thickness, which are defined for abstract graphs. We recall their definitions for easy comparison. The *thickness* of an (abstract) graph $G = (V, E)$ is the smallest $k \in \mathbb{N}$ such that G admits an edge partition $E = E_1 \dot{\cup} \dots \dot{\cup} E_k$ with the property that $G_1 = (V, E_1), \dots, G_k = (V, E_k)$ are each planar graphs. The *geometric thickness* of an (abstract) graph $G = (V, E)$ is the smallest $k \in \mathbb{N}$ such that G admits an edge partition $E = E_1 \dot{\cup} \dots \dot{\cup} E_k$ satisfying that $G_1 = (V, E_1), \dots, G_k = (V, E_k)$ can be simultaneously embedded as plane graphs where the vertex set is mapped to a common labeled point set. The *book thickness* is a restricted version of the geometric thickness where G_1, \dots, G_k are simultaneously embedded on a point set in convex position.

Notice that every k -plane graph, if interpreted as an abstract graph, has geometric thickness at most k , but in addition we are given a specific embedding in the plane in which the decomposition into k plane layers is possible. In other words, the term k -plane graph refers to a geometric object, a drawing, while having geometric thickness k is a property of the underlying abstract graph.

Prior work and organization of the paper. The problem of embedding planar graphs **on a given point set S** , using non-crossing straight line segments with endpoints in S as edges, has been receiving substantial attention in many respects [4, 5, 14]. One of the variants consists of trying to achieve connectivity as high as possible (either for vertices or for edges), on one side finding conditions on the point set that allow or constrain such embeddings, and on the other side developing the algorithmic counterparts of the problem [8, 11]. This is the topic we study in Section 2, considering biplane graphs instead of plane graphs. A closely related family of problems is *graph augmentation*, in which one would like to add new edges, ideally as few as possible, to a given graph in such a way that some desired property is achieved. There has been extensive work on augmenting a disconnected plane graphs to connected one (see [17] for a recent survey) or achieving good connectivity properties [1, 2, 19]. For abstract graphs, this corresponds to the classical connectivity augmentation problem in optimization and it has a rich history. In Section 3, we consider several problems on augmenting plane graphs to biplane supergraphs with higher connectivity.

We conclude in Section 4 with some final remarks and open problems. **Due to space limitations, several proofs have been omitted.**

Finally, we mention a companion paper [13] in which we study combinatorial properties of edge-maximal biplane graphs, and study biplane graphs in $\mathcal{G}_2(S)$ that admit the maximum number of edges and the maximum connectivity over all n -element point sets S .

2 Drawing Biplane Graphs from Scratch

Given a point set S , we would like to construct a graph $G \in \mathcal{G}_2(S)$ with high vertex connectivity $\kappa(G)$. We determine the maximum $\kappa(G)$ that can be attained for every (sufficiently large) point set S . We also consider the special case that the point set S is in convex position.

For comparison, we briefly review analogous results for plane graphs $\mathcal{G}_1(S)$ on a given point set S . Refer to [17] for a survey article. Every set of $n \geq 3$ points in general position admits a spanning cycle (a polygonization of S), which is 2-connected. For points in convex position, every plane graphs has a vertex of degree 2, so in this case $\kappa(G) = 2$ is the best possible value over all $G \in \mathcal{G}_1(S)$. As for the maximum connectivity that can be attained, the vertex connectivity of every plane graph is at most 5, since every planar graph has a vertex of degree not greater than 5. It is known that every set of $n \geq 4$ points not in convex position admits a 3-connected triangulation. Additionally, every set of $n \geq 6$ points whose convex hull is a triangle admits a 4-connected triangulation, provided that a certain condition is satisfied (see [8] for details). No characterization is known for point sets in general position (with arbitrary convex hull) that admit 4- or 5-connected triangulations.

2.1 Point Sets in Convex Position

We begin with the special case of points in convex position. It turns out that all biplane graphs on a point set in convex position are planar. Moreover, both the maximum number of edges and the maximum possible vertex-connectivity are the same as for planar graphs.

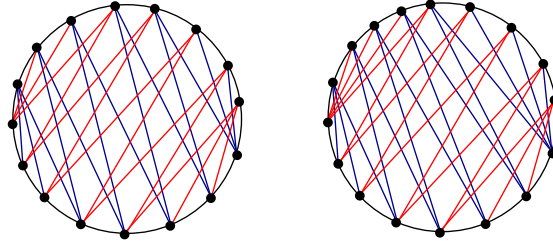


Figure 1: Two 5-connected biplane graphs for points in convex position.

Lemma 1 *Let S be a set of n points in the plane in convex position.*

- (i) *Every graph in $\mathcal{G}_2(S)$ is planar (as an abstract graph).*
- (ii) *If $G = (V, E)$ is a Hamiltonian planar (abstract) graph with n vertices, then it has a geometric realization in $\mathcal{G}_2(S)$.*

Proof. Let $G = (S, E)$ be a biplane graph on a point set S in convex position. By definition, there is an edge partition $E = E_1 \dot{\cup} E_2$ such that both $G_1 = (S, E_1)$ and $G_2 = (S, E_2)$ are plane graphs. To show that G is planar (as an abstract graph), it is enough to construct a crossing-free embedding of G in which the edges are represented by Jordan arcs. Denote the vertices by s_1, \dots, s_n in counterclockwise order along the boundary of $\text{ch}(S)$. Move each vertex s_i to point $s'_i = (\cos(2\pi i/n), \sin(2\pi i/n))$ on the unit circle, and let $S' = \{s'_1, \dots, s'_n\}$. This transformation maps G_1 and G_2 to plane graphs $G'_1 = (S', E'_1)$ and $G'_2 = (S', E'_2)$. Recall that an inversion is a continuous bijection on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ that interchanges the interior and the exterior of the unit circle, but every point on the unit circle is a fixed point. We apply an inversion to map the line segments representing the edges in E'_2 to a set E''_2 of pairwise non-crossing circular arcs in the exterior of the unit circle. Hence $(S', E'_1 \cup E''_2)$ is a crossing-free embedding of graph G , as required.

Let $G = (V, E)$ be a planar Hamiltonian (abstract) graph with n vertices. By Fáry's theorem [10], G can be embedded as a plane graph on some set of n points in the plane. In a straight-line embedding of G , the Hamiltonian cycle is a simple polygon P , and the remaining edges are partitioned into internal and external diagonals of P . This partition yields a **biplane realization of G on any set S of n points in convex position**. Embed the Hamiltonian cycle into the boundary of the convex hull $\text{ch}(S)$. The convex hull edges and the internal diagonals of P form one layer of the biplane graph, and the external diagonals of P form the other layer. \square

However, not every planar graph can be realized as a **biplane graph on a point set** in convex position. For example, there are edge-maximal planar graphs (triangulations) that are not Hamiltonian. These planar graphs have $3n - 6$ edges, which is the maximum number of edges in a **biplane graph on a point set S in convex position**. However, in every edge-maximal graph in $\mathcal{G}_2(S)$, the boundary of the convex hull forms a Hamiltonian cycle. This yields a contradiction.

We can now characterize the maximal vertex connectivity in $\mathcal{G}_2(S)$ when S is in convex position.

Theorem 2 *Let S be a set of n points in convex position.*

- $\mathcal{G}_2(S)$ contains a 4-connected graph if and only if $n \geq 6$.
- $\mathcal{G}_2(S)$ contains a 5-connected graph if and only if $n = 12$ or $n \geq 14$.
- $\mathcal{G}_2(S)$ contains no 6-connected graphs for any $n \in \mathbb{N}$.

Proof. It is well known that every 4-connected (or 5-connected) planar graph G has a Hamiltonian cycle [21]. By Lemma 1(ii), it is enough to establish the existence of a planar graph for a given n .

By Lemma 1(i), every graph in $\mathcal{G}_2(S)$ is planar. Every planar graph on $n \geq 3$ vertices has at most $3n - 6$ edges, and the sum of vertex degrees is at most $6n - 12$. In a k -connected graph, the degree of every vertex is at least k , and the sum of vertex degrees is at least kn . Comparing these bounds, we have $kn \leq 6n - 12$ or $12/(6 - k) \leq n$. It follows that no planar graph is 6-connected, every 5-connected planar graph has at least 12 vertices, and every 4-connected planar graph has at least 6 vertices.

It is easy to see that there is a 4-connected planar graph on n vertices for every $n \geq 6$. Specifically, the 1-skeleton of the octahedron is 4-connected with 6 vertices; and a vertex split operation can increase the number of vertices by one while maintaining 4-connectivity. Barnette [3] and Butler [6] independently designed algorithms for generating all 5-connected triangulations, using simple operations starting from the icosahedron. The 1-skeleton of the icosahedron is 5-connected with 12 vertices, and each operation either splits a vertex of degree 6 or higher, or simultaneously splits two adjacent vertices. Hence there is a 5-connected planar graph for $n = 12$ and for every $n \geq 14$ (but not for $n = 13$). \square

Remark. We have shown that $\mathcal{G}_2(S)$ contains 4- and 5-connected graphs, respectively, if $n \geq 6$ and $n \geq 12$. The existence proof in Theorem 2 can be turned into an $O(n)$ -time algorithm for constructing such biplane graphs. Here we present explicit constructions for 5-connected biplane graphs for points in convex position when $n = 12$ and $n \geq 14$ (Figure 1). Our constructions are formed by the n edges of $\text{ch}(S)$, and $2(n - 3)$ chords of $\text{ch}(S)$ distributed into two layers. Each layer consists of two stars with 3 or 4 leaves each, connected by a zig-zag path. The construction in Figure 1(a) works when n is even and $n \geq 12$, while the one in Figure 1(b) works when n is odd and $n \geq 15$.

2.2 Point Sets in General Position

In this section we find the largest $k \in \mathbb{N}$ such that every sufficiently large point set S in general position admits a k -connected biplane graph. Hutchinson et al. [18] proved that every biplane graph in $\mathcal{G}_2(S)$ has at most $6n - 18$ edges for $n \geq 8$. It follows that every biplane graph contains a vertex of degree 11 or less, hence $k \leq 11$. By Theorem 2, we have $k \leq 5$. We show that every point set S admits a 5-connected biplane graph, provided that S is sufficiently large.

Theorem 3 *Let S be a set of n points in the plane in general position. If n is large enough, then $\mathcal{G}_2(S)$ contains a 5-connected graph.*

Proof outline. Given a set S of n points in general position, we build a **5-connected biplane graph on S as follows**. By the Erdős-Szekeres theorem, if n is large enough, then S contains a subset $S_0 \subset S$ of at least 14 points in convex position. By Theorem 2, S_0 admits a 5-connected biplane graph G_0 . We increment G_0 successively with new vertices from $S \setminus S_0$ in 3 phases, maintaining 5-connectivity. We add the points lying in the interior of $\text{ch}(S_0)$ in the first phase, followed by the vertices of $\text{ch}(S)$ in the second phase, and then all remaining points lying in the exterior of $\text{ch}(S_0)$. We continue with the details. Before the proof of Theorem 3, we state and prove a lemma (Lemma 6) about the incremental steps.

Preliminaries for the incremental steps. When incrementing a 5-connected biplane graph $G = (S, E)$ with a new vertex s , we shall add 5 new edges incident to s and delete at most one of the existing edges of G . To ensure that we maintain 5-connectivity, we use the following well-known properties of graphs.

Property 1 *Let $G = (V, E)$ be a k -connected (abstract) graph. Augment G with a new vertex x joined to k vertices of G . Then the new graph on vertex set $V \cup \{x\}$ is also k -connected.*

Property 2 *Let $G = (V, E)$ be a k -connected (abstract) graph in which vw is an edge. Remove edge vw from G , and augment it with a new vertex x joined to both v, w and to $k - 2$ additional vertices. Then the new graph with vertex set $V \cup \{x\}$ is also k -connected.*

Before incrementing a 5-connected biplane graph $G = (S, E)$ with a new vertex s , we augment $G = (S, E)$ to an edge-maximal biplane graph. Additional edges do not decrease the vertex connectivity. We can rely on the following two structural results for maximal biplane graphs from the companion paper [13].

Lemma 4 [13] *Let $G = (S, E)$ be a maximal biplane graph in $\mathcal{G}_2(S)$. Then there are two triangulations $T_1 = (S, E_1)$ and $T_2 = (S, E_2)$ such that $E = E_1 \cup E_2$.*

Lemma 5 [13] *Let $G = (S, E)$ be a maximal biplane graph in $\mathcal{G}_2(S)$ such that $E = E_1 \cup E_2$, where $T_1 = (S, E_1)$ and $T_2 = (S, E_2)$ are two triangulations. If $e \in E_1 \cap E_2$, then e is flippable in neither T_1 nor T_2 . Furthermore, every maximal biplane graph with $n \geq 4$ vertices is 3-connected.*

The following tool (Property 3) is crucial for increasing the vertex degree of a vertex in a triangulation. This tool is applicable to all triangulations other than the wheel. A *wheel* is a triangulation on n points such that $n - 1$ points are in convex position and one point lies in the interior of $\text{ch}(S)$, the points on the convex hull induce a cycle on the boundary of $\text{ch}(S)$ and the interior point is joined to all other $n - 1$ points. Given an edge $e \in E$ in a triangulation $T = (S, E)$, denote by $Q(e)$ the quadrilateral formed by the two triangles adjacent to e if e is not on the boundary of $\text{ch}(S)$; and $Q(e)$ is undefined if e is an edge of $\text{ch}(S)$. Recall that e is flippable if and only if $Q(e)$ is a convex quadrilateral.

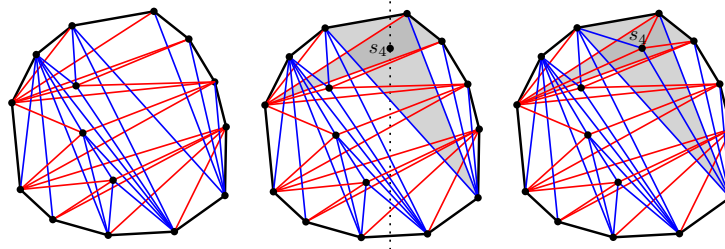


Figure 2: Left: 5-connected biplane graph on 15 points. Middle: point s_4 lies in the interior of two gray triangles, which jointly have 5 distinct vertices. Right: point s_4 is now part of the 5-connected biplane graph.

Property 3 *Let $T = (S, E)$ be a triangulation other than the wheel. Let $s \in S$ be a point in the interior of $\text{ch}(S)$ such that it is adjacent to a vertex on the boundary of $\text{ch}(S)$, and the graph induced by its neighbors in T is a cycle. Then there is a triangle incident to s in which the edge opposite to s is flippable.*

Proof. Denote the neighbors of s in counterclockwise order by $v_0, v_1, \dots, v_{k-1} \in S$, for some $k \geq 3$. Since T is not a wheel, s must have two consecutive neighbors such that the edge between them is not on the boundary of $\text{ch}(S)$. Without loss of generality, assume that v_0 and v_1 are two such neighbors (i.e., v_0v_1 is not an edge of $\text{ch}(S)$). Moreover, we can further assume that v_0 is a vertex of $\text{ch}(S)$.

We proceed by contradiction. Suppose that the edge v_iv_{i+1} is not flippable for any $i = 0, 1, \dots, k - 1$ (where indices are taken modulo k). This implies that all vertices v_0, v_1, \dots, v_{k-1} must be interior vertices, contradicting our initial assumption. Specifically, we prove by induction that for every $i = 0, \dots, k - 1$ the quadrilateral $Q(v_iv_{i+1})$ is defined and has a convex vertex at v_i (where the indices are again taken modulo k). Note that the claim holds for $i = 0$ by the choice of v_0 and v_1 .

The induction step is as follows: assume that $Q(v_iv_{i+1})$ is defined, and has a convex vertex at v_i . Then, v_{i+1} must form a reflex angle (otherwise $Q(v_iv_{i+1})$ would be convex). In particular, point v_{i+1} must be interior, hence $Q(v_{i+1}v_{i+2})$ is defined. Since the neighbors of s induce a cycle, we have $v_iv_{i+2} \notin E$. Since v_{i+1} is a reflex vertex of $Q(v_iv_{i+1})$, it must be a convex vertex of $Q(v_{i+1}v_{i+2})$. This completes the induction step, hence the proof of Property 3. \square

The following lemma is the basis of our incremental construction.

Lemma 6 Let $G = (S, E)$ be a 5-connected biplane graph such that $\text{ch}(S)$ has at least 4 vertices. Denote by $S_{\text{int}} \subset S$ the points lying in the interior of $\text{ch}(S)$, and let $s \notin S$ be a point such that s is in the interior of $\text{ch}(S)$ but in the exterior of $\text{ch}(S_{\text{int}})$. Then a 5-connected biplane graph on $S \cup \{s\}$ can be constructed from $G = (S, E)$ by adding 5 new edges incident to s and deleting at most one edge of E .

Proof. (Sketch) First, augmenting G to a maximal biplane graph \widehat{G} (adding dummy edges if necessary), we can divide \widehat{G} into two triangulations T_1 and T_2 , by Lemma 4. The new point s will lie in the interior of some triangles Δ_1 and Δ_2 in the two triangulations (Δ_1 and Δ_2 may share vertices and edges). Since s lies in the exterior of $\text{ch}(S_{\text{int}})$, at least one vertex of Δ_1 (resp., Δ_2) is on the boundary of $\text{ch}(S)$. Now we can augment T_1 (resp., T_2) with vertex s and three edges joining s to the vertices of Δ_1 (resp., Δ_2) to a new triangulation. Three cases are analyzed, based on the total number of distinct vertices of Δ_1 and Δ_2 . Figure 2 shows the construction when Δ_1 and Δ_2 jointly have 5 or 6 distinct vertices. In the figure, after connecting s to 5 distinct vertices of Δ_1 and Δ_2 , the resulting graph is 5-connected by Property 1, and biplane since the new edges can be inserted into T_1 and T_2 , respectively, without crossings. When the number of distinct vertices is 3 or 4, the constructions are a bit more complicated and are based on flipping some edges of the triangulations to increase the number of edges adjacent to s . □

We are now ready to prove Theorem 3.

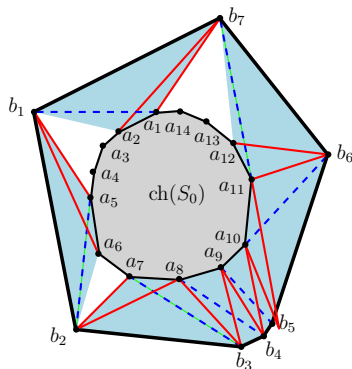


Figure 3: The boundaries of $\text{ch}(S_0)$ and $\text{ch}(S)$ are disjoint.

Proof. [Proof of Theorem 3.] Erdős and Szekeres proved that there is an integer $f(k)$ for every $k \in \mathbb{N}$ such that every set of $f(k)$ points in the plane in general position contains a subset of k points in convex position. They conjectured $f(k) = 2^{k-2} + 1$, and showed $f(k) \geq 2^{k-2} + 1$. The currently best upper bound [20] for $k \geq 7$ is $f(k) \leq \binom{2k-5}{k-2} + 1$. For $k = 14$, this implies that S has a subset of at least 14 points in convex position if $n \geq \binom{2 \cdot 14 - 5}{14 - 2} + 1 = 1352079$. Consider all subsets of S in convex position of maximal cardinality, and let $S_0 \subseteq S$ be one whose convex hull has maximum area. Clearly, we have $|S_0| \geq 14$, and there is no other set $S'_0 \subset S$ in convex position such that $|S'_0| = |S_0|$ and $\text{ch}(S_0) \subset \text{ch}(S'_0)$. By Theorem 2, there is a 5-connected graph $G_0 \in \mathcal{G}_2(S_0)$.

Partition the points in $S \setminus S_0$ into three sets: let $S_{\text{int}} \subset S \setminus S_0$ be the set of points in the interior of $\text{ch}(S_0)$, let $S_{\text{bou}} \subset S \setminus S_0$ be the set of points on the boundary of $\text{ch}(S)$, and let $S_{\text{ext}} \subset S \setminus S_0$ be the set of all remaining points in the exterior of $\text{ch}(S_0)$. We shall construct a 5-connected biplane graph on S by incrementing G_0 successively with the vertices in $S \setminus S_0$ in three phases, adding the points in S_{int} , S_{bou} , and S_{ext} in this order.

Interior points. Denote the points in S_{int} by s_1, \dots, s_k sorted by their x -coordinates (ties are broken arbitrarily). We augment G_0 successively by the vertices s_1, \dots, s_k as follows: For $i = 1, \dots, k$, we construct a **biplane graph G_i on $S_0 \cup \{s_1, \dots, s_i\}$** . Graph G_i is obtained from G_{i-1} by applying Lemma 6. This completes the description of phase 1. By Lemma 6, every edge inserted in this phase is incident

to a point in S_{int} , so all edges between vertices of S_0 were present already in G_0 . Denote by G'_0 the resulting 5-connected biplane graph in $\mathcal{G}_2(S_0 \cup S_{\text{int}})$.

Boundary points. In this phase, we augment G'_0 with the vertices of $\text{ch}(S)$. If $\text{ch}(S_0) = \text{ch}(S)$, then all vertices of $\text{ch}(S)$ are already in G'_0 , and the phase is complete. Assume that $\text{ch}(S_0) \neq \text{ch}(S)$. We define *visibility* for the edges of $\text{ch}(S_0)$: a point s in the exterior of $\text{ch}(S_0)$ *sees* an edge uv of $\text{ch}(S_0)$ if the triangle suw is disjoint from the interior of $\text{ch}(S_0)$; a segment st in the exterior of $\text{ch}(S_0)$ *sees* uv if both s and t sees uv . Note that every exterior point s sees a sequence of consecutive edges of $\text{ch}(S_0)$, but does not see all the edges. By the maximality of S_0 , every exterior vertex of $\text{ch}(S)$ sees at least three edges of $\text{ch}(S_0)$. This property generalizes as follows.

Property 4 *Every k consecutive vertices of $\text{ch}(S)$ in the exterior of $\text{ch}(S_0)$ jointly see at least $k + 2$ edges of $\text{ch}(S_0)$.*

Going back to the main proof, we distinguish two cases based on whether the boundaries of $\text{ch}(S_0)$ and $\text{ch}(S)$ are disjoint or not.

Assume that the boundaries of $\text{ch}(S_0)$ and $\text{ch}(S)$ are disjoint and every two consecutive vertices of $\text{ch}(S)$ see some common edge of $\text{ch}(S_0)$. In this case, every edge of $\text{ch}(S)$ sees at least one edge of $\text{ch}(S_0)$. We show that every $k \geq 2$ consecutive edges of $\text{ch}(S)$ jointly see at least $k + 1$ edges of $\text{ch}(S_0)$. Indeed, k consecutive edges of $\text{ch}(S)$ form a path. The $k - 1$ interior vertices of this path jointly see the same set of edges of $\text{ch}(S_0)$ as the k edges of the path, and by Property 4, they jointly see at least $(k - 1) + 2 = k + 1$ edges of $\text{ch}(S_0)$. In summary, every set of $k \geq 1$ consecutive edges of $\text{ch}(S)$ jointly sees at least k edges of $\text{ch}(S_0)$. By Hall's theorem, we can assign to every edge of $\text{ch}(S)$ a visible edge of $\text{ch}(S_0)$. (See Figure 3.)

Denote the vertices of $\text{ch}(S_0)$ by a_1, \dots, a_p ($p \geq 14$); and the vertices of $\text{ch}(S)$ by b_1, \dots, b_q ($q \geq 3$) in counterclockwise order. For each $i = 1, \dots, q$, if edge $b_i b_{i+1}$ is assigned to $a_j a_{j+1}$, then $b_i b_{i+1} a_{j+1} a_j$ is a convex quadrilateral. If two such quadrilaterals, say $b_i b_{i+1} a_{j+1} a_j$ and $b_{i'} b_{i'+1} a_{j'+1} a_{j'}$, cross (have intersecting interiors), then $b_i b_{i+1}$ also sees $a_{j'+1} a_{j'}$ and $b_{i'} b_{i'+1}$ also sees $a_{j+1} a_j$, thus we can exchange the edges assigned to $b_i b_{i+1}$ and $b_{i'} b_{i'+1}$. We can now assume that the edges of $\text{ch}(S)$ and the assigned edges of $\text{ch}(S_0)$ form interior-disjoint convex quadrilaterals.

We can now augment G'_0 with the vertices b_1, \dots, b_q . In one layer, add all edges of the cycle (b_1, \dots, b_q) . If edge $b_i b_{i+1}$ is assigned to $a_j a_{j+1}$, then join b_i to a_j and a_{j+1} in one layer, and b_{i+1} to a_j in the other layer (where $a_{p+1} = a_1$ and $b_{q+1} = b_1$). Denote the resulting graph by G''_0 . All new edges are disjoint from the interior of $\text{ch}(S_0)$, and the edges in each layer are noncrossing, thus G''_0 is biplane. Each b_i is joined to at least three vertices of the cycle (a_1, \dots, a_p) , which is part of the 5-connected graph G'_0 , and to its two neighbors in the cycle (b_1, \dots, b_q) . It follows that G''_0 is 5-connected. **The case in which the boundaries of $\text{ch}(S_0)$ and $\text{ch}(S)$ are not disjoint is studied in a similar way.**

Exterior points. We sort the points of S_{ext} into a sequence (s_1, \dots, s_k) , which is the reverse of a "removal order," defined as follows. Let $A = S_0 \cup S_{\text{ext}}$ and let $i = |S_{\text{ext}}|$. Until $A \neq S_0$, let s_i be an arbitrary vertex of $\text{ch}(A)$ in S_{ext} , put $A := A \setminus \{s_i\}$, and $i := i - 1$. By construction, s_i is in the exterior of the convex hull of $S_0 \cup \{s_1, \dots, s_{i-1}\}$ for $i = 1, \dots, k$.

We augment G''_0 successively by the vertices s_1, \dots, s_k . For $i = 1, \dots, k$, let G_i denote the **biplane graph on $S_0 \cup \{s_1, \dots, s_i\}$** obtained by Lemma 6.

When all three phases of the algorithm are complete, we obtain a 5-connected biplane graph in $\mathcal{G}_2(S)$. We conclude that a point set S admits a 5-connected biplane graph if there are at least 14 points in convex position in S , which holds for every set of at least 1352079 points. \square

3 Graph Augmentation

In this section we study how the addition of a second layer can improve the vertex connectivity of a graph. Suppose we are given a plane graph $G = (S, E)$ in $\mathcal{G}_1(S)$, and wish to augment it with a set E' of new edges to obtain a biplane graph $G = (S, E \cup E') \in \mathcal{G}_2(S)$ such that the vertex connectivity $\kappa(G)$

is maximal. We are allowed to add new edges in the first layer (augmenting G to a triangulation), or add new edges in the second layer. The more edges G has, the more constrained the problem becomes. In the worst case, we may assume that $G \in \mathcal{G}_1(S)$ is a triangulation, and it is augmented with another triangulation.

In Section 3.1 we show how to augment a triangulation with a second layer in order to make it 4-connected, whenever that is possible. In Section 3.2 we look into the problem of producing an augmentation that uses as few extra edges as possible, that is, a *minimal* augmentation.

3.1 Increasing the connectivity of triangulations

Our first observation is that there are arbitrarily large point sets S and triangulations $T \in \mathcal{G}_1(S)$ such that T cannot be augmented to a 4-connected biplane graph on S . These special triangulations are the wheels and fans. Recall that a *wheel* is a triangulation on n points such that $n - 1$ points are in convex position and one point (the *center* of the wheel) lies in the interior of $\text{ch}(S)$, the points on the convex hull induce a cycle on the boundary of $\text{ch}(S)$ and the interior point is joined to all other $n - 1$ points. A *fan* is a triangulation on n points in convex position, where one point (called *center*) is joined to all other $n - 1$ points. Every wheel is 3-connected and every fan is 2-connected. We show below that their connectivity cannot be increased to 4 by augmenting them with a set of pairwise noncrossing edges.

Lemma 7 *Let $T = (S, E)$ be a wheel or a fan. For every triangulation $T' = (S, E')$, the biplane graph $G = (S, E \cup E')$ is at most 3-connected.*

Recall that every maximal biplane graph on $n \geq 3$ points is 3-connected (Lemma 5), hence Lemma 7 is tight. We show next that wheels and fans are the only exceptions, any other triangulation on $n \geq 4$ vertices can be augmented to a 4-connected biplane graph. To that end, we use the following well-known auxiliary lemma.

Lemma 8 *Let S be a set of $n \geq 5$ points in general position such that u, v , and w , are consecutive vertices of $\text{ch}(S)$. Let T be a triangulation on S including triangle $\Delta = (u, v, w)$ and an adjacent triangle $\Delta_1 = (u, v', w)$. Then, edge uw is flippable in T and, after flipping uw , one of the edges uv' or $v'w$ is also flippable in the new triangulation.*

We also recall the characterization of 2- and 3-vertex cuts in a triangulation $T = (S, E)$ on $n \geq 5$ vertices. A set $\{u, v\} \subset S$ is a 2-vertex cut in T if and only if uv is a chord of $\text{ch}(S)$. A set $\{u, v, w\} \subset S$ is a 3-vertex cut in T if and only if $\{u, v, w\}$ induces a separating triangle or a path between two vertices of $\text{ch}(S)$ via an interior point (such a path is called a *bichord*). In all three cases, the convex hull $\text{ch}(S)$ is split into two nonempty subsets by a chord, a separating triangle or a bichord, such that no edge of T connects vertices in distinct subsets. When we augment a triangulation T to a 4-connected biplane graph G , we must add an edge between subsets of S on opposite sides of each chord, bichord and separating triangle of T . We will say that an edge *properly* crosses a bichord or a separating triangle when the edge crosses it exactly once (a chord is always crossed properly).

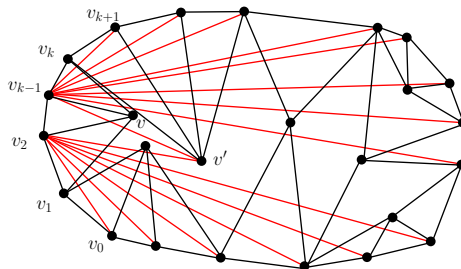


Figure 4: Case of all the vertices belonging to bichords.

We wish to augment a triangulation $T = (S, E)$ on $n \geq 5$ vertices into a 4-connected biplane graph $G = (S, E \cup E')$. Since T is 2-connected, to ensure the 4-connectivity of G it is enough to check that (i)

all vertices have degree at least 4 in G , (ii) every separating triangle and every bichord of T is properly crossed by at least one new edge, and (iii) every chord of T is properly crossed by at least two new edges. In the latter case, if the part of $\text{ch}(S)$ on one side of a chord contains two or more points, then at least two of them must be incident to edges that properly cross the chord (otherwise the chord and a common endpoint of the edges crossing the chord would be a 3-vertex cut).

We are now ready to augment any triangulation other than the wheel or the fan to a 4-connected biplane graph.

Theorem 9 *Let S be a set of $n \geq 6$ points in convex position or a set of $n \geq 5$ points not in convex position. For every triangulation $T = (S, E)$ other than the wheel and the fan, there is a triangulation $T' = (S, E')$ such that $G = (S, E \cup E')$ is 4-connected.*

Proof. (Sketch) We consider two cases depending on the original connectivity of T . Assume T is 3-connected. Then, either a vertex not belonging to any 3-vertex cut exists or every vertex is part of a 3-vertex cut. If there is a vertex v that does not belong to any 3-vertex cut, then we augment T to a 4-connected biplane graph with a star centered at v on the second layer. Otherwise, if every vertex is part of a 3-vertex cut, then all the 3-vertex cuts are bichords. In this case, we can prove that two vertices of $\text{ch}(S)$ (v_2 and v_{k-1} in Figure 4) can be chosen such that T can be augmented to a 4-connected biplane graph with two stars centered at these two vertices and connecting an interior vertex v' and every vertex of $\text{ch}(S)$ to them.

If T is not 3-connected, then T must have a 2-vertex cut because every triangulation is 2-connected. Each 2-vertex cut $\{u, v\}$ corresponds to a chord $uv \in E$ of $\text{ch}(S)$ and the set of chords of $\text{ch}(S)$ decomposes $\text{ch}(S)$ into convex regions, called *cells*. Now we proceed by induction on n , the total number of vertices. **After removing a set of vertices corresponding to a particular cell, the remaining triangulation can be augmented to 4-connected. Then, putting back the removed vertices, the edges on the second layer can be retriangulated to obtain a 4-connected biplane graph on the n vertices.** \square

Note that, by Lemma 7, the fan and the wheel, respectively, are 2- and 3-connected triangulations that cannot be augmented to 4-connected biplane graphs. It is also possible to prove that there are 4-connected triangulations that cannot be augmented to 5-connected biplane graphs.

Theorem 10 *There exist arbitrarily large point sets S and 4-connected triangulations $T = (S, E)$ such that for any other triangulation $T' = (S, E')$, the biplane graph $(S, E \cup E')$ is not 5-connected.*

3.2 Minimal Augmentation

Given a plane graph $G = (S, E)$, we wish to augment G with a minimal set of new edges E' such that we obtain a k -connected biplane graph $G = (S, E \cup E')$ for some target value k . In this section we present an efficient solution (Lemma 12) when $k = 3$ and G is a triangulation. We start with a helpful claim about augmenting a plane tree to 2-edge-connectivity.

Lemma 11 *Given a plane tree $H = (S, E)$ with n vertices and m leaves, let $L \subset S$ be the set of m leaves of H . In $O(n \log n)$ time, one can find a set E' of $\lceil m/2 \rceil$ pairwise noncrossing edges among the leaves of H such that $(S, E \cup E')$ is a 2-edge-connected biplane graph. Moreover, if the m leaves are in convex position, then E' can be found in $O(n)$ time, after computing $\text{ch}(L)$ in $O(m \log m)$ preprocessing time.*

Lemma 12 *Given a triangulation $G = (S, E)$, with $n \geq 3$ vertices, one can find a minimal set of edges E' such that $G' = (S, E \cup E')$ is a 3-connected biplane graph in $O(n)$ time, after computing $\text{ch}(S)$ in $O(n \log n)$ preprocessing time.*

Proof. Triangulation G is 2-connected. Recall that a set $\{u, v\} \subset S$ is a 2-vertex cut if and only if $uv \in E$ is a chord of $\text{ch}(S)$. The biplane graph $G' = (S, E \cup E')$ will be 3-connected if each chord of $\text{ch}(S)$ in E is crossed by at least one edge in E' .

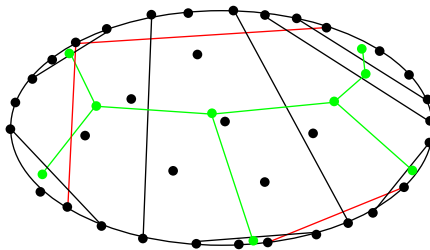


Figure 5: A point set, the associated graph H (green), and the additional edges to obtain 3-connectivity (red).

The chords of $\text{ch}(S)$ in E decompose $\text{ch}(S)$ into convex cells. We construct a dual graph of this decomposition, denoted H , as follows. The nodes of H correspond to the cells, and two nodes are joined by an edge in H if and only if the corresponding cells share a chord. Clearly, H is a tree (see the green edges in Figure 5), and can be easily constructed in $O(n)$ time from G . The leaves of H correspond to *leaf cells*, each leaf ℓ of H is associated with the set of vertices $R_\ell \subset S$ that lie in the cell, excluding the endpoints of the chord on the boundary of the cell. Distinct leaves of H are associated with disjoint vertex sets (i.e., $R_\ell \cap R_{\ell'} = \emptyset$ for $\ell \neq \ell'$). Consider chords that lie on the boundaries of the leaf cells. These chords are in convex position, thus any new edge can cross at most two of them. It follows that we need to add at least $\lceil m/2 \rceil$ new edges, where m denotes the number of leaves of H .

We now show that $\lceil m/2 \rceil$ new edges suffice, and can be computed in linear time. For each leaf ℓ of H , pick a point $v_\ell \in R_\ell$ on the boundary of $\text{ch}(S)$, called the *representative* of ℓ . Embed H in the plane such that every leaf ℓ is embedded at point v_ℓ , and every nonleaf node is embedded at an arbitrary point in the interior of its cell. Clearly, this embedding can be constructed in $O(n)$ time, after computing $\text{ch}(S)$.

By Lemma 11, the embedding of H can be augmented to a 2-edge-connected graph H' by a set E' of $\lceil m/2 \rceil$ noncrossing edges among the leaves (that are representatives from S) in $O(n)$ time, because all the representatives are in convex position. We claim that $(S, E \cup E')$ is 3-connected. Every 2-cut of G is a chord c of $\text{ch}(S)$ that corresponds to an edge e_c of the embedding of H . When the embedding of H is augmented to H' , e_c becomes part of at least one cycle in H' , and any of these cycles will contain exactly one edge from E' . Observe that, if C is one of such cycles and e is the edge of E' in C , then every edge of C (except for e) corresponds to a chord that is crossed by e . In particular, e will cross c . Since H' is 2-edge-connected, every edge of the embedding of H belongs to a cycle in H' , so every chord of $\text{ch}(S)$ is crossed by some edge of E' . \square

The above argument implies that we add one edge for every two leaf cells of H . Associate to each leaf ℓ of H all vertices in R_ℓ and the two endpoints of the chord. Every vertex of $\text{ch}(S)$ is the endpoint of at most two chords that bound leaf cells. Therefore, one can assign to each leaf cell at least two vertices (a vertex of R_ℓ and half of each endpoint of the chord on the boundary of the leaf). It follows that there are at most $\lfloor n/2 \rfloor$ leaf cells. This bound is tight, since the chords of the leaf cells may form a cycle of $n/2$ edges. We obtain an upper bound for the total number of edges to be added in Lemma 12.

Corollary 13 *Every triangulation $G = (S, E)$ on $n \geq 3$ points can be augmented to a 3-connected biplane graph by adding at most $\lceil \lfloor n/2 \rfloor / 2 \rceil = \lceil \frac{n+1}{4} \rceil$ edges.*

4 Conclusions

We have presented several results on constructing **biplane graphs on given sets** to get high connectivity, on augmenting given plane graphs to biplane graphs, again trying to achieve better connectivity, and we have studied several extremal configurations. Improving on our combinatorial bounds is the obvious left open problem. However, we want to conclude with a remark and a mention to a specific open problem:

- Note that the proof of Theorem 3 needs the point set to be very large (roughly 1.3×10^6). However, we might reduce the constant to 137 by using the following result: for any convex polygon P of k vertices, a triangle Δ in the interior of P , and r points inside Δ (for any $r \geq k - 2$), one can construct a plane geometric 4-connected graph on the $k + r + 3$ points [12]. Since this result has neither been published nor referred yet, we decided to use an alternative complete proof.
- Open problem: Given a 3-connected plane graph $G = (S, E)$ with $n \geq 6$ vertices, can we find a minimal set of edges E' such that $G' = (S, E \cup E')$ is a 4-connected biplane graph in polynomial-time?

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