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SPECTRALLY-CONSISTENT REGULARIZATION MODELLING OF WIND FARM BOUNDARY LAYERS

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ABSTRACT

The incompressible Navier-Stokes equations constitute an excellent mathematical modelization of turbulence. Unfortunately, attempts at performing direct simulations are limited to relatively low-Reynolds numbers because of the almost numberless small scales produced by the non-linear convective term. Alternatively, a dynamically less complex formulation is proposed here. Namely, regularizations of the Navier-Stokes equations that preserve the symmetry and conservation properties exactly. To do so, both convective and diffusive term are altered in the same vein. In this way, the convective production of small scales is effectively restrained whereas the modified diffusive term introduces a hyperviscosity effect and consequently enhances the destruction of small scales. In practise, the only additional ingredient is a self-adjoint linear filter whose local filter length is determined from the requirement that vortex-stretching must stop at the smallest grid scale. In the present work, the performance of the above-mentioned recent improvements is assessed through application to homogeneous isotropic turbulence, a turbulent channel flow and a turbulent boundary layer. As a final application, regularization modelling will be applied for large-scale numerical simulation of the atmospheric boundary layer through wind farms.

Keywords: Energy conserving, Hyperviscosity, Regularization modelling, Turbulence, Wind power

NOMENCLATURE

$\mathcal{C}(\underline{u},\underline{v})$	convective operator, $(\underline{u}\cdot\nabla)\underline{v}$
$\mathcal{D}\underline{u}$	diffusive operator, $\nu\Delta\underline{u}$
$H_4(\widehat{g}_k)$	overall damping effect introduced in the k-th Fourier-mode
Q	second invariant of S, $-1/2tr(S^2)$
R	third invariant of S, $-1/3tr(S^3)$
Re	Reynolds number
S	strain tensor, $1/2(\nabla\underline{u}+\nabla\underline{u}^T)$
$f_4^\gamma(\widehat{g}_k)$	damping effect introduced by the $\mathcal{C}_4^\gamma$ regularization in the k-th Fourier-mode
$\widehat{g}_k$	k-th Fourier-mode of the convolution kernel of the filter
h	local grid size
$h_4^\gamma(\widehat{g}_k)$	hyper-viscosity effect introduced in the k-th Fourier-mode
k	k-th Fourier-mode
k_c	smallest grid scale
p	pressure
$\underline{u}$	velocity field
α	$\epsilon/\sqrt{24}$
γ , $\tilde{\gamma}$	parameter in $\mathcal{C}_4^\gamma$ model, $\tilde{\gamma}=1/2(1+\gamma)$
ϵ	filter length
ε	dissipation rate of kinetic energy
λ_i	eigenvalues of S, $\lambda_1\le\lambda_2\le\lambda_3$
λ_Δ	largest non-zero eigenvalue of the Laplacian operator
ν	kinematic viscosity
$\underline{\omega}$	vorticity, $\nabla\times\underline{u}$

Subscripts and Superscripts

$\overline{(\cdot)}$, $(\cdot)'$	symmetric linear filter and its residual
$\widehat{(\cdot)}$	Fourier transform
$(\cdot)^*$	conjugate transpose

1. INTRODUCTION

The incompressible Navier-Stokes (NS) equations form an excellent mathematical model for turbulent flows. In primitive variables they read

$$\partial_t\underline{u}+\mathcal{C}(\underline{u},\underline{u})=\mathcal{D}\underline{u}-\nabla p\,;\qquad \nabla\cdot\underline{u}=0, \tag{1}$$

where $\underline{u}$ denotes the velocity field, p represents the pressure, the non-linear convective term is defined by $\mathcal{C}(\underline{u},\underline{v})=(\underline{u}\cdot\nabla)\,\underline{v}$, and the diffusive term reads $\mathcal{D}\underline{u}=\nu\Delta\underline{u}$, where ν is the kinematic viscosity. Preserving the symmetries of the continuous differential operators when discretizing them has been shown to be a very suitable approach for direct numerical simulation (DNS) (see [1, 2, 3], for instance). Doing so, certain fundamental properties such as the inviscid invariants - kinetic energy, enstrophy (in 2D) and helicity (in 3D) - are exactly preserved in a discrete sense. However, direct simulations at high Reynolds numbers (Re) are not feasible because the convective term produces far too many relevant scales of motion. In the quest for a dynamically less complex formulation we consider regularizations [4, 5, 6] of non-linearity that preserve the symmetry and conservation properties exactly [7]. In this way, the convective production of small scales is effectively restrained in an unconditionally stable manner. In our previous works, we restrict ourselves to the $\mathcal{C}_4$ approximation: the convective term in the NS equations (1) is then replaced by the following $\mathcal{O}(\epsilon^4)$-accurate smooth approximation $\mathcal{C}_4(\underline{u},\underline{v})$ given by

$$\mathcal{C}_4(\underline{u},\underline{v}) = \mathcal{C}(\overline{\underline{u}},\overline{\underline{v}}) + \overline{\mathcal{C}(\overline{\underline{u}},\underline{v}')} + \overline{\mathcal{C}(\underline{u}',\overline{\underline{v}})}, \quad (2)$$

where the prime indicates the residual of the filter, *e.g.* $\underline{u}' = \underline{u} - \overline{\underline{u}}$, which can be explicitly evaluated, and $\overline{(\cdot)}$ represents a symmetric linear filter with filter length ϵ. Therefore, the governing equations result to

$$\partial_t \underline{u}_\epsilon + \mathcal{C}_4(\underline{u}_\epsilon,\underline{u}_\epsilon) = \mathcal{D}\underline{u}_\epsilon - \nabla p_\epsilon; \qquad \nabla\cdot\underline{u}_\epsilon = 0, \quad (3)$$

where the variable names are changed from $\underline{u}$ and p to $\underline{u}_\epsilon$ and p_ϵ, respectively, to stress that the solution of (3) differs from that of (1). Note that the $\mathcal{C}_4$ approximation is also a skew-symmetric operator like the original convective operator. Hence, the same inviscid invariants than the original NS equations are preserved for the new set of partial differential equations (3). The numerical algorithm used to solve the governing equations preserves the symmetries and conservation properties too. In practise, the only additional ingredient is a self-adjoint linear filter [8] whose local filter length is determined from the requirement that vortex-stretching must stop at the smallest grid scale [9]. Altogether, the method constitutes a parameter-free turbulence model that has already been successfully tested for a variety of natural and forced convection configurations (see [7, 9], for instance). However, two main drawbacks have been observed: (i) due to the energy conservation, the model solution tends to display an additional hump in the tail of the spectrum and (ii) for very coarse meshes the damping factor can eventually take very small values. These two issues are addressed in the following section.

2. RESTORING THE GALILEAN INVARIANCE: HYPERVISCOSITY EFFECT

The $\mathcal{C}_4$ regularization preserves all the invariant transformations of the original NS equations, except the Galilean transformation. This is a usual feature for most of the regularizations of the non-linear term [10]. This can always be recover by means of a proper modification of the time-derivative term. With this idea in mind, and following the same principles than in [7], new regularizations have been recently proposed in [11]. Actually, they can be viewed as a generalisation of the regularization methods proposed in [7] where Galilean invariance is partially recovered by means of a modification of the diffusive term. Shortly, by imposing all the symmetries and conservation properties of the original convective operator, $\mathcal{C}(\underline{u},\underline{u})$, and cancelling the second-order terms leads to the following one-parameter fourth-order regularization

$$\mathcal{C}_4^\gamma(\underline{u},\underline{v}) = \frac{1}{2}((\mathcal{C}_4+\mathcal{C}_6)+\gamma(\mathcal{C}_4-\mathcal{C}_6))(\underline{u},\underline{v}). \quad (4)$$

Notice that for $\gamma = 1$ and $\gamma = -1$, $\mathcal{C}_4^\gamma$ becomes respectively the $\mathcal{C}_4$ and $\mathcal{C}_6$ approximations proposed in [7],

$$\mathcal{C}_4(\underline{u},\underline{v}) = \mathcal{C}(\overline{\underline{u}},\overline{\underline{v}}) + \overline{\mathcal{C}(\overline{\underline{u}},\underline{v}')} + \overline{\mathcal{C}(\underline{u}',\overline{\underline{v}})},$$
$$\mathcal{C}_6(\underline{u},\underline{v}) = \mathcal{C}(\overline{\underline{u}},\overline{\underline{v}}) + \mathcal{C}(\overline{\underline{u}},\underline{v}') + \mathcal{C}(\underline{u}',\overline{\underline{v}}) + \overline{\mathcal{C}(\underline{u}',\underline{v}')}.$$

Then, to restore the Galilean invariance we need to replace the time-derivative, $\partial_t\underline{u}_\epsilon$, by the following fourth-order approximation:

$$(\partial_t)_4^\gamma \underline{u}_\epsilon = \partial_t(\underline{u}_\epsilon - 1/2(1+\gamma)\underline{u}_\epsilon'') = \mathcal{G}_4^\gamma(\partial_t\underline{u}_\epsilon), \quad (5)$$

where $\mathcal{G}_4^\gamma(\phi) = \phi - 1/2(1+\gamma)\phi''$. In this case, the new set of PDEs reads

$$(\partial_t)_4^\gamma \underline{u}_\epsilon + \mathcal{C}_4^\gamma(\underline{u}_\epsilon,\underline{u}_\epsilon) = \mathcal{D}\underline{u}_\epsilon - \nabla p_\epsilon. \quad (6)$$

Therefore, Galilean invariance might be restored by simply setting $\gamma = -1$. However, this approach suffers from several practical drawbacks [11]. Another possibility relies on modifying appropriately other terms, *i.e.* viscous dissipation. The energy equation for (6) becomes

$$\frac{d}{dt}(|\underline{u}_\epsilon|^2 - 1/2(1+\gamma)|\underline{u}_\epsilon'|^2) = (\underline{u}_\epsilon, \mathcal{D}\underline{u}_\epsilon) < 0, \quad (7)$$

provided that the filter is self-adjoint, $|\underline{u}|^2 = (\underline{u},\underline{u})$, and the innerproduct of functions is defined in the usual way: $(a,b) = \int_\Omega a\cdot b d\Omega$. Therefore, the modification of time-derivative term (5) constitutes a dissipation model. Recalling that $(\mathcal{G}_4^\gamma)^{-1}(\phi) \approx 2\phi - \mathcal{G}_4^\gamma(\phi) + \mathcal{O}(\epsilon^6)$, we can obtain an energetically almost equivalent set of equations by modifying the viscous diffusive term

$$\partial_t\underline{u}_\epsilon + \mathcal{C}_4^\gamma(\underline{u}_\epsilon,\underline{u}_\epsilon) = \mathcal{D}_4^\gamma\underline{u}_\epsilon - \nabla p_\epsilon, \quad (8)$$

where the linear operator $\mathcal{D}_4^\gamma\underline{u}$ is given by

$$\mathcal{D}_4^\gamma \underline{u} = \mathcal{D}\underline{u} + 1/2(1+\gamma)(\mathcal{D}\underline{u}')'. \quad (9)$$

In this way, we are reinforcing the dissipation by means of a hyperviscosity term. As expected, this basically acts at the tail of the energy spectrum and therefore helps to mitigate the two above-mentioned drawbacks of the original $\mathcal{C}_4$ regularization. Then, to apply the method two parameters still need to be determined; namely, the local filter length, ϵ, and the constant γ. These two issues are addressed in the forthcoming sections 3 and 4, respectively.

3. RESTRAINING THE PRODUCTION OF SMALL SCALES OF MOTION

3.1. *Interscale interactions*

To study the interscale interactions in more detail, we continue in the spectral space. The spectral representation of the convective term in the NS equations is given by

$$\mathcal{C}(\underline{u}, \underline{u})_k = i\Pi(k) \sum_{p+q=k} \hat{u}_p q \hat{u}_q, \quad (10)$$

where $\Pi(k) = I - kk^T/|k|^2$ denotes the projector onto divergence-free velocity fields in the spectral space. Taking the Fourier transform of (8), we obtain the evolution of each Fourier-mode $\hat{u}_k(t)$ of $u_\epsilon(t)$ for the $\{\mathcal{CD}\}_4$ approximation[1]

$$\left(\frac{d}{dt} + h_4^\gamma(\widehat{g}_k)\nu|k|^2\right)\hat{u}_k + i\Pi(k) \sum_{p+q=k} f_4^\gamma\left(\widehat{g}_k, \widehat{g}_p, \widehat{g}_q\right) \hat{u}_p q \hat{v}_q = F_k, \quad (11)$$

where $\widehat{g}_k$ denotes the k-th Fourier-mode of the kernel of the convolution filter, *i.e.*, $\overline{\hat{u}}_k = \widehat{g}_k \hat{u}_k$. The mode $\hat{u}_k$ interacts only with those modes whose wavevectors p and q form a triangle with the vector k. Thus, compared with (10), every triad interaction is multiplied by

$$f_4^\gamma(\widehat{g}_k, \widehat{g}_p, \widehat{g}_q) = (\tilde{\gamma} f_4 + (1-\tilde{\gamma}) f_6)(\widehat{g}_k, \widehat{g}_p, \widehat{g}_q) \quad (12)$$

where $\tilde{\gamma} = 1/2(1+\gamma)$ and f_4 and f_6 are given by

$$f_4(\widehat{g}_k, \widehat{g}_p, \widehat{g}_q) = \widehat{g}_k(\widehat{g}_p + \widehat{g}_q) + \widehat{g}_p\widehat{g}_q - 2\widehat{g}_k\widehat{g}_p\widehat{g}_q \quad (13)$$

$$f_6(\widehat{g}_k, \widehat{g}_p, \widehat{g}_q) = 1 - (1-\widehat{g}_k)(1-\widehat{g}_p)(1-\widehat{g}_q), \quad (14)$$

where $0 < f_n \leq 1$ $(n = 4, 6)$. On the other hand, the k-th Fourier mode of the diffusive term is multiplied by

$$h_4^\gamma(\widehat{g}_k) = 1 + \tilde{\gamma}(1-\widehat{g}_k)^2 \quad (15)$$

where $h_4 \geq 1$. Moreover, since for a generic symmetric convolution filter (see [12], for instance), $\widehat{g}_k = 1 - \alpha^2|k|^2 + \mathcal{O}(\alpha^4)$ with $\alpha^2 = \epsilon^2/24$, the functions f_4^γ and h_4^γ can be approximated by $f_4^\gamma \approx 1 - 1/2(1+\gamma)\alpha^4(|k|^2|p|^2 + |k|^2|q|^2 + |p|^2|q|^2)$ and $h_4 \approx 1 + 1/2(1+\gamma)\alpha^4|k|^4$, respectively. Therefore, the interactions between large scales of motion ($\epsilon|k| < 1$) approximate the NS dynamics up to $\mathcal{O}(\epsilon^4)$. Hence, the triadic interactions between large scales are only slightly altered. All interactions involving longer wavevectors (smaller scales of motion) are reduced. The amount by which the interactions between the wavevector-triple (k, p, q) are lessened depends on the length of the legs of the triangle $k = p + q$. For example, all triadic interactions for which at least two legs are (much) longer than $1/\epsilon$ are (strongly) attenuated; whereas, interactions for which at least two legs are (much) shorter than $1/\epsilon$ are reduced to a small degree only.

[1] Hereafter, for simplicity, the subindex ϵ is dropped.

3.2. *Stopping the vortex-stretching mechanism*

Taking the curl of Eq.(8) leads to

$$\partial_t \underline{\omega} + \mathcal{C}_4^\gamma(\underline{u}, \underline{\omega}) = \mathcal{C}_4^\gamma(\underline{\omega}, \underline{u}) + \mathcal{D}_4^\gamma \underline{\omega}. \quad (16)$$

This equation resembles the vorticity equation that results from the NS equations: the only difference is that $\mathcal{C}$ and $\mathcal{D}$ are replaced by their regularizations $\mathcal{C}_4^\gamma$ and $\mathcal{D}_4^\gamma$, respectively. If it happens that the vortex stretching term $\mathcal{C}_4^\gamma(\underline{\omega}, \underline{u})$ in Eq.(16) is so strong that the dissipative term $\mathcal{D}_4^\gamma \underline{\omega}$ cannot prevent the intensification of vorticity, smaller vortical structures are produced. Left-multiplying the vorticity transport Eq.(16) by $\underline{\omega}$, we can obtain the evolution of $|\underline{\omega}|^2$. In this way, the vortex-stretching and dissipation term contributions to $\partial_t|\underline{\omega}|^2$ result

$$\underline{\omega} \cdot \mathcal{C}_4^\gamma(\underline{\omega}, \underline{u}) \quad \text{and} \quad \underline{\omega} \cdot \mathcal{D}_4^\gamma \underline{\omega}, \quad (17)$$

respectively. In order to prevent local intensification of vorticity, dissipation must dominate the vortex-stretching term contribution at the smallest grid scale, $k_c = \pi/h$. In spectral space, this requirement leads to the following inequality

$$\frac{\hat{\omega}_{k_c} \cdot \mathcal{C}_4^\gamma(\underline{\omega}, \underline{u})_{k_c}^* + \mathcal{C}_4^\gamma(\underline{\omega}, \underline{u})_{k_c} \cdot \hat{\omega}_{k_c}^*}{2\hat{\omega}_{k_c} \cdot \hat{\omega}_{k_c}^*} \leq h_4^\gamma\left(\hat{g}_k\right) \nu k_c^2, \quad (18)$$

where the vortex-stretching term, $\mathcal{C}_4^\gamma(\underline{\omega}, \underline{u})_{k_c}$, is given by

$$\mathcal{C}_4^\gamma(\underline{\omega}, \underline{u})_{k_c} = \sum_{p+q=k_c} f_4^\gamma\left(\widehat{g}_{k_c}, \widehat{g}_p, \widehat{g}_q\right) \hat{\omega}_p i q \hat{u}_q. \quad (19)$$

Note that $f_4^\gamma\left(\widehat{g}_{k_c}, \widehat{g}_p, \widehat{g}_q\right)$ depends on the filter length ϵ and, in general, on the wavevectors p and $q = k_c - p$. This makes very difficult to control the damping effect because f_4^γ cannot be taken out of the summation in (19). To avoid this, filters should be constructed from the requirement that the damping effect of all the triadic interactions at the smallest scale must be virtually independent of the interacting pairs, *i.e.*

$$f_4^\gamma(\widehat{g}_{k_c}, \widehat{g}_p, \widehat{g}_q) \approx f_4^\gamma(\widehat{g}_{k_c}). \quad (20)$$

This is a crucial property to control the subtle balance between convection and diffusion in order to stop the vortex-stretching mechanism. This point was addressed in detail by [8]. Then, the overall damping effect at the smallest grid scale, $H_4(\widehat{g}_{k_c}) = f_4^\gamma(\widehat{g}_{k_c})/h_4^\gamma(\widehat{g}_{k_c})$, follows straightforwardly

$$H_4(\widehat{g}_{k_c}) = \frac{2\nu k_c^2 \hat{\omega}_{k_c} \cdot \hat{\omega}_{k_c}^*}{\hat{\omega}_{k_c} \cdot \mathcal{C}(\omega, u)_{k_c}^* + \mathcal{C}(\omega, u)_{k_c} \cdot \hat{\omega}_{k_c}^*}, \quad (21)$$

with the condition that $0 < H_4(\widehat{g}_{k_c}) \leq 1$.

3.3. *From spectral to physical space*

In the previous subsection we applied our analysis on a spectral space. However, the method needs to be applied on a physical domain in $\mathbb{R}^3$. To that end, here we propose to express the overall damping effect, $H_4(\widehat{g}_{k_c})$, as a function of the invariants of the strain tensor, $S(\underline{u}) = 1/2(\nabla \underline{u} + \nabla \underline{u}^T)$. Recalling that the velocity field, u, is solenoidal ($\nabla \cdot \underline{u} = 0$); $tr(S) = 0$ and the characteristic equation of S reads

$$\lambda^3 + Q\lambda + R = 0, \quad (22)$$

where $R = -1/3tr(S^3) = -det(S) = -\lambda_1\lambda_2\lambda_3$ and $Q = -1/2tr(S^2) = -1/2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)$ are the invariants of S, respectively. We order the eigenvalues of S by $\lambda_1 \leq \lambda_2 \leq \lambda_3$. Let us now consider an arbitrary part of the flow domain Ω with periodic boundary conditions. The innerproduct is defined in the usual way: $(a, b) = \int_\Omega a \cdot b d\Omega$. Then, taking the L^2 innerproduct of (1) with $-\Delta \underline{u}$ leads to the enstrophy equation

$$\frac{1}{2}\frac{d}{dt}|\underline{\omega}|^2 = (\underline{\omega}, \mathcal{C}(\underline{\omega}, \underline{u})) - \nu\left(\nabla\underline{\omega}, \nabla\underline{\omega}\right), \quad (23)$$

where $|\underline{\omega}|^2 = (\underline{\omega}, \underline{\omega})$ and the convective term contribution $(\mathcal{C}(\underline{u}, \underline{\omega}), \underline{\omega}) = 0$ vanishes because of the skew-symmetry of the convective operator. Using the results obtained by [13] and following the same arguments than in [14], it can be shown that the vortex-stretching term can be expressed in terms of the invariant R of $S(u)$

$$(\underline{\omega}, \mathcal{C}(\underline{\omega}, u)) = -\frac{4}{3}\int_\Omega tr(S^3) d\Omega = 4\int_\Omega R d\Omega, \quad (24)$$

and the $L^2(\Omega)$-norm of $\underline{\omega}$ in terms of the invariant Q

$$|\underline{\omega}|^2 = -4\int_\Omega Q d\Omega. \quad (25)$$

Then, the diffusive term can be bounded by

$$(\nabla\underline{\omega}, \nabla\underline{\omega}) = -\left(\underline{\omega}, \Delta\underline{\omega}\right) \leq -\lambda_\Delta\left(\underline{\omega}, \underline{\omega}\right), \quad (26)$$

where $\lambda_\Delta < 0$ is the largest (smallest in absolute value) non-zero eigenvalue of the Laplacian operator Δ on Ω. If we now consider that the domain Ω is a periodic box of volume h, then $\lambda_\Delta = -(\pi/h)^2$. In a numerical simulation h would be related with the local grid size. Then, to prevent a local intensification of vorticity, *i.e.* $|\underline{\omega}|_t \leq 0$, the following inequality must be hold

$$H_4(\widehat{g}_{k_c})\frac{(\underline{\omega}, S\underline{\omega})}{(\underline{\omega}, \underline{\omega})} \leq -\nu\lambda_\Delta, \quad (27)$$

where, in this case, $k_c = \pi/h$. This inequality is the analog to Eq.(21) in physical space. Rayleigh's principle states that

$$\max_{\underline{\omega}\neq 0}\frac{(\underline{\omega}, S\underline{\omega})}{(\underline{\omega}, \underline{\omega})} = \lambda_3, \quad (28)$$

and therefore gives a lower bound for the damping function, $H_4(\widehat{g}_{k_c}) \leq \nu(-\lambda_\Delta/\lambda_3)$. This was the approach consider in our previous work [9]. However, the maximum value is attained only if $\underline{\omega}$ is aligned with the eigenvector corresponding to λ_3, and therefore the convective terms tends to be over-damped. This becomes especially relevant near the walls. In order to overcome this drawback here we propose to rewrite the inequality (27) in terms of the invariants Q and R. From Eqs. (24)-(27) we deduce

$$H_4(\widehat{g}_{k_c}) \leq \nu\lambda_\Delta\frac{Q}{R^+}, \quad (29)$$

where $R^+ = \max\{R, 0\}$ and the overall damping factor $0 < H_4 \leq 1$. Thus, a proper definition of the overall damping factor at the smallest grid scale is given by

$$H_4(\widehat{g}_{k_c}) = \min\left\{\nu\lambda_\Delta\frac{Q}{R^+}, 1\right\}. \quad (30)$$

Notice that the invariant Q is always negative whereas R can be either positive or negative. In terms of the Reynolds number, the quotient of R and Q scales like $R/Q \propto (Re^{3/2})/Re = Re^{1/2}$. Then, recalling that $\lambda_\Delta \propto h^{-2}$, it yields to $H_4(\widehat{g}_{k_c}) \propto h^{-2}Re^{-1}Q/R \propto h^{-2}Re^{-3/2}$. Therefore, we obtain $H_4(\widehat{g}_{k_c}) \rightarrow 1$ if $h \propto Re^{-3/4}$. This shows that the model switches off when h approaches to the smallest scale in a turbulent flow. Another interesting feature of the model is that it automatically switches off ($R \rightarrow 0$) for laminar flows (no vortex-stretching) and 2D flows ($\lambda_2 = 0 \rightarrow R = 0$). The near-wall behaviour of the invariants is given by $R \propto y^3$ and $Q \propto y^0$, respectively, where y is the distance to the wall. Consequently, it results into a model that switches off in the wall.

For convenience, let us now define the ratio between λ_2 and λ_3, $\eta = \lambda_2/\lambda_3$. Note that $\lambda_1 \leq 0$ and $\lambda_3 \geq 0$, whereas the middle eigenvalue, λ_2, can be both positive or negative. Actually, the sign of the invariant $R = -\lambda_1\lambda_2\lambda_3$, λ_2 and η are the same. Then, recalling that the strain tensor is traceless ($tr(S) = 0$), *i.e.* $\lambda_1 + \lambda_2 + \lambda_3 = 0$, the first eigenvalue can also be written in terms of λ_3 and η: $\lambda_1 = -(1+\eta)\lambda_3$. Then, the ratio Q/R results into $(-Q/R)^{-1} = \lambda_{QR} = (1+\eta)\eta/(\eta^2+\eta+1)\lambda_3$. Here, λ_{QR} can be viewed as the rate of amplification of vorticity at the smallest grid scale. Then, assuming that $|\eta| \ll 1$, $\lambda_{QR} \approx \eta\lambda_3 = \lambda_2$ and therefore it is consistent with the preferential vorticity alignment with the intermediate eigenvector (see the work by [15] and references therein).

4. ON THE DETERMINATION OF γ

A criterion to determine the local filter length, ϵ, has been presented in the previous section. Then, the only parameter that still needs to be determined in Eq.(9) is the constant γ. As stated before, by simply setting $\gamma = -1$, the $\mathcal{C}_4^\gamma$ becomes the sixth-order accurate $\mathcal{C}_6$ regularization and the Galilean invariance is restored without introducing any additional modification in the dissipation term. However, the $\mathcal{C}_6$ approximation itself suffers from a fundamental drawback. Namely, the overall method relies on the fact that Eq.(20) is approximately satisfied and therefore, the damping factor $\mathcal{C}_4^\gamma$ can be taken out of the summation in (19). This is not the case of f_6: notice that since $\widehat{g}_0 = 1$, $f_6(\widehat{g}_{k_c}, \widehat{g}_{k_c}, \widehat{g}_0) = 1$ irrespectively of the value of $\widehat{g}_{k_c}$.

At this point, the 'optimal' value of γ could be determined by means of a trial-and-error numerical procedure. Alternatively, the constant γ can be obtained by assuming that the smallest grid scale $k_c = \pi/h$ lies within the inertial range for a classical Kolmogorov energy spectrum $E(k) = C_K \varepsilon^{2/3} k^{-5/3}$. In such a case, and recalling that $\widehat{g}_k = 1 - \alpha^2|k|^2 + \mathcal{O}(\alpha^4)$, the total dissipation for $k_T \le k \le k_c$ can be approximated by the contribution of the following two terms

$$\mathcal{D}_\nu \equiv \nu \int_{k_T}^{k_c} k^2 E(k) dk, \quad (31)$$

$$\mathcal{D}_\nu^{''} \equiv \nu \int_{k_T}^{k_c} k^4 \alpha^4 E(k) dk, \quad (32)$$

where $\mathcal{D}_\nu$ is the physical viscous dissipation and $\mathcal{D}_\nu^{''}$ is the additional dissipation introduced by the hyperviscosity term, $(\mathcal{D}u')'$. Hence, integration for a Kolmogorov energy spectrum, the total dissipation within the range $k_T \le k \le k_c$ is given by

$$\mathcal{D}_\nu + \tilde{\gamma}\mathcal{D}_\nu^{''} = \frac{3\nu}{16} C_K \varepsilon^{2/3} \left\{ \left(4 + \tilde{\gamma}\alpha^4 k_c^4\right) k_c^{4/3} - \left(4 + \tilde{\gamma}\alpha^4 k_T^4\right) k_T^{4/3} \right\}, \quad (33)$$

where $\tilde{\gamma} = 1/2(1+\gamma)$ has been introduced here for the sake of simplicity. At the tail of the spectrum the following

$$\tilde{H}_4 \approx \frac{\mathcal{D}_\nu + \tilde{\gamma}\mathcal{D}_\nu^{''}}{\varepsilon}, \quad (34)$$

represents the ratio between the total dissipation and the energy transferred from scales larger than k_T to the tail of the spectrum. Let us assume that $\tilde{H}_4 = \mathcal{O}(H_4(\widehat{g}_{k_c}))$ where the overall damping at the smallest grid scale, $H_4(\widehat{g}_{k_c})$, is given by Eq.(30). However, at this point it is more suitable to express it in terms of the invariant Q. To do so, we simply notice that the three roots of the characteristic cubic equation (22) can be computed analytically:

$$\lambda_i = -|S|\sqrt{\frac{1}{3}} \cos\left(\frac{\theta}{3} - \frac{2\pi(i-1)}{3}\right) \qquad i = 1, 2, 3, \quad (35)$$

where $|S| = \sqrt{-4Q}$ and the angle θ is given by

$$\theta = \arccos\left(\frac{1}{2}R \Big/ \sqrt{\left(-\frac{1}{3}Q\right)^3} \right). \quad (36)$$

Since S is symmetric, the eigenvalues must be real-valued, $\lambda_i \in \mathbb{R}$, and, therefore, the invariants Q and R must satisfy

$$27R^2 + 4Q^3 \le 0 \quad (37)$$

Hence, $\theta \in [0, \pi]$ and the ratio $R^+/-Q$ can be bounded in terms of the invariant Q

$$0 \le \left(\frac{R^+}{-Q}\right) \le \sqrt{\frac{4}{27}}\sqrt{-Q}, \quad (38)$$

then, plugging this into Eq.(30) leads to

$$1 \ge H_4(\widehat{g}_\pi) \ge -\sqrt{\frac{27}{4}} \frac{\lambda_\Delta \nu}{\sqrt{-Q}}. \quad (39)$$

On the other hand, for a classical Kolmogorov energy spectrum, the ensemble averaged invariant Q is approximately given by

$$< Q > = -\frac{1}{4}\int_0^{k_c} k^2 E(k) dk \approx -\frac{3}{16} C_K \varepsilon^{2/3} k_c^{4/3}. \quad (40)$$

Finally, combining Eqs.(39) and (40), the energy balance given by Eq.(34) results

$$\frac{-12\lambda_\Delta \nu \varepsilon}{\sqrt{C_K \varepsilon^{2/3} k_c^{4/3}}} \lesssim \mathcal{D}_\nu + \tilde{\gamma}\mathcal{D}_\nu^{''}. \quad (41)$$

Then, plugging Eq.(33) and rearranging terms leads to

$$1 \lesssim \frac{-C_K^{3/2} k_c^2}{32\lambda_\Delta} \left\{ \left(4 + \tilde{\gamma}\alpha^4 k_c^4\right) - \left(4 + \tilde{\gamma}\alpha^4 k_T^4\right) \left(\frac{k_T}{k_c}\right)^{4/3} \right\} k_c^2. \quad (42)$$

Recalling that $\lambda_\Delta = -(\pi/h)^2$, $k_c = \pi/h$ and $\alpha \approx k_c^{-1}$, the previous expression simplifies

$$1 \lesssim \frac{C_K^{3/2}}{32} \left\{ (4+\tilde{\gamma}) - \left(4 + \tilde{\gamma} \left(\frac{k_T}{k_c} \right)^4 \right) \left(\frac{k_T}{k_c} \right)^{4/3} \right\}. \quad (43)$$

Since $k_c > k_T$, we can consider than $4 \gg \tilde{\gamma}(k_T/k_c)^4$, we get a proper bound for $\tilde{\gamma}$,

$$\tilde{\gamma} \gtrsim 4 \left\{ 8C_K^{-3/2} - \left(1 - \left(\frac{k_T}{k_c} \right)^{4/3} \right) \right\}$$
$$\approx 4 \left(8C_K^{-3/2} - 1 \right). \quad (44)$$

Hence, for a Kolmogorov constant of $C_K \approx 1.58$ [16] it leads to a lower limit of $\tilde{\gamma} \approx 12.1$ ($\gamma \approx 23.2$). At this point, numerical experiments are needed to confirm this estimation; also to provide more accurate values. To do so, simulation of isotropic turbulence at $Re_\lambda \approx 72$ has been chosen as a first test-case. The code is pseudo-spectral and uses the $3/2$ dealiasing rule. Filters proposed in [8] are applied in spectral space. The total amount of energy in the first two modes is kept constant following the approach proposed in [17]. Figure 1 displays the results for a box size of 16^3 for different values of $\tilde{\gamma}$ from 0 up to 30. As expected, the original hump displayed for $\tilde{\gamma} = 0$ attenuates for increasing values of $\tilde{\gamma}$. Moreover, the lower bound for $\tilde{\gamma}$ given by Eq.(44) is in a fairly good agreement with these numerical tests. Even more important, for $\tilde{\gamma}$ bigger than a certain value, the results are virtually independent on the value of $\tilde{\gamma}$.

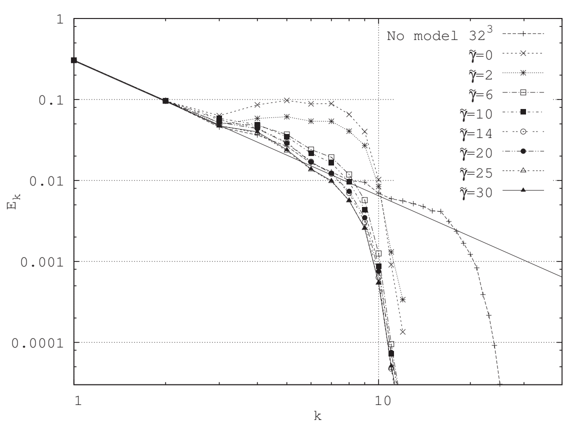

Figure 1. Energy spectra at $Re_\lambda \approx 72$ for different values of $\tilde{\gamma}$ from 0 up to 30.

Figure 2 displays the results obtained for a box size of 64^3 and $Re_\lambda \approx 202$. In this case, the energy-containing and dissipative scales are clearly separated by an inertial range. Again, the hump at the tail of the spectrum attenuates for increasing values of $\tilde{\gamma}$. More importantly, the inertial range is well predicted only for those cases with $\tilde{\gamma} \gtrsim 14$, in relatively good agreement with the lower bound given by Eq.(44). These are still preliminary results and more simulations are needed to confirm these conclusions.

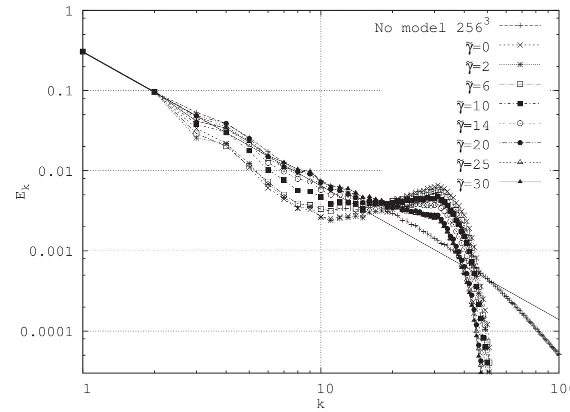

Figure 2. Energy spectra at $Re_\lambda \approx 202$ for different values of $\tilde{\gamma}$ from 0 up to 30.

5. TOWARDS THE SIMULATION OF WIND FARM BOUNDARY LAYERS

The final goal of this work is to perform numerical simulation of atmospheric boundary layer through wind farms. This challenging application can only be achieved by combining the most advanced numerical techniques with reliable turbulence models. Regarding the former, regularization of the Navier-Stokes equations have already been successfully tested for several configurations (see [7, 9], for instance). Furthermore, it is expected that the proposed modifications will result into relevant improvements. With regard to the numerical techniques, the above-described pseudo-spectral algorithm is the most appropriate choice for this type of configurations. Preliminary results for a three-dimensional homogeneous isotropic turbulence have been presented in the previous section. The next test-case is a turbulent channel flow at $Re_\tau = 180$. First results are displayed in Figure 3 together with the classical results obtained by J.Kim *et al.* [18]. This configuration was successfully tested for a $\mathcal{C}_4$ regularization in [7]. Those results also showed an additional hump at the tail of the spectrum. It is expected that the additional hyperviscosity term will improve the results. This will be presented and discussed during the conference.

6. CONCLUDING REMARKS AND FUTURE RESEARCH

Since DNS simulations are not feasible for real-world applications the $\{\mathcal{CD}\}_4^\gamma$-regularization of the NS equations has been proposed as a simulation shortcut: the convective and diffusive operators in the NS equations (1) are replaced by the $\mathcal{O}(\epsilon^4)$-accurate smooth approximation given by Eq.(4) and Eq.(9), respectively. The symmetries and conservation properties of the original convective term are exactly

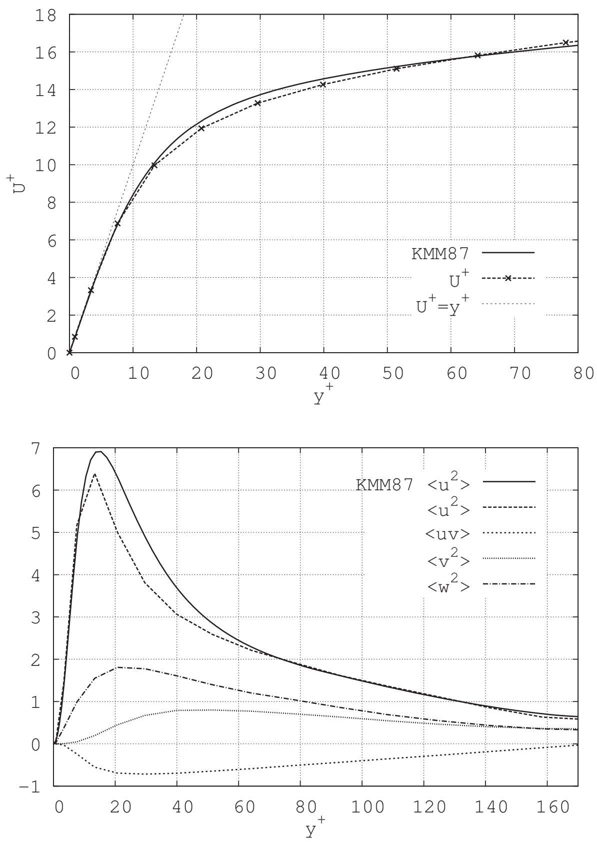

Figure 3. Results for a turbulent channel flow at $Re_\tau = 180$ and grid size $32 \times 32 \times 32$. Top: mean streamwise velocity profile. Bottom: some turbulent statistics. Reference data corresponds with the results published in [18].

preserved. Doing so, the production of smaller and smaller scales of motion is restrained in an unconditionally stable manner. In this way, the new set of equations is dynamically less complex than the original NS equations, and therefore more amenable to be numerically solved. The only additional ingredient is a self-adjoint linear filter whose local filter length is determined from the requirement that vortex-stretching must be stopped at the scale set by the grid. This can be easily satisfied in spectral space via Eq.(21) provided that discrete filter satisfies Eq.(20), *i.e.* the triadic interactions at the smallest scale are virtually independent of the interacting pairs. This was addressed in detail by [8]. However, in physical space it becomes more cumbersome. To circumvent this, here a criterion based on the two invariants, R and Q, of the local strain tensor is used. Doing so, the expected behaviour of a turbulence model is achieved: it switches off (*i.e.* $H_4 = 1$) for laminar flows (no vortex-stretching), 2D flows ($R = 0$) and near the walls.

In the present paper, the parameter γ of Eq.(8) is approximately bounded by assuming a Kolmorogov energy spectrum. This has been addressed in Section 4 where the following bound has been determined:

$$\tilde{\gamma} \gtrsim 4\left(8C_K^{-3/2} - 1\right), \qquad (45)$$

where $\tilde{\gamma} = 1/2(1+\gamma)$ and C_K is the Kolmogorov constant. Preliminary simulations for homogeneous isotropic turbulence seems to confirm the adequacy of the bound given by Eq.(45). In this way, the proposed method constitutes a parameter-free turbulence model suitable for complex geometries and flows. Apart from homogeneous isotropic turbulence numerical results evaluating the performance of the $\{\mathcal{CD}\}_4^\gamma$ method for wall-bounded configurations will be presented during the conference. Namely, a turbulent channel flow and a turbulent boundary layer. As a final application, regularization modelling will be applied for large-scale numerical simulation of the atmospheric boundary layer through wind farms. The analysis of these $\{\mathcal{CD}\}_4^\gamma$ regularization models is also part of our future research plans.

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