Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

EXISTENCE OF PERIODIC SOLUTIONS WITH NONCONSTANT SIGN IN A CLASS OF GENERALIZED ABEL EQUATIONS

Josep M. Olm

Departament de Matemàtica Aplicada IV,
Universitat Politècnica de Catalunya, Av. Esteve Terradas 5,
08860 Castelldefels, Spain

Xavier Ros-Oton

Departament de Matemàtica Aplicada I,
Universitat Politècnica de Catalunya, Diagonal 647,
08028 Barcelona, Spain

(Communicated by the associate editor name)

Abstract. This article provides sufficient conditions for the existence of periodic solutions with nonconstant sign in a family of polynomial, non-autonomous, first-order differential equations that arise as a generalization of the Abel equation of the second kind.

1. Introduction.

The family of polynomial, non-autonomous, first-order Ordinary Differential Equations (ODE) that answer to

$$\dot{x} = \sum_{i=0}^{n} a_i(t)x^i \tag{1}$$

are know as Abel-like [1] or generalized Abel equations [2] because, when $n = 3$, they reduce to the Abel ODE of the first kind [3]. Due to its well known connection with the number of lymit cycles of planar, polynomial systems and, consequently, with Hilbert's 16th problem [4], considerable research interest has been devoted to study the existence of periodic solutions in (1). Remarkable works in this field include, among others, [1, 2, 5, 6, 7, 8, 9, 10] and the references therein, which report estimates on the number of periodic solutions of (1)-like ODE under certain assumptions on the coefficients $a_i(t)$ with different degrees of tightness.

Recently, a further generalization of (1), namely,

$$\dot{x} = x^m \sum_{i=0}^{n} a_i(t)x^i, \quad m \in \mathbb{Z}, \tag{2}$$

has been investigated in [11]. The study of periodic solutions in this class of ODE is specifically interesting because of its application to estimate the number of limit cycles in a class of planar vector fields that include the so-called *rigid systems*.

2000 *Mathematics Subject Classification.* Primary: 34C25, 34C07; Secondary: 34C05.

Key words and phrases. generalized Abel differential equation, periodic solutions.

The first author is partially supported by the spanish Ministerio de Educación (MEC) under project DPI2010-15110. The second author is supported by grants MTM2011-27739-C04-01 (Spain) and 2009SGR345 (Catalunya).

However, as it happens in most of the above cited works, attention is focused only on limit cycles with definite sign.

It is an interesting fact that (2) admits negative powers of m; indeed, Abel ODE of the second kind [3] may arise from (1) by setting $m = -1$, $n = 2$. The existence of periodic solutions in (1) with $m \in \mathbb{Z}^-$ is a problem with few records in the scientific literature, mainly because for nontrivial limit cycles with constant sign the change $x \to x^{-1}$ transforms (2) to (1) for certain values of m and n. Nevertheless, even less reports have been published dealing with periodic solutions with nonconstant sign [12], a case which cannot be tackled through the change $x \to x^{-1}$.

This article generalizes the results in [12] for Abel ODE of the second kind with the obtention of sufficient conditions that guarantee the existence of periodic solutions with nonconstant in (2)-like generalized Abel ODE. Moreover, it is shown that the particularization of the derived conditions in the general case to Abel ODE of the second kind is sharper than the obtained in [12] for specific situations. In turn, the analysis comes in addition to the results derived in [11] for limit cycles with definite sign in (2).

Let us then consider the class of polynomial, non-autonomous, first-order differential equations of the form

$$x^m \dot{x} = \sum_{i=0}^{n} a_i(t) x^i, \quad a_i(t) \in \mathcal{C}^1([0,T]), \quad i = 0, \dots, n, \quad m \in \mathbb{Z}^+. \tag{3}$$

Notice that if a solution of (3) has nonconstant sign, then its zeros are also zeros of $a_0(t)$. Hence, the search of periodic solutions with nonconstant sign in (3) only makes sense when $a_0(t)$ itself has nonconstant sign.

The main result of the paper reads as:

Theorem 1.1. *Let m, n be positive integers, and let $a_0(t), ..., a_n(t)$ be $\mathcal{C}^1$, T-periodic functions. Assume that $a_0(t)$ has at least one zero in $[0,T]$ and that one of the following conditions is satisfied:*

i) m is odd and there exists $\beta > 0$ such that:

$$\frac{T}{\beta} \max_{t \in \mathbb{R}} \{-\dot{a}_0(t)\} + \max_{t \in \mathbb{R}} \left\{ -a_1(t) + \sum_{i=2}^{n} |a_i(t)| \beta^{i-1} \right\} + \frac{\beta^m}{T} < 0. \tag{4}$$

ii) m is even and there exists $\beta > 0$ such that:

$$\frac{T}{\beta} \max_{t \in \mathbb{R}} |\dot{a}_0(t)| + \max_{t \in \mathbb{R}} \left\{ a_1(t) + \sum_{i=2}^{n} |a_i(t)| \beta^{i-1} \right\} + \frac{\beta^m}{T} < 0. \tag{5}$$

Then equation (3) has a T-periodic solution that has the sign of $(-1)^m a_0(t)$, and it is also $\mathcal{C}^1$.

It is worth remarking that, by fixing a value for β, (4)-(5) boils down to tighter but easily checkable conditions that depend solely on the coefficients $a_0(t), ..., a_n(t)$. A much simpler, yet also tighter, alternative to (4)-(5) that is independent of n may be obtained as follows: the existence of $M := \max_{t \in \mathbb{R}} \{|a_2(t)|, \dots, |a_n(t)|\} \in \mathbb{R}_0^+$ is guaranteed by the continuous and T-periodic character of $a_i(t)$; then, setting $\beta = \frac{1}{2}$, it is found that (4)-(5) is fulfilled if

$$\min_{t \in \mathbb{R}} \left\{ (-1)^{m+1} a_1(t) \right\} > M + 2T \max_{t \in \mathbb{R}} |\dot{a}_0(t)| + \frac{1}{2^m T}.$$

One can also derive conditions for the existence of periodic solutions that depend only on $a_0(t)$ and $a_1(t)$ for a subset of specific generalized Abel equations of the form (3), namely, those with $a_{2i} = 0$, $i \geq 1$. Let us consider the ODE:

$$x^m \dot{x} = a_0(t) + \sum_{i=1}^{k} a_{2i-1}(t) x^{2i-1}, \quad a_{2i-1}(t) \in \mathcal{C}^1([0,T]), \quad i = 1, \ldots, k. \tag{6}$$

Theorem 1.2. *Let k, m be positive integers, and let $a_0(t), a_1(t), ..., a_{2k-1}(t)$ be $\mathcal{C}^1$, T-periodic functions such that $(-1)^{m+1} a_{2i-1}(t) \geq 0$, $i = 1, ..., k$. Assume that $a_0(t)$ has at least one zero in $[0,T]$ and that one of the following conditions is satisfied:*

i) m is odd and

$$\min_{t\in\mathbb{R}} \{a_1(t)\} > (m+1) T^{\frac{m-1}{m+1}} \left[\frac{1}{m} \max_{t\in\mathbb{R}} \{-\dot{a}_0(t)\} \right]^{\frac{m}{m+1}}. \tag{7}$$

ii) m is even and

$$\min_{t\in\mathbb{R}} \{-a_1(t)\} > (m+1) T^{\frac{m-1}{m+1}} \left[\frac{1}{m} \max_{t\in\mathbb{R}} |\dot{a}_0(t)| \right]^{\frac{m}{m+1}}. \tag{8}$$

Then, equation (3) has a T-periodic solution that has the sign of $(-1)^m a_0(t)$, and it is also $\mathcal{C}^1$.

It follows from the proofs of Theorems 1.1 and 1.2 that, unless $m = 1$ and all the zeros of $a_0(t)$ are simple, the construction of a T-periodic solution, $x^*(t)$, arising from both results requires the use of the Center Manifold Theorem. In such cases, there may exist a family of periodic solutions of (3) or (6) with the same sign as $(-1)^m a_0(t)$. However, even if $x^*(t)$ is non unique, there do not exist other T-periodic solutions with nonconstant sign in (3) or (6) sharing only some of the zeros of $a_0(t)$:

Theorem 1.3. *Let the assumptions of Theorem 1.1 (resp. Theorem 1.2) be fulfilled. Then, any $\mathcal{C}^1$, T-periodic solution of (3) (resp. (6)) with at least one zero in $[0,T]$ has the sign of $(-1)^m a_0$. Moreover, if $m = 1$ and all the zeros of $a_0(t)$ are simple, then (3) has a unique $\mathcal{C}^1$, T-periodic solution with nonconstant sign.*

Finally, it is worth pointing out two issues. On the one hand, the study on periodic solutions with nonconstant sign of (3) or (6) might be complemented with an analysis of the possible existence of nontrivial limit cycles with definite sign in some specific situations. Unfortunately, [11] is not useful for this purpose because a key assumption therein is $a_0(t) a_n(t) \neq 0$, for all t. However, notice that the above suggested change of variables $x \to x^{-1}$ transforms (3) into

$$\dot{x} = -x^{m+2-n} \sum_{i=0}^{n} a_i(t) x^{n-i}.$$

When $m + 2 \geq n$ this ODE follows the pattern (1), and one could then use results available in the literature that do not require definite sign for the coefficient of x^{m+2}, i.e. for $a_0(t)$.

On the other hand, two important aspects that are not studied in the paper are stability and numerical approximation of the T-periodic solutions with nonconstant sign, which are left as open problems for further research. Possible starting points for these investigations might be the stability analysis procedure for Abel ODE of the 2nd kind, i.e. with $m = 1$ and $n = 2$ in (3), followed in [12], and the approximations of periodic solutions with constant sign for Abel ODE of the 2nd kind in the normal form developed in [13] and [14] using Galerkin and iterative schemes, respectively.

The remainder of the paper is organized as follows. The proofs of Theorems 1.1, 1.2 and 1.3 are in Sections 2, 3 and 4, respectively. Finally, Section 5 is devoted to compare the conditions for the existence of periodic solutions with nonconstant sign in Abel equations of the second kind derived in [12] with the one provided in Theorem 1.1.

2. **Proof of Theorem 1.1.** Let us first recall a result from [12] which is essential in subsequent demonstrations.

Lemma 2.1 ([12])**.** *Consider the ODE*

$$\dot{x} = S(t,x), \quad S : \Omega \to \mathbb{R}, \tag{9}$$

where $\Omega := \mathbb{R} \times \mathbb{R}^*$, $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and S is a locally Lipschitz function. Assume that $p, q \in \mathbb{R}$ and let $r := \{(t,x) : \ x = pt + q\}$ be a straight line of slope p, which splits $\mathbb{R}^2$ into the half planes $\Omega_r^+ = \{(t,x) : x > pt + q\}$, $\Omega_r^- = \{(t,x) : \ x < pt + q\}$. Finally, let $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2$.

(i) Assume that $S(t,x) > p$ for all $(t,x) \in [t_1,t_2) \times \mathbb{R}^* \cap r$. Then, any maximal solution $x(t)$ of (9) defined for all $t \in I_\omega \subseteq \mathbb{R}$ with $(t_1, x(t_1)) \in \overline{\Omega_r^+}$ is such that $(t, x(t)) \in \Omega_r^+$, for all $t \in (t_1,t_2) \cap I_\omega$.

(ii) Assume that $S(t,x) < p$ for all $(t,x) \in [t_1,t_2) \times \mathbb{R}^* \cap r$. Then, any maximal solution $x(t)$ of (9) defined for all $t \in I_\omega \subseteq \mathbb{R}$ with $(t_1, x(t_1)) \in \overline{\Omega_r^-}$ is such that $(t, x(t)) \in \Omega_r^-$, for all $t \in (t_1,t_2) \cap I_\omega$.

Remark 1. Recall that the T-periodicity and $\mathcal{C}^1$ character of $a_0(t)$ implies that $\min\{\dot{a}_0(t)\} \le 0$ and $\max\{\dot{a}_0(t)\} \ge 0$. Furthermore, the assumptions of Theorems 1.1 and 1.2 entail that $(-1)^m a_1(t) < 0$.

Theorem 1.1 considers a case in which $a_0(t)$ has nonconstant sign. The next Lemmas study the behavior of the solutions of (3) in an open interval (a,b) where a and b are two consecutive zeros of $a_0(t)$.

Lemma 2.2. *Let the assumptions of Theorem 1.1 be fulfilled, and let also*

$$S(x,t) = x^{-m} \sum_{i=0}^{n} a_i(t) x^i. \tag{10}$$

If a and b are two consecutive zeros of $a_0(t)$, then there exists $p > 0$ such that $S(t,x) > p$ for each $(t,x) \in r_p := \{(t,x) : x = p(t-b)\}$ with $a < t < b$.

Proof. Letting $x = p(t-b)$, with $p > 0$, in (10) one gets:

$$\begin{aligned} S(t,x) &= \frac{a_0(t)}{p^m (t-b)^m} + \frac{a_1(t)}{p^{m-1}(t-b)^{m-1}} + \cdots + \frac{a_n(t)}{p^{m-n}(t-b)^{m-n}} \\ &= p^{1-m}(t-b)^{1-m} \left[\frac{a_0(t)}{p(t-b)} + a_1(t) + \cdots + a_n(t) p^{n-1} (t-b)^{n-1} \right]. \end{aligned}$$

Assume that $t \in (a,b)$. Then, by the Mean Value Theorem, for each t there exists $\xi \in (t,b)$ such that

$$\dot{a}_0(\xi) = \frac{a_0(t) - a_0(b)}{t-b} = \frac{a_0(t)}{t-b}, \tag{11}$$

this yielding

$$S(t,x) = p^{1-m}(t-b)^{1-m}\left[p^{-1}\dot{a}_0(\xi) + a_1(t) + \sum_{i=2}^{n} a_i(t)p^{i-1}(t-b)^{i-1}\right]. \qquad (12)$$

Let m be even. Then, as $(t-b)^{1-m} < 0$ in (a,b), we have from (12) that

$$S(t,x) \geq -\left(p|t-b|\right)^{1-m}\left[p^{-1}\max\{\dot{a}_0\} + a_1(t) + \sum_{i=2}^{n} |a_i(t)|p^{i-1}T^{i-1}\right]. \qquad (13)$$

Alternatively, when m is odd it results that $(t-b)^{1-m} > 0$ in (a,b), and from (12):

$$\begin{aligned} S(t,x) &\geq \left(p|t-b|\right)^{1-m}\left[p^{-1}\min\{\dot{a}_0(t)\} + a_1(t) - \sum_{i=2}^{n} |a_i(t)|p^{i-1}T^{i-1}\right] \\ &= -\left(p|t-b|\right)^{1-m}\left[p^{-1}\max\{-\dot{a}_0(t)\} - a_1(t) + \sum_{i=2}^{n} |a_i(t)|(pT)^{i-1}\right] \quad (14) \end{aligned}$$

Hence, denoting

$$\Lambda = \begin{cases} \max\{-\dot{a}_0(t)\} & \text{if } m \text{ is odd} \\ \max|\dot{a}_0(t)| & \text{if } m \text{ is even,} \end{cases}$$

(13) and (14) can be merged as:

$$S(t,x) \geq -\left(p|t-b|\right)^{1-m}\left[p^{-1}\Lambda + (-1)^m a_1(t) + \sum_{i=2}^{n} |a_i(t)|(pT)^{i-1}\right],$$

and selecting $p = \frac{\beta}{T}$, where β is given by condition (4), we obtain:

$$\begin{aligned} S(t,x) &\geq -\left(p|t-b|\right)^{1-m}\left[\frac{T}{\beta}\Lambda + (-1)^m a_1(t) + \sum_{i=2}^{n} |a_i(t)|\beta^{i-1}\right] \\ &\geq -\left(pT\right)^{1-m}\left[\frac{T}{\beta}\Lambda + \max\left\{(-1)^m a_1(t) + \sum_{i=2}^{n} |a_i(t)|\beta^{i-1}\right\}\right] \\ &> \left(pT\right)^{1-m}\frac{\beta^m}{T} = \frac{(pT)^m}{(pT)^{m-1}T} = p \end{aligned}$$

□

Remark 2. Notice that there is no loss of generality in assuming that $(-1)^m a_0(t)$ be negative in (a,b) because, otherwise, the change of variables $(t,x) \mapsto ((-1)^m t, -x)$ reduces (3) to

$$\dot{x} = x^{-m}\sum_{i=0}^{n} \hat{a}_i(t)x^i,$$

with $\hat{a}_i(t) = (-1)^{i+1}a_i((-1)^m t)$ for all $t \in (a,b)$, while conditions (4) and (5) remain invariant.

In fact, this is the point that requires the establishment of different conditions depending on the parity of m, since the sign of $\dot{a}_0$ is conserved under the change of variables $(t,x) \mapsto ((-1)^m t, -x)$ when m is odd, but it is reversed when m is even.

The hypotheses of Theorem 1.1 are assumed to be fulfilled throughout the remainder of the section.

Lemma 2.3. *Let $a, b \in \mathbb{R}$ be such that $a_0(a) = a_0(b) = 0$, with $(-1)^m a_0(t) < 0$ for all $t \in (a, b)$. Then, any negative solution $x(t)$ of (3) defined on $[t_1, t_2)$, with $t_1 \geq a$ and $x(t_1)$ sufficiently close to 0, can be extended to $[t_1, b)$.*

Proof. The ODE (3) may be written as

$$\dot{x} = S(t, x) = x^{-m} \sum_{i=0}^{n} a_i(t) x^i \tag{15}$$

in $\Omega^- := \mathbb{R} \times \mathbb{R}^-$. Let us denote as $x(t)$ a solution of (15) with $x(t_1) < 0$. Let us also consider that $I_\omega = (\omega_-, \omega_+)$, with $\omega_- < t_1$, be its maximal interval of definition and assume that $\omega_+ < b$. Hence, $t \to \omega_+$ means that either $x(t) \to -\infty$ or $x(t) \to 0$.

On the one hand, applying Lemma 2.1 to the straight line

$$r_p := \{(t, x) : x = p(t - b)\},$$

where p is given by Lemma 2.2, we obtain that when $x(t_1) > p(t_1 - b)$ then $x(t) > p(t - b)$ for $t \in (t_1, \omega_+)$, and therefore, $x(t) \not\to -\infty$ for $t \to \omega_+ < b$.

On the other hand, let us take $c \in \mathbb{R}$, $t_1 < c < \omega_+$ and select $\delta \in \mathbb{R}^+$ small enough, in such a way that

$$S(t, -\delta) = \delta^{-m} \left[(-1)^m a_0(t) + \delta \sum_{i=1}^{n} (-1)^{m+i} \delta^{i-1} a_i(t) \right] < 0, \quad \forall t \in [c, \omega_+].$$

The existence of δ is ensured by the assumption $(-1)^m a_0(t) < 0$ in $[c, \omega_+]$. Defining now $r_\delta := \{(t, x) : \ x + \delta = 0\}$, it is straightforward that $S(t, x) < 0$, for all $(t, x) \in [c, \omega_+] \times \mathbb{R}^- \cap r_\delta$ and, by Lemma 2.1.ii, $x(t) + \delta < 0$ for $t \in (c, \omega_+)$, that is, $x(t) \not\to 0$.

Therefore, the assumption $\omega_+ < b$ is contradictory: it has to be $\omega_+ \geq b$ and the solution $x(t)$ is defined in $[t_1, b)$. □

Lemma 2.4. *Let $a, b \in \mathbb{R}$ be such that $a_0(a) = a_0(b) = 0$, with $(-1)^m a_0(t) < 0$ for all $t \in (a, b)$. Then, there exists a $\mathcal{C}^1$ solution $x^*(t)$ of (3) in (a, b), which is negative and such that*

$$x^*(t) \to 0 \quad \textit{and} \quad \dot{x}^*(t) \to \begin{cases} \frac{a_1(a)}{2} - \sqrt{\frac{a_1(a)^2}{4} + \dot{a}_0(a)} & \textit{if} \quad m = 1 \\ -\frac{\dot{a}_0(a)}{a_1(a)} & \textit{if} \quad m \geq 2 \end{cases} \quad \textit{when} \quad t \to a^+.$$

Proof. The ODE (3) may be transformed into the planar, generalized Liénard system [15]:

$$\frac{dt}{ds} = x^m, \tag{16}$$

$$\frac{dx}{ds} = a_0(t) + a_1(t)x + \cdots + a_n(t)x^n. \tag{17}$$

It is worth remarking that, when $x \neq 0$, the portrait of the integral curves of (3) and the phase plane of (16)-(17) are coincident.

The jacobian matrix of (16)-(17) in $(a, 0)$ is:

$$\begin{pmatrix} 0 & mx^{m-1} \\ \dot{a}_0(a) & a_1(a) \end{pmatrix}_{x=0}.$$

Recall also that $a_1(a) \neq 0$. Hence, the local analysis of the equilibrium when $m = 1$ is essentially different from the case $m = 2$.

The proof for $m = 1$ runs parallel to that of Lemma 4 in [12], which is carried out therein for $n = 2$ but can be straightforwardly extended to deal with a generic, positive integer n.

Let us then focus on $m \geq 2$. In such a scenario, the eigenvalues of the non-hyperbolic critical point $(t, x) = (a, 0)$ are

$$\lambda_{su}^a = a_1(a), \quad \lambda_c^a = 0.$$

Notice that the sign of λ_{su}^a depends on the parity of m. Moreover, the associated invariant subspaces of the linearized system are

$$\mathbb{E}_{su}^a = \text{span}\,\{(0,1)\}\,, \quad \mathbb{E}_c^a = \text{span}\,\{(a_1(a), -\dot{a}_0(a))\}\,.$$

Hence, by the Center Manifold Theorem [16], there exists a $\mathcal{C}^1$ invariant curve, tangent to $\mathbb{E}_c^a$ at $(a, 0)$, with slope $-\frac{\dot{a}_0(a)}{a_1(a)}$. Moreover, since $(-1)^m a_0(t) < 0$ and $(-1)^m a_1(t) < 0$ in (a, b), it follows that $-\frac{\dot{a}_0(a)}{a_1(a)} \leq 0$.

If $-\frac{\dot{a}_0(a)}{a_1(a)} < 0$, this orbit lies on the subsets $\mathcal{A}^+ := \{(t,x) :\ t < a,\ x > 0\}$ and $\mathcal{A}^- := \{(t,x) :\ t > a,\ x < 0\}$ when $t \neq a$. The branch of the manifold that lies in $\mathcal{A}^-$ is a negative, $\mathcal{C}^1$ solution $x^*(t)$ of (3) in $(a, a+\epsilon)$, $\epsilon > 0$, that satisfies $x^*(t) \to 0$ and $\dot{x}^*(t) \to -\frac{\dot{a}_0(a)}{a_1(a)}$ when $t \to a^+$.

If $-\frac{\dot{a}_0(a)}{a_1(a)} = 0$, then this orbit matches a $\mathcal{C}^1$ solution $x^*(t)$ of (3) that satisfies $x^*(t) \to 0$ and $\dot{x}^*(t) \to 0$ when $t \to a^+$. Thus, let us see that this orbit lies in $\mathcal{A}^-$ for $t > a$. For, let us denote such an orbit as $x = h(t)$, with $h(a) = \dot{h}(a) = 0$ and satisfying

$$h(t)\left[a_n(t)h(t)^{n-1} + \cdots + a_2(t)h(t) + a_1(t) - h(t)^{m-1}\dot{h}(t)\right] = -a_0(t).$$

As

$$a_n(a)h(a)^{n-1} + \cdots + a_2(a)h(a) + a_1(a) - h(a)^{m-1}\dot{h}(a) = a_1(a) \neq 0,$$

then

$$\text{sign}\left[a_n(t)h(t)^{n-1} + \cdots + a_2(t)h(t) + a_1(t) - h(t)^{m-1}\dot{h}(t)\right] = \text{sign}\,(a_1(t))$$

for $t-a$ small enough; consequently, h and $-a_0(t)a_1(t)$ have the same sign in a neighborhood of $(a, 0)$, and taking into account that $(-1)^m a_0(t) < 0$ and $(-1)^m a_1(t) < 0$, we obtain that $h(t) < 0$ for $0 < t - a << 1$.

Notice finally that, by Lemma 2.3, $x^*(t)$ is defined in (a, b). □

Lemma 2.5. *Let $a, b \in \mathbb{R}$ be such that $a_0(a) = a_0(b) = 0$, with $(-1)^m a_0(t) < 0$, for all $t \in (a, b)$. Then, there exists a $\mathcal{C}^1$ solution $x^*(t)$ of (3) in (a, b), which has the sign of $(-1)^m a_0(t)$, and is such that*

$$x^*(t) \to 0 \quad \textit{and} \quad \dot{x}^*(t) \to \begin{cases} \frac{a_1(a)}{2} - \sqrt{\frac{a_1(a)^2}{4} + \dot{a}_0(a)} & \textit{if} \quad m = 1 \\ -\frac{\dot{a}_0(a)}{a_1(a)} & \textit{if} \quad m \geq 2 \end{cases} \quad \textit{when} \quad t \to a^+,$$

$$x^*(t) \to 0 \quad \textit{and} \quad \dot{x}^*(t) \to \begin{cases} \frac{a_1(b)}{2} - \sqrt{\frac{a_1(b)^2}{4} + \dot{a}_0(b)} & \textit{if} \quad m = 1 \\ -\frac{\dot{a}_0(b)}{a_1(b)} & \textit{if} \quad m \geq 2 \end{cases} \quad \textit{when} \quad t \to b^-.$$

Proof. Let $x^*(t)$ be the solution of (3) featured in Lemma 2.4. Then, let us focus the attention on the behavior for $t \to b^-$.

Firstly, by Lemmas 2.2 and 2.1 we know that there exists p such that $x^*(t) > p(t-b)$ for all $t \in (a,b)$. But since $x^*(t)$ is negative in (a,b), then $p(t-b) < x^*(t) < 0$ and, taking limits for $t \to b^-$, it is immediate that $x^*(t) \to 0$.

Secondly, consider the planar, autonomous system (16)-(17), which is equivalent to (3) in (a,b). The situation is equivalent to the proof of Lemma 2.4. Hence, we refer the reader to the proof of Lemma 5 in [12] for $m=1$ and we focus the attention on the case $m \geq 2$.

When $m \geq 2$, $(t,x) = (b,0)$ is a non-hyperbolic critical point with eigenvalues

$$\lambda_{su}^b = a_1(b), \quad \lambda_c^b = 0,$$

the associated invariant subspaces of the linearized system being

$$\mathbb{E}_{su}^b = \text{span}\{(0,1)\}, \quad \mathbb{E}_c^b = \text{span}\{(a_1(b), -\dot{a}_0(b))\}.$$

Since we have a negative solution x^* which tends to zero when $t \to b^-$, then it has to be either a center manifold or the stable/unstable manifold. But the stable/unstable manifold is tangent to the line $\{t = b\}$, and since our solution satisfies $p(t-b) < x^*(t) < 0$, then it has to be a center manifold, and therefore $\dot{x}^*(t) \to -\frac{a_0(b)}{a_1(b)}$ when $t \to b^-$. □

Let us now complete the proof of Theorem 1.1. Recalling that $a_0(t)$ has, at least, one zero in $[0,T]$ by hypothesis, let $t_0 \in \mathbb{R}$ be such that $a_0(t_0) = 0$ and define

$$Z := \{t \in [t_0, t_0+T] : \ a_0(t) = 0\}.$$

Let also $\mathcal{P}$ and $\mathcal{N}$ stand for the sets of maximal intervals in $[t_0, t_0+T]$ where $(-1)^m a_0(t)$ is positive and negative, respectively. Let also $I_i = (a_i, b_i)$, $a_i, b_i \in Z$, denote an interval of $\mathcal{P} \cup \mathcal{N}$. Lemma 2.5 ensures that, for every I_i, there exists a $\mathcal{C}^1$ solution $x_i^*(t)$ of (3) on I_i, which has the sign of $(-1)^m a_0(t)$, and is such that $x_i^*(t) \to 0$ when $t \to a_i^+$ and also when $t \to b_i^-$. Hence,

$$x^*(t) = \begin{cases} x_i^*(t) & \text{if} \quad t \in I_i \\ 0 & \text{if} \quad t \in Z, \end{cases} \tag{18}$$

is indeed a continuous solution of (3) in $\mathbb{R}$ which is also $\mathcal{C}^1$ in every open interval I_i.

Let us finally prove that $x^*(t)$ is $\mathcal{C}^1$ for all $t_i \in Z$. When $m = 1$, this follows immediately from [12]. Alternatively, when $m \geq 2$ the graph of $x^*(t)$ in a neighborhood of t_i is the orbit of a center manifold of (16)-(17), so $x^*(t)$ is $\mathcal{C}^1$ in t_i (see the discussion in the proof of Lemma 2.4). Therefore, the T-periodic extension of x^* is a $\mathcal{C}^1$ solution of (3) defined in $\mathbb{R}$.

□

3. **Proof of Theorem 1.2.**

It follows from Remark 2 that the sign of $a_{2i-1}(t)$ is invariant under the change of variables $(t,x) \mapsto ((-1)^m t, -x)$. In turn, this change also keeps invariant conditions (7) and (8). Hence, the proof of Theorem 1.2 follows equivalently to that of Theorem 1.1, the only difference being in Lemma 2.2, which has to be replaced by:

Lemma 3.1. *Let the assumptions of Theorem 1.2 be fulfilled, and let also*

$$S(t,x) = x^{-m}\left[a_0(t) + \sum_{i=1}^{k} a_{2i-1}(t) x^{2i-1}\right]. \tag{19}$$

If a and b are two consecutive zeros of $a_0(t)$, then there exists $p > 0$ such that

$$S(t, x) > p$$

for each $(t, x) \in r_p := \{(t, x) : x = p(t - b)\}$, with $a < t < b$.

Proof. Letting $x = p(t - b)$ in (19), with $p > 0$, one gets:

$$\begin{aligned} S(t,x) &= \frac{a_0(t)}{p^m(t-b)^m} + \frac{a_1(t)}{p^{m-1}(t-b)^{m-1}} + \cdots + \frac{a_{2k-1}(t)}{p^{m-2k+1}(t-b)^{m-2k+1}} \\ &= p^{1-m}(t-b)^{1-m}\left[\frac{a_0(t)}{p(t-b)} + a_1(t) + \cdots + a_{2k-1}(t)p^{2k-2}(t-b)^{2k-2}\right]. \end{aligned}$$

Now, as in Lemma 2.2, the Mean Value Theorem guarantees that, for every $t \in (a, b)$ there exists $\xi \in (a, b)$ such that (11) is verified.

Hence

$$S(t,x) = p^{1-m}(t-b)^{1-m}\left[p^{-1}\dot{a}_0(\xi) + a_1(t) + \sum_{i=1}^{k-1} a_{2i+1}(t)p^{2i}(t-b)^{2i}\right] =$$

$$= (p|t-b|)^{1-m}\left[p^{-1}(-1)^{m+1}\dot{a}_0(\xi) + (-1)^{m+1}a_1(t) + \sum_{i=1}^{k-1}(-1)^{m+1}a_{2i+1}(t)p^{2i}|t-b|^{2i}\right].$$

But since $(-1)^{m+1}a_{2i+1} \geq 0$ then it is immediate that

$$S(t,x) \geq p^{1-m}|t-b|^{1-m}\left[p^{-1}(-1)^{m+1}\dot{a}_0(\xi) + (-1)^{m+1}a_1(t)\right].$$

Therefore, we want to select p such that

$$p^{-1}(-1)^{m+1}\dot{a}_0(\xi) + (-1)^{m+1}a_1(t) > p^m(t-b)^{m-1},$$

and hence it suffices to set p in such a way that

$$-p^{-1}\Lambda + \min\{(-1)^{m+1}a_1(t)\} > p^m T^{m-1}, \tag{20}$$

where

$$\Lambda = \begin{cases} \max\{-\dot{a}_0(t)\} & \text{if } m \text{ is odd} \\ \max|\dot{a}_0(t)| & \text{if } m \text{ is even.} \end{cases}$$

For, applying the classical inequality between the arithmetic mean and the geometric mean to the $m + 1$ numbers

$$z_1 = p^m T^{m-1}, \quad z_2 = \cdots = z_{m+1} = \frac{1}{m}p^{-1}\Lambda,$$

we obtain

$$\frac{z_1 + \cdots + z_{m+1}}{m+1} \geq (z_1 \cdots z_{m+1})^{\frac{1}{m+1}},$$

that is:

$$p^m T^{m-1} + p^{-1}\Lambda \geq (m+1)T^{\frac{m-1}{m+1}}\left(\frac{\Lambda}{m}\right)^{\frac{m}{m+1}};$$

moreover, the equality holds if and only if $z_1 = z_2 = \cdots = z_{m+1}$, i.e.

$$p = \left(\frac{\Lambda}{T^{m-1}m}\right)^{\frac{1}{m+1}}.$$

Thus, one can choose p satisfying (20) if and only if

$$\min\{(-1)^{m+1}a_1(t)\} > (m+1)T^{\frac{m-1}{m+1}}\left(\frac{\Lambda}{m}\right)^{\frac{m}{m+1}}.$$

□

4. **Proof of Theorem 1.3.** When $m = 1$, the claimed results follow from the proof of Theorem 2 in [12], which can be easily extendend to the case $n \geq 2$.

Assume now that $m \geq 2$, and let $x(t)$ be a $\mathcal{C}^1$ solution of (3) (resp. (6)) with nonconstant sign and non identically 0 (otherwise the result is trivial). As stated in Section 1, the zeros of $x(t)$ are also zeros of $a_0(t)$. Then, let $I_u = (a, b)$ be an interval where $x(t) \neq 0$, for all $t \in I_u$, and such that $x(a) = x(b) = 0$. It is no loss of generality to assume that $x(t) < 0$ in I_u (recall Remark 2), which yields $\dot{x}(a) \leq 0$.

It has been noticed in the proof of Lemma 2.4 that (16)-(17) possesses at least two invariant manifolds in $(a, 0)$, one which is stable/unstable, and another one (or more) which is (are) center manifold(s). However, the stable/unstable manifold is tangent to the vector $(0, 1)$, and hence the $\mathcal{C}^1$ solution $x(t)$ has to coincide with the solution curve corresponding to a center manifold orbit and, consequently, with the periodic solution $x^*(t)$ guaranteed by Theorem 1.1 (resp. Theorem 1.2), from $t = a$ to the next zero of $a_0(t)$, and this zero has to be in $t = b$. It is therefore proved that $x(t) = x^*(t)$, for all t such that $x(t) \neq 0$. Furthermore, when $x(t) = 0$, then $a_0(t) = 0$ and $x^*(t) = 0$, which implies that $x(t) = x^*(t)$, for all $t \in \mathbb{R}$.

5. **The Abel ODE of the second kind.** Abel ODE of the second kind appear setting $m = 1$ and $n = 2$ in (3), that is, they answer to

$$x\dot{x} = a_0(t) + a_1(t)x + a_2(t)x^2. \tag{21}$$

The existence of periodic solutions with nonconstant sign in (21) was studied in [12], and its existence was guaranteed under a certain restriction on the coefficients, $a_i(t)$. In this section we will compare this constraint with that provided in Theorem 1.1.

The main result in [12] is:

Theorem 5.1 ([12]). *Let $a_0(t), a_1(t), a_2(t)$ be $\mathcal{C}^1$, T-periodic functions. If $a_0(t)$ has at least one zero in $[0, T]$ and*

$$\min_{t\in\mathbb{R}} \left\{|a_1(t)|^2\right\} > -4 \min_{t\in\mathbb{R}} \{\dot{a}_0(t)\} \cdot \left[1 + T \max_{t\in\mathbb{R}} \{|a_2(t)|\}\right], \tag{22}$$

then (21) has a T-periodic solution that has the sign of $-a_0(t)a_1(t)$, and it is also $\mathcal{C}^1$.

Notice that (22) demands $a_1(t) \neq 0$, whereas (4) requires $a_1(t) > 0$ according to Remark 1. However, for this specific situation it results that (4) is slightly sharper than (22), as established in next Proposition.

Proposition 1. *Let $a_0(t), a_1(t), a_2(t)$ be $\mathcal{C}^1$, T-periodic functions. Assume that $a_0(t)$ has at least one zero in $[0, T]$ and also that (22) is verified. Then, there exists $\beta > 0$ such that*

$$\frac{T}{\beta} \max_{t\in\mathbb{R}} \{-\dot{a}_0(t)\} + \max_{t\in\mathbb{R}} \{-a_1(t) + \beta|a_2(t)|\} + \frac{\beta}{T} < 0. \tag{23}$$

Proof. It is immediate that the existence of $\beta > 0$ such that

$$-\frac{T}{\beta} \min_{t\in\mathbb{R}} \{\dot{a}_0(t)\} - \min_{t\in\mathbb{R}} \{a_1(t)\} + \beta \max_{t\in\mathbb{R}} \{|a_2(t)|\} + \frac{\beta}{T} < 0 \tag{24}$$

guarantees the fulfillment of (23). Notice that (24) is equivalent to

$$\min_{t\in\mathbb{R}} \{a_1(t)\} > \beta \left(\frac{1}{T} + \max_{t\in\mathbb{R}} \{|a_2(t)|\}\right) - \frac{T}{\beta} \min_{t\in\mathbb{R}} \{\dot{a}_0(t)\}.$$

Nevertheless, the inequality between arithmetic and geometric means yields

$$\beta\left(\frac{1}{T}+\max_{t\in\mathbb{R}}\{|a_2(t)|\}\right)-\frac{T}{\beta}\min_{t\in\mathbb{R}}\{\dot{a}_0(t)\}\geq 2\sqrt{-T\min_{t\in\mathbb{R}}\{\dot{a}_0(t)\}\left(\frac{1}{T}+\max_{t\in\mathbb{R}}\{|a_2(t)|\}\right)},$$

with equality iff

$$\beta\left(\frac{1}{T}+\max_{t\in\mathbb{R}}\{|a_2(t)|\}\right)=-\frac{T}{\beta}\min_{t\in\mathbb{R}}\{\dot{a}_0(t)\}$$

i.e. iff

$$\beta=\sqrt{\frac{-T^2\min_{t\in\mathbb{R}}\{\dot{a}_0(t)\}}{1+T\max_{t\in\mathbb{R}}\{|a_2(t)|\}}}. \tag{25}$$

Then, the resulting condition is

$$\min_{t\in\mathbb{R}}\{a_1(t)\}>2\sqrt{-\min_{t\in\mathbb{R}}\{\dot{a}_0(t)\}\left(1+T\max_{t\in\mathbb{R}}\{|a_2(t)|\}\right)}$$

which is equivalent to

$$\min_{t\in\mathbb{R}}\left\{|a_1(t)|^2\right\}>-4\min_{t\in\mathbb{R}}\{\dot{a}_0(t)\}\left(1+T\max_{t\in\mathbb{R}}\{|a_2(t)|\}\right).$$

because of the assumption $a_1(t)>0$. As this last is also verified by hypothesis, the result follows with β selected as in (25). □

Example. Notice that for any (21)-like Abel ODE with

$$a_1(t)=5-\sin^2(\pi t) \qquad \text{and} \qquad a_2(t)=\cos^2(\pi t),$$

condition (22) becomes $-\min_{t\in\mathbb{R}}\{\dot{a}_0(t)\}<2$, while condition (23) with $\beta=1$ boils down to $-\min_{t\in\mathbb{R}}\{\dot{a}_0(t)\}<3$. Hence, whenever

$$2\leq-\min_{t\in\mathbb{R}}\{\dot{a}_0(t)\}<3$$

the ODE does not satisfy (22) but (23).

REFERENCES

[1] P.J. Torres, *Existence of closed solutions for a polynomial first order differential equation*, J. Math. Anal. Applic., **328** (2007), 1108–1116.

[2] M.A.M. Alwash, *Periodic solutions of Abel differential equations*, J. Math. Anal. Applic., **329** (2007), 1161–1169.

[3] A.D. Polyanin and V.F. Zaitsev, "Handbook of exact solutions for ordinary differential equations," 2^{nd} edition, Chapman & Hall/CRC, Boca Raton, 2003.

[4] S. Smale, *Mathematical problems for the next century*, Math. Intelligencer, **20** (1998), 7–15.

[5] A. Gasull, J. Llibre, *Limit cycles for a class of Abel equations*, SIAM J. Math. Anal., **21** (1990), 1235-1244.

[6] Yu. Ilyashenko, *Hilbert-type numbers for Abel equations, growth and zeros of holomorphic functions*, Nonlinearity, **13** (2000), 1337–1342.

[7] M.J. Álvarez, A. Gasull, H. Giacomini, *A new uniqueness criterion for the number of periodic orbits of Abel equations*, J. Differential Equations, **234** (2007), 161–176.

[8] J.L. Bravo, J. Torregrosa, *Abel-like differential equations with no periodic solutions*, J. Math. Anal. Applic., **342** (2008), 931–942.

[9] J.L. Bravo, M. Fernández and A. Gasull, *Limit cycles for some Abel equations having coefficients without fixed signs*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., **19** (2009), 3869–3876.

[10] M.A.M. Alwash, *Polynomial differential equations with small coefficients*, Discrete Continuous Dynam. Systems - A, **25** (2009), 1129–1141.

[11] N.H.M. Alkoumi, P.J. Torres, *Estimates on the number of limit cycles of a generalized Abel equation*, Discrete Continuous Dynam. Systems - A, **31** (2011), 25–34.

[12] J.M. Olm, X. Ros-Oton, T.M. Seara, *Periodic solutions with nonconstant sign in Abel equations of the second kind*, J. Math. Anal. Appl., **381** (2011), 582–589.

[13] E. Fossas and J.M. Olm, *Galerkin method and approximate tracking in a non-minimum phase bilinear system*, Discrete Continuous Dynam. Systems - B, **7** (2007), 53–76.

[14] J.M. Olm and X. Ros-Oton, *Approximate tracking of periodic references in a class of bilinear systems via stable inversion*, Discrete Continuous Dynam. Systems - B, **15** (2011), 197–215.

[15] A. Gasull, H. Giacomini, *A New Criterion for Controlling the Number of Limit Cycles of Some Generalized Liénard Equations*, J. Differential Equations, **185** (2002), 54–73.

[16] A. Kelley, *The stable, center-stable, center, center-unstable and unstable manifolds*, J. Differential Equations, **3** (1967), 546–570.

[17] J. Guckenheimer and P. Holmes, "Nonlinear oscillations, dynamical systems and bifurcations of vector fields", 2^{nd} edition, Springer-Verlag, New York, 1985.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: `josep.olm@upc.edu`
E-mail address: `xavier.ros.oton@upc.edu`