Locating domination in graphs and their complements[S]

C. Hernando[a], **M. Mora**[b], **I. M. Pelayo**[c]

(a) *Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya,*
E-mail: `carmen.hernando@upc.edu`

(a) *Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya,*
E-mail: `merce.mora@upc.edu`

(a) *Departament de Matemàtica Aplicada III, Universitat Politècnica de Catalunya,*
E-mail: `ignacio.m.pelayo@upc.edu`

Resumen. A dominating set S of a graph G is called *locating-dominating*, *LD-set* for short, if every vertex v not in S is uniquely determined by the set of neighbors of v belonging to S. Locating-dominating sets of minimum cardinality are called LD-codes and the cardinality of an LD-code is the *location-domination number*. An LD-set of a graph G is *global* if S is an LD-set of both G and its complement, $\overline{G}$. In this work, we give some relations between the locating-dominating sets and location-domination number in a graph and its complement.

Palabras clave. *graph, domination, location, complement graph*

1 Introduction

Many problems involving detection devices can be modeled with graphs. Detection devices and the objects or intruders to be detected occupy some vertices of a graph. We are interested in finding the minimum number of devices needed according to the type of devices and the necessity of locating the intruder. This gives rise to consider locating and dominating sets. Locating-dominating sets can be used to determine the location of an object in a graph if devices can detect only objects in its neighborhood and the object cannot occupy the same vertex as detection devices.

Let $G = (V, E)$ be a simple graph, not necessarily connected. The *(open) neighborhood* of a vertex $v \in V$ is $N_G(v) = \{u \in V : uv \in E\}$. We denote the complement of G by $\overline{G}$. The distance between vertices $v, w \in V$ is denoted by

[S] Partially supported by projects MTM2009-07242, MTM2012-30951, Gen. Cat. DGR 2009SGR1040, ESF EUROCORES programme EUROGIGA-ComPoSe IP04-MICINN, MTM2011-28800-C02-01, Gen. Cat. DGR 2009SGR1387

$d_G(v,w)$. We write $N(u)$ or $d(v,w)$ if the graph G is clear from the context. The diameter $diam(G)$ is the maximum distance between any two vertices of G. Let $S = \{x_1, \ldots, x_k\}$ be a set of vertices and let $v \in V \setminus S$. The ordered k-tuple $c_S(v) = (d(v,x_1), \ldots, d(v,x_k))$ is called the vector of *metric coordinates* of v with respect to S. For further notation see [2].

A set $D \subseteq V$ is a *dominating set* if for every vertex $v \in V \setminus D$, $N(v) \cap D \neq \emptyset$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set is *global* if it is a dominating set of both G and its complement graph, $\overline{G}$. The minimum cardinality of a global dominating set of G is the *global domination number* of G, denoted with $\gamma_g(G)$ [4]. If T is a subset of V and $v \in V$, we say that v *dominates* T if $T \subseteq N(v)$.

A set $S = \{x_1, \ldots, x_k\} \subseteq V$ is a *locating set* if for every pair of distinct vertices $u, v \in V$, $c_S(u) \neq c_S(v)$. The *location number* (also called the *metric dimension*) $\beta(G)$ is the minimum cardinality of a locating set of G [3, 7]. A locating set of cardinality $\beta(G)$ is called a β-*code*.

A set $S \subseteq V$ is a *locating-dominating set*, *LD-set* for short, if S is a dominating set such that for every two different vertices $u, v \in V \setminus S$, $N(u) \cap S \neq N(v) \cap S$. The *location-domination number* of G, $\lambda(G)$ is the minimum cardinality of a locating-dominating set. A locating-dominating set of cardinality $\lambda(G)$ is called an *LD-code* [8]. We have $\beta(G) \leq \lambda(G)$ and $\gamma(G) \leq \lambda(G)$. A complete and regularly updated list of papers on locating dominating codes is to be found in [6].

In this paper we introduce *global LD-sets* and the *global location-domination number*. Concretely, we give some general results and study them in a family of graphs, the *blockcactus*. The family of blockcactus is interesting because it contains cycles, trees, complete graphs and unicyclic graphs (see Figure 1). We omit proofs because of limited space.

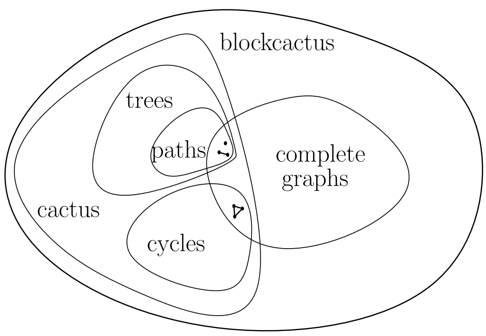

Fig. 1: Families of blockcactus

2 Global locating domination

A set of vertices of a graph G is a *global LD-set* if S is an LD-set of both G and its complement, $\overline{G}$. The *global location-domination number* of G, denoted by $\lambda_g(G)$, is the minimum cardinality of a global LD-set. Notice that global LD-sets and the global location-domination number are the same in the graphs G and $\overline{G}$. An LD-code S of G is *global* if it is a global LD-set, i.e. S is an LD-code of G and an LD-set of $\overline{G}$. Next, we give some results involving global locating domination, some of them already proved in [5].

Proposition 1 ([5]). *If G is a graph with a global LD-set, then $\lambda(\overline{G}) \leq \lambda(G)$.*

Proposition 2 ([5]). *If $S \subseteq V$ is an LD-set of a graph $G = (V, E)$, then S is a global LD-set of G if and only if there is no vertex in $V \setminus S$ dominating S.*

Corollary 1. *If S is an LD-set of a graph G, then S is a global LD-set if and only if S is a global dominating set.*

Proposition 3 ([5]). *If $S \subseteq V$ is an LD-set of a graph $G = (V, E)$ such that there exists a vertex $u \in V \setminus S$ dominating S, then $S \cup \{u\}$ is an LD-set of $\overline{G}$.*

Corollary 2 ([5]). *For every graph G, $|\lambda(G) - \lambda(\overline{G})| \leq 1$.*

According to the preceding result, we have that $\lambda(G) = \lambda(\overline{G}) + 1$ or $\lambda(\overline{G}) = \lambda(G) + 1$ or $\lambda(G) = \lambda(\overline{G})$ and the three cases are possible. For example, it is easy to see that complete graphs K_n of order $n \geq 2$ satisfy $\lambda(\overline{K_n}) = \lambda(K_n) + 1$, the star $K_{1,n-1}$ of order at least 3 satisfies $\lambda(K_{1,n-1}) = \lambda(\overline{K_{1,n-1}})$, and the double star $K_2(r, s)$, $r, s \geq 2$, obtained by joining the central vertices of two stars $K_{1,r}$ and $K_{1,s}$ respectively, satisfies $\lambda(K_2(r, s)) = \lambda(\overline{K_2(r, s)}) + 1$.

Proposition 4. *For any graph $G = (V, E)$,*

$$\max\{\lambda(G), \lambda(\overline{G})\} \leq \lambda_g(G) \leq \min\{\lambda(G), \lambda(\overline{G})\} + 1.$$

Corollary 3. *Let $G = (V, E)$ be a graph.*

If $\lambda(G) \neq \lambda(\overline{G})$, then $\lambda_g(G) = \max\{\lambda(G), \lambda(\overline{G})\} = \min\{\lambda(G), \lambda(\overline{G})\} + 1$.

If $\lambda(G) = \lambda(\overline{G})$, then $\lambda_g(G) \in \{\lambda(G), \lambda(G) + 1\}$, and both possibilities are feasible.

Next, we give some examples to illustrate all possibilities given in the preceding corollary. Complete graphs of order n, $n \geq 2$, satisfy $\lambda(K_n) \neq \lambda(\overline{K_n})$ and $\lambda_g(K_n) = \lambda(\overline{K_n}) = \lambda(K_n) + 1$; stars of order n, $n \geq 3$, satisfy $\lambda(K_{1,n-1}) = \lambda(\overline{K_{1,n-1}}) = \lambda_g(K_{1,n-1}) = n - 1$ and the cycle C_5 satisfies $\lambda(C_5) = \lambda(\overline{C_5}) = 2$ and $\lambda_g(C_5) = 3$.

3 Blockcactus

A *block* of a graph is a maximal connected subgraph with no cut vertices. A graph G is a *cactus* if all its blocks are cycles or complete graphs of order at most 2. Cactus graphs are characterized as those graphs such that two different cycles share at most one vertex. A *blockcactus* is a connected graph such that all its blocks are cycles or complete graphs. It is important to notice that complete graphs, cycles and trees are blockcactus (see Figure 1). In this section we characterize blockcactus satisfying $\lambda(G) < \lambda(\overline{G})$. By Corollary 1, this inequality is possible only in graphs with no global LD-sets.

Proposition 5. *If G is a graph with a non-global LD-set, then G is a connected graph containing a vertex of eccentricity at most* 2*.*

Corollary 4. *If G is a graph satisfying $\lambda(G) < \lambda(\overline{G})$, then G is a connected graph with diameter at most* 4*.*

We will refer in this section to some special families of graphs such as paths of order n, P_n; cycles of order n, C_n; the complete graph of order n, K_n; the *paw* (or 3-pan); the *bull*; the *banner* (or 4-pan), P; the *complement of the banner*, $\overline{P}$; the butterfly and the corner (see Figure 2).

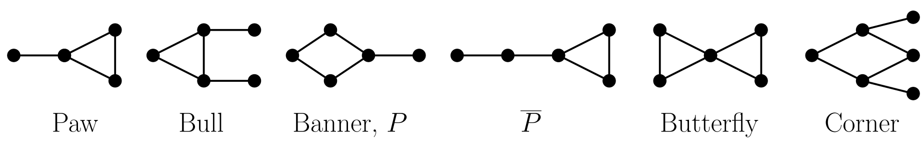

Fig. 2: Some special graphs.

In [1], all 16 non-isomorphic graphs with $\lambda(G) = 2$ are given. After carefully examining all cases, the following result is obtained.

Proposition 6. *Let $G = (V, E)$ be a blockcactus such that $\lambda(G) = 2$. Then $\lambda(G) < \lambda(\overline{G})$ if and only if G is isomorphic to a cycle of order 3, the paw, the butterfly or the complement of a banner.*

Next, some necessary conditions are given for a blockcactus with $\lambda(G) \geq 3$ to have at least a non-global LD-set.

Proposition 7. *Let $G = (V, E)$ be a blockcactus and $S \subseteq V$ a non-global LD-set of G. Suppose that vertex $u \in V$ satisfies $S \subseteq N(u)$. Then G can be obtained by identifying the vertex u of some copies of each of the following graphs (See Figure 3):*

a) u is adjacent to every vertex of a complete graph K_r, $r \geq 1$, and each one of the vertices of K_r is adjacent to at most a new vertex of degree 1;

b) u is a vertex of a cycle of order 4, and each neighbor of u is adjacent to at most a new vertex of degree 1;

c) u is a vertex of a cycle of order 5.

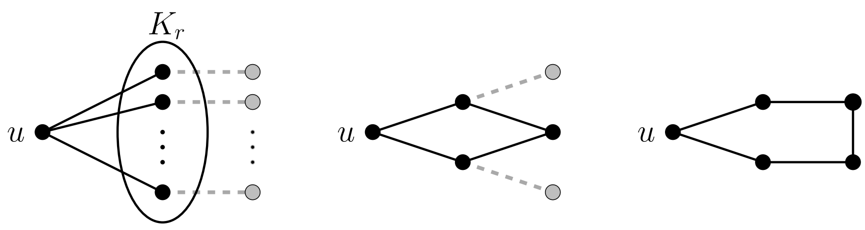

Fig. 3: Subgraphs of a blockcactus having a non global LD-set. Gray vertices are optional.

In the next proposition we characterize those blockcactus not containing a global LD-code of order at least 3. Assume that G and H is a pair of graphs whose vertex sets are disjoint. The *union* $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The *join* $G + H$ has $V(G) \cup V(H)$ as vertex set and $E(G) \cup E(H) \cup \{uv : u \in v(G) \text{ and } v \in V(H)\}$ es edge set.

Proposition 8. *Let $G = (V, E)$ be a blockcactus such that $\lambda(G) \geq 3$, then G does not contain any global LD-code if and only if G is isomorphic to one of the following graphs (see Figure 4):*

a) $K_1 + (K_1 \cup K_r)$, $r \geq 3$;

b) the graph obtained by joining one vertex of K_2 with a vertex of a complete graph of order $r + 1$, $r \geq 3$;

c) $K_1 + (K_{r_1} \cup \cdots \cup K_{r_t})$, where $t \geq 1$, $r_1, \ldots, r_t \geq 2$ and $r_1 \geq 3$ if $t = 1$, and $r_1 \geq 3$ or $r_2 \geq 3$ if $t = 2$;

d) the graph obtained by joining a vertex of K_2 with one of the vertices of degree 2 of a corner;

e) if we consider the graph $K_1 + (K_{r_1} \cup \cdots \cup K_{r_t})$ and t' copies of a corner, with $t + t' \geq 2$ and $r_1, \ldots, r_t \geq 2$, the graph obtained by identifying the vertex of K_1 with one of the vertices of degree 2 of each copy of the corner.

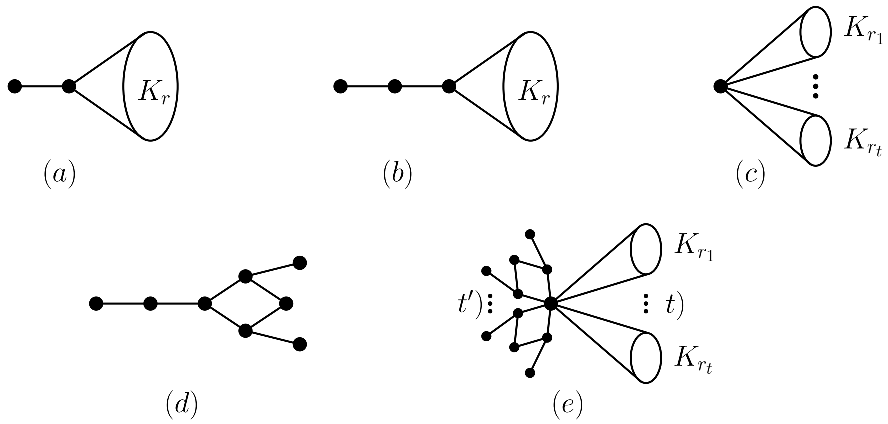

Fig. 4: Blockcactus not containing a global LD-code. Graphs of the first row satisfy $\lambda(G) < \lambda(\overline{G})$.

Blockcactus of order at most 2 are the complete graphs, K_1 and K_2. For those graphs we have $\lambda(K_1) = \lambda(\overline{K_1}) = 1$ and $\lambda(K_2) = 1 < 2 = \lambda(\overline{K_2})$. If G is a blockcactus of order at least 3, we have obtained the following characterization.

Proposition 9. *If $G = (V, E)$ is a blockcactus of order at least 3, then $\lambda(G) < \lambda(\overline{G})$ if and only if G is isomorphic to one of the following graphs (see Figure 4):*

a) $K_1 + (K_1 \cup K_r)$, $r \geq 2$;

b) the graph obtained by joining one vertex of K_2 with a vertex of a complete graph of order $r + 1$, $r \geq 2$;

c) $K_1 + (K_{r_1} \cup \cdots \cup K_{r_t})$, $t \geq 1$, $r_1, \ldots, r_t \geq 2$.

Notice that complete graphs of order at least 3 are included in the family of graphs described under $c)$ when $t = 1$.

Corollary 5. *Every tree T of order at least 3 satisfies $\lambda(\overline{T}) \leq \lambda(T)$.*

4 Bipartite graphs

In this section we study the relation between the location-domination number of a graph G and its complement, $\overline{G}$, when G is a bipartite graph. Let $G = (V, E)$ be a bipartite graph of order n with partite sets V_1 and V_2. Suppose that $|V_1| = r$, $|V_2| = s$, and $2 \leq r \leq s$.

Proposition 10. *If G is not connected, then $\lambda(\overline{G}) \leq \lambda(G)$.*

Proposition 11. *If G contains an LD-code S with vertices in both partite sets, then $\lambda(\overline{G}) \leq \lambda(G)$.*

Proposition 12. *If V_2 is an LD-code of G, then $\lambda(\overline{G}) \leq \lambda(G)$.*

Proposition 13. *If V_1 is an LD-code and $s > 2^r - 1$, then $\lambda(G) \geq \lambda(\overline{G})$.*

Proposition 14. *If V_1 is a non-global LD-code and $2^{r-1} + 2 \leq s \leq 2^r - 1$, then $\lambda(G) < \lambda(\overline{G})$.*

Proposition 15. *If V_1 is a non-global LD-code and $s \leq 2^{r-1} + 1$, then there exist bipartite graphs satisfying both inequalities, $\lambda(\overline{G}) \leq \lambda(G)$ and $\lambda(G) < \lambda(\overline{G})$.*

Referencias

[1] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, Locating-dominating codes: Bounds and extremal cardinalities, *Appl. Math. Comput*, (2013), accepted.

[2] Chartrand, G., Lesniak, L., *Graphs and Digraphs*, Fourth Edition, CRC Press, Boca Raton (FL), (2005).

[3] F. Harary and R. A. Melter, On the metric dimension of a graph, *Ars Combinatoria*, **2** (1976) 191–195.

[4] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker, New York, 1998.

[5] C. Hernando, M. Mora, I. M. Pelayo, Nordhaus-Gaddum bounds for locating domination, *Eur. J. Combin.*, (2012) accepted.

[6] A. Lobstein, Watching systems, identifying, locating-dominating ans discriminating codes in graphs,
http://www.infres.enst.fr/ lobstein/debutBIBidetlocdom.pdf

[7] P. J. Slater, Leaves of trees, *Congressus Numerantium*, **14** (1975) 549–559.

[8] P. J. Slater, Dominating and reference sets in a graph, *J. Math. Phys. Sci.* **22** (1988) 445–455.