Analysis of behavior of the eigenvalues and eigenvectors of singular linear systems

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Abstract: Let $E(p)\dot{x} = A(p)x + B(p)u$ be a family of singular linear systems smoothly dependent on a vector of real parameters $p = (p_1, \dots, p_n)$. In this work we construct versal deformations of the given differentiable family under an equivalence relation, providing a special parametrization of space of systems, which can be effectively applied to perturbation analysis. Furthermore in particular, we study the behavior of a simple eigenvalue of a singular linear system family $E(p)\dot{x} = A(p)x + B(p)u$.

Key–Words: Singular linear systems, Eigenvalues, Perturbation, Versal deformation.

1 Introduction

Let us consider a finite-dimensional singular linear time-invariant system

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0, \tag{1}$$

where x is the state vector, u is the input (or control) vector, $E, A \in M_n(\mathbb{C})$, $B \in M_{n\times m}(\mathbb{C})$, and $\dot{x} = dx/dt$. We will represent the systems as a triples of matrices (E, A, B), and we will denote by $\mathcal{M}$, the set of this kind of triples. In the case where $E = I_n$ the system is standard denoted by a pair (A, B).

Singular systems (also called diferential/algebraic systems, descriptor systems or generalized systems), are found in engineering systems such as electrical, chemical processing circuit or power systems among others, and they have attracted interest in recent years.

The transfer matrix of the system (E, A, B) relating the inputs to the outputs is given by $H(s) = (sE - A)^{-1}B$, so there is a singular matrix pencil $(sE-A)$ associated with the system. Eigenvalues corresponding to the matrix pencil $(sE - A)$ associated with the system (1) play an important role in analysis of the stability of the solutions. The values of eigenvalues can correspond to frequencies of vibration, or critical values of stability parameters, or energy levels of atoms.

Sometimes it is possible to change the value of some eigenvalues introducing proportional and derivative feedback controls in the system. The values of the eigenvalues that can not be modified by any feedback (proportional or derivative), correspond to the eigenvalues of the singular pencil $(sE - A\ B)$, that we will simply call eigenvalues of the triple (E, A, B).

Perturbation theory of linear systems has been extensively studied over the last 50 years starting from the works of Rayleigh and Schrodinger [13]. It is a tool for efficiently approximating the influence of small perturbations on different properties of the unperturbed system.

It is well known that if λ_0 is an eigenvalue of a singular system $E\dot{x} = Ax + Bu$ for almost all perturbations of the system, the perturbed system has no eigenvalues (see [4], for example). We are interested in analyzing the perturbation of the eigenvalues in the case where not all eigenvalues disappear under small perturbations.

When the perturbed system depends on parameters, we say that we have a family of systems. Additionally, if the parameters of the family vary slightly in a neighborhood of a fixed value, we says that we have a deformation.

The study of all deformations is sometimes reduced to the study of the only one from which the rest derives. This new family must be richer, in some sense, than any other, giving us all possible bifurcations of the original system. This kind of deformations is called versal.

The Arnold technique [2] to construct a versal deformation of a differentiable family of square matrices under conjugation has been generalized by several authors to matrix pencils under the strict equivalence [5, 10], pairs or triples of matrices under the action of the general linear group [17], pairs of matrices under the feedback similarity [6].

Versal deformations provide a special parametrization of matrix spaces, which can be effectively applied to perturbation analysis and investigation of complicated objects like singularities and bifurcations in multi-parameter dynamical systems.

In the sequel we will denote by $\mathcal{M} = \{(E, A, B) \mid E, A \in M_n(\mathbb{C}), B \in M_{n\times n}(\mathbb{C})\}$ the set of singular systems.

2 Versal deformations

First, we recall the definition of versal deformation. Let M be a smooth manifold (in our particular setup $M = \mathcal{M}$).

Definition 1 *Let $\mathcal{U}_0$ be a neighborhood of the origin of $\mathbb{C}^\ell$. A deformation $\varphi(\lambda)$ of x_0 is a smooth mapping*

$$\varphi : \mathcal{U}_0 \longrightarrow M$$

such that $\varphi(0) = x_0$. The vector $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \mathcal{U}_0$ is called the parameter vector.

The deformation $\varphi(\lambda)$ is also called *differentiable family* of elements of M.

Let $\mathcal{G}$ be a Lie group acting smoothly on M. We denote the action of $g \in \mathcal{G}$ on $x \in M$ by $g \circ x$.

Definition 2 *The deformation $\varphi(\lambda)$ of x_0 is called versal if any deformation $\varphi'(\xi)$ of x_0, where $\xi = (\xi_1, \dots, \xi_k) \in \mathcal{U}'_0 \subset \mathbb{C}^k$ is the parameter vector, can be represented in some neighborhood of the origin as*

$$\varphi'(\xi) = g(\xi) \circ \varphi(\phi(\xi)), \quad \xi \in \mathcal{U}''_0 \subset \mathcal{U}'_0, \qquad (2)$$

where $\phi : \mathcal{U}''_0 \longrightarrow \mathbb{C}^\ell$ and $g : \mathcal{U}''_0 \longrightarrow \mathcal{G}$ are differentiable mappings such that $\phi(0) = 0$ and $g(0)$ is the identity element of $\mathcal{G}$. Expression (2) means that any deformation $\varphi'(\xi)$ of x_0 can be obtained from the versal deformation $\varphi(\lambda)$ of x_0 by an appropriate smooth change of parameters $\lambda = \phi(\xi)$ and an equivalence transformation $g(\xi)$ smoothly depending on parameters.

A versal deformation having minimal number of parameters is called *miniversal*.

The following result was proved by Arnold [2] in the case where $\mathrm{Gl}(n;\mathbb{C})$ acts on $M_n(\mathbb{C})$, and was generalized by Tannenbaum [17] in the case where a Lie group acts on a complex manifold. This provides the relationship between a versal deformation of x_0 and the local structure of the orbit.

Theorem 3 *[17]*

1. *A deformation $\varphi(\lambda)$ of x_0 is versal if and only if it is transversal to the orbit $\mathcal{O}(x_0)$ at x_0.*
2. *Minimal number of parameters of a versal deformation is equal to the codimension of the orbit of x_0 in M, $\ell = \operatorname{codim} \mathcal{O}(x_0)$.*

Let $\{v_1, \dots, v_\ell\}$ be a basis of any arbitrary complementary subspace $(T_{x_0}\mathcal{O}(x_0))^c$ to $T_{x_0}\mathcal{O}(x_0)$ (for example, $(T_{x_0}\mathcal{O}(x_0))^\perp$).

Corollary 4 *The deformation*

$$x : \mathcal{U}_0 \subset \mathbb{C}^\ell \longrightarrow M, \quad x(\lambda) = x_0 + \sum_{i=1}^{\ell} \lambda_i v_i \qquad (3)$$

is a miniversal deformation.

We are interested in constructing a versal deformation of a given differentiable family of singular systems $(E(p), A(p), B(p)) \in \mathcal{M}$.

In order to construct a versal deformation of a system (E_0, A_0, B_0) in the given family, we define the following equivalence relation.

Definition 5 *Two triples (E', A', B') and (E, A, B) in $\mathcal{M}$ are called equivalent if, and only if, there exist matrices $P, Q \in Gl(n;\mathbb{C})$, $R \in Gl(m;\mathbb{C})$, $K_1, K_2 \in M_{m\times n}(\mathbb{C})$ such that*

$$(E', A', B') = (QEP + QBK_1, QAP + QBK_2, QBR), \qquad (4)$$

or in a matrix form

$$\begin{pmatrix} E' & A' & B' \end{pmatrix} = Q \begin{pmatrix} E & A & B \end{pmatrix} \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ K_1 & K_2 & R \end{pmatrix}.$$

It is easy to check that this relation is an equivalence relation.

Let $\mathcal{G} = \{(P, Q, R, K_1, K_2) \in Gl(n;\mathbb{C}) \times Gl(n;\mathbb{C}) \times Gl(m;\mathbb{C}) \times M_{m\times n}(\mathbb{C}) \times M_{m\times n}(\mathbb{C})\}$ be a set. Notice that this set is a Lie group.

The system $(E, A, B) \in \mathcal{M}$, for which there exists a matrix K_1 such that $E + BK_1$ is invertible is called standarizable, and in this case there exist matrices P, Q, K_1 such that $QEP + QBK_1 = I_n$. Consequently the equivalent system is standard. Notice that the standarizable character is invariant under the equivalence relation being considered.

If the original system is standard and if we want to preserve this condition under the equivalence relation we restrict the operation to the subgroup $\mathcal{G}_1 = \{(P, P^{-1}, R, 0, K_2) \in Gl(n;\mathbb{C}) \times Gl(n;\mathbb{C}) \times Gl(m;\mathbb{C}) \times M_{m\times n}(\mathbb{C}) \times M_{m\times n}(\mathbb{C})\} \subset \mathcal{G}$, so obtaining alternatively the feedback similarity relation defined over the set of standard systems.

The equivalence relation (4) can be seen as the action of the Lie group $\mathcal{G}$ on $\mathcal{M}$ in the following manner.

$$\begin{array}{l}\alpha : \mathcal{G} \times \mathcal{M} \to \mathcal{M}\\ (g, x) \to (QEP + QBK_1, QAP + QBK_2, QBR)\end{array} \tag{5}$$

where $g = (P, Q, R, K_1, K_2)$, and $x = (E, A, B)$. (From now on and if confusion is not possible, we will make use this reduced notation).

Given a triple $x_0 = (E_0, A_0, B_0) \in \mathcal{M}$ we define the map

$$\alpha_{x_0}(g) = \alpha(g, x_0). \tag{6}$$

The equivalence class of the triple x_0 with respect to the $\mathcal{G}$-action, called the $\mathcal{G}$-orbit of x_0, is the range of the function α_{x_0} and is denoted by

$$\mathcal{O}(x_0) = \operatorname{Im} \alpha_{x_0} = \{\alpha_{x_0}(g) \mid g \in \mathcal{G}\}. \tag{7}$$

Let us denote by $T_e\mathcal{G}$ the tangent space to the manifold $\mathcal{G}$ at the unit element e. It is well known that

$$\begin{aligned} T_e\mathcal{G} &= (M_{n\times n}(\mathbb{C}))^2 \times M_{m\times m}(\mathbb{C}) \times (M_{m\times n}(\mathbb{C}))^2,\\ T_{x_0}\mathcal{M} &= \mathcal{M}.\end{aligned}$$

Proposition 6 *Let $d\alpha_{x_0} : T_e\mathcal{G} \longrightarrow \mathcal{M}$ be the differential of α_{x_0} at the unit element e. Then*

$$\begin{array}{l} d\alpha_{x_0}(G) =\\ (EP + QE + BK_1, AP + QA + BK_2, BR + QB),\end{array} \tag{8}$$

where $G = (Q, P, R, K_1, K_2) \in T_e\mathcal{G}$.

Clearly, $T_{x_0}\mathcal{O}(x_0) = \operatorname{Im} d\alpha_{x_0} \subset \mathcal{M}$.

In order to apply Theorem 3 we will try to obtain a complementary subspace of $T_{x_0}\mathcal{O}(x_0)$, in particular an orthogonal subspace. For that we need to consider a scalar product.

Hermitian product in $\mathcal{M}$ to be dealt with in this paper is the following:

$$\begin{array}{l}\langle x_1, x_2\rangle =\\ \operatorname{trace}(E_1E_2^*) + \operatorname{trace}(A_1A_2^*) + \operatorname{trace}(B_1B_2^*),\end{array} \tag{9}$$

where $x_i = (E_i, A_i, B_i) \in \mathcal{M}$, and A^* denotes the conjugate transpose of the matrix A.

A description of $T_{x_0}\mathcal{O}(x_0)^\perp$ for $x_0 \in \mathcal{M}$ can be easily deduced.

Proposition 7 *Let $x_0 = (E, A, B) \in \mathcal{M}$ be a triple of matrices. Then $(X, Y, Z) \in T_{x_0}\mathcal{O}(x_0)^\perp$ if and only if*

$$\left.\begin{aligned} EX^* + AY^* + BZ^* &= 0\\ X^*E + Y^*A &= 0\\ X^*B &= 0\\ Y^*B &= 0\\ Z^*B &= 0\end{aligned}\right\}. \tag{10}$$

We will use the description of the orthogonal complementary subspace to the tangent space to the orbit of x_0 obtained in Proposition 7 for explicitly obtaining miniversal deformations.

Proposition 8 *Let $x_0 = (E, A, B)$ be a triple of matrices and $\{u_1, \dots, u_r\}$ be a basis of the vector subspace $T_{x_0}\mathcal{O}(x_0)^\perp$. Then the map defined by*

$$\varphi(\lambda_1, \dots, \lambda_r) = x_0 + \lambda_1u_1 + \dots + \lambda_ru_r$$

is a miniversal deformation with respect to the $\mathcal{G}$-action.

Example 1 Let $\left(\begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}0&0\\0&3\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right)$ be a system. Solving the system (10), we obtain the following "orthogonal" miniversal deformation:

$$\left(\begin{pmatrix}0&0\\-3\lambda_1&1-3\lambda_2\end{pmatrix}, \begin{pmatrix}0&0\\\lambda_1&3+\lambda_2\end{pmatrix}, \begin{pmatrix}1\\3\lambda_1\end{pmatrix}\right).$$

Now we can obtain a minimal miniversal deformation by selecting a basis of a complementary space to the tangent space to the orbit in the following manner: for the first vector, we put 1 in λ_1 in the matrix $\begin{pmatrix}0&0\\\lambda_1&3+\lambda_2\end{pmatrix}$ and zero for all the other parameters in this matrix and we put zero in all parameters in the other matrices. For the second vector, we put 1 in λ_2 in the same matrix and zero for the other all parameters in this matrix and we put zero in all parameters in the other matrices, obtaining the following minimal miniversal deformation:

$$\left(\begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}0&0\\\lambda_1&3+\lambda_2\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right).$$

In this reduced form it is easy to analyze the eigenvalues of the perturbed systems. If $\lambda_1 \neq 0$, the perturbed system has no eigenvalues whereas if $\lambda_1 = 0$ the system has only one eigenvalue, concretely $3 + \lambda_2$.

The versal deformation can be effectively applied to perturbation analysis of eigenvalues of a given system. If we have a family of systems in a neighborhood of a system we can relate it with a miniversal deformation in the following manner.

First of all we consider the *stabilizer* of x_0 under the $\mathcal{G}$-action, this set is defined as the null-space of the function $\alpha_{x_0} - x_0$. This is denoted by

$$\mathcal{S}(x_0) = \operatorname{Ker}(\alpha_{x_0} - x_0) = \{g \in \mathcal{G} \mid g \circ x_0 = x_0\}. \tag{11}$$

The differentiability of the mapping α_{x_0} ensures that $\mathcal{S}(x_0)$ is a smooth submanifold of $\mathcal{G}$.

As in the case of orbits, we can consider $T_e\mathcal{S}(x_0)^\perp$ with respect a Hermitian product in the space $T_e\mathcal{G}$. Concretely we are going to consider in this paper the following

$$\begin{array}{l}\langle y_1, y_2\rangle = \\ \operatorname{trace}(Q_1Q_2^*) + \operatorname{trace}(P_1P_2^*) + \operatorname{trace}(R_1R_2^*)+ \\ \operatorname{trace}(K_{1_1}K_{1_2}^*) + \operatorname{trace}(K_{2_1}K_{2_2}^*),\end{array} \tag{12}$$

where $y_i = (Q_i, P_i, R_i, K_{1_i}, K_{2_i}) \in T_e\mathcal{G}$.

Theorem 9 *The tangent space to the stabilizer of the system x_0 and the corresponding normal complementary subspace with respect to $T_e\mathcal{G}$ can be found in the following form*

1. $T_e\mathcal{S}(x_0) = \operatorname{Ker} d\alpha_{x_0} \subset T_e\mathcal{G}$,
2. $(T_e\mathcal{S}(x_0))^\perp = \operatorname{Im} d\alpha_{x_0}^* \subset T_e\mathcal{G}$.

Corollary 10 *The mappings $d\alpha_{x_0}$ and $d\alpha_{x_0}^*$ define one-to-one correspondences between the subspaces $T_{x_0}\mathcal{O}(x_0)$ and $(T_e\mathcal{S}(x_0))^\perp$:*

$$T_{x_0}\mathcal{O}(x_0) \underset{d\alpha_{x_0}}{\overset{d\alpha_{x_0}^*}{\rightleftarrows}} (T_e\mathcal{S}(x_0))^\perp.$$

As a consequence obtain the following.

Theorem 11 *If $\varphi(\gamma)$ is a miniversal deformation and values of the mapping $g(\xi)$ are restricted to belong to a smooth submanifold $\mathcal{R} \subset \mathcal{G}$, which is transversal to $\mathcal{S}(x_0)$ at e and has the minimal dimension $\dim \mathcal{R} = \operatorname{codim} \mathcal{S}(x_0)$, then the mappings $\phi(\xi)$ and $g(\xi)$ in representation (2) are uniquely determined by $\varphi'(\xi)$.*

Let $\{r_1, \dots, r_d\}$ be a basis for $T_e\mathcal{S}(x_0)^\perp$, we have the following.

Corollary 12 *The functions $g(\xi)$ and $\phi(\xi)$ in the versal deformation reduction (2) are uniquely determined if the mapping $g(\xi)$ is taken in the form*

$$g(\xi) = e + \sum_{i=1}^{d} r_i\mu_i(\xi),$$

where $\mu(\xi)$ are smooth functions in $\mathbb{C}$ such that $\mu_j(0) = 0$, for $i = 1, \dots d$.

3 Perturbation analysis of simple eigenvalues of standard systems

For a more comprehensive analysis, we begin studying the case of standard systems. So, we consider systems in the form $\dot{x} = Ax + Bu$ with $A \in M_n(\mathbb{C})$ and $B \in M_{n\times m}(\mathbb{C})$. Hence, we will write the systems as a pair of matrices (A, B).

Remember that $\lambda_0 \in \mathbb{C}$ is an eigenvalue of the system if and only if there exists a non-zero vector v_0 such that

$$\left.\begin{array}{ll} A^tv_0 & = \lambda_0 v_0 \\ B^tv_0 & = 0 \end{array}\right\},$$

and v_0 is called eigenvector of the system for this eigenvalue.

The eigenvalues of the system (A, B) correspond to the eigenvalues of the associate singular pencil $(sI - A\ B)$ and the eigenvectors correspond to the left eigenvectors of the pencil.

Remark 13 *The vector v_0 is an eigenvector of A^t corresponding to the eigenvalue λ_0, So, λ_0 is an eigenvalue of the matrix A, and the corresponding eigenvector u_0 is a left eigenvector of the matrix A^t.*

Definition 14 *An eigenvalue λ_0 of a system (A, B) is called simple if it is simple as eigenvalue of A.*

Observe that an eigenvalue of A is not necessarily an eigenvalue of (A, B).

Proposition 15 ([11]) *If λ_0 is a simple eigenvalue of (A, B). Then we can choose an eigenvector u_0 of A and an eigenvector v_0 of (A, B) such that $u_0^tv_0 \neq 0$.*

Sometimes an eigenvalue of (A, B) is not simple but there can exists a feedback such that the resulting closed-loop system has a simple eigenvalue as we can be seen in the following example.

Example 2 Let $(A, B) = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}, \begin{pmatrix} 1 \\ 0\end{pmatrix}\right)$ be a system. We observe that $\lambda = 1$ is an eigenvalue of (A, B) being a double eigenvalue of A. Taking the feedback $\begin{pmatrix} 1 & 0\end{pmatrix}$, the closed-loop system is $\left(\begin{pmatrix} 2 & 1 \\ 0 & 1\end{pmatrix}, \begin{pmatrix} 1 \\ 0\end{pmatrix}\right)$. It is not difficult to observe that $\lambda = 1$ is an eigenvalue for this system and a simple eigenvalue of $A + BF$. Observe that $\lambda = 2$ is a simple eigenvalue of $A + BF$ but not an eigenvalue of $(A + BF, B)$.

Let λ_0 be a multiple eigenvalue of A^t which is a simple eigenvalue of $A^t + K^tB^t$ for some feedback K.

Proposition 16 *Let λ_0 be an eigenvalue and v_0 be a corresponding eigenvector of (A, B). Then λ_0 is an eigenvalue and v_0 it is corresponding eigenvector of $(A + BK, B)$ for all K.*

Proof: If $A^t v_0 = \lambda_0 v_0$ and $B^t v_0 = 0$ then $K^t B^t v_0 = 0$ and $(A^t + K^t B^t)v_0 = \lambda_0 v_0$.

Reciprocally, if $(A^t + K^t B^t)v_0 = \lambda_0 v_0$ and $B^t v_0 = 0$, then $A^t v_0 = A^t v_0 + K^t B^t v_0 - K^t B^t v_0 = (A^t + K^t B^t)v_0 - K^t B^t v_0 = \lambda_0 v_0$. □

Corollary 17 *Let $K \in M_{m\times n}(\mathbb{C})$ be a feedback such that μ_0 is an eigenvalue of $A^t + K^t B^t$ and w_0 is a corresponding eigenvector. If μ is not an eigenvalue of (A, B), then $B^t w \neq 0$.*

Let (A, B) be a linear system and assume that the matrices A, B smoothly depend on the vector $p = (p_1, \dots, p_r)$ of real parameters. The function $(A(p), B(p))$ is called a multi-parameter family of linear systems. Eigenvalues of linear system functions are continuous functions $\lambda(p)$ of the vector of parameters. In this section, we are going to study the behavior of a simple eigenvalue of the family of linear systems $(A(p), B(p))$.

Let us consider a point p_0 in the parameter space and assume that $\lambda(p_0) = \lambda_0$ is a simple eigenvalue of $(A(p_0), B(p_0)) = (A_0, B_0)$, and $v(p_0) = v_0$ is an eigenvector, i.e.

$$\left.\begin{array}{ll} A_0^t v_0 & = \lambda_0 v_0 \\ B_0^t v_0 & = 0 \end{array}\right\}.$$

Equivalently

$$\left.\begin{array}{rl} (A_0^t + K^t B^t)v_0 & = \lambda_0 v_0 \\ B_0^t v_0 & = 0 \end{array}\right\}, \ \forall K \in M_{m\times n}(\mathbb{C}).$$

Now, we are going to review the behavior of a simple eigenvalue $\lambda(p)$ of the family of standard linear systems.

The eigenvector $v(p)$ corresponding to the simple eigenvalue $\lambda(p)$ determines a one-dimensional null-subspace of the matrix operator $\begin{pmatrix} A^t \\ B^t \end{pmatrix}$ smoothly dependent on p. Hence, the eigenvector $v(p)$ can be chosen as a smooth function of the parameters. We will try to obtain an approximation by means of their derivatives.

We write the eigenvalue problem as

$$\left.\begin{array}{r} (A^t(p) + K^t(p)B^t(p))v(p) = \lambda(p)v(p) \\ B^t(p)v(p) = 0. \end{array}\right\}, \tag{13}$$

or equivalently

$$\left.\begin{array}{r} A^t(p)v(p) = \lambda(p)v(p) \\ B^t(p)v(p) = 0. \end{array}\right\}. \tag{14}$$

Taking the derivatives with respect to p_i, we have

$$\left.\begin{array}{r} \frac{\partial A^t(p)}{\partial p_i}v(p) + A^t(p)\frac{\partial v(p)}{\partial p_i} = \frac{\partial \lambda}{\partial p_i}v(p) + \lambda(p)\frac{\partial v(p)}{\partial p_i} \\ \frac{\partial B^t(p)}{\partial p_i}v(p) + B^t(p)\frac{\partial v(p)}{\partial p_i} = 0 \end{array}\right\}.$$

At the point p_0, we obtain

$$\left.\begin{array}{r} \left(\frac{\partial A^t(p)}{\partial p_i} - \frac{\partial \lambda}{\partial p_i}I_n\right)_{|p_0} v_0 = (\lambda_0 I_n - A^t(p_0))\frac{\partial v(p)}{\partial p_i}_{|p_0} \\ \frac{\partial B^t(p)}{\partial p_i}_{|p_0} v_0 + B^t(p_0)\frac{\partial v(p)}{\partial p_i}_{|p_0} = 0 \end{array}\right\}. \tag{15}$$

This is a linear equation system for the unknowns $\frac{\partial \lambda}{\partial p_i}$ and $\frac{\partial v(p)}{\partial p_i}$, where the matrix $\lambda_0 I_n - A^t(p_0)$ is singular with rank equal to $n - 1$ because λ_0 is a simple eigenvalue.

Lemma 18 *[11] The matrix $\lambda_0 I_n - A^t(p_0) - u_0 u_0^t$ is invertible.*

Under the same conditions we have the following

Proposition 19 *The system (15) has a solution if and only if*

$$\left.\begin{array}{r} u_0^t\left(\frac{\partial \lambda}{\partial p_i}_{|p_0} I_n - \frac{\partial A^t(p)}{\partial p_i}_{|p_0}\right)v_0 = 0 \\ \frac{\partial B^t(p)}{\partial p_i}_{|p_0} v_0 + B^t(p_0)\frac{\partial v(p)}{\partial p_i}_{|p_0} = 0 \end{array}\right\} \tag{16}$$

where u_0 is a left eigenvector for the simple eigenvalue λ_0 of the matrix A^t.

Proof: From the first equation in (16) we obtain a solution for $\frac{\partial \lambda}{\partial p_i}_{|(\lambda_0;p_0)}$:

$$\frac{\partial \lambda}{\partial p_i}(u_0^t v_0) = u_0^t \frac{\partial A^t(p)}{\partial p_i}v_0,$$

$$\frac{\partial \lambda}{\partial p_i} = \frac{u_0^t \frac{\partial A^t(p)}{\partial p}v_0}{u_0^t v_0}. \tag{17}$$

We can choice u_0 in such away that $u_0^t v_0 = 1$.

Replacing this solution in the first equation in (15) we obtain

$$\frac{\partial v(p)}{\partial p_i} = (\lambda_0 I_n - A^t(p_0) - u_0 u_0^t)^{-1}\left(\frac{\partial A^t(p)}{\partial p_i}_{|p_i} - \frac{\partial \lambda}{\partial p_i}I_n\right)v_0.$$

Now we need to see if this expression verifies the second equation of (15). □

Taking the partial derivative $\partial^2/\partial p_i\partial p_j$ on both sides of both equations in the eigenvalue problem (13), we have:

$$\left.\begin{array}{l} \frac{\partial^2 A^t(p)}{\partial p_i\partial p_j}v(p)+\frac{\partial A^t(p)}{\partial p_i}\frac{\partial v(p)}{\partial v(p)}\partial p_j+\\ \quad\frac{\partial A^t(p)}{\partial p_j}\frac{\partial v(p)}{\partial v(p)}\partial p_i+A^t(p)\frac{\partial^2 v(p)}{\partial p_i\partial p_j}=\\ \\ \frac{\partial^2\lambda(p)}{\partial p_i\partial p_j}v(p)+\frac{\partial\lambda(p)}{\partial p_i}\frac{\partial v(p)}{\partial p_j}+\frac{\partial\lambda(p)}{\partial p_j}\frac{\partial v(p)}{\partial p_i}+\\ \quad\lambda(p)\frac{\partial^2 v(p)}{\partial p_i\partial p_j}\frac{\partial^2 B^t}{\partial p_i\partial P_j}v(p)+\\ \quad\frac{\partial B^t(p)}{\partial p_i}\frac{\partial v(p)}{\partial p_j}+\frac{\partial B^t(p)}{\partial p_j}\frac{\partial v(p)}{\partial p_i}+B^t\frac{\partial^2 v(p)}{\partial p_i\partial p_j}=0 \end{array}\right\}.$$

At p_0 and premultiplying the previous equation by u_0^t, we can deduce the following expression for derivatives $\frac{\partial^2\lambda(p)}{\partial p_i\partial p_j}_{|p_0}$

$$\begin{array}{l} \frac{\partial^2\lambda(p)}{\partial p_i\partial p_j}_{|p_0}u_0^t v_0=\\ u_0^t\frac{\partial^2 A^t(p)}{\partial p_i\partial p_j}_{|p_0}v_0+u_0\frac{\partial A^t(p)}{\partial p_j}_{|p_0}\frac{\partial v(p)}{\partial p_i}_{|p_0}+\\ \quad u_0\frac{\partial A^t(p)}{\partial p_i}_{|p_0}\frac{\partial v(p)}{\partial p_j}_{|p_0}-\\ u_0^t\frac{\partial\lambda(p)}{\partial p_j}_{|p_0}\frac{\partial v(p)}{\partial p_i}_{|p_0}-u_0^t\frac{\partial\lambda(p)}{\partial p_i}_{|p_0}\frac{\partial v(p)}{\partial p_j}_{|p_0}. \end{array}$$

Once $\frac{\partial^2\lambda}{\partial p_i\partial p_j}_{|p_0}$ we can deduce the values of $\frac{\partial^2 v(p)}{\partial p_i\partial p_j}_{|p_0}$

$$\begin{array}{l} \frac{\partial^2 v(p)}{\partial p_i\partial p_j}_{|p_0}=(A^t(p)-\lambda(p)I-u_0u_0^t)^{-1}\cdot\\ \quad\Big(\frac{\partial^2\lambda(p)}{\partial p_i\partial p_j}v(p)+\frac{\partial\lambda(p)}{\partial p_i}\frac{\partial v(p)}{\partial p_j}\\ \qquad+\frac{\partial\lambda(p)}{\partial p_j}\frac{\partial v(p)}{\partial p_i}-\frac{\partial^2 A^t(p)}{\partial p_i\partial p_j}v(p)-\\ \qquad\quad\frac{\partial A^t(p)}{\partial p_i}\frac{\partial v(p)}{\partial p_j}-\frac{\partial A^t(p)}{\partial p_j}\frac{\partial v(p)}{\partial p_i}\Big). \end{array}$$

Example 3 Consider now, the following differentiable family of standard systems

$$\left(\begin{pmatrix}0&0&0\\0&3+p_1&p_2\\0&p_2&4+p_1\end{pmatrix},\begin{pmatrix}1+p_1\\0\\0\end{pmatrix}\right).$$

At $p_0=(0,0)$, the matrix A has $\lambda=3$ as a simple eigenvalue (and $\lambda=4$). The corresponding left and right eigenvectors are $v_0=(0,1,0)$ and $u_0=(0,1,0)$.

$A^t(p)=A(p)$,

$$\frac{\partial A^t}{\partial p_1}=\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix},\qquad\frac{\partial B^t}{\partial p_1}=\begin{pmatrix}1\\0\\0\end{pmatrix},$$

$$\frac{\partial A^t}{\partial p_2}=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\qquad\frac{\partial B^t}{\partial p_2}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

$$\frac{\partial\lambda(P)}{\partial p_1}=\begin{pmatrix}0&1&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}=1,$$

$$\frac{\partial\lambda(P)}{\partial p_2}=\begin{pmatrix}0&1&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}=0.$$

$$\begin{array}{l}\frac{\partial^2\lambda(p)}{\partial p_1^2}_{|p_0}=2\begin{pmatrix}0&1&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}0\\-1\\0\end{pmatrix}\\ \qquad-2\begin{pmatrix}0&1&0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}=-2-2=-4.\end{array}$$

$$\begin{array}{l}\frac{\partial^2\lambda(p)}{\partial p_2^2}_{|p_0}=2\begin{pmatrix}0&1&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0\\1\\-1\end{pmatrix}\\ \qquad-2\begin{pmatrix}0&1&0\end{pmatrix}\begin{pmatrix}0\\1\\-1\end{pmatrix}=-4,\end{array}$$

$$\frac{\partial^2\lambda(p)}{\partial p_1\partial p_2}_{|p_0}=\frac{\partial^2\lambda(p)}{\partial p_2\partial p_1}_{|p_0}=0.$$

$$\begin{array}{l}\frac{\partial v(p)}{\partial p_1}_{|p_0}=\left(\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}-\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}\right)^{-1}\cdot\\ \left(\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right)\begin{pmatrix}0\\1\\0\end{pmatrix}=\\ \begin{pmatrix}0\\-1\\0\end{pmatrix},\end{array}$$

$$\begin{array}{l}\frac{\partial v(p)}{\partial p_2}_{|p_0}=\left(\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}-\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}\right)^{-1}\\ \left(\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right)\begin{pmatrix}0\\1\\0\end{pmatrix}=\\ \begin{pmatrix}0\\1\\-1\end{pmatrix}.\end{array}$$

$$\begin{array}{l}\frac{\partial B^t(p)}{\partial p_1}_{|p_0}v(p_0)+B^t(p)\frac{\partial v(p)}{\partial p_1}_{|p_0}=\\ \begin{pmatrix}1&0&0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}+\begin{pmatrix}1&0&0\end{pmatrix}\begin{pmatrix}0\\-1\\0\end{pmatrix}=(0)\end{array}$$

$$\begin{array}{l}\frac{\partial B^t(p)}{\partial p_2}_{|p_0}v(p_0)+B^t(p)\frac{\partial v(p)}{\partial p_2}_{|p_0}=\\ \begin{pmatrix}1&0&0\end{pmatrix}\begin{pmatrix}0\\1\\-1\end{pmatrix}=(0).\end{array}$$

Then,
$$\lambda(p)=3+p_1-4p_1^2-2p_2^2+O(3),$$
$$v(p)=\begin{pmatrix}0\\1\\0\end{pmatrix}+\begin{pmatrix}0\\1\\0\end{pmatrix}p_1+\begin{pmatrix}0\\1\\-1\end{pmatrix}p_2+O(2).$$

4 Perturbation analysis of simple eigenvalues of singular systems

Now we consider singular systems in the form $E\dot{x} = Ax + Bu$ where $E, A \in M_n(\mathbb{C})$ and $B \in M_{n\times m}(\mathbb{C})$. This will be written we will write as triple of matrices (E, A, B).

Let $M(\lambda) = (\lambda E - A, B)$ be a matrix pencil associated with the triple (E, A, B), λ_0 is an eigenvalue of (E, A, B), if $\operatorname{rank} M(\lambda_0) < \operatorname{rank} M(\lambda)$. In the case where the matrix pencil $\lambda E - A$ is regular this is equivalent to $\det(\lambda_0 E - A) = 0$.

$v_0 \in \mathbb{C}^n$ is an eigenvector corresponding to the eigenvalue λ_0, if $(\lambda_0 E^t - A^t)v_0 = 0$ and $B^t v_0 = 0$.

Proposition 20 *Let λ_0 be an eigenvalue and v_0 an associated eigenvector of the (E, A, B). Then λ_0 is an eigenvalue and v_0 an associated eigenvector of $(E + BK_1, A + BK_2, B)$ for all K.*

Suppose that matrices E, A, B, defining the singular system, smoothly depend on the vector $p = (p_1, \dots, p_r)$ of real parameters. The function $(E(p), A(p), B(p))$ is called a multi-parameter family of singular systems.

We write the eigenvalue problem as

$$\left.\begin{aligned}(\lambda E^t(p) - A^t(p))v(p) &= 0\\ B^t(p)v(p) &= 0\end{aligned}\right\}$$

or equivalently

$$\left.\begin{aligned}(\lambda(E^t(p) + K_1(p)B^t(p)) - (A^t(p)+&\\ K_2^t(p)B^t(p)))v(p) &= 0\\ B^t(p)v(p) &= 0\end{aligned}\right\}$$

Taking derivatives we obtain

$$\left.\begin{aligned}\left(\frac{\partial\lambda}{\partial p_i}E^t(p) + \lambda\frac{\partial E^t(p)}{\partial p_i} - \frac{\partial A^t(p)}{\partial p_i}\right)v(p)+&\\ (\lambda E^t(p) - A^t(p))\frac{\partial v(p)}{\partial p_i} &= 0\\ \frac{\partial B^t(p)}{\partial p_i}v(p) + B^t(p)\frac{\partial v(p)}{\partial p_i} &= 0\end{aligned}\right\}$$

At the point (λ_0, p_0) the result is

$$\left.\begin{aligned}\Bigg(\left(\frac{\partial\lambda}{\partial p_i}E^t(p) + \lambda\frac{\partial E^t(p)}{\partial p_i} - \frac{\partial A^t(p)}{\partial p_i}\right)v(p)+&\\ (\lambda E^t(p) - A^t(p))\frac{\partial v(p)}{\partial p_i}\Bigg)_{|(\lambda_0,p_0)} &= 0\\ \left(\frac{\partial B^t(p)}{\partial p_i}v(p) + B^t(p)\frac{\partial v(p)}{\partial p_i}\right)_{|(\lambda_0,p_0)} &= 0\end{aligned}\right\}$$

Premultiplying the first equality by u_0^t we have

$$\left.\begin{aligned}u_0^t\left(\frac{\partial\lambda}{\partial p_i}E^t(p) + \lambda_0\frac{\partial E^t(p)}{\partial p_i} - \frac{\partial A^t(p)}{\partial p_i}\right)_{|(\lambda_0,p_0)}v_0 &=0\\ \frac{\partial B^t(p)}{\partial p_i}_{|(\lambda_0,p_0)}v_0 + B_0^t\frac{\partial v(p)}{\partial p_i}_{|(\lambda_0,p_0)} &=0\end{aligned}\right\}$$

$$\left.\begin{aligned}&\frac{\partial\lambda}{\partial p_i}_{|(\lambda_0,p_0)}u_0^tE^t(p_0)v_0 =\\ &\qquad -\lambda_0u_0^t\frac{\partial E^t(p)}{\partial p_i}_{|(\lambda_0,p_0)}v_0 + u_0^t\frac{\partial A^t(p)}{\partial p_i}_{|(\lambda_0,p_0)}v_0\\ &\frac{\partial B^t(p)}{\partial p_i}_{|(\lambda_0,p_0)}v_0 + B_0^t\frac{\partial v(p)}{\partial p_i}_{|(\lambda_0,p_0)} = 0\end{aligned}\right\}$$

Suppose that $\operatorname{rank}(\lambda_0E(p_0) - A(p_0)) = n-1$. In this case, we can chose u_0 in such a way that $u_0^tv_0 \neq 0$.

Using the normalization condition $u_0^tv(p) = 1$ is possible because the function $u_0^tv(p)$ in $p = p_0$ is non zero. Then, we have that $u_0^t\frac{\partial v(p)}{\partial p_i}_{|(\lambda_0,p_0)} = 0$.

Lemma 21 *There exists a left eigenvector such that $u_0^tE(p_0)v_0 \neq 0$.*

Proof: Take into account that λ_0 is a simple eigenvalue $E^tv_0 + (\lambda E^tv_1 - A^tv_1) \neq 0$ for all vector v_1. Taking $v_1 = 0$, we have that $E^tv_0 \neq 0$.

If $u_0^tE^tv_0 = 0$, we have that $Eu_0, Au_0 \in [v_0]^\perp$. So u_0 is an eigenvector of the linear map $(\lambda_0E - A)_{|[v_0]^\perp}$ for the zero eigenvalue, but zero is a simple eigenvalue of $\lambda_0E - A$. □

Lemma 22 *The matrix $T_0 = \lambda_0E^t(p_0) - A^t(p_0) + u_0u_0^t$ is invertible.*

Proof: $u_0u_0^t$ is a symmetric map of rank 1, u_0 is an eigenvector of eigenvalue $\|u_0\|^2$ and $[u_0]^\perp$ is the null-space.

Given $w \in \operatorname{Ker} T_0$, we can write $w = \alpha u_0 + w_1$ with $w_1 \in [u_0]^\perp$. Then $0 = T_0w$ and

$$\begin{aligned}&0 = u_0^tT_0w =\\ &u_0^t(\lambda_0E^t(p_0) - A^t(p_0) + u_0u_0^t)(\alpha u_0 + w_1) =\\ &u_0^t(u_0u_0^t)(\alpha u_0 + w_1) = \alpha(u_0^tu_0)^2.\end{aligned}$$

Then $\alpha = 0$ and $w = w_1 \in \operatorname{Ker} u_0u_0^t$. Consequently $(\lambda_0E^t(p_0) - A^t(p_0))w = 0$. Taking into account that λ_0 is a simple eigenvalue, we have $w = w_1 = \beta v_0 \in [u_0]^\perp$. Finally, condition $u_0^tv_0(p_0) \neq 0$ implies $\beta = 0$ and T_0 is invertible. □

5 Bifurcation of double eigenvalues with single eigenvector for standard systems

Let us consider an arbitrary family of linear systems $(A(p), B(p))$. Let p_0 be a point in the parameter space where the matrix $A(p_0) = A_0$ has a double nonderogatory eigenvalue λ_0. Let v_0, v_1 be the left Jordan chain of length 2 corresponding to λ_0, i.e.

$$A_0^t v_0 = \lambda_0 v_0, \quad A_0^t v_1 = \lambda_0 v_1 + v_0 \qquad (18)$$

where $w_0 = \alpha v_0$ and $w_1 = \alpha v_1 + \beta v_0$ with $\alpha \neq 0$.

Let u_0 and u_1 be the left Jordan chain corresponding to A_0^t), that is

$$\begin{array}{r} u_0^t A_0^t = \lambda_0 u_0^t, \\ u_1^t A^t = \lambda_0 u_1^t + u_0^t \end{array}$$

or equivalently

$$\begin{array}{r} A_0 u_0 = \lambda_0 u_0, \\ A_0 u_1 = \lambda_0 u_1 + u_0. \end{array}$$

Lemma 23 *Under the previous conditions, the following condition holds*

$$u_0^t v_0 = 0, \quad u_0^t v_1 \neq 0$$

Proof: From $A_0^t(v_1) = \lambda_0 v_1 + v_0$, we have, $\lambda_0 u_0 t v_1 = u_0^t A_0^t v_1 = \lambda_0 u_0^t v_1 + u_0^t v_0$. So $u_0^t v_0 = 0$.

For second condition, we consider $\bar{v}_0, \bar{v}_1$ giving an orthonormal basis of $[v_0, v_1]$ obtained by Gram-Schmidt process. Now we complete it to a basis $\{\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_{n-1}\}$ of the whole space.

In this orthonormal basis, the matrix of A_0^t is

$$\begin{pmatrix} \lambda_0 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & \lambda_0 & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & a_{n3} & \dots & a_{nn} \end{pmatrix}$$

and the matrix of A_0 is

$$\begin{pmatrix} \lambda_0 & 0 & 0 & \dots & 0 \\ a_{12} & \lambda_0 & 0 & \dots & 0 \\ a_{13} & a_{23} & a_{33} & \dots & a_{n3} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \dots & a_{nn} \end{pmatrix}$$

We observe that $[v_0, v_1]^\perp$ is an A_0-invariant subspace.

If $u_0^t v_0 = 0$, we have that $u_0 \in [v_0, v_1]^\perp$ and, taking into account that $A_0 u_0 = \lambda_0 u_0$, we have that λ_0 is an eigenvalue of multiplicity at least three. In conclusion, $u_0^t v_1 \neq 0$. $\square$

We have three possibilities:

1. λ_0 is not an eigenvalue of the pair $(A(p_0), B(p_0)) = (A_0, B_0)$, i.e. $B_0^t v_0 \neq 0$.
2. λ_0 is a simple eigenvalue of the pair $(A(p_0), B(p_0))$, i.e. $B^t v_0 = 0$ and $B^t v_1 \neq 0$.
3. λ_0 is a double nonderogatory eigenvalue of the pair $(A(p_0), B(p_0))$, i.e. $B^t v_0 = 0$ and $B^t v_1 = 0$.

Example 4

1. Consider a family of systems $(A(p), B(p))$ with $A_0^t = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ and $B_0^t = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$. The eigenvalues of A^t are $\lambda_0 = 3$ (double) and $\lambda_1 = 0$ (simple). A Jordan chain for λ_0 is $v_0^t = (0, 0, 1)$, $v_1 = (0, 1, 0)$. As $B_0^t v_0 = 0$ and $B_0^t v_1 \neq 0$, then λ_0 is a simple eigenvalue of (A_0, B_0). For λ_1 its eigenvector $w = (9, -3, 1)$ satisfies $B^t w \neq 0$. So λ_1 is not eigenvalue for (A_0, B_0).
2. Consider now a family of systems $(A(p), B(p))$ with $A_0^t = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $B_0^t = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$. In this case, the eigenvalues of A_0^t are: $\lambda_0 = 2$ being simple but not being an eigenvalue of (A_0, B_0), and; $\lambda_1 = 1$ being double for A_0 and also double for (A_0, B_0).

In order to analyze the behavior of two eigenvalues $\lambda(p)$ that merge to λ_0 at p_0, we consider a perturbation of the parameter along a smooth curve $p = p(\varepsilon)$, where $\varepsilon \geq 0$ is a small real perturbation parameter and $p(0) = p_0$.

Along the curve $p(\varepsilon) = (p_1(\varepsilon), \ldots, p_r(\varepsilon))$ we have a one parameter matrix family $(A(p(\varepsilon)), B(p(\varepsilon)))$, which can be represented in the form of Taylor expansion

$$\begin{array}{ll} A(p(\varepsilon)) & = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \ldots \\ B(p(\varepsilon)) & = B_0 + \varepsilon B_1 + \varepsilon^2 B_2 + \ldots \end{array} \qquad (19)$$

with $A_0 = A(p_0)$, $A_1 = \sum_{i=1}^r \frac{\partial A(p(\varepsilon))}{\partial p_i} \frac{dp_i}{d\varepsilon}$,
$A_2 =$
$\frac{1}{2}\left(\sum_{i=1}^r \frac{\partial A(p(\varepsilon))}{\partial p_i} \frac{d^2 p_i}{d\varepsilon^2} + \sum_{i,j=1}^r \frac{\partial^2 A(p(\varepsilon))}{\partial p_i \partial p_j} \frac{dp_i}{d\varepsilon} \frac{dp_j}{d\varepsilon}\right)$,
$B_0 = B(p_0)$,
$B_1 = \sum_{i=1}^r \frac{\partial A(p(\varepsilon))}{\partial p_i} \frac{dp_i}{d\varepsilon}$,
$B_2 =$
$\frac{1}{2}\left(\sum_{i=1}^r \frac{\partial B(p(\varepsilon))}{\partial p_i} \frac{d^2 p_i}{d\varepsilon^2} + \sum_{i,j=1}^r \frac{\partial^2 B(p(\varepsilon))}{\partial p_i \partial p_j} \frac{dp_i}{d\varepsilon} \frac{dp_j}{d\varepsilon}\right)$
where the derivatives are evaluated at p_0.

If λ_0 is a double eigenvalue of $A^t(p_0)$ having a unique eigenvector v_0 up to a non-zero scaling factor with $B^t(p_0)v_0 = 0$, the perturbation theory (see [13], for example) assumes that the double eigenvalue λ_0 generally splits into a pair of simple eigenvalues λ under perturbation of the pair of matrices $(A(p_0), B(p_0))$. These eigenvalues λ and the corresponding eigenvectors v can be represented in the form of the Puiseux series:

$$\begin{array}{l} \lambda = \lambda_0 + \varepsilon^{1/2}\lambda_1 + \varepsilon\lambda_2 + \varepsilon^{3/2}\lambda_3 + \varepsilon^2\lambda_4 + \dots \\ v = v_0 + \varepsilon^{1/2}w_1 + \varepsilon w_2 + \varepsilon^{3/2}w_3 + \varepsilon^2 w_4 + \dots \end{array} \tag{20}$$

Substituting (20) into (19), we obtain

$$\begin{array}{ll} A_0^t v_0 & = \lambda_0 v_0 \\ B_0^t v_0 & = 0 \end{array} \tag{21}$$

$$\begin{array}{ll} A_0^t w_1 & = \lambda_0 w_1 + \lambda_1 v_0 \\ B_0^t w_1 & = 0 \end{array} \tag{22}$$

$$\begin{array}{ll} A_0^t w_2 + A_1^t v_0 & = \lambda_0 w_2 + \lambda_1 w_1 + \lambda_2 v_0 \\ B_0^t w_2 + B_1^t v_0 & = 0 \end{array} \tag{23}$$

$$\begin{array}{ll} A_0^t w_3 + A_1^t w_1 & = \lambda_0 w_3 + \lambda_1 w_2 + \lambda_2 w_1 + \lambda_3 v_0 \\ B_0^t w_3 + B_1^t w_1 & = 0 \\ & \vdots \end{array} \tag{24}$$

Equation (21) is satisfied because v_0 is an eigenvector corresponding to the eigenvalue λ_0. By comparing equation (22) with (18), we observe that $w_1 = \lambda_1 v_1 + \beta v_0$ is a solution and we take $w_1 = \lambda_1 v_1$.

To find the value of λ_1, we premultiply equation (23) by u_0^t and, using the given value for w_1, we have

$$\lambda_1^2 u_0^t v_1 = u_0^t A_1^t v_0$$

Taking into account that $u_0^t v_1 \neq 0$, we obtain λ_1:

$$\lambda_1 = \sqrt{\frac{u_0^t A_1^t v_0}{u_0^t v_1}}.$$

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