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ANALYSIS OF CONTROL PROPERTIES OF CONCATENATED CONVOLUTIONAL CODES

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Abstract
In this paper we consider two models of concatenated convolutional codes from the perspective of linear systems theory. We present an input-state-output representation of these models and we study the conditions for control properties for concatenated convolutional codes.

Key words
Convolutional codes, linear systems, control.

1 Introduction

In coding theory, concatenated codes form a class of error-correcting codes that are obtained by combining an inner code and an outer code. They were conceived in 1966 by Dave Forney as a solution to the problem of finding a code that has both exponentially decreasing error probability with increasing block length and polynomial-time decoding complexity. Concatenated codes became widely used in space communications in the 1970s. In this paper we study two kinds of concatenated convolutional codes (serial and parallel) using linear systems theory.

It is well known that convolutional codes can be described using a quite more general theory, the linear systems theory over finite fields (see [Rosenthal, Schumacher and York, 1996], [Rosenthal and York, 1999], [Rosenthal and Smarandache, 1999] for example).

The aim of this work is to give a input-output representation of a concatenated (serial and parallel) convolutional, and deduce conditions for control properties as controllability, observability as well output observability. The control properties are relied to the minimality of strict equivalent encoders. Concretely, token minimality in the sense that encoders use the smallest number of memory elements, among the encoders having the same set of possible output sequences leads to that it is not necessary to have two loops with the same output in the state diagram, it suffices one of such loop. This condition is described by means output observability.

The connection between coding theory and linear systems theory help us to well-understand the properties of convolutional codes. In fact, the concepts of controllability and observability arising in linear systems theory can be translated into the context of convolutional codes (see [Rosenthal, Schumacher and York, 1996; Gluesing-Luerssen, 2005] for example), leading to a correspondence between observability character of the linear system and the noncatastrophicity character of the corresponding encoder.

It is known that transfer of biological information can be modeled as a communication channel with the DNA sequence as the input and the amino acid sequence which forms protein as the channel output, so can be described as a convolutional code (see [May *et al.*, 1999; Schneider] for example). Convolutional codes produces encoded blocks based on present and past information bits or blocks. The model that considers is that genetic operations such as initiation and translation may involve ?decisions? which are based on immediate past and immediate future information. The advantage of this modeling is allows error correction. Analysis of multigenes can be treated as a concatenated codes, gene sequences are concatenated into a supergene alignment.

Also noteworthy that one of the greatest difficulties of using coding theory for transmission of information is that, in an effort to approximate the theoretical limit for the capacity of Shannon's channel, there is the need to constraint the length of the convolutional code. The way to solve this kind of problem is using a concatenation procedure, in particular serial or parallel concatenation.

This paper is structured as follows. In section 2 we review the concept of convolutional codes under linear systems point of view and we review the concepts of controllability, observability and output observability. In section 3 the serial and parallel concatenation are developed and finally, in section 4 control properties of concatenated convolutional codes are presented.

2 Preliminaries

Throughout the paper, we denote by $\mathbb{F} = GF(q)$ the Galois field of q elements and $\bar{\mathbb{F}}$ the algebraic closure of $\mathbb{F}$.

A convolutional code $\mathcal{C}$ of rate k/n and degree δ, called an (n, k, δ)-code, can be given by the input-state-output representation (see [Hutchinson, Rosenthal and Smarandache, 2005; Rosenthal, Schumacher and York, 1996; Rosenthal and York, 1999])

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \\ v(t) &= \begin{pmatrix} y(t) \\ u(t) \end{pmatrix}, \quad x(0) = 0, \end{aligned} \tag{1}$$

where for each instant t , $x(t) \in \mathbb{F}^{\delta}$ is the state vector, $u(t) \in \mathbb{F}^k$ is the information vector, $y(t) \in ?\mathbb{F}^{n-k}$ is the parity vector and $v(t)$ is a codeword of $\mathcal{C}$. In that case, $\mathcal{C}$ is said to be generated by the quadruple of matrices (A, B, C, D) and we will denote it by $\mathcal{C}(A, B, C, D)$.

The encoder matrix of the code is defined as the transfer matrix of the system (1):

$$G(s) = C(sI_{\delta} - A)^{-1}B + D. \tag{2}$$

Here A, B, C and D are matrices of sizes $\delta \times \delta$, $\delta \times k$, $(n-k)? \times \delta$ and $(n-k)? \times k$, respectively, that is, (A, B, C, D) is a minimal representation and it is characterized through the condition that the pair (A, B) is controllable, i.e.,

$$\text{rank}\ \begin{pmatrix} B\ AB? \ldots ?A^{\delta-1}B \end{pmatrix} = \delta$$

or equivalently (see [Hautus, 1969]),

$$\text{rank}\ \begin{pmatrix} zI - A\ B \end{pmatrix} = \delta,\ \forall z? \in \overline{\mathbb{F}}.$$

A pair (A, C) is said to be an observable pair if (A^t, C^t) is a controllable pair, i.e.,

$$\text{rank} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{\delta-1} \end{pmatrix} = \delta$$

or equivalently (see [Hautus, 1969]),

$$\text{rank} \begin{pmatrix} zI - A \\ C \end{pmatrix} = \delta,\ \forall z? \in \overline{\mathbb{F}}.$$

We define a convolutional code $\mathcal{C}(A, B, C, D)$ to be observable if one and therefore every encoder $G(z)$ is right prime. If $G(z)$ is an encoder of an observable convolutional code, then $G(z)$ is necessarily a non-catastrophic encoder. The following result shows that if (A, B) is controllable, then the observability of the pair (A, C) ensures that the linear system (1) describes a non-catastrophic convolutional encoder.

Lemma 2.1 ([Rosenthal and York, 1999]). *Assume that the matrices (A, B) form a controllable pair. The convolutional code $\mathcal{C}(A, B, C, D)$ defined through* (1) *represents an observable convolutional code if and only if (A, C) forms an observable pair.*

Ch. Fragouli and R. D. Wesel [Fragouli and Wesel, 1999] give the following definition of output observable for standard systems.

Definition 2.1. *The system (A, B, C, D) is said to be output observable if the state sequence $\{x_0, x_1, \ldots, x_{n-1}\}$ is uniquely determined by the knowledge of the output sequence $\{y_0, y_1, \ldots, y_{n-1}\}$ for a finite number of steps $n - 1$.*

Taking into account that

$$\begin{aligned} y_0 &= Cx_0 + Du_0 \\ &\vdots \\ y_k &= CA^k x_0 + CA^{k-1}Bu_0 + \ldots + CABu_{k-2} + CBu_{k-1}, \\ &\vdots \end{aligned}$$

the output observability is characterized by the following proposition.

Proposition 2.1 ([Fragouli and Wesel, 1999]). *The system (A, B, C, D) is output observable if and only if the matrix $T \in M_{(n-k)\delta \times (\delta(k+1))}(\mathbb{F})$ defined as*

$$\begin{pmatrix} C & D & & & \\ CA & CB & D & & \\ CA^2 & CAB & CB & D & \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ CA^{\delta} & CA^{\delta-1}B & CA^{\delta-2}B & ?\ldots & CB\ D \end{pmatrix}$$

has full row rank.

If we want to preserve the delay condition we need to add the condition that the matrix D has full rank (condition for delay preserving).

Finally, in terms of the input-state-output representation (1), the free distance of a convolutional code $\mathcal{C}$, that is, the minimum Hamming distances between any two code sequences of $\mathcal{C}$, can be characterized as (see [Hutchinson, Rosenthal and Smarandache, 2005])

$$d_{free}(\mathcal{C}) = \lim_{j\to\infty} d_j^c(\mathcal{C}), \qquad (3)$$

where

$$d_j^c(\mathcal{C}) = \min_{u(0)\neq 0}\left\{\sum_{t=0}^{j} wt(u(t)) + \sum_{t=0}^{j} wt(y(t))\right\}$$

is the j-th column distance of the convolutional code $\mathcal{C}$, for $j = 0, 1, 2, \ldots$.

3 Concatenation

In this section we introduce the following models of concatenation of two convolutional codes.

The first model considered is the following.

Let $\mathcal{C}_{\text{o}}(A_1, B_1, C_1, D_1)$ and $\mathcal{C}_{\text{i}}(A_2, B_2, C_2, D_2)$ be convolutional codes, called outer code, and inner code respectively. Let $x_1(t)$, $u_1(t)$, and $y_{(1)}(t)$ be the state vector, the information vector and the parity vector of $\mathcal{C}_{\text{o}}(A_1, B_1, C_1, D_1)$, and let $x_2(t)$, $u_2(t)$, and $y_2(t)$ be the state vector, the information vector and the parity vector of $\mathcal{C}_{\text{i}}(A_2, B_2, C_2, D_2)$, respectively.

The outer code $\mathcal{C}_{\text{o}}$ and the inner code $\mathcal{C}_{\text{i}}$ are serialized, one after the other, so that the input information $u_2 = y_1(t)$. Consequently

$$\begin{aligned} x_1(t+1) &= A_1x_1(t) + B_1u_1(t)\\ x_2(t+1) &= A_2x_2(t) + B_2C_1x_1(t) + B_2D_1u_1(t)\\ y_2(t) &= C_2x_2(t) + D_2C_1x_1(t) + D_2D_1u_1(t)\end{aligned}$$

That is to say the concatenated code is $\mathcal{C}(A, B, C, D)$ with

$$A = \begin{pmatrix} A_1 & 0\\ B_2C_1 & A_2\end{pmatrix},\ B = \begin{pmatrix} B_1\\ B_2D_1\end{pmatrix},\quad C = \begin{pmatrix} D_2C_1 & C_2\end{pmatrix},\ D = D_2D_1.$$

If $\mathcal{C}_0(A_1, B_1, C_1, D_1)$ is a (m, k, δ_1)-code and $\mathcal{C}_i(A_2, B_2, C_2, D_2)$ is a $(n, m-k, \delta_2)$-code, then $\mathcal{C}(A, B, , D)$ is a $(n - m + 2k, k, \delta_1 + \delta_2)$-code.

Proposition 3.1. *The transfer matrix defining the matrix encoder of the serial concatenated code is:*

$$G(s) = G_2(s)G_1(s)$$

being $G_1(s)$ and $G_2(s)$ the transfer matrices corresponding to the codes $\mathcal{C}_0(A_1, B_1, C_1, D_1)$ and $\mathcal{C}_i(A_2, B_2, C_2, D_2)$, respectively.

Proof.

$$\begin{pmatrix} sI_{\delta_1} - A_1 & 0\\ -B_2C_1 & sI_{\delta_2} - A_2\end{pmatrix}^{-1} = \begin{pmatrix} (sI_{\delta_1}-A_1)^{-1} & 0\\ (sI_{\delta_2}-A_2)^{-1}B_2C_1(sI_{\delta_1}-A_1)^{-1} & (sI_{\delta_2}-A_2)^{-1}\end{pmatrix}$$

So,

$$\begin{aligned} &(D_2C_1\ C_2)\begin{pmatrix} sI_{\delta_1} - A_1 & 0\\ -B_2C_1 & sI_{\delta_2} - A_2\end{pmatrix}^{-1}\begin{pmatrix} B_1\\ B_2D_1\end{pmatrix} =\\ &D_2C_1(sI_{\delta_1} - A_1)^{-1}B_2C_1(sI_{\delta_1} - A_1)^{-1}B_1+\\ &C_2(sI_{\delta_2} - A_2)^{-1}B_2D_1 + D_2D_1 =\\ &G_2(s)G_1(s)\end{aligned}$$

The second model presented is the parallel concatenation. Let $\mathcal{C}_1(A_1, B_1, C_1, D_1)$ and $\mathcal{C}_2(A_2, B_2, C_2, D_2)$ be convolutional codes. Let $x_1(t)$, $u_1(t)$, and $y^{(1)}(t)$ be the state vector, the information vector and the parity vector of $\mathcal{C}_1(A_1, B_1, C_1, D_1)$, and let $x_2(t)$, $u_2(t)$, and $y_2(t)$ be the state vector, the information vector and the parity vector of $\mathcal{C}_2(A_2, B_2, C_2, D_2)$, respectively.

Proposition 3.2. *The transfer matrix defining the matrix encoder of the parallel concatenated code is:*

$$G(s) = G_1(s) + G_2(s)$$

being $G_1(s)$ and $G_2(s)$ the transfer matrices corresponding to the codes $\mathcal{C}_1(A_1, B_1, C_1, D_1)$ and $\mathcal{C}_2(A_2, B_2, C_2, D_2)$, respectively.

Proof.

$$\begin{pmatrix} sI_{\delta_1} - A_1 & 0\\ 0 & sI_{\delta_2} - A_2\end{pmatrix}^{-1} = \begin{pmatrix} (sI_{\delta_1} - A_1)^{-1} & 0\\ 0 & (sI_{\delta_2} - A_2)^{-1}\end{pmatrix}$$

So,

$$\begin{aligned} &(C_1\ C_2)\begin{pmatrix} sI_{\delta_1} - A_1 & 0\\ 0 & sI_{\delta_2} - A_2\end{pmatrix}^{-1}\begin{pmatrix} B_1\\ B_2\end{pmatrix} =\\ &C_1(sI_{\delta_1} - A_1)^{-1}B_1 + C_2(sI_{\delta_2} - A_2)^{-1}B_2 =\\ &G_1(s) + G_2(s).\end{aligned}$$

Both codes are concatenated in a parallel form, so that the input information $u_2(t) = u_1(t) = u(t)$ and the final parity vector $y(t) = y_1(t) + y_2(t)$. Consequently

$$\begin{aligned} x_1 &= A_1x_1(t) + B_1u(t) \\ x_2 &= A_2x_2(t) + B_2u(t) \\ y(t) &= C_1x_1(t) + C_2x_2(t) + (D_1 + D_2)u(t) \end{aligned}$$

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},\ B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

$$C = \begin{pmatrix} C_1 & C_2 \end{pmatrix},\ D = D_1 + D_2.$$

If $\mathcal{C}_1(A_1, B_1, C_1, D_1)$ is a (n, k, δ_1)-code and $\mathcal{C}_2(A_2, B_2, C_2, D_2)$ is a (n, k, δ_2)-code, then $\mathcal{C}(A, B, , D)$ is a $(n, k, \delta_1 + \delta_2)$-code.

4 Control Properties of Concatenated Codes

In this section, we establish conditions on the linear systems with matrices (A_i, B_i, C_i, D_i) of the inner convolutional code $\mathcal{C}_i$ in order to obtain an observable convolutional code with a minimal representation from the different models of concatenation introduced in Section 3, that is, a representation with the pair (A, B) controllable and the pair (A, C) observable.

4.1 Serial Concatenated Case

Following Hautus theorem
a) the serial concatenated system (A, B, C, D) is controllable if and only if the matrix

$$\begin{pmatrix} zI_{\delta_1} - A_1 & 0 & B_1 \\ -B_2C_1 & zI_{\delta_2} - A_2 & B_2D_1 \end{pmatrix}$$

has full row rank $(= \delta_1 + \delta_2)$, for all $z? \in \overline{\mathbb{F}}$.
b) the serial concatenated system (A, B, C, D) is observable if and only if the matrix

$$\begin{pmatrix} zI_{\delta_1} - A_1 & 0 \\ -B_2C_1 & zI_{\delta_2} - A_2 \\ D_2C_1 & C_2 \end{pmatrix}$$

has full column rank $(= \delta_1 + \delta_2)$, for all $z? \in \overline{\mathbb{F}}$.
Therefore we have the following propositions.

Proposition 4.1. *A necessary condition for controllability of serial concatenated code is that the pair (A_1, B_1) be controllable.*

This condition is not sufficient as we can see in the following example.

Example 4.1. *Let us have the following two realizations (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) with $A_1 = (1)$, $B_1 = (1)$, $C_1 = (1)$, $D_1 = (1)$, and $A_2 = (0)$, $B_2 = (1)$, $C_2 = (1)$, $D_2 = (1)$.*
Both systems are controllable but the serial concatenated system (A, B, C, D) with

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ C = \begin{pmatrix} 1 & 1 \end{pmatrix},\ D = (1).$$

is not controllable because of

$$\text{rank}\begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 1 < 2.$$

Proposition 4.2. *A necessary condition for observability of serial concatenated code is that the pair (A_2, C_2) be observable.*

Example 4.2. *Let us have the following two realizations (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) with $A_1 = (0)$, $B_1 = (1)$, $C_1 = (1)$, $D_1 = (1)$, and $A_2 = (1)$, $B_2 = (1)$, $C_2 = (1)$, $D_2 = (1)$.*
Both systems are observable but the serial concatenated system (A, B, C, D) with

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ C = \begin{pmatrix} 1 & 1 \end{pmatrix},\ D = (1),$$

is not observable because of

$$\text{rank}\begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 1 < 2.$$

Suppose now that $k \geq \delta_1 + \delta_2$, the we have the following proposition

Proposition 4.3. *If the matrix*

$$\begin{pmatrix} B_1 \\ B_2D_1 \end{pmatrix}$$

has full rank, then the system (A, B, C, D) is controllable.

Corollary 4.1. *With the same hypothesis than 4.3, if the matrix*

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

has full rank, then the system (A, B, C, D) is controllable.

Proof. Because of $k \geq \delta_1 + \delta_2$, if the matrix $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ has full rank we have rank $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \delta_1 + \delta_2$, so:

$$\begin{aligned}
&\delta_1 + \delta_2 = \\
&\text{rank}\begin{pmatrix} B_1 \\ B_2 D_1 \end{pmatrix} = \text{rank}\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}\begin{pmatrix} I_k & \\ & D_1 \end{pmatrix} \leq \\
&\text{rank}\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} = \text{rank}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \delta_1 + \delta_2.
\end{aligned}$$

Remark 4.1. *The equality* $\text{rank}\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} = \text{rank}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ *is true because* $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ *has full row rank.*

Suppose now that $n - k \geq \delta_1 + \delta_2$, the we have the following proposition.

Proposition 4.4. *If the matrix*

$$\begin{pmatrix} D_2 C_1 & C_2 \end{pmatrix}$$

has full rank, then the system (A, B, C, D) *is observable.*

4.2 Parallel Concatenated Case

The controllability matrix of the parallel concatenate code is

$$\begin{pmatrix} B_1 & A_1 B_1 & \dots & A_1^{\delta_1+\delta_2-1} B_1 \\ B_2 & A_2 B_2 & \dots & A_1^{\delta_1+\delta_2-1} B_2 \end{pmatrix}$$

Proposition 4.5. *A necessary condition for controllability of parallel concatenated system is that the pairs* (A_2, B_1) *and* (A_2, B_2) *are controllable*

Obviously, this condition it is not sufficient as we can seen in the following example:

Example 4.3. *Let* $A_i = (0)$, $B_i = (1)$, $C_i = (1)$ *and* $D_i = (1)$ *for* $i = 1, 2$, *the parallel concatenated code is*

$$A = 0, B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 \end{pmatrix}, D = (2).$$

It is obvious that this code is not controllable, but both codes (A_i, B_i, C_i, D_i) *are controllable.*

With respect observability we have that, the observability matrix of the parallel concatenated code is

$$\begin{pmatrix} C_1 & C_2 \\ C_1 A_1 & C_2 A_2 \\ \vdots & \vdots \\ C_1 A_1^{\delta_1+\delta_2-1} & C_2 A_2^{\delta_1+\delta_2-1} \end{pmatrix}$$

Proposition 4.6. *A necessary condition for observability of parallel concatenated system is that the pairs* (A_1, C_1) *and* (A_2, C_2) *are observable*

The same codes in the previous example serve to prove that the converse of this proposition is not true.

Finally, we analyze the output observability character of parallel concatenated codes.

Consider now a parallel concatenated code $\mathcal{C}(A, B, C, D)$ obtained from the concatenation of the equal codes $\mathcal{C}_1(A_1, B_1, C_1, D_1) = \mathcal{C}_2(A_2, B_2, C_2, D_2)$.

The output observability matrix of this parallel concatenated code is

$$\begin{pmatrix}
C_1 & C_1 & 2D_1 & & & \\
C_1 A_1 & C_1 A_1 & 2C_1 B_1 & 2D_1 & & \\
C_1 A_1^2 & C_1 A_1^2 & 2C_1 A_1 B_1 & 2C_1 B_1 & 2D_1 & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \\
C_1 A_1^{2\delta_1} & C_1 A_1^{2\delta_1} & 2C_1 A_1^{2\delta_1-1} B_1 & \dots & 2C_1 B_1 & 2D_1
\end{pmatrix}$$

and the rank of this matrix coincides with the rank of

$$\begin{pmatrix}
C_1 & D_1 & & & & \\
C_1 A_1 & C_1 B_1 & D_1 & & & \\
C_1 A_1^2 & C_1 A_1 B_1 & C_1 B_1 & D_1 & & \\
\vdots & \ddots & \ddots & & & \\
C_1 A_1^{2\delta_1} & C_1 A_1^{2\delta_1-1} B_1 & \dots & C_1 A_1 B_1 & C_1 B_1 & D_1
\end{pmatrix}$$

Notice that the submatrix

$$T_\delta = \begin{pmatrix}
C_1 & D_1 & & & & \\
C_1 A_1 & C_1 B_1 & D_1 & & & \\
C_1 A_1^2 & C_1 A_1 B_1 & C_1 B_1 & D_1 & & \\
\vdots & \ddots & \ddots & & & \\
C_1 A_1^{\delta_1} & C_1 A_1^{\delta_1-1} B_1 & \dots & C_1 A_1 B_1 & C_1 B_1 & D_1
\end{pmatrix}$$

corresponds to the output observability matrix of the $\mathcal{C}_1(A, B, C, D)$ code.

Therefore, is having the following proposition.

Proposition 4.7. *A necessary condition for output observability of the parallel concatenated code* $\mathcal{C}(A, B, C, D)$ *is that the code* $\mathcal{C}_1(A, B, C, D)$ *be output observable.*

Calling T_i the matrix

$$T_i = \begin{pmatrix}
C_1 & D_1 & & & & \\
C_1 A_1 & C_1 B_1 & D_1 & & & \\
C_1 A_1^2 & C_1 A_1 B_1 & C_1 B_1 & D_1 & & \\
\vdots & \ddots & \ddots & & & \\
C_1 A_1^{i} & C_1 A_1^{i-1} B_1 & \dots & C_1 A_1 B_1 & C_1 B_1 & D_1
\end{pmatrix}$$

for all $i \geq \delta$, we have the following theorem.

Theorem 4.1. *Suppose that the code* $\mathcal{C}_1(A, B, C, D)$ *is output observable. A necessary condition for output observability of the parallel concatenated code* $\mathcal{C}(A, B, C, D)$ *is*

$$\text{rank}\, T_{\delta+1} - \text{rank}\, T_\delta = n - k.$$

Proof. Following [García-Planas and Magret, 1999], for all $i \geq \delta$ the relation

$$\operatorname{rank} T_{i+1} - \operatorname{rank} T_i = \ell \,(\text{constant}).$$

5 Conclusions

In this paper a detailed look at the algebraic structure of concatenated (serial and parallel) convolutional codes using techniques of linear systems theory has been made. Conditions for controllability, observability and output-observability has been obtained. These results are of interest in various fields such as for example in the study of the genetics.

Using convolutional codes for biological information processing systems can lead to the development of powerful methods for identifying and manipulating protein coding sequences within a genome as well as further our understanding of translation regulatory mechanisms. The knowledge of the output observability of the system ensures the ability of an external input to move the output from any initial condition to any final condition in a finite time interval.

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