A general approach for computing residues of partial-fraction expansion of transfer matrices

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Abstract: This paper deals with the description of a general method for calculating the residues of a linear system. Considering, physical models, it is well-assumed that the system described only presents simple eigenvalues, or at least simple-complex eigenvalues. However, as demonstrated in this paper, it is not completely true for all the real systems, and a method to evaluate the residues for these cases is required. In this paper, a methodology for computing the residues, even with the existence of multiple eigenvalues (described by their Jordan normal form) is developed and presented. Moreover, the calculation of the residues is applied to analyze the output-controllability of dynamic systems. Finally, some real examples are presented to validate the methodologies proposed.

Key–Words: Eigenvalues, Jordan normal form, output-controllability, residues.

1 Introduction

It is well known that for many physical problem description, the state space representation is used

$$\left.\begin{array}{ll} \dot{X} & = AX + Bu \\ Y & = CX \end{array}\right\}, \tag{1}$$

which input-output relationship can be given by the transfer function

$$G(s) = C(sI - A)^{-1}B = \sum_{i=1}^{r}\sum_{j=1}^{m_i} \frac{R_{\lambda_i j}}{(s-\lambda_i)^j}.$$

Matrices R_i are known as residues of the transfer functions, and their knowledge have some interest since provide the gain of the transfer function from input to output as well as reveal which inputs have the largest influence on the output, among other information.

The importance of residues knowledge can be reflected in the variety of classic (see [1, 12, 13, 15] for example) and recent (for example [8, 14, 16, 17]) publications which can be found. Concretely, P. Navratil and L. Pekar in [16] use the residues calculation to study the problem of decoupling considering multi-inputmulti-output (MIMO) systems.

In the literature (see, e.g., [12]), there are several studies explaining how to obtain the residues of A matrix, for the case where A only has simple eigenvalues; However, it is not described for the general case, it means when the matrix A has eigenvalues (both real and complex) of algebraic multiplicity greater than one. It is worth to say that generally the matrices have simple eigenvalues (at least the complex), but not all mathematical models that represents physical problems are generic. In this paper, algorithms for obtaining the residues for the general case are presented. In [13], it is developed a calculation method for partial fraction expansion of transfer matrices which uses a Vandermonde matrix formed by the eigenvalues of the matrix of the system, however the method requires to calculate the powers of the matrix A, making it though and hard to develop.

Other authors as L. De Tommasi, M. de Magistris, D. Deschrijver and T. Dhaene in [17] present an algorithm for the identification of a positive real rational transfer matrix of a MIMO system from frequency domain data samples. The algorithm is based on the combination of least-squares pole identification by the Vector Fitting algorithm and residue identification.

This paper aims to give a general method to obtain the residues to any system. Moreover, a new approach to analyze output-controllability using residues is presented. Finally, some application examples where the general method is required are shown in the paper.

This paper is organized as follows. In Section 2, the classical procedure for computing the residues is presented. Different methods for reducing the size of the system are explained in Section 3. In Section 4, a general method for computing the residues is developed. The method is also developed for composite systems in Section 5. In Section 6, a relationship between the residues and the output-controllability concept is presented. Finally, in Section 7, some application examples are done to show the real requirement of the general method proposed.

2 Preliminaries

Consider the general state space system presented in equation (1)

$$\left.\begin{array}{ll}\dot{X} & = AX + Bu \\ Y & = CX\end{array}\right\},$$

where $A \in M_n(\mathbb{C})$, $B \in M_{n\times m}(\mathbb{C})$, and $C \in M_{p\times n}(\mathbb{C})$.

The transfer matrix can be calculated from the Laplace transform of the state equations

$$G(s) = C(sI - A)^{-1}B$$

Usually for the analysis of dynamic systems, it is necessary to find the partial fraction expansion in terms of the individual modes.

Let λ_i, $i = 1, \ldots, r$ be the eigenvalues of A with multiplicities m_i respectively. Remembering that the inverse of a square matrix can be obtained from its adjunct matrix (or adjugate matrix) which is the transpose of the matrix formed by the cofactors of elements ([7]). It can be stated that

$$(sI - A)^{-1} = \frac{A_0 s^{n-1} + A_1 s^{n-2} + \ldots + A_{n-2}s + A_{n-1}}{(s-\lambda_1)^{m_1}(s-\lambda_2)^{m_2}\ldots(s-\lambda_r)^{m_r}}$$

$$A_0 s^{n-1} + A_1 s^{n-2} + \ldots + A_{n-2}s + A_{n-1} = \mathrm{Adj}\,(sI - A)$$

$$(s-\lambda_1)^{m_1}(s-\lambda_2)^{m_2}\ldots(s-\lambda_r)^{m_r} = \det(sI - A).$$

Then, decomposing into simple fractions:

$$(sI - A)^{-1} = \sum_{i=1}^{r}\sum_{j=1}^{m_i}\frac{K_{\lambda_i j}}{(s-\lambda_i)^j}, \tag{2}$$

where $K_{\lambda_i j}$ are the matrix residues of the partial fraction expansion. It will be written simply K_{ij} in order to avoid confusion.

Then, by multiplying the simple fraction decomposition of the inverse (in (2)) by input and output matrices, the transfer matrix can be defined as a partial fraction expansion,

$$C(sI - A)^{-1}B = \sum_{i=1}^{r}\sum_{j=1}^{m_i}\frac{R_{\lambda_i j}}{(s-\lambda_i)^j},$$

where $R_{\lambda_i j}$ are the matrix residues which will be written simply R_{ij} if confusion is not possible, and it will be denoted simply as R_i in the case where $m_i = 1$ for all i.

Example

Let

$$\left.\begin{array}{ll}\dot{X} & = AX + Bu, \\ Y & = CX\end{array}\right\}$$

be a system with

$$A = \begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}, B = \begin{pmatrix}2\\ 1\end{pmatrix}, C = \begin{pmatrix}3 & 4\end{pmatrix},$$

$$(sI - A)^{-1} = \begin{pmatrix}\frac{1}{s-2} & \frac{1}{(s-2)^2}\\ 0 & \frac{1}{(s-2)}\end{pmatrix},$$

$$C(sI - A)^{-1}B = \frac{10}{s-2} + \frac{3}{(s-2)^2}.$$

Then $K_{11} = I$, $K_{12} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$,
and $R_{11} = 10$, $R_{12} = 3$.

3 Reduction

3.1 Reduction to the SISO systems

Let $C \in M_{p\times n}(\mathbb{C})$ and $B_{n\times m}(\mathbb{C})$ be the output and input matrices respectively, then $C = \begin{pmatrix}C_1\\ \vdots\\ C_p\end{pmatrix}$ and $B = \begin{pmatrix}B_1 & \ldots & B_m\end{pmatrix}$ with $C_k \in M_{1\times n}(\mathbb{C})$ and $B_\ell \in M_{n\times 1}(\mathbb{C})$ for $i = k, \ldots, p$ and $\ell = 1, \ldots, m$.

Then the transfer matrix $G(s) = C(sI - A)^{-1}B$ can be partitioned in the following manner

$$(C_i)(sI - A)^{-1}(B_j) = (C_i(sI - A)^{-1}B_j) \tag{3}$$

Consequently

$$C(sI - A)^{-1}B = \left(\sum_{i=1}^{r}\sum_{j=1}^{m_i}\frac{(R_{k\lambda_{ij}\ell})}{(s-\lambda_i)^j}\right).$$

So, the study can be reduced simply to the study of single input single output systems.

3.2 Reduction to the canonical reduced from

Another reduction can be achieved if an equivalence relation which preserves the transfer matrix is considered, allowing to consider the state matrix in a simpler form.

Let $S \in Gl(n;\mathbb{C})$ be such that $J = S^{-1}AS$ where J is the Jordan canonical reduced form of the matrix, that is to say:

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_r \end{pmatrix},$$

$$J_i = \begin{pmatrix} J_{i_1} & & \\ & \ddots & \\ & & J_{i_{s_i}} \end{pmatrix}, J_{i_j} = \lambda_i I + N \text{ where}$$

$$N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & \ddots & \ddots & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \in M_{n_{i_j}}(\mathbb{C}),$$

$\lambda_1, \dots, \lambda_r$ the distinct eigenvalues of A with multiplicities m_i respectively, $n_{i_1} + \dots + n_{i_{s_i}} = m_i$, and $s_i = \dim \operatorname{Ker}(\lambda_i I - A)$.

Remark 1 *In the case where $\lambda_i \neq \lambda_j$ for all $i \neq j$ the matrix J is obviously diagonal and it will be written simply as $J = D$.*

Using the expression $A = SJS^{-1}$ in the transfer matrix $G(s)$ the following proposition can be obtained.

Proposition 2 *The transfer matrix is invariant under basis change in the state space on the system.*

Proof:

Let $x = Sx_1$ be such that $J = S^{-1}AS$, then

$$G(s) = C(sI - A)^{-1}B = CS(sI - J)^{-1}S^{-1}B.$$

□

Then, taking $C' = CS$ and $B' = S^{-1}B$, the SISO case in its Jordan reduced form can be considered.

4 Computation of residues

For simplicity and because of the a diagonalizable system is the most extended, the case where the matrix A is diagonalizable, is analyzed first.

4.1 Diagonalizable case

i) Simple eigenvalues

In the case where the matrix has simple eigenvalues, it is well known that

Proposition 3

$$R_i = Cv_iu_iB$$

where v_i is a right eigenvector (column vector) and $\bar{u}_i$ left eigenvector (row vector) which are chosen in such a way that $u_iv_i = 1$.

Corollary 4 *Let $G(s)$ be the transfer matrix of a MIMO system. Then, the residue matrix R_i corresponding to the eigenvalue λ_i is*

$$R_i = (R_{ki\ell}) = (C_k v_i u_i B_\ell).$$

ii) Multiple eigenvalues

Supposing now that the eigenvalue λ_i appears with multiplicity m_i; in this case

$$C'(sI - D)^{-1}B' = C' \begin{pmatrix} \frac{1}{s-\lambda_1} I_{m_1} & & \\ & \ddots & \\ & & \frac{1}{s-\lambda_r} I_{m_r} \end{pmatrix} B', \tag{4}$$

then, the following result is obtained

Lemma 5

$$\begin{aligned} R_{11} &= c'_1 b'_1 + \dots + c'_{m_1} b'_{m_1}, \\ &\vdots \\ R_{r1} &= c'_{m_1+\dots+m_{r-1}+1} b'_{m_1+\dots+m_{r-1}+1} + \dots + c'_n b'_n. \end{aligned}$$

Theorem 6 *Let A be a diagonalizable matrix, $\lambda_1, \dots, \lambda_r$ the different eigenvalues of A with multiplicity m_i for all $i = 1, \dots, r$ and S the matrix of corresponding eigenvectors. The rows of S^{-1} are the corresponding left eigenvectors of the matrix A. Thus,*

$$\begin{aligned} R_{11} &= \textstyle\sum_{\ell=1}^{m_1} Cv_\ell u_\ell B, \ \dots, \ R_{r1} \\ &= \textstyle\sum_{\ell=m_1+\dots+m_{r-1}+1}^{n} Cv_\ell u_\ell B. \end{aligned}$$

Remark 7 *In the case where only one R_{i1} is of interest, to invert the matrix S is not necessary to obtain the corresponding left eigenvectors. It suffices to obtain the right eigenvectors $v_{i_1}, \dots, v_{i_{m_i}}$ (column vectors) and then to select the left eigenvectors $u_{i_1}, \dots, u_{i_{m_i}}$ (row vectors) in such a way that*

$$\begin{pmatrix} u_{i_1} \\ \vdots \\ u_{i_{m_i}} \end{pmatrix} \begin{pmatrix} v_{i_1} & \dots & v_{i_{m_i}} \end{pmatrix} = I_r.$$

Corollary 8 *Let $G(s)$ be the transfer matrix of a MIMO system. Then, the residue matrix R_i corresponding to the eigenvalue λ_i is defined as1*

$$R_{i1} = (R_{ji1k}) = (\sum_{\ell=1}^{m_i} C_j v_\ell u_\ell B_k).$$

4.2 General case

The matrix $(sI - J)^{-1}$ in the transfer matrix can be decomposed into blocks in the following manner

$$
\begin{array}{l}
(sI - J)^{-1} = \\
\begin{pmatrix} sI_i - J_i & & \\ & \ddots & \\ & & sI_i - J_i \end{pmatrix}^{-1} = \\
\begin{pmatrix} (sI - J_1)^{-1} & & \\ & \ddots & \\ & & (sI - J_r)^{-1} \end{pmatrix}
\end{array}
$$

and

$$
\begin{array}{l}
(sI_i - J_i)^{-1} = \\
\begin{pmatrix} (sI_{i_1} - J_{i_1})^{-1} & & \\ & \ddots & \\ & & (sI_{i_{m_i}} - J_{i_{m_i}})^{-1} \end{pmatrix}.
\end{array}
$$

i) Case of non-derogatory matrices with single eigenvalue

Suppose the matrix J has only one block, that is to say that the matrix A has a unique eigenvalue λ with $\dim \operatorname{Ker}(A - \lambda I) = 1$. Then $J = S^{-1}AS$, with $J = \lambda I + N \in M_n(\mathbb{C})$.

Then

$$
(sI - J)^{-1} = \frac{1}{s - \lambda} I_n + \ldots + \frac{1}{(s - \lambda)^n} N^{n-1},
$$

and the following result is obtained.

Lemma 9

$$
\begin{array}{ll}
R_{11} & = c'_1 b'_1 + \ldots + c'_n b'_n \\
R_{12} & = c'_1 b'_2 + \ldots + c'_{n-1} b'_n \\
\vdots & \\
R_{1n} & c'_1 b'_n.
\end{array}
$$

Hence, as a consequence, the following proposition is obtained.

Proposition 10

$$
\begin{array}{ll}
R_{11} & = \sum_{i=1}^{n} C v_i u_i B = c_1 b_1 + \ldots c_n b_n \\
R_{12} & = \sum_{i=1}^{n-1} C v_i u_{i+1} B \\
\vdots & \\
R_{1n} & = C v_1 u_n B.
\end{array}
$$

Proof:

$$
CS(sI - J)^{-1} S^{-1} B = CS(\textstyle\sum_i^n \frac{1}{(s-\lambda)^i} N^{i-1}) S^1 B = \textstyle\sum_i^n CS \frac{1}{(s-\lambda)^i} N^{i-1} S^{-1} B.
$$

Then the matrix K_{11} is

$$
CS \frac{1}{s - \lambda} I_n S^{-1} B = \frac{1}{s - \lambda} CSS^{-1} B = \frac{1}{s - \lambda} CB.
$$

□

Corollary 11 *Let $G(s)$ be the transfer matrix of a MIMO system. Then, the residue matrix R_1 corresponding to the single eigenvalue λ is*

$$
R_{11} = (R_{j11k}) = CB.
$$

ii) Case non-derogatory matrix with r eigenvalues

Suppose that A has a r eigenvalues λ_i with multiplicity m_i respectively, and $\dim \operatorname{Ker}(A - \lambda_i) = 1$ for all $i = 1, \ldots, r$.

So, $A = S^{-1} J S$ with $J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_r \end{pmatrix}$ and $J_i = \lambda_i I + N \in M_{m_i}(\mathbb{C})$.

Then

$$
(sI - J)^{-1} = \begin{pmatrix} sI_{m_1} - J_1 & & \\ & \cdots & \\ & & sI_{m_r} - J_r \end{pmatrix}^{-1}, \tag{5}
$$

and

$$
(sI_{m_i} - J_i)^{-1} = \tfrac{1}{s-\lambda_i} I_{m_i} + \ldots + \tfrac{1}{(s-\lambda_i)^{m_i}} N_{m_i}^{m_i - 1},
$$

$(I_{m_i}, N_{m_i} \in M_{m_i}(\mathbb{C}))$.

The matrix shown in (5) can be decomposed in the following manner

$$
\Pi_1 + \ldots + \Pi_r
$$

with

$$
\Pi_1 = \begin{pmatrix} (sI_{m_1} - J_1)^{-1} & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}
$$

$\vdots$

$$
\Pi_r = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & (sI_{m_r} - J_r)^{-1} \end{pmatrix}.
$$

As a consequence, the following result is obtained.

Proposition 12

$$R_{\lambda_i 1} = CS \begin{pmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & I_{m_i} & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} S^{-1}B,$$

$1 \leq i \leq n$.

Remark 13 *If only the residue of a specific eigenvalue is of interest, it is not necessary to obtain the complete Jordan basis for the matrix A. It suffices to obtain a sub-basis $(v_{i1}, \dots, v_{im_i})$ (column vectors) corresponding to the eigenvalue block and the corresponding left sub-basis $(u_{i1}, \dots, u_{im_i})$ (row vectors) such that $\begin{pmatrix} v_{i1} & \dots & v_{im_i} \end{pmatrix} \begin{pmatrix} u_{i1} \\ \vdots \\ u_{im_i} \end{pmatrix} = I_{m_i}$, as we can proof in the following manner.*

Let $S = \begin{pmatrix} v_{11} & \dots & v_{1m_1} & \dots & v_{r1} & \dots & v_{rm_r} \end{pmatrix}$ be the Jordan basis and $S^{-1} = \begin{pmatrix} u_{11} \\ \vdots \\ u_{1m_1} \\ \vdots \\ u_{r1} \\ \vdots \\ u_{rm_r} \end{pmatrix}$ the corresponding left Jordan basis such that $SS^{-1} = I_n$.

So, calling

$$\begin{pmatrix} V_1 & \dots & V_r \end{pmatrix} = \begin{pmatrix} v_{11} & \dots & v_{1m_1} & \dots & v_{r1} & \dots & v_{rm_r} \end{pmatrix}$$

$$P_i = \begin{pmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & I_{m_i} & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}$$

$$\begin{pmatrix} U_1 \\ \vdots \\ U_r \end{pmatrix} = \begin{pmatrix} u_{11} \\ \vdots \\ u_{1m_1} \\ \vdots \\ u_{r1} \\ \vdots \\ u_{rm_r} \end{pmatrix}$$

we have

$$\begin{pmatrix} V_1 & \dots & V_r \end{pmatrix} P_i \begin{pmatrix} U_1 \\ \vdots \\ U_r \end{pmatrix} = \begin{pmatrix} v_{i1} & \dots & v_{im_i} \end{pmatrix} \begin{pmatrix} u_{i1} \\ \vdots \\ u_{im_1} \end{pmatrix} = I_{m_i}.$$

iii) Case derogatory matrix with single eigenvalue

Suppose now the matrix A is equivalent to $J = \text{diag}\,(J_1, \dots, J_s)$, with $J_i = \lambda I_i + N_i \in M_{n_i}$, $n_1 + \dots + n_s = n$. Without loss of generality, it can be considered $n_1 \geq \dots \geq n_s$.

Calling $N = \text{diag}\,(N_1, \dots, N_s)$, we have

$$\begin{aligned} (sI - J)^{-1} &= \begin{pmatrix} (sI_1 - J_1)^{-1} & & \\ & \ddots & \\ & & (sI_s - J_s)^{-1} \end{pmatrix} \\ &= \textstyle\sum_1^s \frac{1}{(s-\lambda)^i} N^{i-1} \end{aligned}$$

(observe that $N^i = 0$ for all $i \geq n_1$).

Then, the following result is obtained

Lemma 14

$$R_{11} = c_1' b_1' + \dots + c_n' b_n'.$$

And as a consequence,

Proposition 15

$$R_{11} = CB = c_1 b_1 + \dots + c_n b_n.$$

Finally, the case derogatory matrix with multiple eigenvalues is a simple corollary.

5 Residues of Composite systems

In physical or engineering problems, a system is sometimes built by interconnecting some other systems.

Let

$$\left.\begin{array}{ll}\dot{X}_i &= A_iX_i + B_iu_i\\ Y_i &= C_iX_i\end{array}\right\},\quad \text{for } i = 1, 2,$$

be two systems.

These systems can be connected in different ways, the most common are the following.
i) Serialized one after the other, so that the input information $u_2 = Y_1(t)$. Consequently

$$\left.\begin{array}{ll}\dot{X} &= \begin{pmatrix}A_1 & 0\\ B_2C_1 & A_2\end{pmatrix}\begin{pmatrix}X_1\\ X_2\end{pmatrix} + \begin{pmatrix}B_1\\ 0\end{pmatrix}u\\ Y &= \begin{pmatrix}0 & C_2\end{pmatrix}\begin{pmatrix}X_1\\ X_2\end{pmatrix}.\end{array}\right\}$$

Proposition 16 *Let $\dot{X}_i = A_iX_i + B_iu_i$, $Y_i = C_iX_i$ for $i = 1, 2$ be two systems. If*

$$\sum_{i=1}^{r}\sum_{j=1}^{m_i}\frac{R^{(1)}_{\lambda^{(1)}_i,j}}{(s-\lambda^{(1)}_i)^j}$$

and

$$\sum_{i=1}^{s}\sum_{j=1}^{n_i}\frac{R^{(2)}_{\lambda^{(2)}_i,j}}{(s-\lambda^{(2)}_i)^j}$$

are the fractional expansion of the transfer matrices of the given systems, then the fractional expansion of the serial concatenated system is

$$\sum_{i=1}^{r}\sum_{j=1}^{m_i}\frac{R^{(1)}_{\lambda^{(1)}_i,j}}{(s-\lambda^{(1)}_i)^j}\cdot\sum_{i=1}^{s}\sum_{j=1}^{n_i}\frac{R^{(2)}_{\lambda^{(2)}_i,j}}{(s-\lambda^{(2)}_i)^j}.$$

Proposition 17 *Suppose that matrices A_1 and A_2 have simple eigenvalues $\lambda^{(1)}_1,\dots,\lambda^{(1)}_{n_1}$ and $\lambda^{(2)}_1,\dots,\lambda^{(2)}_{n_2}$ with $\lambda^{(1)}_i \neq \lambda^{(2)}_j$ for all $1 \leq i \leq n_1$, $1 \leq j \leq n_2$. Then*

$$R_{\lambda^{(1)}_i} = \frac{R^{(1)}_iR^{(2)}_j}{\lambda^{(1)}_i-\lambda^{(2)}_j},\quad R_{\lambda^{(2)}_j} = \frac{R^{(1)}_iR^{(2)}_j}{\lambda^{(2)}_j-\lambda^{(1)}_i}.$$

Proof:
The fractional expansion of the transfer matrix is

$$\left(\sum_i^{n_1}\frac{R^{(1)}_i}{s-\lambda^{(1)}_i}\right)\cdot\left(\sum_j^{n_2}\frac{R^{(1)}_j}{s-\lambda^{(2)}_j}\right) = \frac{\frac{R^{(1)}_iR^{(2)}_j}{\lambda^{(1)}_i-\lambda^{(2)}_j}}{s-\lambda^{(1)}_i} + \frac{\frac{R^{(1)}_iR^{(2)}_j}{\lambda^{(2)}_j-\lambda^{(1)}_i}}{s-\lambda^{(2)}_j}.$$

□

ii) The second model considered in this work is the parallel connection. This type of connection is of special interest particularly the so-called interleaver parallel concatenation (see [2], [3], and [10] for example).

$$\left.\begin{array}{ll}\dot{X} &= \begin{pmatrix}A_1 & 0\\ 0 & A_2\end{pmatrix}\begin{pmatrix}X_1\\ X_2\end{pmatrix} + \begin{pmatrix}B_1\\ B_2\end{pmatrix}u\\ Y &= \begin{pmatrix}C_1 & C_2\end{pmatrix}\begin{pmatrix}X_1\\ X_2\end{pmatrix}.\end{array}\right\}$$

Proposition 18 *Let $\dot{X}_i = A_iX_i + B_iu_i$, $Y_i = C_iX_i$ for $i = 1, 2$ be two systems. If*

$$\sum_{i=1}^{r}\sum_{j=1}^{m_i}\frac{R^{(1)}_{\lambda^{(1)}_i,j}}{(s-\lambda^{(1)}_i)^j}$$

and

$$\sum_{i=1}^{s}\sum_{j=1}^{n_i}\frac{R^{(2)}_{\lambda^{(2)}_i,j}}{(s-\lambda^{(2)}_i)^j}$$

are the fractional expansion of the transfer matrices of the given systems, then the fractional expansion of the parallel concatenated system is

$$\sum_{i=1}^{r}\sum_{j=1}^{m_i}\frac{R^{(1)}_{\lambda^{(1)}_i,j}}{(s-\lambda^{(1)}_i)^j}+\sum_{i=1}^{s}\sum_{j=1}^{n_i}\frac{R^{(2)}_{\lambda^{(2)}_i,j}}{(s-\lambda^{(2)}_i)^j}.$$

Corollary 19 *i) If $\lambda^{(1)}_i \neq \lambda^{(2)}_j$ then*

$$R_{\lambda^{(k)}_{i,j}} = R^{(k)}_{\lambda^{(k)}_{i,j}}$$

ii) If $\lambda^{(1)}_i = \lambda^{(2)}_j$ for some i, j then

$$R_{\lambda^{(k)}_{i,j}} = R^{(1)}_{\lambda^{(1)}_{i,j}} + R^{(2)}_{\lambda^{(2)}_{i,j}}$$

6 Residues and output-controllability

The knowledge of the residues of a system can be used to analyze the character of output-controllability of this system.

The output controllability concept generally means that the system under study can steer output of dynamical system independently of its state vector. More concretely,

Definition 20 *Dynamical system (1) is said to be output controllable if for every $y(0)$ and every vector $y_1 \in \mathbb{R}^p$, there exist a finite time t_1 and control $u_1(t) \in \mathbb{R}^m$, that transfers the output from $y(0)$ to $y_1 = y(t_1)$.*

For a linear continuous-time system, alike in (1), described by matrices A, B, and C, the output controllability matrix can be defined:

$$\begin{array}{l} oC(A,B,C) = \\ \begin{pmatrix} CB & CAB & \dots & CA^{n-1}B \end{pmatrix}, \end{array} \tag{6}$$

and the following result is obtained.

Theorem 21 *Dynamical system 1 is output controllable if and only if* $\operatorname{rank} oC(A,B,C) = p$.

Proposition 22 *The output controllability characteristic is invariant under basis change in the state space form of the system.*

Proof:
Let $x = Sx_1$ be such that $J = S^{-1}AS$, and calling $C' = CS$ and $B' = S^{-1}B$, then

$$\operatorname{rank} oC(A,B,C) = \operatorname{rank} oC(J,B',C').$$

□

So, the matrix A in its Jordan reduced form can be considered.

$$J^i = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_r^i \end{pmatrix} = \begin{pmatrix} (\lambda_1+N_1)^i & & \\ & \ddots & \\ & & (\lambda_r+N_r)^i \end{pmatrix}.$$

Proposition 23 *Let J a non derogatory matrix with a single eigenvalue. Then*

$$\operatorname{rank} oC = \operatorname{rank} \begin{pmatrix} R_{11} & R_{12} & \dots & R_{1n} \end{pmatrix}.$$

Proof:
Matrix J is in the form $\lambda I + N$. Now, it suffices to observe that

$$CJ^\ell B = C(\sum_{j=0}^{\ell} \binom{\ell}{j} \lambda^j N^j)B$$

and

$$C(sI-J)^{-1}B = \sum_{j=0}^{n-1} \frac{CN^jB}{(s-\lambda)^{j+1}}.$$

□

Corollary 24 *Let A a matrix with a single eigenvalue. Then*

$$\operatorname{rank} oC = \operatorname{rank} \begin{pmatrix} R_{11} & R_{12} & \dots & R_{1n} \end{pmatrix}.$$

Proof:
It suffices to apply propositions (22) and (23). □

Proposition 25 *Let A be a matrix having n simple eigenvalues $\lambda_1, \dots, \lambda_n$. Then*

$$\begin{array}{l} \operatorname{rank} oC(A,B,C) = \\ \operatorname{rank} \begin{pmatrix} \sum_{i=1}^n R_{i1} & \sum_{i=1}^n \lambda_i R_{i1} & \dots & \sum_{i=1}^n \lambda_i^{n-1} R_{i1} \end{pmatrix} \end{array}$$

Proof:
It suffices to consider the system in its diagonal reduced form. □

Example
Let

$$G(s) = \frac{s-1}{s^3 - 6s^2 + 11s - 6}$$

be the transfer matrix of a system. Then, the output controllability matrix is defined as

$$\begin{array}{rl} oC & = \begin{pmatrix} \sum_{i=1}^3 R_{i1} & \sum_{i=1}^3 \lambda_i R_{i1} & \sum_{i=1}^3 \lambda_i^2 R_{i1} \end{pmatrix} \\ & = \begin{pmatrix} 0 & 1 & 5 \end{pmatrix} \end{array}$$

If a representation of this system is considered, as for example (A,B,C,D) with

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix},$$

and $D = (0)$, it is easy to prove that the controllability matrix is

$$\begin{array}{rl} oC(A,B,C) & = \begin{pmatrix} CB & CAB & CA^2B \end{pmatrix} \\ & = \begin{pmatrix} 0 & 1 & 5 \end{pmatrix}. \end{array}$$

Taking another representation as for example (A,B,C,D) with

$$A = \begin{pmatrix} 6 & -11 & 6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$C = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}$,
and $D = 0$, to obtain the output controllability matrix is easy.

$$oC(A, B, C) = \begin{pmatrix} 0 & 1 & 5 \end{pmatrix}$$

It is worth to remark that the output controllability matrix does not depends on the representation.

A more general result is the following.

Theorem 26 *Let $G(s)$ be a system with eigenvalues $\lambda_1, \dots, \lambda_r$. Then the output controllability matrix is*

$$oC(A, B, C) = \\ \Big(\sum_{i=1}^r R_{i1} \quad \sum_{i=1}^r \lambda_i R_{i1} + \sum_{i=1}^r R_{i2} \quad \dots \\ \sum_{i=1}^r \lambda_i^{n-1} R_{i1} + \sum_{i=1}^r (n-1)\lambda_i^{n-2} R_{i2} + \dots \\ + \sum_{i=1}^r R_{in} \Big)$$

The proof is analogous to the particular cases previously presented.

Remark 27 *Notice that proposition (23) is a direct corollary of this theorem.*

7 Application examples

In this section, two examples of physical problems are presented which highlight the necessity to know the residues of the transfer function corresponding to no simple eigenvalues.

7.1 Exemple 1: Synchronous machine infinite bus (SMIB)

This example is based on a classical power system problem that can be found in [12], where a simplified synchronous machine against an infinite bus is presented. The scheme of the system under study is shown in Figure 1. Applying Taylor's method to the mechanical equations, the linearized system equations

$$\left.\begin{array}{ll} \dot{A} & = AX + Bu \\ Y & = CX \end{array}\right\}$$

can be described as follows:

$$\left.\begin{array}{l} \begin{pmatrix} \Delta\dot{\omega}_r \\ \Delta\dot{\delta} \end{pmatrix} = \begin{pmatrix} -\frac{K_D}{2H} & -\frac{K_s}{2H} \\ \omega_0 & 0 \end{pmatrix} \begin{pmatrix} \Delta\omega_r \\ \Delta\delta \end{pmatrix} + \begin{pmatrix} \frac{1}{2H} \\ 0 \end{pmatrix} \Delta T_m \\ Y = X \end{array}\right\} \tag{7}$$

where

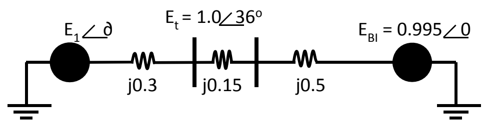

Figure 1: Synchronous machine infinite bus electrical scheme

$H = 3.5$

$K_s = \frac{E' E_B}{X_r} \cos\delta_0 = 0.757,$

$\omega_0 = 120\pi$

and K_D as parameter.

In order to obtain the eigenvalues of the matrix A, the characteristic equation can be computed.

$$\left.\begin{array}{r} \lambda^2 + 0.143 K_D \lambda + 40.79 = 0. \Leftrightarrow \\ \lambda^2 + 2\xi \cdot \omega_n \lambda + \omega_n^2 = 0. \end{array}\right\}$$

$$\lambda = \xi\omega_n \pm j\omega_n \sqrt{\xi^2 - 1}$$

So,
$\omega_n = \sqrt{40.79} = 6.387 rad/s = 1.0165 Hz,$
$\xi = \frac{0.143 K_D}{2 \cdot 6.387} = 0.0112 K_D.$

In the case where $\xi = 1$, the matrix A has a double eigenvalue with single eigenvector, that is to say the Jordan equivalent form is

$$J = \begin{pmatrix} -6.387 & 1 \\ 0 & -6.387 \end{pmatrix},$$

and the generalized (right) eigenvectors are

$$S = \begin{pmatrix} -6.387 & 1 \\ 377 & 0 \end{pmatrix}.$$

Then

$$R_{11} = \begin{pmatrix} \frac{1}{2H} \\ 0 \end{pmatrix},$$

and

$$R_{12} = \begin{pmatrix} -\frac{6.387}{2H} \\ \frac{377}{2H} \end{pmatrix}.$$

Remark 28 *Notice that the left generalized eigenvectors are compute as S^{-1}.*

7.2 Example 2: Multiple vehicle system

The following example is based on one that can be found in [4], four identical vehicles moving in a single lane are considered (Figure 2). The dynamical equations corresponding to the problem of maintaining the distance between adjacent vehicles at a predetermined value h_0 are given by:

$$\dot{X} = AX + Bu$$

with

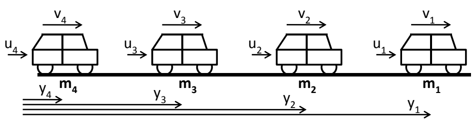

Figure 2: Multiple vehicle system representation

$$A = \begin{pmatrix} -k/m & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k/m & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -k/m & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -k/m \end{pmatrix}$$

$$B = \begin{pmatrix} 1/m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1/m & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/m & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/m \end{pmatrix}$$

where:

$X = \begin{pmatrix} \bar{v}_1 & y_{12} & \bar{v}_2 & y_{23} & \bar{v}_3 & y_{34} & \bar{v}_4 \end{pmatrix}^t$,

$u = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 \end{pmatrix}^t$,

with $\bar{v}_i = v_i - v_0$, $y_{i,i+1} = y_i - y_{i+1} - h$ and v_i the velocities, y_i the positions and u_i the applied force of the ith vehicle.

In this case the eigenvalues of the matrix A are 0 and $-k/m$ with multiplicities 3 and 4 respectively.

The matrices of residues are:

For $\lambda = 0$

$$R_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1/k & -1/k & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1/k & -1/k & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/k & -1/k \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For $\lambda = -k/m$

$$R_{21} = \begin{pmatrix} 1/m & 0 & 0 & 0 \\ -1/k & 1/k & 0 & 0 \\ 0 & 1/m & 0 & 0 \\ 0 & -1/k & 1/k & 0 \\ 0 & 0 & 1/m & 0 \\ 0 & 0 & -1/k & 1/k \\ 0 & 0 & 0 & 1/m \end{pmatrix}$$

So, the output controllability matrix is

$$oC(A,B,C) = \\ \Big(R_{11} + R_{21} \quad -k/mR_{21} \quad k^2/m^2R_{21} \quad -k^3/m^3R_{21} \\ k^4/m^4R_{21} \quad -k^5/m^5R_{21} \quad k^6/m^6R_{21}\Big)$$

and

$$\operatorname{rank} oC(A,B,C) = \\ \operatorname{rank} \Big(R_{11} + R_{21} \quad -k/mR_{21}\Big) =$$

$$\operatorname{rank} \begin{pmatrix} \frac{1}{m} & 0 & 0 & 0 & -\frac{k}{m^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{m} & -\frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{m} & 0 & 0 & 0 & -\frac{k}{m^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{m} & -\frac{1}{m} & 0 \\ 0 & 0 & \frac{1}{m} & 0 & 0 & 0 & -\frac{k}{m^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{m} & -\frac{1}{m} \\ 0 & 0 & 0 & \frac{1}{m} & 0 & 0 & 0 & -\frac{k}{m^2} \end{pmatrix} \\ = 7 = p.$$

So, the system is output-controllable.

Observe that in this case, output-controllability coincides with controllability of the system. That is obviously true when as in this case, the C is a nonsingular square matrix.

8 Conclusions

The conventional method to compute the residues up to now has been presented. Moreover, a general approach for computing the residues of dynamical systems has been proposed in the paper. This method has introduced the assumption of existence of non simple eigenvalues, as occurs up to now. Moreover, a relationship between the residues and the output-controllability concept is given. Finally, some application examples have been presented including a simple power systems (Synchronous Machine Infinite Bus) and a multiple vehicle system, in order to show that the development of this general method is really required.

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