Classification of Monogenic Invariant Subspaces and Uniparametric Linear Control Systems *

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Abstract

The classification of the invariant subspaces of an endomorphism has been an open problem for a long time, and it is a "wild" problem in the general case. Here we obtain a full classification for the monogenic ones. Some applications are derived: in particular, canonical forms for uniparametric linear control systems, non necessarily controllable, with regard to linear changes of state variables.

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1 Introduction

In this paper we tackle the classification of the invariant subspaces of an endomorphism in the particular case that they are "monogenic", that is to say, spanned by one vector and its successive images. For the general case, [9] shows that it is a so called "wild" problem when the degree of the minimal polynomial is greater than 6.

Here, the classification of monogenic subspaces is fully solved (3.8) by means of the "marked" and "perturbation" indices because they determine a canonical matrix for the endomorphism (3.7). Moreover, we list all possible indices (3.14) and we compute them by means of ranks (3.15).

The names of the indices highlight that any monogenic subspace appears as a perturbation of a marked one (those having a Jordan basis extendible to a Jordan basis of the whole space). More

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precisely, any monogenic subspace appears in the miniversal deformation in [3] of a marked one (3.10). This fact confirms the relevance of this kind of subspaces, as announced in [7].

The marked and perturbation indices are defined (3.2) by means of the L-R sequence associated to each invariant subspace in [1]. Indeed, the key tool in this paper is the geometrical approach there to the Carlson problem. We recall that it asks for conditions to ensure when three given Segre characteristics can be realized as the ones corresponding to an endomorphism , its restriction to an invariant subspace and the one induced in the quotient space. The L-R sequences give an implicit answer to this problem ([8], [1]), but they do not characterize the equivalence class of a general invariant subspace, whereas we show that it is so for monogenic ones.

The above results can be interpreted as the simultaneous classification of a square matrix and a vector with regard to changes of bases (4.2). In other words, given a square matrix we find canonical coordinates for each vector among those corresponding to Jordan bases (4.3). Indeed, this interpretation allows an easy presentation of our results (4.6).

Moreover, some applications follow from this interpretation: we obtain canonical forms (5.1, 5.3) for linear control uniparametric systems (non necessarily controllable) with regard to changes of basis in the state space; we improve the reduced forms in [4] for bimodal piecewise dynamical systems (5.5). Concerning the former application, we point out that different canonical forms have been obtained for the controllable case [10]. Here we present two canonical forms for uncontrollable uniparametric systems: in (5.1) the state matrix is the Jordan form and the control column as a J-vector, determined by the marked and perturbation indices; in (5.3) both matrices are in control form.

The paper is organized as follows. In section 2 we recall the basic definitions and results concerning invariant subspaces which will be used in the sequel. In particular, the Carlson problem, the L-R sequences and the techniques in [1] based in the double Jordan filtration which will be the key tool in our reasonings.

In section 3 we focus in the monogenic case. Firstly, we introduce the marked and perturbation indices for our L-R sequences associated to a monogenic subspace (3.1, 3.2). Next, in (3.5, 3.7) we construct a matrix reduced form which shows that they give a full classification (3.8). Finally, the second matrix reduced form in (3.12) makes easier to compute them (3.15).

In section 4 we reformulate the above result as classifying vectors with regard to a fixed endomorphism (4.3). Section 5 contains the applications to control and bimodal systems (5.1, 5.3, 5.5).

2 Invariant Subspaces

We recall some definitions and results concerning the classification of invariant subspaces.

Definition 2.1 *Let E be a n-dimensional vector space over $\mathbb{C}$, and f an endomorphism. A subspace $V \subset E$ is called* invariant *(or f-invariant) if $f(V) \subset V$. We write (V, f) too.*

Or, equivalently if the matrix of f in any basis of E adapted to V (that is to say, a basis of E obtained by extending one of V) has the form

$A = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}$, *where* $A_1 \in \mathbb{C}^{h\times h}, h = dim(V)$.

Then, A_1 *is the matrix of the restriction* $\hat{f}$ *of* f *to* V *in the corresponding basis of* V*, and* A_2 *the one of the quotient endomorphism* $\tilde{f}$ *of* E/V *in the induced basis of* E/V.

In particular, V *is called* marked *if there is some Jordan basis for* $\hat{f}$ *which can be extended to a Jordan basis for* f*. See [5] for a matricial characterization.*

Definition 2.2 *Two invariant subspaces* (V, f) *and* (V', f') *are called* equivalent *if there is* $\varphi \in Aut(E)$ *such that* $\varphi(V) = V'$ *and* $\varphi \circ f = f' \circ \varphi$.

If A *and* A' *are the matrices of* f *and* f' *in adapted bases to the subspace* V *and* V' *respectively, it is equivalent to the existence of a matrix* S *such that:*

$$S = \begin{pmatrix} S_1 & S_3 \\ 0 & S_2 \end{pmatrix}, \quad A' = S^{-1}AS,$$

where $S_1 \in \mathbb{C}^{h\times h}, h = dim(V) = dim(V')$.

Bearing in mind the decomposition

$$V = \oplus_\lambda (Ker(f - \lambda I)^{m_\lambda} \cap V)$$

where λ runs over the eigenvalues of f, we can restrict ourselves to f being nilpotent.

Moreover, we focus in the *Jordan dense* invariant subspaces, because the generalization is obvious:

Definition 2.3 *An invariant subspace* (V, f) *is called* Jordan dense *if there is not a Jordan Chain* $U = [u, (f - \lambda I)(u), \dots, (f - \lambda I)^n(u)]$ *such that* $U \cap V = \{0\}$.

It is obvious that the equivalence relation in definition 2.2 preserves the Jordan type of the matrices A, A_1 and A_2. We will confirm in a moment that this triple does not characterize the equivalence classes (different classes having the same triple can exist). Previously we remark that these three Jordan types are not independent. The so called Carlson's problem ask for conditions characterizing the compatible triples, that is to say, those which occur for some invariant subspace. Next theorem gives an implicit answer in terms of the existence of a so-called Littlewood-Richardson sequence (L-R sequence).

Definition 2.4 *A* partition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m, 0, \dots)$, *will be any non increasing finite sequence of non negative integers*

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m > 0$$

where $\ell(\alpha) = m$ *is his* length *and* $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_m$ *is his* weight.

Its conjugate partition α^* *is defined by* $\alpha^*_j = \#\{1 \leq i \leq \ell(\alpha) : \alpha_i \geq j\}$.

If f is a nilpotent endomorphism, the *Weyr characteristic* is $\alpha = (\dim Kerf, \dim Kerf^2 - \dim Kerf\,, \dim Kerf^3 - \dim Kerf^2\,, \dots)$ and its conjugate partition is the *Segre characteristic*, formed by the sizes of Jordan blocks.

Theorem 2.5 *[8], [2] Let α, γ, β be three partitions with $|\alpha| = n$, $|\gamma| = d$, $|\beta| = n-d$, $l(\alpha) = m$. The following conditions are equivalent:*

- **(I)** *There are a nilpotent endomorphism $f \in End(E)$ having Weyr characteristic α and a f-invariant subspace V such that the restriction $\hat{f}$ and the quotient $\tilde{f}$ applications have Weyr characteristic γ and β respectively.*
- **(II)** *There is a finite sequence of partitions $\gamma^0, \gamma^1, \dots, \gamma^m$ such that $\gamma^0 = \gamma$, $\gamma^m = \alpha$, and for all $i, j \geq 1$:*
  - *(a)* $|\gamma^j| - |\gamma^{j-1}| = \beta_j$
  - *(b)* $\gamma_i^j \geq \gamma_i^{j-1} \geq \gamma_{i+1}^j$
  - *(c)* $\sum_{\ell \leq i}(\gamma_\ell^{j+1} - \gamma_\ell^j) \leq \sum_{\ell \leq i-1}(\gamma_\ell^j - \gamma_\ell^{j-1})$

  taking $\gamma_0^{j-1} = 0$, $j \geq 1$.

The sequence $\gamma^0, \gamma^1, \dots, \gamma^m$ appearing in (II) is called a Littlewood-Richardson sequence.

The geometric proof of 2.5 in [2] gives an explicit computation of the L-R sequence for an invariant subspace which we recall in lemma 2.7. The construction for the converse $(II) \Rightarrow (I)$ will be used in the next section.

Definition 2.6 *Given a nilpotent endomorphism f and an invariant subspace V, we consider the* double Jordan filtration

$$
\begin{array}{cccccccccccc}
\dots & f(V) & \subset & V & = & V & \subset & f^{-1}(V) & \subset \dots \subset & f^{-m}(V) & = & E \\
 & \| & & \| & & \| & & \| & & \| & & \| \\
\dots & V_m^{-1} & \subset & Ker\widehat{f^m} & = & V_m^0 & \subset & V_m^1 & \subset \dots \subset & V_m^m & = & Kerf^m \\
 & \cup & & & & \cup & & \cup & & \cup & & \\
 & \vdots & & & & \vdots & & \vdots & & \vdots & & \\
 & \cup & & & & \cup & & \cup & & \cup & & \\
\dots & V_i^{-1} & \subset & Ker\widehat{f^i} & = & V_i^0 & \subset & V_i^1 & \subset \dots \subset & V_i^m & = & Kerf^i \\
 & \cup & & & & \cup & & \cup & & \cup & & \\
 & \vdots & & & & \vdots & & \vdots & & \vdots & & \\
 & \cup & & & & \cup & & \cup & & \cup & & \\
\dots & V_1^{-1} & \subset & Ker\widehat{f} & = & V_1^0 & \subset & V_1^1 & \subset \dots \subset & V_1^m & = & Kerf
\end{array}
$$

defined by $V_i^j \doteq Ker\ f^i \cap f^{-j}(V)$, $1 \leq i \leq m$, $|j| \leq m$.

Notice that $V_i^j = V_i^i$ if $i \leq j$, $V_i^{-j} = f^j(V_{j+i})$ if $j > 0$ and $V_i^{j-1} \cap V_{i-1}^j = V_{i-1}^{j-1}$.

Lemma 2.7 *[2] Let $f \in End(E)$ be nilpotent, $V \subset E$ be an invariant subspace and (α, γ, β) as in theorem 2.5. Notice that with the notation in 2.5:*

- $\alpha_i = \dim(V_i^m) - \dim(V_{i-1}^m), 1 \leq i \leq m$
- $\gamma_i = \dim(V_i^0) - \dim(V_{i-1}^0), 1 \leq i \leq m$
- $\beta_j = \dim(V_m^j) - \dim(V_m^{j-1}), 1 \leq j \leq m.$

Then the L-R sequence in theorem 2.5(II) is given by

$$\gamma_i^j = \dim(V_i^j) - \dim(V_{i-1}^j), 1 \leq i \leq m, 0 \leq j \leq m.$$

That is to say, the partitions $\gamma^0 = \gamma, \gamma^1, \ldots, \gamma^m = \alpha$ are the Weyr characteristics of the restriction of f to the invariant subspaces $V, f^{-1}(V), \ldots, f^{-m}(V) = E$ respectively.

Hence, not only the triple (α, γ, β) but also the L-R sequence in theorem 2.5(II) are preserved by the equivalence relation 2.2. However, the following examples show that they do not characterize the equivalent classes.

Example 2.8 *A compatible triple (α, γ, β) as in theorem 2.5(I) can be realized by different L-R sequences. For example, given $E = [e_1, e_2, \ldots, e_6]$ and $V = [e_4, e_5, e_6]$, the compatible triple $\gamma = (2, 1)$, $\alpha = (3, 2, 1)$, $\beta = (2, 1)$, could be realized by the L-R sequences $\gamma^1 = (3, 2)$ or $\gamma^1 = (3, 1, 1)$, corresponding to the endomorphisms $e_1 \to e_2 \to e_4 \to 0$, $e_3 \to 0$, $e_5 \to e_6 \to 0$ and $e_1 \to e_2 \to 0$, $e_3 \to e_4 \to e_5 \to 0$, $\to e_6 \to 0$ respectively.*

Example 2.9 *A L-R sequence can be realized by non-equivalent invariant subspaces. For example, $\gamma = (2, 1), \gamma^1 = (3, 2), \alpha = (3, 2, 1)$ above corresponds also to the endomorphism $e_1 \to e_3 + e_5$, $e_2 \to e_4 \to 0$, $e_3 \to 0$, $e_5 \to e_6 \to 0$. V is marked with regard to the endomorphism in example 2.8, but it is not for the one here.*

In the next section one shows that the L-R sequences characterize the equivalent classes in the particular case of V being spanned by a Jordan chain, that is to say, $\gamma = (1, 1, \ldots, 1)$. Moreover, the possible L-R sequences for a given α are easily characterized and computed, so that the β partitions compatible with α, γ are also easily obtained.

3 The monogenic case

From now on, we restrict ourselves to f-invariant subspaces $V \subset E$ spanned by only one vector u and their images, that is to say:

$$V = [u, f(u), f^2(u), \ldots, f^{n-1}(u)]$$

We will show that for this class of subspaces the equivalence classes are determined by its L-R sequence (see example 2.8). Moreover we prove that the L-R sequences are easily described in terms of the so-called "marked" and "perturbation" indices.

Firstly, we consider f being nilpotent. Then, if $\dim(V) = h$, we have $V = [u, f(u), f^2(u), \ldots, f^{h-1}(u)]$ or equivalently

$$\gamma = (1, \ldots, 1, 0, \ldots, 0), |\gamma| = h = \dim(V).$$

The following proposition allows to define the "marked" and "perturbation" indices (these names will be justified in remark 3.10).

From now on, given two integers $p < q$ we denote $[p : q] \doteq \{i \in \mathbb{Z} : p \leq i \leq q\}$ and N_p will be a $p \times p$ nilpotent square matrix with ones in the below-diagonal and zeros in the rest of entries.

Proposition 3.1 *Let $f \in End(E)$ be nilpotent, $V \subset E$ be an f-invariant monogenic subspace and $\gamma = \gamma^0, \gamma^1, \ldots, \gamma^m = \alpha$ be its L-R sequence (see lemma 2.7). Then:*

1. *There is an integer $0 \leq s \leq m - h$ such that $\gamma_i^{i-1} = 1$ for $1 < i \leq h + s$ and $\gamma_i^{i-1} = 0$ for $i > h + s$.*
2. *For $i \in [h : h+s]$ we define $j(i)$ as the only integer $0 \leq j(i) < i$ such that $\gamma_i^{j(i)} - \gamma_i^{j(i)-1} = 1$. Then, $j(i) < j(i+1)$.*

Clearly, these integers determine the L-R sequence.

Proof.

If $\gamma_i = 1$ for $1 \leq i \leq h$, the condition (b) in 2.5 (II) implies that $1 = \gamma_1 = \gamma_2^1 = \cdots = \gamma_i^{i-1} \geq \gamma_{i+1}^i \geq \cdots \geq \gamma_m^{m-1} \geq \gamma_{m+1}^m = 0$. Let $h + s$ be the last natural i such that $\gamma_i^{i-1} = 1$.

Applying again (b) in 2.5 (II), for $h < i \leq h + s$ we have $0 = \gamma_i \leq \gamma_i^1 \leq \cdots \leq \gamma_i^{i-1} = 1$. Let $j(i)$ be the first natural j such that $\gamma_i^j = 1$, then $0 = \gamma_i^{j(i)-1} \geq \gamma_{i+1}^{j(i)} \geq 0$ and we conclude that $j(i) < j(i+1)$.

∎

Definition 3.2 *With the above notation, we call s the* marked index *of V and the sequence $j_i = j(h + s - i + 1), 1 \leq i \leq s$ its* perturbation indices.

Remark 3.3 *Notice that not all sequences $j(i)$ are possible. For example, $j(i) - j(i-1) = 1$ requires only $\alpha_i > 0$ but $j(i) - j(i-1) > 1$ requires $\alpha_{j(i)-1} - \alpha_{j(i)} > 0$ too.*

Example 3.4 *Let $\gamma = (1, 1, 1, 1, 1)$, $\alpha = (4, 4, 4, 3, 3, 3, 2, 2, 1, 1)$. The indices $s = 5$ and $j_1 = j(10) = 9, j_2 = j(9) = 7, j_3 = j(8) = 5, j_4 = j(7) = 4, j_5 = j(6) = 1$ are possible and the L-R sequence is summarized in the following table:*

	γ									α
10									1	1
9								1	1	1	1
8						1	1	1	2	2	2
7					1	1	1	2	2	2	2
6		1	1	1	1	1	3	3	3	3	3
5	1	1	1	1	1	3	3	3	3	3	3
4	1	1	1	1	3	3	3	3	3	3	3
3	1	1	1	4	4	4	4	4	4	4	4
2	1	1	4	4	4	4	4	4	4	4	4
1	1	4	4	4	4	4	4	4	4	4	4
i/j	0	1	2	3	4	5	6	7	8	9	10

Let us see that a matrix of f in basis of E adapted to V can be constructed in a natural way starting from the marked and perturbation indices of V, so that , they characterize its equivalent class.

Lemma 3.5 *Let*$T \doteq \{i \in [h+1 : h+s] : j(i) - j(i-1) > 1\}$. *We could obtain vectors* $v_i \in V_i^{j(i)}$, $\tilde{v}_i \neq 0$, *for* $i \in [h : h+s]$ *and* $w_i \in V_{j(i)-1}^{j(i)-1}$, $\tilde{w}_i \neq 0$ *for* $i \in T$ *such that* $f(v_i) = v_{i-1}$ *for* $i \in [h+1 : h+s] \setminus T$ *and* $f(v_i) = v_{i-1} + w_i$ *for* $i \in T$.

Proof. Given $v_i \in V_i^{j(i)} \setminus (V_{i-1}^{j(i)} + V_i^{j(i)-1})$, we have $f(v_i) \in V_{i-1}^{j(i)-1}$ but $f(v_i) \notin V_{i-2}^{j(i)-1}$ and $f(v_i) \notin V_{i-1}^{j(i)-2}$. Therefore let $i \in [h+1 : h+s]$: if $i \notin T$, then $f(v_i) \notin V_{i-2}^{j(i)-1} + V_{i-1}^{j(i)-2}$ and we define $v_{i-1} \doteq f(v_i)$; if $i \in T$ we define $v_{i-1} \in V_{i-1}^{j(i-1)}$, $w_i \in V_{j(i)-1}^{j(i)-1}$ such that $f(v_i) = v_{i-1} + w_i$ and obviously $v_{i-1} \notin V_{i-2}^{j(i-1)} + V_{i-1}^{j(i-1)-1}$ and $w_i \notin V_{j(i)-2}^{j(i)-1} + V_{j(i)-1}^{j(i)-2}$. By recurrence, one defines $v_i \in V_i^{j(i)} \setminus (V_{i-1}^{j(i)} + V_i^{j(i)-1})$ for $h \leq i \leq h+s$ and $w_i \in V_{j(i)-1}^{j(i)-1} \setminus (V_{j(i)-2}^{j(i)-1} + V_{j(i)-1}^{j(i)-2})$ for $i \in T$ (notice that $v_h \in V_h$ is a generator of V).

∎

Example 3.6 *Following with the example 3.4, the next table shows the "place" of the vectors* v_i, w_i *above:*

10										v_{10}
9								v_9		
8						v_8			w_{10}	
7					v_7					
6		v_6					w_9			
5	v_5									
4										
3				w_7						
2										
1										
	0	1	2	3	4	5	6	7	8	9

Theorem 3.7 *Let $f \in End(E)$ be nilpotent, and $V \subset E, \dim(V) = h$, be a monogenic Jordan dense f-invariant subspace having marked and perturbation indices $s, j_1, j_2, \dots, j_s$. Then there is a basis of E adapted to V such that the matrix of f in this basis is:*

$$diag(N_h, N_s, N_{j(i_1)-1}, \dots, N_{j(i_t)-1}) + M$$

where $i_1 > i_2 > \dots > i_t$ are the indices in T (see lemma 3.5); $J_0 = h + s$, $J_l = J_{l-1} + j(i_l) - 1$ for $1 \leq l \leq t$; and M is a matrix whose the only non-zero entries are ones in the $(1, J_0), (J_0 + 1, J_0 + h + 1 - i_1), \dots, (J_{t-1} + 1, J_0 + h + 1 - i_t)$ positions. In particular, V is generated by the first vector of the basis.

Proof.

Such a basis is given by:

- For $1 \leq i \leq h$, $e_i \doteq f^{i-1}(v_h)$ (it is a Jordan basis of V),
- for $h < i \leq h + s$, $e_i \doteq v_{J_0+h+1-i}$,
- For $1 \leq l \leq t, 1 \leq i \leq j(i_l) - 1$, $e_{J_{l-1}+i} \doteq f^{i-1}(w_{i_l})$.

Notice that $\dim(E) = J_t = h + s + \sum_{1 \leq l \leq t} j(i_l) - t$. ■

Corollary 3.8 *Let $f \in End(E)$ be nilpotent. Two invariant monogenic subspaces $V, V' \subset E$ are equivalent if and only if* $\dim(V) = \dim(V')$ *and they have the same marked and perturbation indices.*

Example 3.9 *Following with the example in 3.4 and 3.6, in the basis:*

- $v_5, f(v_5), \dots, f^4(v_5)$,
- $v_{10}, v_9, v_8, v_7, v_6$,
- $w_{10}, f(w_{10}), \dots, f^7(w_{10})$,
- $w_9, f(w_9), \dots, f^5(w_9)$,
- $w_7, f(w_7), f^2(w_7)$,

the subspace V is generated by v_5, and the matrix is:

$$
\left(\begin{array}{c|c|c|c|c}
\begin{matrix}0&&&&\\1&0&&&\\&1&0&&\\&&1&0&\\&&&1&0\end{matrix} & \begin{matrix}0&0&0&0&1\\&&&&\\&&&&\\&&&&\\&&&&\end{matrix} & & & \\
\hline
 & \begin{matrix}0&&&&\\1&0&&&\\&1&0&&\\&&1&0&\\&&&1&0\end{matrix} & & & \\
\hline
 & \begin{matrix}1&0&0&0&0\\&&&&\\&&&&\\&&&&\\&&&&\\&&&&\\&&&&\\&&&&\end{matrix} & \begin{matrix}0&&&&&&&\\1&0&&&&&&\\&1&0&&&&&\\&&1&0&&&&\\&&&1&0&&&\\&&&&1&0&&\\&&&&&1&0&\\&&&&&&1&0\end{matrix} & & \\
\hline
 & \begin{matrix}0&1&0&0&0\\&&&&\\&&&&\\&&&&\\&&&&\\&&&&\end{matrix} & & \begin{matrix}0&&&&&\\1&0&&&&\\&1&0&&&\\&&1&0&&\\&&&1&0&\\&&&&1&0\end{matrix} & \\
\hline
 & \begin{matrix}0&0&0&1&0\\&&&&\\&&&&\end{matrix} & & & \begin{matrix}0&&\\1&0&\\&1&0\end{matrix}
\end{array}\right)
$$

Remark 3.10 *Notice that the matrices obtained in theorem 3.7 are a particular case of the miniversal deformation of a marked matrix obtained in [3]. Indeed, s determines the central marked matrix and $j_1, j_2, \ldots, j_s$ determine the non-zero miniversal parameters.*

Definition 3.11 *The matrix in theorem 3.7 will be called the* marked perturbed *(MP) reduced form of f.*

The marked and perturbation indices can be computed by means of standard algebraic algorithms:

Proposition 3.12 *Let $f \in End(E)$ be nilpotent, and $V \subset E, \dim(V) = h$, be a monogenic Jordan dense f-invariant subspace having marked and perturbation indices $s, j_1, j_2, \ldots, j_s$ and $i_1 > i_2 > \cdots > i_t$ are the indices in T (see lemma 3.5).*

Then there is a Jordan basis of E such that the matrix of f in this basis is $diag(N_{h+s}, N_{j(i_1)-1}, \ldots, N_{j(i_t)-1})$ and the matrix of the components of a generator of V is $(E_0, E_1, \ldots, E_t)$ where $E_0 \in \mathbb{C}^{(h+s)\times 1}$, $E_l \in \mathbb{C}^{(j(i_l)-1)\times 1}, 1 \le l \le t$ are zero matrices with one 1 in the rows $s+1$ and $i_l - h$ respectively.

Proof.

A Jordan basis is given by

- For $1 \le i \le h+s$, $e_i \doteq f^{i-1}(v_{h+s})$,

- For $1 \leq l \leq t, 1 \leq i \leq j(i_l) - 1$, $e_{J_{l-1}+i} \doteq -f^{i-1}(w_{i_l})$.

In this basis the generator of V, v_h has the expression $v_h = f^s(e_1) + \sum_{1 \leq l \leq t} f^{i_l-h-1}(e_{J_{l-1}+1})$.

■

Example 3.13 *Following with the example in 3.4, 3.6 and 3.9, in the Jordan basis:*

- $v_{10} \to v_9 + w_{10} \to v_8 + w_9 + f(w_{10}) \to v_7 + f(w_9) + f^2(w_{10}) \to v_6 + w_7 + f^2(w_9) + f^3(w_{10}) \to v_5 + f(w_7) + f^3(w_9) + f^4(w_{10}) \to f(v_5) + f^2(w_7) + f^4(w_9) + f^5(w_{10}) \to f^2(v_5) + f^5(w_9) + f^6(w_{10}) \to f^3(v_5) + f^7(w_{10}) \to f^4(v_5)$,
- $-w_{10} \to -f(w_{10}) \to \cdots \to -f^7(w_{10})$,
- $-w_9 \to -f(w_9) \to \cdots \to -f^5(w_9)$,
- $-w_7 \to -f(w_7) \to -f^2(w_7)$,

the matrices of f and of the generator v_5 of V are respectively:

$$
\left(\begin{array}{cccccccccc|cccccccc|cccccc|ccc}
0 \\
1 & 0 \\
& 1 & 0 \\
&& 1 & 0 \\
&&& 1 & 0 \\
&&&& 1 & 0 \\
&&&&& 1 & 0 \\
&&&&&& 1 & 0 \\
&&&&&&& 1 & 0 \\
&&&&&&&& 1 & 0 \\
\hline
&&&&&&&&&& 0 \\
&&&&&&&&&& 1 & 0 \\
&&&&&&&&&&& 1 & 0 \\
&&&&&&&&&&&& 1 & 0 \\
&&&&&&&&&&&&& 1 & 0 \\
&&&&&&&&&&&&&& 1 & 0 \\
&&&&&&&&&&&&&&& 1 & 0 \\
&&&&&&&&&&&&&&&& 1 & 0 \\
\hline
&&&&&&&&&&&&&&&&&& 0 \\
&&&&&&&&&&&&&&&&&& 1 & 0 \\
&&&&&&&&&&&&&&&&&&& 1 & 0 \\
&&&&&&&&&&&&&&&&&&&& 1 & 0 \\
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\hline
&&&&&&&&&&&&&&&&&&&&&&&& 0 \\
&&&&&&&&&&&&&&&&&&&&&&&& 1 & 0 \\
&&&&&&&&&&&&&&&&&&&&&&&&& 1 & 0
\end{array}\right),
\left(\begin{array}{c}
0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\
\hline
0\\0\\0\\0\\1\\0\\0\\0\\
\hline
0\\0\\0\\1\\0\\0\\
\hline
0\\1\\0
\end{array}\right)
$$

Corollary 3.14 *Let $f \in End(E)$ be nilpotent,there are so many classes of monogenic invariants subspaces as sequences of pairs of Jordan blocks and rows in them such their sizes and rows form strictly decreasing sequences, the first one with successive differences more than one.*

Proof. In proposition 3.12 we have seen that the sequences of the sizes of Jordan blocks and the rows are $s + h, j(i_1) - 1, \ldots, j(i_t) - 1$ and $s + 1, i_1 - h, \ldots, i_t - h$ respectively and they keep the required conditions.

Proposition 3.15 *Let $f \in End(E)$ be nilpotent, $u \in E$ and $V = [u, f(u), \dots, f^{n-1}(u)], \dim(V) = h$. Being $d_q \doteq \max\{k \in [0 : m-1] : f^q(u) \in Im(f^k)\}$ for $0 \le q < h$, usually called the deep of $f^q(u)$ (notice that $d_{q+1} - d_q \ge 1$), we consider $R \doteq \{0\} \cup \{q \in [1 : h-1] : d_q - d_{q-1} > 1\}$. Let $q_0 > q_1 > \dots > q_r = 0$ be the indices in R.*

Then, the marked and perturbation indices of V are given by:

- $t = r$ *and* $i_l = d_{q_l} - q_l + h + 1$ *for* $l \in [1 : t]$ *(see lemma 3.5 and theorem 3.7),*
- $j(i_l) = i_l + q_{l-1} - h$ *for* $l \in [1 : t]$,
- $s = d_{q_0} - q_0$,
- $j(i) = j(i-1) + 1$ *if* $i \notin T$.

Proof. Obviously, the deep of $f^q(u)$ and of $f^q(v_h)$ defined in proposition 3.1 are the same for every q. The expression of v_h in proposition 3.12 shows that the deep of successive images increases more than 1 when the exponent is $j(i_l) - i_l + h$ for $t \ge l \ge 1$ (so, $r = t$). Then we define $q_{l-1} \doteq j(i_l) - i_l + h$ for $1 \le l \le t-1$ and the corresponding deep will be the sum $d_{q_l} = (i_l - h - 1) + q_l$ and $d_{q_0} = s + q_0$.

Finally, the definition of T in proposition 3.12 implies that $j(i) = j(i-1) + 1$ if $i \notin T$. ■

Example 3.16 *Following with the example in 3.4, 3.6, 3.9 and 3.13; if A and b are the matrices of the endomorphism and the generator in some basis respectively, we organize the next table of the ranks of $(A^k, A^q b)$ that we must compute in this case:*

	b	Ab	A^2b	A^3b	A^4b	$A^5b = 0$
A	23					23
A^2	20					19
A^3		16				15
A^4			12			12
A^5			9			9
A^6			7			6
A^7				4		4
A^8				3		2
A^9					1	1
A^{10}					1	0

$\text{rank}(A^4 b) = 1$ *implies that* $h = 5$ *and*

q	d		$i_l = d_{q_l} - q_l + 6$	$j(i_l) = i_l + q_{l-1} - 5$
0	1	$q_3 = 0$	7	4
1	2			
2	5	$q_2 = 2$	9	7
3	7	$q_1 = 3$	10	9
4	9	$q_0 = 4$	$s = d_{q_0} - q_0 = 5$	

4 The canonical J-form of a vector

Proposition 3.12 can be reformulated as follows: given a fixed endomorphism, for each vector v one can select a Jordan basis for f in such a way that the coordinates of v become as simple as possible. In other words, one has canonical coordinates of v among those for Jordan bases of f. Let us precise these ideas.

Definition 4.1 *Given a fixed $f \in End(E)$ and its Jordan matrix J, two vectors $v, v' \in E$ are called* f-equivalent *(or* J-equivalent*) if there is $\varphi \in Aut(E)$ such that $\varphi^{-1} \circ f \circ \varphi = f$, $f(v) = v'$. Equivalently, if there is $S \in M_n(\mathbb{C})$ non-singular such that:*

$$S^{-1} J S = J, \qquad S^{-1} v = v'.$$

It is obvious that:

Lemma 4.2 *In the above conditions, the vectors v, v' are f-equivalent if and only if the f-invariant subspaces $V = [v, f(v), \ldots, f^{n-1}(v)]$ and $V' = [v', f(v'), \ldots, f^{n-1}(v')]$ are f-equivalent.*

Hence, we can reformulate proposition 3.12 as follows:

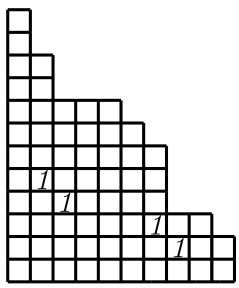
Corollary 4.3 *Let $f \in End(E)$, J be its Jordan matrix and $v \in E$. Assume that the monogenic f-invariant subspace $V = [v, f(v), \ldots, f^{n-1}(v)]$ is Jordan dense and let $s, j_1, j_2, \ldots, j_s$ and $i_1 > i_2 > \cdots > i_t$ as in proposition 3.12. Then the components $(E_0, E_1, \ldots, E_t)$ are canonical representative of v with regard to the f-equivalence. If V is not Jordan dense, it suffices to add zero components for the complementary Jordan chains.*

Clearly, two vectors v, v' are f-equivalent if and only if they have the same f-representative above.

Definition 4.4 *In the conditions of the above corollary, the f-representative of v there will be called its* canonical f-form *(or J-form)*

Example 4.5 *In example 3.13 the column matrix is the canonical J-form of the generator v_5 of V.*

Remark 4.6 *The canonical f-form for a vector v can be sketched as follows:*



where the columns mean Jordan chains and the only non-zero components are placed in cells $s+1, i_1-h, \dots, i_t-h$ *counting from the top (deep plus one). Notice that the dimension of* $V=[v, f(v), \dots, f^{n-1}(v)]$ *is given by the highest non-zero component.*

These possible placements are determined by the following rules:

- *Non consecutive in chains having the same length or differing in 1.*
- *The heights and deeps must be decreasing .*

These rules allow to list all possible f*-classes.*

5 Application to control and bimodal systems

The results of section 4 can be reformulated from the point of view of control systems, giving a canonical form of uniparametric non-controllable linear control systems, with regard to changes of basis in the state variables.

As a second application, we improve the reduced form in [4] for a bimodal continuous linear piecewise dynamical system, with regard to changes of basis which preserve the separating hyperplane.

We recall that a linear control system is given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad A \in M_n(\mathbb{R}), B \in M_{n\times p}(\mathbb{R})$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^p$ are respectively the state and control variables. A change of basis $\bar{x} = S^{-1}x$ in the state space transforms the above equation into

$$\dot{\bar{x}}(t) = (S^{-1}AS)\bar{x}(t) + (S^{-1}B)u(t).$$

Canonical forms in this sense have been obtained ([10]) for controllable systems, that is to say, when $\operatorname{rank}(B, AB, A^2B, \dots, A^{n-1}B) = n$. The above corollary 4.3 gives an alternative canonical form for uniparametric systems (that is, $p = 1$), non necessarily controllable, which we will call its *J-canonical form.*

Corollary 5.1 *Let us consider the linear control system*

$$\dot{x}(t) = Ax(t) + bu(t), \quad A \in M_n(\mathbb{R}), b \in \mathbb{R}.$$

There is a change of basis $\bar{x} = S^{-1}x$ *in the state space such that the equation is transformed in the following canonical form*

$$\dot{\bar{x}}(t) = J\bar{x}(t) + S^{-1}bu(t),$$

where J *is the Jordan form of* A *and* $S^{-1}b$ *is the canonical J-form of* b*. We will refer to it as the J-canonical form of a uniparametric control linear system.*

Example 5.2 *The matrices in example 3.13 can be viewed as the canonical J-form of an uniparametric control linear system.*

The following alternative reduced form separates the controllable and uncontrollable parts:

Proposition 5.3 *Let $f \in End(E)$ be nilpotent, and $V \subset E, \dim(V) = h$, be a monogenic Jordan dense f-invariant subspace having marked and perturbation indices $s, j_1, j_2, \ldots, j_s$, $P \doteq \{h+s\} \cup \{i \in [h+1 : h+s-1] : i+1 \in T\}$ (see lemma 3.5 and notice that $card(P) = card(T)+1$). If we denote by $r_1 > r_2 > \cdots > r_{t+1}$ the indices in P and we define $K_0 = h, K_q = K_{q-1} + j(r_q)$ for $1 \leq q \leq t+1$, there is a basis of E in which the matrix of f is $diag(N_h, N_{j(r_1)}, \ldots, N_{j(r_{t+1})}+F$ where F is a zero matrix with ones in the $(1, K_{q-1} + r_q - h)$ position for $1 \leq q \leq t+1$.*

Proof.

Such a basis is:

1. For $1 \leq i \leq h$, $e_i \doteq f^{i-1}(v_h)$ (it is a Jordan basis of V),

2. For $1 \leq q \leq t+1$,

   - $e_{K_{q-1}+i} \doteq f^{i-1}(v_{r_q})$ if $1 \leq i \leq r_q - h$,
   - $e_{K_{q-1}+r_q-h+1} \doteq f^{r_q-h}(v_{r_q}) - v_h$ if $r_q - h < j(r_q)$
   - $e_{K_{q-1}+i} \doteq f^{i-r_q+h-1}(e_{K_{q-1}+r_q-h+1}))$ if $r_q - h + 1 < i \leq j(r_q)$.

■

Example 5.4 *Following with the example in 3.4, 3.6, 3.9 and 3.13, in the basis:*

- $v_5, f(v_5), \ldots, f^4(v_5)$,
- $v_{10} \to v_9 + w_{10} \to v_8 + w_9 + f(w_{10}) \to v_7 + f(w_9) + f^2(w_{10}) \to v_6 + w_7 + f^2(w_9) + f^3(w_{10}), f(w_7) + f^3(w_9) + f^4(w_{10}) \to f^2(w_7) + f^4(w_9) + f^5(w_{10}) \to f^5(w_9) + f^6(w_{10}) \to f^7(w_{10})$,
- $v_9 \to v_8+w_9 \to v_7+f(w_9) \to v_6+w_7+f^2(w_9), f(w_7)+f^3(w_9) \to f^2(w_7)+f^4(w_9) \to f^5(w_9)$,
- $v_8 \to v_7 \to v_6 + w_7, f(w_7) \to f^2(w_7)$,
- v_6,

the matrices of f and of the components of a generator of V, v_5) are respectively:

$$
\left(\begin{array}{c|c|c|c|c}
\begin{array}{ccccc} 0 &&&& \\ 1 & 0 &&& \\ & 1 & 0 && \\ && 1 & 0 & \\ &&& 1 & 0 \end{array} &
\begin{array}{ccccccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ &&&&&&&& \\ &&&&&&&& \\ &&&&&&&& \\ &&&&&&&& \end{array} &
\begin{array}{ccccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ &&&&&& \\ &&&&&& \\ &&&&&& \\ &&&&&& \end{array} &
\begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \\ &&&& \\ &&&& \\ &&&& \\ &&&& \end{array} &
\begin{array}{c} 1 \\ \\ \\ \\ \end{array} \\
\hline
 &
\begin{array}{ccccccccc} 0 &&&&&&&& \\ 1 & 0 &&&&&&& \\ & 1 & 0 &&&&&& \\ && 1 & 0 &&&&& \\ &&& 1 & 0 &&&& \\ &&&& 1 & 0 &&& \\ &&&&& 1 & 0 && \\ &&&&&& 1 & 0 & \\ &&&&&&& 1 & 0 \end{array} & & & \\
\hline
 & &
\begin{array}{ccccccc} 0 &&&&&& \\ 1 & 0 &&&&& \\ & 1 & 0 &&&& \\ && 1 & 0 &&& \\ &&& 1 & 0 && \\ &&&& 1 & 0 & \\ &&&&& 1 & 0 \end{array} & & \\
\hline
 & & &
\begin{array}{ccccc} 0 &&&& \\ 1 & 0 &&& \\ & 1 & 0 && \\ && 1 & 0 & \\ &&& 1 & 0 \end{array} & \\
\hline
 & & & & 0
\end{array}\right),
\quad
\left(\begin{array}{c}
1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline
0
\end{array}\right)
$$

Finally, we apply corollary 5.1 to improve the reduced forms in [4] for a bimodal continuous linear dynamical system. We recall that such systems are given by

$$\begin{array}{ll} \dot{x}(t) = A_1 x(t) + b & \text{if} x_1 \leq 0 \\ \dot{x}(t) = A_2 x(t) + b & \text{if} x_1 \geq 0 \end{array} \quad \text{where } A_1, A_2 \in M_n(\mathbb{R}) \text{ and } b \in M_{n\times 1}(\mathbb{R}).$$

A change of basis $\bar{x} = S^{-1}x$ is called admissible if the hyperplanes $x_1 = k$ are preserved, that is to say, if

$$S = \begin{pmatrix} 1 & 0 \\ U & T \end{pmatrix}, \quad T \in Gl_{n-1}(R).$$

In [4] one proves that there exists an admissible change of basis such that

$$S^{-1}A_1S = \begin{pmatrix} K_1 & 0 \\ \overline{A}_1 & J \end{pmatrix}, S^{-1}A_2S = \begin{pmatrix} K_2 & 0 \\ \overline{A}_2 & J \end{pmatrix}$$

where

$$K_1 = \begin{pmatrix} a_1 & 1 & 0 & \dots & 0 & 0 \\ a_2 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & & \dots & \dots \\ a_{r-1} & 0 & 0 & \dots & 0 & 1 \\ a_r & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} \alpha_1 & 1 & 0 & \dots & 0 & 0 \\ \alpha_2 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & & \dots & \dots \\ \alpha_{r-1} & 0 & 0 & \dots & 0 & 1 \\ \alpha_r & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$\overline{A}_1 = \begin{pmatrix} \overline{a}_{r+1} & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & & \dots & \dots \\ \overline{a}_n & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \quad \overline{A}_2 = \begin{pmatrix} \overline{\alpha}_{r+1} & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & & \dots & \dots \\ \overline{\alpha}_n & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

and

$J \in M_{n-r}(\mathbb{C})$ is a Jordan matrix, whereas non reduction is achieved for b:

$$(S^{-1}b)^t = (b_1 \; \overline{b}_2 \; \dots \; \overline{b}_n).$$

The above results allow us to improve this reduced form for the bimodal system:

Corollary 5.5 *In the above conditions, there is an admissible change of basis*

$$S' = \begin{pmatrix} I & 0 \\ 0 & Q \end{pmatrix}, \quad Q \in Gl_{n-r}(\mathbb{C})$$

such that

$$Q^{-1}(\overline{b}_{r+1} \; \dots \; \overline{b}_n)^t$$

is the J-canonical form of $(\overline{b}_{r+1} \; \dots \; \overline{b}_n)^t$.

It makes easier, for example, to apply the criteria in [4] for the controllability of the bimodal system.

References

[1] A. Compta; J. Ferrer, *A Geometric Approach to the Carlson Problem*, SIAM Journal on Matrix Analysis Appl. Vol. 22, n. 1(2000), pp. 258-275.

[2] A. Compta; J. Ferrer, *Matricial Realizations of the Solutions of the Carlson Problem*,Linear Algebra Appl., 353 (2002), pp. 197–206.

[3] A. Compta; J. Ferrer; F. Puerta, *Miniversal deformations of marked matrices*, Linear Algebra Appl., 361 (2003), p. 181–201.

[4] J. Ferrer; M. D. Magret; M. Peña, *Bimodal Piecewise Linear Dynamical Systems. Reduced Forms*, International Journal of Bifurcation and Chaos., Vol. 20, n. 9 (2010), p. 2795–2808.

[5] J. Ferrer; F. Puerta; X. Puerta, *Geometric Characterization and Classification of Marked Subspaces*, Linear Algebra Appl., 235 (1996), . 15–34.

[6] I. Gohberg; M.A. Kaashoek, *Unsolved Problems in Matrix and Operator Theory II. Partial Multiplicities for Products*, Integral Equations and Operator Theory, vol. 2 /1 (1979), p. 116–120.

[7] I. Gohberg; P. Lancaster; L. Rodman, *Invariant Subspaces of Matrices with Applications.* Wiley, New York (1986).

[8] T. Klein, *The Multiplication of Schur Functions and Extensions of p-modules*, Journal London Math. Soc., 43 (1968), p. 280–284.

[9] C.M. Ringel; M. Schmidmeier, *Invariant subspaces of nilpotent operators. I.*, Journal Reine Angew. Math. Band, 614 (2008), p. 1–52.

[10] A. Tannenbaum, *Invariance and System Theory: Algebraic and Geometric Aspects*, LNM, n. 845, Springer, 1981.

[11] R.C. Thompson, *Smith Invariants of a Product of Integral Matrices*, Contemporary Mathematics. AMS, vol. 47 (1985), p. 401–435.