Periodic orbits of planar integrable birational maps.

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A general approach, with examples from works with

G. Bastien, A. Cima, I. Gálvez, A. Gasull, M. Rogalski, X. Xarles.

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We are interested in the study of periodic orbits of *planar integrable birational maps*. That is

- Planar rational: $F:\mathcal{U}\subseteq\mathbb{R}^2\longrightarrow\mathcal{U}$ where $F(x,y)=\left(\frac{f_1}{f_2},\frac{f_3}{f_4}\right)$ where f_i are polynomials.
- Birational: The inverse map F^{-1} is also rational.
- Integrable: There is a function V which is a *first integral* of F in $\mathcal{U}$. That is if

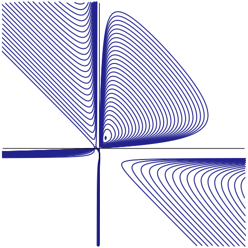
$$V(F(x,y))=V(x,y),\ \text{for all } (x,y) \text{ in } \mathcal{U}.$$

In addition, we will consider that V is a *rational function*

$$V(x,y)=\frac{P(x,y)}{Q(x,y)}.$$

So the map preserves a foliation of the plane given by *algebraic curves*

$$\mathcal{F}_h=\{P(x,y)-h\,Q(x,y)=0,h\in\mathrm{Im}(V)\}.$$



They appear in Number theory and Algebraic Geometry as automorphisms of algebraic curves.

The Lyness map $F_a(x,y) = \left(y, \frac{a+y}{x}\right)$ associated to $x_{n+2} = \frac{a+x_{n+1}}{x_n}$.

It has the with first integral

$$V(x,y) = \frac{(x+1)(y+1)(x+y+a)}{xy}.$$

So it preserves the foliation given by

$$\mathcal{F}_h := \{(x+1)(y+1)(x+y+a) - hxy = 0\}$$

Among a large literature, see for instance Barbeau et al. 1995; Bastien & Rogalski, 2004a; Beukers & Cushman, 1998; Duistermaat, 2010; Esch & Rogers, 2001; Zeeman, 1996.

They appear as reduction of differential–difference soliton equations. For instance QRT maps, introduced in Quispel et al. 1989, see also Duistermaat, 2010.

DΔ Nonlinear Schrödinger equation.

Soliton equation: $-i\frac{d}{dt}u_n = u_{n+1} - 2u_n + u_{n-1} + \frac{1}{2}|u_n|^2(u_{n+1} + u_{n-1})$

Reduction: $u_n(t) = x_n \exp(i\omega t)$

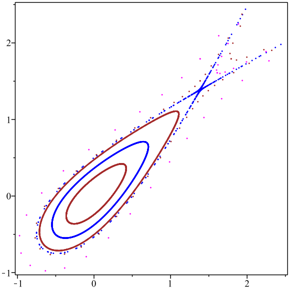
Map: $F(x,y) = \left(y, -x + \frac{(\omega+2)y}{1+\frac{1}{2}y^2}\right)$ a type of Gumovski-Mira map, also in the Mc Millan family of maps, and QRT.

First integral $V(x,y) = x^2y^2 - 2(\omega+2)xy + 2x^2 + 2y^2$

$\Leftrightarrow \mathcal{F}_h := \{x^2y^2 - 2(\omega+2)xy + 2x^2 + 2y^2 - h = 0\}$

Are there planar integrable birational maps with nonrational first integrals?

$$F(x,y)=\left(\frac{x\left(a+y+day+bx+dy^2+cxy\right)}{(a+y)(bx+a+y)},\frac{\left(abex+adxy+bexy+cx^2y+dxy^2+a^2e+2aey+bx^2+ey^2+ax+xy\right)y}{(cxy+bx+a+y)(bx+a+y)}\right)$$



The map is associated to 5-periodic Lyness difference equations

$$x_{n+2}=\frac{a_n+x_{n+1}}{x_n},\ \text{with } a_n=\{a,b,c,d,e\}.$$

They are non rationally integrable, in fact non meromorphically integrable, for a generic set of parameters Cima *et al.* 2013ab, although numerically show integrable “features”.

Our objective:

To show the relationship between the algebraic-geometric properties of the invariant foliation $\mathcal{F}$ and the dynamics of the map F.

More precisely, we will focus on the case that $\mathcal{F}$ is *generically* elliptic.

The topics:

- The *locus* of periodic orbits.
- Existence of *rational* periodic orbits.
- The *rotation number function* and the *set of periods* of a map or a family.
- And two short digressions.

The examples (our results):

- The Lyness Map $F_a(x,y) = \left(y, \frac{a+y}{x}\right)$. Gasull *et al.* 2012.
- The composition map $F_{b,a}(x,y) := (F_b \circ F_a)(x,y) = \left(\frac{a+y}{x}, \frac{a+bx+y}{xy}\right)$ associated to 2-periodic Lyness recurrences. Bastien *et al.* 2013.

A natural phase space to study planar birational maps is

$$\mathbb{C}P^2 = \{[x : y : z] \neq [0 : 0 : 0],\, x, y, z \in \mathbb{C}\}/\sim$$

where $[x_1 : y_1 : z_1] \sim [x_2 : y_2 : z_2]$ if and only if $[x_1 : y_1 : z_1] = \lambda[x_2 : y_2 : z_2]$ for $\lambda \neq 0$.

The points $[x : y : \mathbf{1}]$ are called *affine* points and the points $[x : y : \mathbf{0}]$ are called *infinity* points.

The Lyness map $F(x, y) = \left(y, \dfrac{a+y}{x}\right)$ can be seen as the map

$$\tilde{F}([x : y : z]) = [xy : az^2 + yz : xz]$$

except for the points $[x : 0 : 0]$, $[0 : y : 0]$ and $[0 : -a : 1]$.

The invariant real planar foliation $C_h = \{(x+1)(y+1)(x+y+a) - hxy = 0\}$ is now

$$\widetilde{C}_h := \{(x+z)(y+z)(x+y+az) - hxyz = 0\},$$

In general we will drop $\sim$

Algebraic curves in $\mathbb{C}P^2$ are Riemann surfaces and are characterized by their *genus*.

$g = 0$ $g = 1,$ $g = 2$

The *degree-genus* formula states that for a given *irreducible* curve $C_h \in \mathcal{F}$:

$$g = \frac{(d-1)(d-2)}{2} - \sum_{p \in \mathrm{Sing}(C_h)} \frac{m_p(m_p - 1)}{2}$$

For instance the Lyness' Foliation

$$\widetilde{\mathcal{F}_h} := \{(x+z)(y+z)(x+y+az) - hxyz = 0\},$$

has degree $d = 3$, so generically the curves have genus $g = 1$, and the singular curves have genus $g = 0$.

If our map is NOT Globally periodic, $\mathcal{F}$ is a genus 0 or 1-foliation.

Proposition.

A birational map in $\mathcal{U} \subseteq \mathbb{K}^2$ with a rational first integral V, such that the curves $\{V = c\}$ are non-singular with with genus $g > 1$, must be globally periodic.

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Proposition.

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Proof:

- If the curves $\{V = c\}$ have generically genus $g > 1$ then there exists an open set $\mathcal{V} \subseteq \mathcal{U}$ foliated by curves of these type.
- By Hurwitz Theorem *on each of these curves* the map must be periodic.

Theorem (Hurwitz, 1893)

The group of birational maps on a non-singular algebraic curve in $\mathbb{K}^2$ of genus $g > 1$ is finite and of order at most $84(g-1)$.

- So F is *pointwise periodic* on $\mathcal{V}$, and hence by the Montgomery Theorem F must be *globally periodic* on $\mathcal{V}$.

Theorem (Montgomery, 1937)

Pointwise periodic homeomorphisms in connected metric spaces are globally periodic.

- Since F is *rational* then it must be periodic on the whole $\mathbb{K}^2$ except at the points where its iterates are not well defined. ∎

An elliptic curve is a projective algebraic curve of genus 1.

Curves of genus 1 are birationally isomorphic to *smooth* cubic curves.

In the case that $\mathcal{F}$ is given (generically) by *elliptic curves*, then the *group structure* of the elliptic foliation characterizes the dynamics of any birational map preserving it.

Theorem (after Jogia, Roberts, Vivaldi. 2006.)

Any birational map F that preserves an elliptic curve C_h, can be expressed in terms of the group law of the curve as either

- $F_{|C_h} : P \mapsto P + Q$ where $Q \in C_h$ or
- $F_{|C_h} : P \mapsto i(P) + Q$ where i (and F) is GP of orders 2,3,4 or 6.

where $+$ denotes the inner sum of C_h.

What is the *group structure*? what is this *inner sum*?

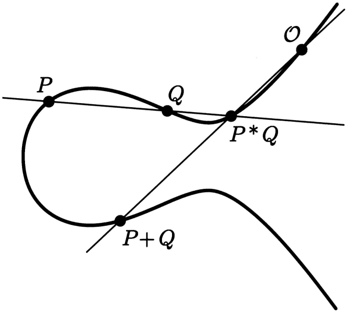
Take two points P and Q in a nonsingular cubic C

(1) Select a point $\mathcal{O}$ to be the neutral element.

(2) Take the third intersection point $P * Q$.

(3) The point $P + Q$ is defined as $P + Q = \mathcal{O} * (P * Q)$.

$(C, +, \mathcal{O})$, is an *abelian group*.



Group law with an affine neutral element $\mathcal{O}$

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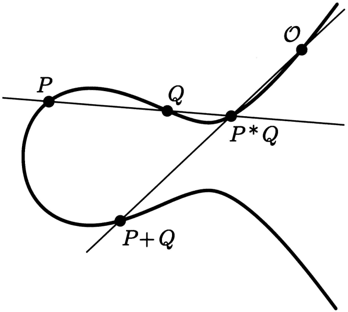

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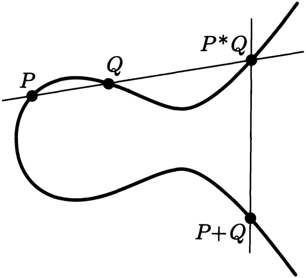

Group law with $\mathcal{O} = [0 : 1 : 0]$

Theorem (Jogia *et al.* 2006.)

Any birational map F that preserves an elliptic curve C_h, there is a choice of $\mathcal{O}$ such that

- $F_{|C_h} : P \mapsto P + Q$ where $Q \in C_h$ or
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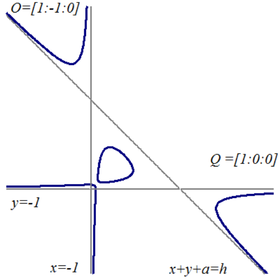


Taking $\mathcal{O} := [1 : -1 : 0]$, on each elliptic level the Lyness map is

$$F(P) = P + \mathbf{[1{:}0{:}0]}$$

$F_{b,a}(x,y) := \left(\frac{a+y}{x}, \frac{a+bx+y}{xy}\right)$ preserves $C_h := \{(bx+a)(ay+b)(ax+by+ab) - hxy = 0\}$.
Setting $\mathcal{O} := V = [0 : 1 : 0]$, on each elliptic level

$$F_{b,a}(P) = P + \mathbf{[1{:}0{:}0]}$$

If $F_{|C_h} : P \mapsto P + Q$, the map is *conjugate to a rotation* with *rotation number* $\theta(h)$.

Because

$(C_h(\mathbb{R}), +) \cong \{(e^{it}, \pm 1);\ t \in [0, 2\pi)\};$ with the operation $(e^{it}, v) \cdot (e^{is}, w) = (e^{i(t+s)}, v\,w)$

A birational map F on $(C_h, +, \mathcal{O})$ is

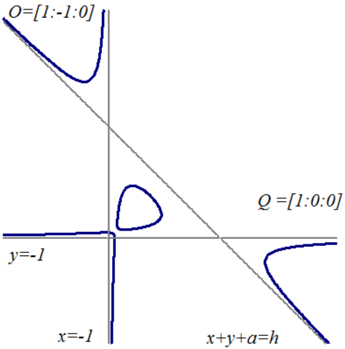
$$F^{\mathbf{n}} : P \longrightarrow P + \mathbf{n}\,\mathbf{Q}$$

Hence C_h is full of p-periodic orbits $\Leftrightarrow$ $p \cdot Q = \mathcal{O}$.

Q is called a *torsion point* of C_h.

In summary:

- $Q \in \mathrm{Tor}(C_h) \Rightarrow$ all the orbits in $C_h(\mathbb{R})$ are periodic.
- $Q \notin \mathrm{Tor}(C_h) \Rightarrow$ the orbits of F fill densely the connected components of $C_h(\mathbb{R})$.



Each map

$$F_a([x:y:z]) = [xy : az^2 + yz + xz]$$

preserves

$$C_h = \{(x+z)(y+z)(x+y+az) - hxyz\}$$

Elliptic except for $h \in \{0, a-1, h_c^{\pm}\}$.

Taking $\mathcal{O} = [1:-1:0]$ and $Q = [1:0:0]$. Observe that both $\mathcal{O}, Q \in C_h$ for all C_h.

$$F_{|C_h} : [x:y:z] \mapsto [x:y:z] + [\mathbf{1}:\mathbf{0}:\mathbf{0}] \Leftrightarrow F^{\mathbf{n}}_{|C_h} : [x:y:z] \mapsto [x:y:z] + \mathbf{n}[\mathbf{1}:\mathbf{0}:\mathbf{0}]$$

Observe that there exists a p-periodic orbit iff $p \cdot [1:0:0] = \mathcal{O} = [1:-1:0]$

When $a(a-1)\neq 0$, using the group operation on each elliptic curve, we get

$$2Q=[-1:0:1],\ 3Q=[0:-a:1],$$
$$4Q=\left[-a:\frac{ah-a+1}{a-1}:1\right],$$

and

$$-Q=[0:1:0],\ -2Q=[0:-1:1],\ -3Q=[-a:0:1],$$
$$-4Q=\left[\frac{ah-a+1}{a-1}:-a:1\right],$$
$$-5Q=\left[\frac{-a^2-ah+2a-1}{a(a-1)}:\frac{ah-a+1}{a-1}:1\right],$$

- Are there 4 periodic orbits on the elliptic levels? **NO**

$$4Q=\left[-a:\frac{ah-a+1}{a-1}:\mathbf{1}\right]\neq[1:-1:\mathbf{0}]=\mathcal{O}\,!!$$

If $a=1$, then $4Q=[0:h:0]\neq[1:-1:0]=\mathcal{O}$, and no 4-periodic orbits on the genus-0 levels (very easy).

- Are there 9-periodic orbits on the elliptic levels? **YES**

$$4Q=\left[-\mathbf{a}:\frac{ah-a+1}{a-1}:1\right]=\left[\frac{\mathbf{-a^2-ah+2a-1}}{\mathbf{a(a-1)}}:\frac{ah-a+1}{a-1}:1\right]=-5Q\Leftrightarrow$$

The 9-periodic orbits are in C_{h_9} where $h_9=\dfrac{(a-1)(a^2-a-1)}{a}$

The *group structure* of the $\mathcal{F}$ plays a crucial role when searching *rational periodic orbits*

A *rational orbit* is an orbit such that all the iterates have *rational coordinates*.

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Our motivation was:

Conjecture (Zeeman, 1996; Bastien and Rogalski, 2004a)

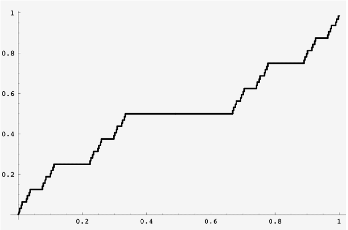
There is no *rational* periodic orbit of period 9 for the Lyness recurrence $x_{n+2} = \frac{a+x_{n+1}}{x_n}$.

Where are the periodic orbits for birational maps on elliptic curves?

``If we wish study possible period with a computer, it is easier to work with rational numbers. so, we suppose that a is rational, and that the point (u_1, u_0) is rational. With the use of a computer and a program of calculation with fractions, is it possible to see periodic points? **Only in few cases!**'' Bastien and Rogalski, 2004b

A numerical digression: where are the periodic orbits?

Our maps F restricted to C_h can be interpreted as a *smooth one-parameter family of diffeomorphism* of $\mathbb{S}^1$. Thus *generically* the graph of the rotation function $\theta(h)$ has a *devil's staircase* shape:



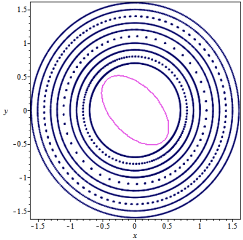
The existence of an *open interval* I_m where $\theta(h) \equiv q/p$, is a consequence of the *existence of a hyperbolic p*-periodic orbit.

$$\Downarrow$$

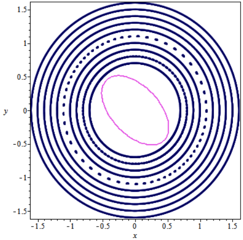
Generically, periodic orbits are numerically visible.

In our case $\theta(h)$ is *piecewise analytic*, but even when $\theta(h)$ is analytic if we have **NOT** a birational map on elliptic curves you see periodic orbits, e.g. *proper Poncelet maps*.

But even when $\theta(h)$ is analytic we find P.O.: playing with a proper *Poncelet map*.



1000 iterates of a Poncelet map



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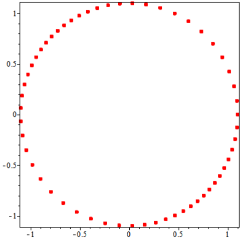
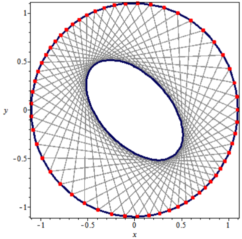


A 57-periodic orbit after 20000 iterates



The Poncelet process

Poncelet Maps are not birational, but using *Lie symmetries* $\theta(h)$ is analytic, Cima *et al.* 2010

The points with rational coordinates in $(C_h, +, \mathcal{O})$ form a *subgroup* denoted by $C_h(\mathbb{Q})$.

Theorem (Mordell, 1922 + Mazur, 1978)

If E is a non-singular cubic, then $(E(\mathbb{Q}), +)$ is either the *finitely-generated abelian group*

$$E(\mathbb{Q}) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$$

or

$$E(\mathbb{Q}) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}/\mathbf{p} \text{ where } 1 \leq p \leq 10 \text{ or } \mathbf{p} = \mathbf{12}$$

or

$$E(\mathbb{Q}) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}/\mathbf{2} \oplus \mathbb{Z}/\mathbf{p} \text{ where } \mathbf{p} = \mathbf{2}, \mathbf{4}, \mathbf{6}, \mathbf{8}$$

Remember that $F_{|C_h} : P \longrightarrow P + Q$ will be periodic $\Leftrightarrow Q \in \mathrm{Tor}(C_h)$

$$\Downarrow$$

Corollary

Any birational map on $C_h(\mathbb{R})$ only can have rational periodic orbits of periods $1, 2, 3, 4, 5, 6, 7, 8, 9, 10,$ ~~11~~$, 12.$

That's why we "don't see" periodic orbits.

Theorem

For any $p \in \{1, 2, 3, \not{4}, 5, 6, 7, 8, 9, 10, \not{11}, 12\}$ there are $a \in \mathbb{Q}^+ \cup \{0\}$ and rational initial conditions x_0, x_1 such that the sequence generated by $x_{n+2} = (a + x_{n+1})/x_n$ is p-periodic. Moreover these values of p are the only possible minimal periods for rational initial conditions and $a \in \mathbb{Q}$.

A counterexample to Zeeman's Conjecture: taking $a = 7$ and the initial condition $x_0 = 3/2$, $x_1 = 5/7$, we have

$$\frac{3}{2} \longrightarrow \frac{5}{7} \longrightarrow \frac{36}{7} \longrightarrow 17 \longrightarrow \frac{14}{3} \longrightarrow \frac{35}{51} \longrightarrow \frac{28}{17} \longrightarrow \frac{63}{5} \longrightarrow \frac{119}{10} \longrightarrow \frac{3}{2} \longrightarrow \frac{5}{7} \longrightarrow \ldots$$

In fact, we only need to find one rational 9-periodic point to find an infinite number of them.

Proposition

If $a \in \mathbb{Q}^+$ is s.t. $\exists$ an initial condition $(x_0, x_1) \in \mathbb{Q}^+ \times \mathbb{Q}^+$ s.t. the sequence is 9-periodic, then there *infinite* rational ones. In fact these points *fill densely* the elliptic curve C_{h_9}

$$(x+1)(y+1)(x+y+a) - \underbrace{\frac{(a-1)(a^2-a+1)}{a}}_{h_9} xy = 0$$

Proof: Suppose that we have $a \in \mathbb{Q}^+$, such that $Q = [1:0:0] \in \text{Tor}(C_{h_9})$ ($\Leftrightarrow$ 9-P.O.)

• If $P = (x, y) \in \mathbb{Q}^+ \times \mathbb{Q}^+$ is on the oval of C_{h_9} then $(2k+1)P \Rightarrow$ is also on it.

Notice that $(2k+1) \cdot P$ are not iterates of F.

• The the points $(2k+1)P$ are rational and fill densely the oval (i.e. $(2k+1)P \neq \mathcal{O}$) because all 9-torsion points are not in $\mathbb{Q}^+ \times \mathbb{Q}^+$,

Remember that $(E(\mathbb{R}), +) \cong \{e^{it} : t \in [0, 2\pi)\} \times \{1, -1\};$ with the operation $(e^{it}, v) \cdot (e^{is}, w) = (e^{i(t+s)}, vw)$

Theorem

There are an *infinite* number of values $a \in \mathbb{Q}^+$ and initial conditions $x_0(a), x_1(a) \in \mathbb{Q}^+$ giving rise to 9-periodic orbits for the Lyness Eq.

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Proof:

- We want to find infinitely many points $(x(a), y(a), a) \in (\mathbb{Q}^+ \times \mathbb{Q}^+ \times \mathbb{Q}^+) \cap S_a$, where

$S_a := \{(a; x, y) : \; a(x+1)(y+1)(x+y+a) - (a-1)(a^2-a+1)xy = 0, \; x > 0, y > 0, a > a_*\}.$

and a_* is the infimum s.t. C_{h_9} has an oval in $\mathbb{R}^{2,+}$.

- The points in S_a s.t. $x + y = 23/4$, are in an elliptic curve isomorphic to

$$\mathcal{E} : \quad Y^2 = X^3 - \frac{1288423179}{71639296} X + \frac{8775405707427}{303177500672}.$$

- If we find ONE *valid* rational point $R \in \mathcal{E}$ NOT in the torsion, then $k \cdot R$ gives an infinite number.

Remember that $(E(\mathbb{R}), +) \cong \{e^{it} : t \in [0, 2\pi)\} \times \{1, -1\};$ with the operation $(e^{it}, v) \cdot (e^{is}, w) = (e^{i(t+s)}, vw)$

- Using MAGMA we found our seed:

$$R = \left(\frac{18243}{8464}, \frac{81}{184} \right)$$

- Recovering the values $(x(a), y(a), a)$ corresponding to the points kR, we get the result. $\blacksquare$

Universality of the Lyness example. Gasull *et al.* 2012.

Theorem (Lyness normal form)

The family of elliptic curves $C_{a,h} = \{(x+1)(y+1)(x+y+a) - hxy = 0\}$ over any field $\mathbb{K}$ (not of char. 2 or 3) together with the points $\mathcal{O} = [1:-1:0]$ and $Q = [1:0:0]$, is the universal family of elliptic curves with a point of order n, $n \geq 5$ (including $n = \infty$).

For any elliptic curve $E(\mathbb{K})$ with a point R of order $n \geq 5$, $\exists\,!$ values $a_{(E,R)}, h_{(E,R)} \in \mathbb{K}$ and a unique isomorphism between E and $C_{a_{(E,R)},h_{(E,R)}}$ sending the *zero* of E to $\mathcal{O}$ and R to Q.

$$\begin{array}{ccc} E(\mathbb{K}) & \stackrel{\cong}{\longrightarrow} & C_{a,h} \\ R & \longrightarrow & [1:0:0] \\ \mathcal{O} & \longrightarrow & [1:-1:0] \end{array}$$

The known results on elliptic curves with a point of order greater than 4 also holds in Lyness curves.

See http://web.math.pmf.unizg.hr/~duje/tors/generic.html

Proof:

• Any elliptic curve having a point R that is not a 2 or a 3 torsion point can be brought to the *Tate normal form*

$$Y^2Z + (1-c)XYZ - bYZ^2 = X^3 - bX^2Z.$$

where R is sent to $(0,0)$.

• The change of variables

$$X = bz, \quad Y = bc(y+z), \quad Z = c(x+y) + (c+1)z$$

and the relations

$$h = -\frac{b}{c^2}, \quad a = \frac{c^2+c-b}{c^2},$$

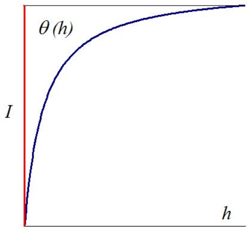
show that the curves $C_{a,h} = \{(x+z)(y+z)(x+y+az) - hxyz = 0\}$ and the *Tate normal form* are equivalent.

• The case $c = 0$ corresponds to a curve with a 4-torsion point. $\blacksquare$

7. THE SET OF PERIODS.

Proposition

A birational map preserving a foliation of elliptic curves is either *rigid* or it has an infinite number of periods.



Proof:

- From JRV Theorem *on each elliptic curve* C_h our map is conjugate to a rotation.
- The rotation number function $\theta(h)$ is *piecewise continuous* for $h \in \mathrm{Im}(V)$.
- *Generically* $\theta(h)$ is not constant $\Rightarrow$ $\exists$ *rotation interval* I. $\blacksquare$

For all the irreducible $\frac{q}{p} \in I$, $\exists$ periodic orbits of F of minimal period p.

You can *construct* p_0 such that $q/p \in I$ for all $p > p_0$,

and then check if there is any missing period if I is not optimal.

Which are the periods of a particular F? $\Leftrightarrow$ Which are the irreducible fractions in I?

- It is possible to *construct* p_0 s.t. for any $r > p_0$ there exists an irreducible fraction $q/r \in I$.
- A finite checking determines for which values of $p \leq p_0$ there exists $q/p \in I$.
- Still the forbidden periods must be detected if $I \neq \mathrm{Im}(\theta(h))$.

To construct this value p_0 we can use, among other methods, this one:

Lemma (Cima *et al.* 2007)

Consider a non empty interval (c, d)*;*

Let $p_1 = 2, p_2 = 3, p_3, \ldots, p_n, \ldots$ *be all the prime numbers.*

- *Let* p_{m+1} *be the smallest prime number satisfying that* $p_{m+1} > \max(3/(d-c), 2)$,
- *Given any prime number* p_n, $1 \leq n \leq m$, *let* s_n *be the smallest natural number such that* $p_n^{s_n} > 4/(d-c)$.
- *Set* $p_0 := p_1^{s_1-1} p_2^{s_2-1} \cdots p_m^{s_m-1}$.

Then, for any $p > p_0$ *there exists an irreducible fraction* $q/p \in (c, d)$.

We study the set of periods of the map

$$F_{b,a}(x,y) := \left(\frac{a+y}{x}, \frac{a+bx+y}{xy}\right)$$

which preserves the (generically) elliptic curves

$$C_h := \{(bx+a)(ay+b)(ax+by+ab) - hxy = 0\}$$

This map appears of the composition map associated to the the 2-periodic Lyness' equations

$$u_{n+2} = \frac{a_n + u_{n+1}}{u_n} \text{ where } a_n = \begin{cases} a & \text{for} \quad n = 2\ell+1, \\ b & \text{for} \quad n = 2\ell. \end{cases}$$

Indeed, $F_{b,a} = F_b \circ F_a$

$$(u_1, u_2) \xrightarrow{F_a} (u_2, u_3) \xrightarrow{F_b} (u_3, u_4) \xrightarrow{F_a} (u_4, u_5) \xrightarrow{F_b} (u_5, u_6) \xrightarrow{F_a} \cdots$$

Setting $\mathcal{O} := V$, on each elliptic level $(C_h, +, V)$:

$$F_{b,a}(P) = P + \mathbf{[1{:}0{:}0]}$$

Theorem.

Consider the family $F_{b,a}$ with $a, b \in \mathbb{R}$.

(a) If $(a, b) \neq (1, 1)$, then $\exists\, p_0(a, b) \in \mathbb{N}$, generically computable, s.t. for any $p > p_0(a, b)$ $\exists$ at least a curve C_h filled by p–periodic orbits.

(b) The set of periods of the family $\{F_{b,a},\ a, b \in \mathbb{R}\}$ contains all minimal periods except 2, 3.

The set of periods when we restrict ourselves to the case $a, b > 0$ and $x, y > 0$ contains all minimal periods except 2, 3, 4, 6, 10.

$$\Downarrow$$

Corollary.

Consider the 2–periodic Lyness' recurrence for $a, b > 0$ and positive initial conditions u_1 and u_2.

(a) If $(a, b) \neq (1, 1)$, then $\exists\, p_0(a, b) \in \mathbb{N}$, generically computable, s.t. for any $p > p_0(a, b)$ $\exists$ continua of initial conditions giving $2p$–periodic sequences.

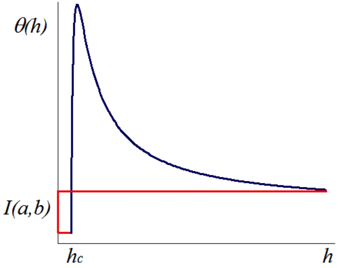
(b) The set of minimal periods arising when $(a, b) \in (0, \infty)^2$ and positive initial conditions are considered contains all the even numbers except 4, 6, 8, 12, 20.

If $a \neq b$, then it does not appear any odd period, except 1.

Proposition

Fixing $a, b > 0 \Rightarrow \exists\, h_c$ such that $\forall\, h > h_c$ C_h is elliptic.

The dynamics of $F_{b,a}$ restricted to C_h is *conjugate to a rotation* with *rotation number* $\theta(h)$.



Proposition.

(a) $\lim_{h\to+\infty} \theta(h) = \frac{2}{5}$

(b) $\lim_{h\to h_c} \theta(h) = \sigma(a,b) = \frac{1}{2\pi} \arccos\left(\frac{1}{2}\left[-2 + \frac{1}{x_c\, y_c}\right]\right).$

Hence we have, generically, a rotation interval

$I(a,b) := \left\langle \sigma(a,b), \frac{2}{5} \right\rangle.$

- (a) is proved using the parametrization of the elliptic curves given by the Weierstrass $\wp$ function as in Bastien *et al.* 2004a.
- Taking the family $a = b^2$ we found that:

$$\bigcup_{b>0} I(b^2, b) = \left(\frac{1}{3}, \frac{1}{2}\right) \setminus \left\{\frac{2}{5}\right\} \subset \bigcup_{a>0,\, b>0} I(a,b) \subset \bigcup_{a>0,\, b>0} \mathrm{Image}\left(\theta\left(h_c, +\infty\right)\right).$$

- It can be proved that there are $q/p \in (1/3, 1/2)$ with all denominators except 2, 3, 4, 6 and 10.
- We get the result by studying the energy levels using

$$p \cdot \mathcal{O} = p \cdot [1:0:0] = [0:1:0] \text{ for } p = 2, 3, 4, 6, 10.$$

What happens if $\mathcal{F} = \{\underbrace{P - hQ = 0}_{C_h}\}$ has genus 0, or C_h is a genus 0 curve?

Each genus 0 curve C_h admits a rational parametrization

$$\begin{array}{ccccccc} \mathbb{R} & \longrightarrow & C_h & \stackrel{F}{\rightarrow} & C_h & \longrightarrow & \mathbb{R} \\ t & \longrightarrow & (x(t), y(t)) & \rightarrow & F(x(t), y(t)) & \longrightarrow & t' \end{array}$$

Hence our birational map F on each curve C_h admits a representation given by the one-dimensional birational map $t \rightarrow t'$.

The only one-dimensional birational maps are the *Möbius* transformations

$$t \longrightarrow \frac{a\,t + b}{c\,t + d}$$

which are well known.

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THANK YOU!