LYUBEZNIK TABLE OF SEQUENTIALLY COHEN-MACAULAY RINGS

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Abstract. We prove that sequentially Cohen-Macaulay rings in positive characteristic, as well as sequentially Cohen-Macaulay Stanley-Reisner rings in any characteristic, have trivial Lyubeznik table. Some other configurations of Lyubeznik tables are also provided depending on the deficiency modules of the ring.

1. Introduction

Relying on the finiteness of Bass numbers of local cohomology modules, G. Lyubeznik [12] introduced a set of numerical invariants of local rings containing a field as follows:

Theorem/Definition 1.1. *Let A be a local ring containing a field k, so that its completion $\widehat{A}$ admits a surjective ring homomorphism $R \stackrel{\pi}{\twoheadrightarrow} \widehat{A}$ from a regular local ring $(R, \mathfrak{m}, k)$ of dimension n and set $I := \ker(\pi)$. Then, the Bass numbers*[1]

$$\lambda_{p,i}(A) := \mu_p(\mathfrak{m}, H_I^{n-i}(R)) = \mu_0(\mathfrak{m}, H_{\mathfrak{m}}^p(H_I^{n-i}(R)))$$

depend only on A, i and p, but neither on R nor on π.

We refer to these invariants as *Lyubeznik numbers* and they are known to satisfy the following properties:

(i) $\lambda_{p,i}(A) = 0$ if $i > d$.
(ii) $\lambda_{p,i}(A) = 0$ if $p > i$.
(iii) $\lambda_{d,d}(A) \neq 0$.

where $d = \dim A$. Therefore we can collect them in the so-called *Lyubeznik table*:

$$\Lambda(A) = \begin{pmatrix} \lambda_{0,0} & \cdots & \lambda_{0,d} \\ & \ddots & \vdots \\ & & \lambda_{d,d} \end{pmatrix}$$

The key point in the construction of Lyubeznik numbers is the fact that the local cohomology modules $H_{\mathfrak{m}}^p(H_I^{n-i}(R))$ are injective as R-modules (see [8], [12], [13]). Thus, $H_{\mathfrak{m}}^p(H_I^{n-i}(R))$ is isomorphic to a finite direct sum of e copies of the injective hull $E_R(R/\mathfrak{m})$. One may define the multiplicity $e(-)$ of this module as the integer e so the Lyubeznik numbers are

$$\lambda_{p,i}(A) = e(H_{\mathfrak{m}}^p(H_I^{n-i}(R))).$$

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[1] The last equality follows from [12, Lemma 1.4].

Local cohomology modules have a natural structure over the ring of k-linear differential operators $D_{R|k}$ (see [12], [13]). One may check that the multiplicity of $H^p_{\mathfrak{m}}(H^{n-i}_I(R))$ is nothing but its length as $D_{R|k}$-module. Namely,

$$\lambda_{p,i}(A) = e(H^p_{\mathfrak{m}}(H^{n-i}_I(R))) = \text{length}_{D_{R|k}}(H^p_{\mathfrak{m}}(H^{n-i}_I(R))).$$

In particular, the multiplicity $e(-)$ is an additive function in the category of $D_{R|k}$-modules, i.e if $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ is a short exact sequence then $e(M_2) = e(M_1) + e(M_3)$.

The following property of Lyubeznik numbers will play a crucial role in our main result. It was shown to us by R. García-López in a graduate course [6] but we will sketch the proof for the sake of completeness:

(iv) *Euler characteristic formula:*

$$\sum_{0 \leq p,i \leq d} (-1)^{p-i} \lambda_{p,i}(A) = 1.$$

Proof. Consider Grothendieck's spectral sequence

$$E_2^{p,n-i} = H^p_{\mathfrak{m}}(H^{n-i}_I(R)) \Longrightarrow H^{p+n-i}_{\mathfrak{m}}(R).$$

We define the Euler characteristic of the E_2-term with respect to the multiplicity e as

$$\chi_e(E_2^{\bullet,\bullet}) = \sum_{p,i} (-1)^{p+n-i} e(E_2^{p,n-i}).$$

We can also define the Euler characteristic of the graded R-module $H^{\bullet}_{\mathfrak{m}}(R)$ as

$$\chi_e(H^{\bullet}_{\mathfrak{m}}(R)) = \sum_j (-1)^j e(H^j_{\mathfrak{m}}(R)).$$

It is a general fact of the theory of spectral sequences that $\chi_e(E_2^{\bullet,\bullet}) = \chi_e(H^{\bullet}_{\mathfrak{m}}(R))$ due to the additivity of the multiplicity.

Therefore, since $e(E_2^{p,n-i}) = e(H^p_{\mathfrak{m}}(H^{n-i}_I(R))) = \lambda_{p,i}(A)$ and $e(H^n_{\mathfrak{m}}(R)) = 1$ we get

$$\chi_e(E_2^{\bullet,\bullet}) = \sum_{0 \leq p,i \leq d} (-1)^{p+n-i} \lambda_{p,i}(A) = (-1)^n = \chi_e(H^{\bullet}_{\mathfrak{m}}(R))$$

and the result follows. □

The first example one may think of Lyubeznik tables is when there is only one local cohomology module different from zero. Indeed, assume that $H^r_I(R) = 0$ for all $r \neq \operatorname{ht} I$. Then, using Grothendieck's spectral sequence we obtain a *trivial Lyubeznik table.*

$$\Lambda(R/I) = \begin{pmatrix} 0 & \cdots & 0 \\ & \ddots & \vdots \\ & & 1 \end{pmatrix}$$

This situation is achieved, among others, in the following cases:

- R/I is Cohen-Macaulay and contains a field of positive characteristic.
- R/I is Cohen-Macaulay and I is a squarefree monomial ideal in any characteristic.

Remark 1.1. When R/I is Cohen-Macaulay containing a field of characteristic zero the previous result is no longer true. For example, consider the ideal generated by the 2×2 minors of a generic 2×3 matrix. Its Lyubeznik table was computed in [1]:

$$\Lambda(R/I) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 1 \\ & & & 0 & 0 \\ & & & & 1 \end{pmatrix}$$

We point out that K. I. Kawasaki already proved in [9] that the highest Lyubeznik number $\lambda_{d,d}$ of a Cohen-Macaulay ring (or even S_2) is always one.

In this note we will prove that the previous result still holds true replacing the Cohen-Macaulay property for sequentially Cohen-Macaulay, in particular, assuming that we may have more than one local cohomology module different from zero. In the spirit of [19], we give a unified proof of both cases using the theory of modules over skew-polynomial rings. We point out that the case of squarefree monomial ideals is already treated in a joint work with K. Yanagawa [3] using the description of Lyubeznik numbers of squarefree monomial ideals in terms of the linear strands of the Alexander dual ideal given in [2]. Finally, in the last section, we use the same techniques to provide some configurations of Lyubeznik tables depending on the deficiency modules of the ring.

Sequentially Cohen-Macaulay modules were introduced by R. Stanley [18] in the graded case but extended later on to the local case. We present here an homological characterization, due to C. Peskine (see [7]) in the graded case and P. Schenzel [16] in the local case (see also [5]), that will be useful for our purposes. We decided to consider just the case of regular local rings to keep the same framework as in the rest of the paper.

Theorem/Definition 1.2. *Let* $(R, \mathfrak{m})$ *be a regular local ring of dimension* n*. Then, an* R*-module* M *is sequentially Cohen-Macaulay if and only if for all* $0 \leq i \leq \dim M$ *we have that* $\operatorname{Ext}_R^{n-i}(M, R)$ *is zero or Cohen-Macaulay of dimension* i*.*

Throughout this work we will freely use some standard facts about local cohomology modules. We refer to [4] for any unexplained terminology.

2. Finitely generated unit $R[\Theta; \varphi]$-modules

Let $(R, \mathfrak{m})$ be a regular local ring of dimension n containing a field k. Throughout the rest of the paper we will assume that we have a flat local endomorphism $\varphi : R \longrightarrow R$ satisfying, for a given ideal $I \subseteq R$, the condition:

$(*)$ The ideals $\{\varphi^t(I)R\}_{t \geq 0}$ form a descending chain cofinal with the chain $\{I^t\}_{t \geq 0}$.

Notice that in this case, by the dimension formula, we have that $\varphi^t(\mathfrak{m})R$ is $\mathfrak{m}$-primary. The main examples we are going to consider are:

- *Positive characteristic case:* When R contains a field of positive characteristic, the Frobenius endomorphism $\varphi = F$ satisfies $(*)$ for any ideal $I \subseteq R$ (see [8], [13]) and is flat by the celebrated theorem of E. Kunz [10].
- *Squarefree monomial ideals case:* Consider the polynomial ring $R = k[x_1, \dots, x_n]$ over a field k in any characteristic. The k-linear endomorphism $\varphi(x_i) = x_i^2$ is flat and satisfies $(*)$ for any squarefree monomial ideal (see [11]).

G. Lyubeznik [13] developed his theory of F-modules in positive characteristic building upon these properties for the Frobenius map. One may give a slightly extended theory associated to the morphism $\varphi : R \longrightarrow R$ instead of the Frobenius but, for our purposes, we are just going to use some very basic facts. First we will consider a new R-module structure on R given by $r \cdot s := \varphi(r)s$. One denotes this R-module $\varphi_* R$. Let Φ be the functor on the category of R-modules defined as

$$\Phi(M) = \varphi_* R \otimes_R M$$

with R-module structure given by $r \cdot (s \otimes m) = \varphi(r)s \otimes m$, for $r, s \in R$ and $m \in M$. Notice that we can also construct the e-th iterations Φ^e in the usual way.

Let $R[\Theta; \varphi]$ be the skew polynomial ring which is the free left R-module $\bigoplus_{e \geq 0} R\Theta^e$ with multiplication $\Theta r = \varphi(r)\Theta$. In fact we have

$$R[\Theta; \varphi] = R\langle\Theta\rangle / \langle \Theta r - \varphi(r)\Theta \mid r \in R \rangle.$$

To give a $R[\Theta; \varphi]$-module structure on a R-module $\mathcal{M}$ is equivalent to fix a R-linear map $\theta_{\mathcal{M}} : \mathcal{M} \longrightarrow \Phi(\mathcal{M})$. We say that $\mathcal{M}$ is a *unit* $R[\Theta; \varphi]$*-module* if $\theta_{\mathcal{M}}$ is an isomorphism.

Given a finitely generated R-module M and a R-linear map $\beta : M \longrightarrow \Phi(M)$ one can obtain a unit $R[\Theta; \varphi]$-module

$$\mathcal{M} := \operatorname{Gen}(M) = \varinjlim (M \xrightarrow{\beta} \Phi(M) \xrightarrow{\Phi(\beta)} \Phi^2(M) \xrightarrow{\Phi^2(\beta)} \cdots)$$

just because

$$\Phi(\mathcal{M}) := \operatorname{Gen}(\Phi(M)) = \varinjlim (\Phi(M) \xrightarrow{\Phi(\beta)} \Phi^2(M) \xrightarrow{\Phi^2(\beta)} \Phi^3(M) \xrightarrow{\Phi^3(\beta)} \cdots) = \mathcal{M}$$

We say that $\mathcal{M}$ is a *finitely generated unit* $R[\Theta; \varphi]$*-module* if it can constructed in this way[2]. Moreover, if the generating morphism β is injective, we say that M is a *root* of $\mathcal{M}$. The main example we are going to consider, that was already treated by A. Singh and U. Walther in [19], is the case of local cohomology modules.

As it was already stated in [19], the flatness of the morphism φ implies that Φ is an exact functor and it also follows that $\Phi^e(\operatorname{Ext}^i_R(R/I, R)) \cong \operatorname{Ext}^i_R(R/\varphi^e(I)R, R)$. We also

[2] A (F-finite) F-module in the sense of G. Lyubeznik [13] is a (finitely generated) unit $R[\Theta; F]$-module.

have a commutative diagram

$$\begin{array}{ccccccc} \cdots \longrightarrow & \mathrm{Ext}^i_R(R/\varphi^e(I)R,R) & \longrightarrow & \mathrm{Ext}^i_R(R/\varphi^{e+1}(I)R,R) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & \\ \cdots \longrightarrow & \Phi^e(\mathrm{Ext}^i_R(R/I,R)) & \longrightarrow & \Phi^{e+1}(\mathrm{Ext}^i_R(R/I,R)) & \longrightarrow \cdots \end{array}$$

where the maps in the top row are induced by the natural surjection

$$R/\varphi^{e+1}(I)R \longrightarrow R/\varphi^e(I)R$$

and the vertical maps are isomorphisms. Taking into account property $(*)$, the limit of the top row is the local cohomology module $H^i_I(R)$. We conclude that local cohomology modules are finitely generated unit $R[\Theta;F]$-modules and the generating morphism

$$\beta : \mathrm{Ext}^i_R(R/I,R) \longrightarrow \mathrm{Ext}^i_R(R/\varphi(I)R,R)$$

is induced by the natural surjection $R/\varphi(I)R \longrightarrow R/I$.

Remark 2.1. Under this terminology, [19, Thm. 2.8] states that $\mathrm{Ext}^i_R(R/I,R)$ is a root of $H^i_I(R)$ when the induced morphism $\overline{\varphi} : R/I \longrightarrow R/I$ is pure.

3. Main result

The description of local cohomology modules given in Section 2 allow us to obtain the main result of this note, but first we consider the following vanishing result for Bass numbers that is a mild generalization of [8, Thm. 3.3].

Lemma 3.1. *Let $(R,\mathfrak{m})$ be a regular local ring of dimension n containing a field k and $\varphi : R \longrightarrow R$ a flat local endomorphism satisfying $(*)$ for an ideal $I \subseteq R$. Given $p,i \in \mathbb{N}$, if $H^p_{\mathfrak{m}}(\mathrm{Ext}^{n-i}_R(R/I,R)) = 0$ then $\mu_p(\mathfrak{m}, H^{n-i}_I(R)) = 0$.*

Proof. Using flat base change for local cohomology and the fact that $\varphi^t(\mathfrak{m})R$ is $\mathfrak{m}$-primary we have:

$$\Phi^e(H^p_{\mathfrak{m}}(\mathrm{Ext}^{n-i}_R(R/I,R))) \cong H^p_{\mathfrak{m}}(\Phi^e(\mathrm{Ext}^{n-i}_R(R/I,R))) \cong H^p_{\mathfrak{m}}(\mathrm{Ext}^{n-i}_R(R/\varphi^e(I)R,R)).$$

Therefore, since local cohomology commutes with inductive limits

$$H^p_{\mathfrak{m}}(H^{n-i}_I(R)) \cong H^p_{\mathfrak{m}}(\varinjlim \Phi^e(\mathrm{Ext}^{n-i}_R(R/I,R))) \cong \varinjlim \Phi^e(H^p_{\mathfrak{m}}(\mathrm{Ext}^{n-i}_R(R/I,R)))$$

we get the desired result. □

Theorem 3.2. *Let $(R,\mathfrak{m})$ be a regular local ring of dimension n containing a field k and $\varphi : R \longrightarrow R$ a flat local endomorphism satisfying $(*)$ for an ideal $I \subseteq R$ such that R/I is sequentially Cohen-Macaulay. Then the Lyubeznik table of R/I is trivial.*

Remark 3.3. The completion with respect to the maximal ideal of a sequentially Cohen-Macaulay ring is sequentially Cohen-Macaulay [16, Thm. 4.9] but the converse does not hold as P.Schenzel showed in [16, Ex. 6.1] using Nagata's example [15, Ex.2]. Lyubeznik numbers does not depend on the completion so we can just assume that the completion of R/I is sequentially Cohen-Macaulay in the hypothesis of Theorem 3.2.

Proof. If R/I is sequentially Cohen-Macaulay then we have that $\mathrm{Ext}_R^{n-i}(R/I, R)$ is zero or Cohen-Macaulay of dimension i. Therefore $H_{\mathfrak{m}}^p(\mathrm{Ext}_R^{n-i}(R/I, R)) = 0$ for all $p \neq i$. It follows from Lemma 3.1 that the possible non-zero Lyubeznik numbers are $\lambda_{i,i}(R/I)$, i.e. those in the main diagonal of the Lyubeznik table. Using properties (iii) and (iv) of Lyubeznik numbers we have $\lambda_{0,0} + \cdots + \lambda_{d,d} = 1$ and $\lambda_{d,d} \neq 0$ so we must have a trivial Lyubeznik table. ☐

Specializing to the cases considered at the beginning of Section 2 we obtain:

Corollary 3.4. *Let $(R, \mathfrak{m})$ be a regular local ring containing a field k. Then the Lyubeznik table of R/I is trivial in the following cases:*

- *R/I is sequentially Cohen-Macaulay and contains a field of positive characteristic.*
- *R/I is sequentially Cohen-Macaulay and I is a squarefree monomial ideal.*

Remark 3.5. As it was already pointed out in [2], the converse statement does not hold. For example consider the ideal in $k[\![x_1, \dots, x_9]\!]$:

$$I = (x_1, x_2) \cap (x_3, x_4) \cap (x_5, x_6) \cap (x_7, x_8) \cap (x_9, x_1) \cap (x_9, x_2) \cap (x_9, x_3) \cap (x_9, x_4) \cap (x_9, x_5) \cap \cap (x_9, x_6) \cap (x_9, x_7) \cap (x_9, x_8).$$

R/I has a trivial Lyubeznik table but is not sequentially Cohen-Macaulay. We remark that $H_I^r(R)$ does not vanish for $r = 2, 3, 4, 5$.

4. Some partial vanishing results

A way to measure the deviation of R/I from being Cohen-Macaulay is through the *deficiency modules*

$$K^i(R/I) := \mathrm{Ext}_R^{n-i}(R/I, R).$$

In this sense, sequentially Cohen-Macaulay rings form a class where these modules are well understood. The methods developed in the previous section suggest that some configurations of Lyubeznik tables could be described depending on the behavior of these modules.

In this direction we recall the following notion developed by P. Schenzel in [17]: We say that R/I is *canonically Cohen-Macaulay* (CCM for short) if the canonical module $K^d(R/I)$ is Cohen-Macaulay.

Proposition 4.1. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension n containing a field k and $\varphi : R \longrightarrow R$ a flat local endomorphism satisfying $(*)$ for an ideal $I \subseteq R$ such that R/I is canonically Cohen-Macaulay. Then, $\lambda_{i,d}(R/I) = 0$ for all $i < d$.*

Proof. The canonical module $K^d(R/I) := \mathrm{Ext}_R^{n-d}(R/I, R)$ is Cohen-Macaulay of dimension d by [17, Prop. 2.3], so the result follows from Lemma 3.1. ☐

For a general description of the highest Lyubeznik number we refer to [14], [20] where $\lambda_{d,d}(R/I)$ is described as the number of connected components of the Hochster-Huneke graph of the completion of the strict Henselianization of R/I.

Examples of CCM modules include Cohen-Macaulay and sequentially Cohen-Macaulay modules among others (see [17, Ex.3.2]). Using Theorem 3.2 we have that $\lambda_{d,d}(R/I) = 1$ in these cases but, of course, we may find examples where this number is larger. For instance, the ideal $I = (x_1, x_2) \cap (x_3, x_4)$ in $k[\![x_1, x_2, x_3, x_4]\!]$ satisfies that R/I is CCM and its Lyubeznik table is

$$\Lambda(R/I) = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 2 \end{pmatrix}$$

This example can be seen as a particular case of the following result.

Proposition 4.2. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension n containing a field k and $\varphi : R \longrightarrow R$ a flat local endomorphism satisfying $(*)$ for an ideal $I \subseteq R$ such that R/I is unmixed and* $\operatorname{depth} K^i(R/I) \geq i - 1$ *for $0 \leq i < d$. Then, its Lyubeznik table is of the form*

$$\Lambda(R/I) = \begin{pmatrix} 0 & \lambda_{0,1} & \cdots & 0 & 0 \\ & 0 & \cdots & 0 & 0 \\ & & \ddots & \vdots & \vdots \\ & & & \lambda_{d-2,d-1} & 0 \\ & & & 0 & 0 \\ & & & & \lambda_{d,d} \end{pmatrix}$$

where $\lambda_{0,1} + \cdots + \lambda_{d-2,d-1} = \lambda_{d,d} - 1$. In particular, the Lyubeznik table is trivial when the highest Lyubeznik number is 1.

Proof. By [17, Thm.4.4] we have that R/I is CCM and $K^i(R/I)$ is either zero or Cohen-Macaulay of dimension $i - 1$. Then, using Lemma 3.1 we get the desired Lyubeznik table. The rest of the proof follows from the Euler characteristic property of Lyubeznik numbers. □

Another large class of CCM rings discussed in [17, §6] is the case of simplicial affine semigroup rings. In the sequel, let $R = k[x_1, \ldots, x_n]$ be the polynomial ring with n variables over a field k. Let S be a finitely generated submonoid of $\mathbb{N}^n$. The affine semigroup $k[S]$ of S over k is the subring of R generated by all monomials $x^s := x_1^{s_1} \cdots x_n^{s_n}$, $s \in S$. We say that S is simplicial if there is a homogeneous system of parameters of $k[S]$ with $d = \dim k[S]$ elements.

Proposition 4.3. *Let $k[S]$ be a simplicial affine semigroup ring of codimension* 2*, i.e. $n = d + 2$ and $\varphi : R \longrightarrow R$ a flat endomorphism satisfying $(*)$ for the ideal of vanishing of $k[S]$. If the number of generators of this ideal is $m \leq 3$, its Lyubeznik table is trivial. Otherwise it is of the form*

$$\Lambda(k[S]) = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ & \ddots & \vdots & \vdots \\ & & \lambda_{d-2,d-1} & 0 \\ & & 0 & 0 \\ & & & \lambda_{d,d} \end{pmatrix}$$

where $\lambda_{d-2,d-1} = \lambda_{d,d} - 1$.

Proof. By [17, Thm.6.5] we have that $k[S]$ is Cohen-Macaulay if and only if $m \leq 3$. When $m > 3$ we have that $K^i(k[S]) = 0$ for all $0 \leq i < d-1$ and $K^{d-1}(k[S]) = 0$ is Cohen-Macaulay of dimension $d-2$. Then, using Lemma 3.1 we get the desired Lyubeznik table. □

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