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A NEW PROCEDURE TO SMOOTH AND UNTANGLE MESHES ON PARAMETERIZED SURFACES

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Abstract. *We present a technique to extend any distortion (quality) measure for planar meshes to meshes on parameterized surfaces. The resulting distortion (quality) measure is expressed in terms of the parametric coordinates of the nodes. This extended distortion (quality) measure can be used to check the quality and validity of both triangle and quadrilateral surface meshes. We also apply it to simultaneously smooth and untangle surface meshes by minimizing the extended distortion measure. The minimization is performed in terms of the parametric coordinates of the nodes and therefore, the nodes always lie on the surface. Finally, we include several examples to illustrate the applicability of the proposed technique. Specifically, we extend several Jacobian-based measures, and we us them to smooth and untangle triangle and quadrilateral meshes on CAD surfaces.*

1 INTRODUCTION

During the last decades, the Finite Element Method (FEM) has become one of the most-used techniques in applied sciences and engineering. The application of the method requires a previous discretization of the geometry. Moreover, the accuracy of the FEM simulations depends on the quality of this discretization. On the one hand, this discretization has to be composed by elements of the correct size to capture properly the geometry details. On the other hand, this discretization has to be composed by well-shaped elements that satisfy certain geometrical requirements. One of the most-succesful techniques to improve the quality of a given mesh is to relocate inner nodes while maintaining the connectivity of the mesh (to smooth the mesh). However, in practical applications, some smoothing methods can lead to final meshes that contain inverted elements. This issue is usually triggered when the mesh boundary contains concave features. If that is the case, few smoothing methods can repair the inverted elements (untangle) and therefore, the mesh is not valid. To address this issue, there are several methods specialized to untangle the mesh. Note that the proper combination of an untangling method with a smoothing technique would provide the desired valid and high-quality mesh.

A wide range of smoothing algorithms based on a geometric reasoning have been developed to smooth planar meshes, *e.g.*, see [1, 2]. However, these algorithms are not designed to maximize a given quality measure. A family of quality measures placed within an algebraic framework has been introduced in [3, 4, 5]. They are based on an affine mapping between an ideal element and the physical one. Hence, the Jacobian matrix of the defined affine mapping contains the distortion information of the physical element. Later, in reference [6], it was proposed a smoothing method based on an optimization of these measures. In fact, this optimization procedure is transformed into a continuous minimization problem. However, these optimization methods are still not able to untangle inverted elements. Afterwards, [7] introduced a modification of the procedures developed in [6], in which the untangling of the mesh is achieved together with the smoothing procedure. The optimization of the new objective function can simultaneously untangle and smooth a mesh, saving time and effort in order to obtain the final mesh.

These ideas are of the major importance for surface meshes. That is, since most of the meshing algorithms are hierarchic procedures, the quality of the final 3D mesh is directly related to the quality of the previously generated surface mesh. Therefore, special attention is required on repairing and improving the quality of surface meshes. It is also important to highlight, that the formulation of a smoothing or untangling technique on a surface is more complex than on a volume, because it requires to deal with the constrain of moving the nodes on the surface.

To ensure that the nodes move on the surface, it is required to select a representation for the surface geometry. From all the possible surface representations, CAD models are the preferred representation for industrial applications. In addition, CAD entities can provide some advantages for formulating a relocation technique. For instance, in CAD models, the representation is obtained by a surface parameterization. Thus, using the parameterization, we can ensure that the nodes of the smoothed mesh are on the surface. Specifically, we can move the nodes on the parameter space and then use the parameterization to map them on the surface, avoiding any projection process.

Two main approaches have been proposed to relocate nodes on surface meshes. On the one hand, several methods compute an ideal location of the optimized node, that can be off the surface, and then relocate the nodes on the surface [8, 9, 10, 11, 12]. On the other hand, there also exist several methods that obtain an ideal location of the nodes directly on the surface

[13, 14, 15]. These methods, express the optimization procedure in terms of the parametric coordinates of an approximated representation of the original surface. We also compute the optimal location directly on the surface. However, we propose to quantify the distortion (quality) of the element in terms of the coordinates on the parametric space of the CAD surface. An optimization approach based on the proposed distortion ensures that the nodes always lie on the surface, since the whole process is developed in the parametric space of the original surface.

2 DISTORTION AND QUALITY FOR ELEMENTS ON PARAMETERIZED SURFACES

In this section, we first develop an analytical formulation to extend any quality measure for planar triangles to triangular meshes on a parameterized surface. As a result, we obtain a quality measure expressed in the two coordinates of the parametric space of the surface. Then, we extend this technique to quadrilateral elements.

2.1 Preliminaries on planar quality measures

Let η be a distortion measure for planar elements, with image $[1, \infty)$, taking value 1 for an ideal configuration of the element, and value ∞ when it is degenerated or tangled. Let q be the corresponding quality measure, defined as

$$q = \frac{1}{\eta}. \tag{1}$$

The image of the quality measure q is $[0, 1]$, taking value 1 for ideal configurations and 0 for degenerated or tangled ones. These measures for planar elements presented can be expressed as the mappings

$$\eta \ : \ \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow [1, \infty) \subset \mathbb{R}, \tag{2}$$
$$q \ : \ \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow [0, 1] \subset \mathbb{R}. \tag{3}$$

In this work we consider two Jacobian-based quality measures, namely the *shape* and the *Oddy* quality measures [3, 4, 5]. Let $\boldsymbol{\phi}$ be the mapping between the ideal (equilateral for triangles and square for quadrilateral elements) and the physical element, see Figure 1. This mapping can be expressed as:

$$\boldsymbol{\phi} : t_I \xrightarrow{\boldsymbol{\psi}_0^{-1}} t_R \xrightarrow{\boldsymbol{\psi}} t.$$

where $\boldsymbol{\psi}_0$ is the mapping between the reference and the ideal element and $\boldsymbol{\psi}$ is the mapping between the reference and the physical elelemnt.

The Jacobian of the affine mapping $\boldsymbol{\phi}$ contains information about the deviation of the physical element with respect to the ideal. Hence, the distortion measure of the physical element is defined in terms of $\mathbf{S}(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2) = \mathbf{D}\boldsymbol{\phi}$. Using these mappings the *Shape distortion measure* is defined as:

$$\eta_{sh}(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2) \ = \ \frac{\|\mathbf{S}(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2)\|^2}{2|\sigma(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2)|}, \tag{4}$$

where $\|\cdot\|$ is the Frobenius norm, and $\sigma(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2) = \det(\mathbf{S}(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2))$. This distortion measure quantifies the deviation of the shape of the physical triangle with respect to the ideal

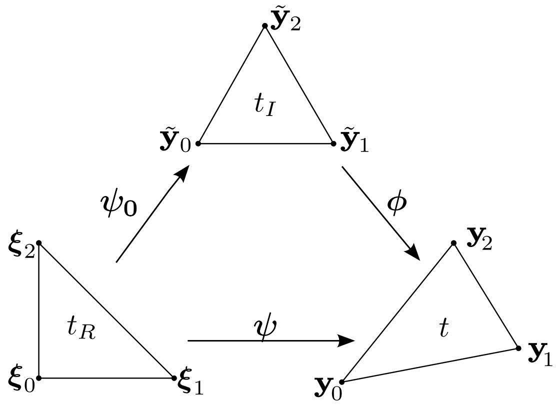

Figure 1: Mappings between the reference, the ideal and the physical elements.

shape. To incorporate the untangling capability to the optimization method, see Section 3 we replace σ in (4) by

$$\sigma_*(\sigma;\delta) = \frac{1}{2}\left(\sigma + \sqrt{\sigma^2 + 4\delta^2}\right), \tag{5}$$

where δ is a numerical parameter that has to be determined [7].

Similarly, the *Oddy measure* is defined as:

$$\eta_{od}(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2) = \frac{3}{2}\sigma^{-2}\left(\|\mathbf{S}^T\mathbf{S}\|^2 - \frac{1}{3}\|\mathbf{S}\|^4\right), \tag{6}$$

where analogously to than for the shape distortion measure, we can replace σ by σ_* to optimize tangled meshes.

2.2 Measures for triangles on parametric coordinates

Given a distortion and its associated quality measure for triangles in the plane, our goal is to extend these measure to triangles with the vertices on a parameterized surface, Σ. Assume that the surface Σ is parameterized by a continuously differentiable and invertible mapping

$$\begin{array}{rrcl} \varphi: & \mathcal{U} \subset \mathbb{R}^2 & \longrightarrow & \Sigma \subset \mathbb{R}^3 \\ & \mathbf{u} = (u, v) & \longmapsto & \mathbf{x} = \varphi(\mathbf{u}). \end{array} \tag{7}$$

To evaluate the quality of a triangle t_Σ with vertices on a surface Σ, we first express the vertices as the image by the parameterization φ of the corresponding parametric coordinates in $\mathcal{U}$. Since t_Σ is planar, but it is immersed in $\mathbb{R}^3$, we define the quality of the physical triangle as the quality of a geometrically equivalent triangle t on $\mathbb{R}^2$. Once in $\mathbb{R}^2$, the proposed formulation allows to extend any existent distortion and quality measure for planar elements.

In order to define a quality measure in terms of the parametric coordinates of the three vertices of the triangle, we define the mapping

$$\begin{array}{rrcl} \tilde{\varphi}: & \mathcal{U} \times \mathcal{U} \times \mathcal{U} & \longrightarrow & \Sigma \times \Sigma \times \Sigma \\ & (\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2) & \longmapsto & (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2) = (\varphi(\mathbf{u}_0), \varphi(\mathbf{u}_1), \varphi(\mathbf{u}_2)). \end{array} \tag{8}$$

This mapping transforms a triangle $t_\mathcal{U} = (\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2)$ in the parametric space $\mathcal{U}$, to a triangle $t_\Sigma = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2)$ with the nodes on the surface Σ determined by φ, see Figure 2. Since t_Σ defines a plane in $\mathbb{R}^3$, we can map t_Σ to a geometrically equivalent triangle in $\mathbb{R}^2$. That is,

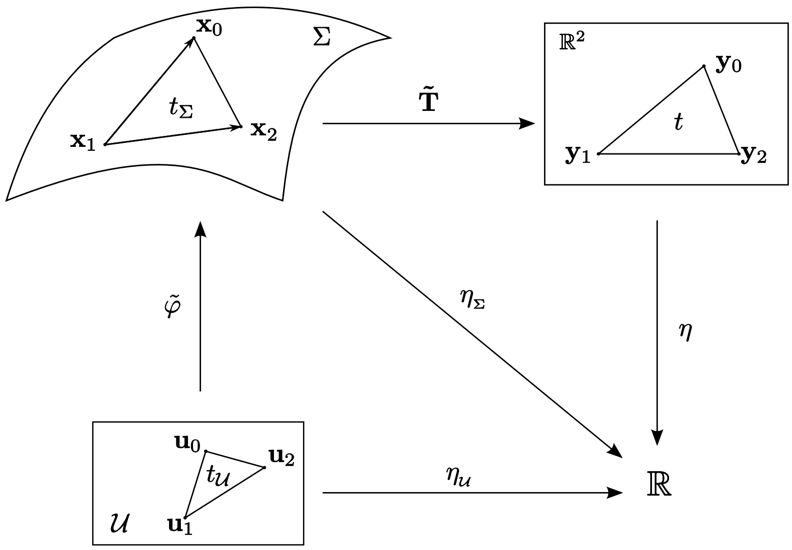

Figure 2: Diagram of mappings involved in the definition of the quality measure.

we can define a mapping $\tilde{\mathbf{T}}$ from $\Sigma \times \Sigma \times \Sigma$ to $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$. To define $\tilde{\mathbf{T}}$, we consider an auxiliary linear mapping $\mathbf{T}$ from $\mathbb{R}^3$ to the plane. The domain of this mapping is expressed in the canonical basis of $\mathbb{R}^3$, and the image is expressed in terms of a new 2D orthogonal basis determined by a combination of two edges of the triangle. Let

$$\begin{aligned} \mathbf{e}_1 &:= \mathbf{x}_2 - \mathbf{x}_1, \\ \mathbf{e}_2 &:= \mathbf{x}_0 - \mathbf{x}_1, \end{aligned} \tag{9}$$

be the vectors determined by two edges of the triangle. Then, we define

$$\begin{aligned} \tilde{\mathbf{e}}_1 &:= \frac{\mathbf{e}_1}{\|\mathbf{e}_1\|}, \\ \tilde{\mathbf{e}}_2 &:= \gamma\, \tilde{\mathbf{e}}_{2,0}, \qquad \text{with} \quad \tilde{\mathbf{e}}_{2,0} := \frac{\mathbf{e}_2 - (\mathbf{e}_2^T \cdot \tilde{\mathbf{e}}_1)\, \tilde{\mathbf{e}}_1}{\|\mathbf{e}_2 - (\mathbf{e}_2^T \cdot \tilde{\mathbf{e}}_1)\, \tilde{\mathbf{e}}_1\|}, \end{aligned}$$

as the two orthonormal vectors of the new basis, where γ is defined to ensure a well oriented orthonormal basis. Specifically, we define γ as:

$$\gamma := \frac{(\tilde{\mathbf{e}}_1 \times \tilde{\mathbf{e}}_{2,0}) \cdot \mathbf{n}}{|(\tilde{\mathbf{e}}_1 \times \tilde{\mathbf{e}}_{2,0}) \cdot \mathbf{n}|} = \frac{\det(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_{2,0}, \mathbf{n})}{|\det(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_{2,0}, \mathbf{n})|},$$

where $\mathbf{n} \equiv \mathbf{n}(\mathbf{x}_1) = \frac{\partial \varphi}{\partial u}(u_1, v_1) \times \frac{\partial \varphi}{\partial v}(u_1, v_1)$ is the normal to the surface at $\mathbf{x}_1 = \varphi(u_1, v_1)$. Note that $\gamma = \pm 1$, being 1 for counter-clockwise oriented triangles, and -1 for clockwise oriented ones.

Now, we can define $\mathbf{T}$ as

$$\begin{aligned} \mathbf{T}:\ \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ \mathbf{x} &\longmapsto \mathbf{M} \cdot (\mathbf{x} - \mathbf{x}_1), \end{aligned} \tag{10}$$

where $\mathbf{M} = (\tilde{\mathbf{e}}_1 \ \ \tilde{\mathbf{e}}_2)^T$ is a 2×3 matrix. In addition, we define $\tilde{\mathbf{T}}$ as:

$$\begin{aligned} \tilde{\mathbf{T}}:\quad \Sigma \times \Sigma \times \Sigma &\longrightarrow \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \\ t_\Sigma = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2) &\longmapsto t = (\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2) = (\mathbf{T}(\mathbf{x}_0), \mathbf{T}(\mathbf{x}_1), \mathbf{T}(\mathbf{x}_2)), \end{aligned} \tag{11}$$

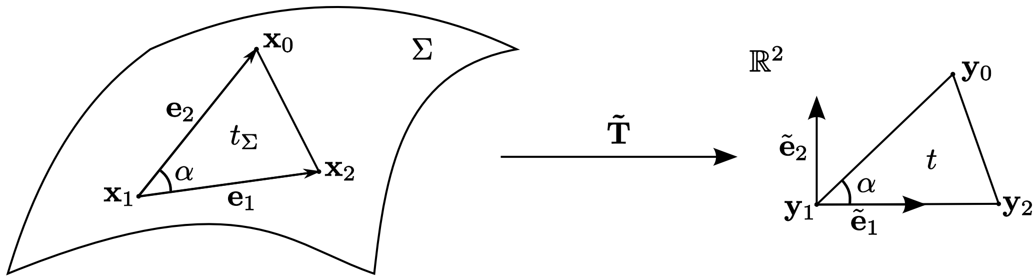

Figure 3: Vector edges $\mathbf{e}_1$ and $\mathbf{e}_2$ for a triangle $t_\Sigma = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2)$ on a surface Σ, and diagram of function $\tilde{\mathbf{T}}$.

see Figure 3. Hence, we can express the distortion measure for a triangle t_Σ on the surface as:

$$\begin{array}{rcccccc} \eta_\Sigma : & \Sigma \times \Sigma \times \Sigma & \xrightarrow{\tilde{\mathbf{T}}} & \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 & \xrightarrow{\eta} & \mathbb{R} \\ & (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2) & \longmapsto & \tilde{\mathbf{T}}(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2) & \longmapsto & \eta(\tilde{\mathbf{T}}(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2)). \end{array}$$

That is, as the composition

$$\eta_\Sigma = \eta \circ \tilde{\mathbf{T}} : \Sigma \times \Sigma \times \Sigma \longrightarrow [1, \infty). \tag{12}$$

Note that η_Σ is a distortion measure on Σ, since it is the composition of a planar distortion measure η, and a change of variable of the plane where t_Σ lies. Moreover, the reciprocal of η_Σ,

$$q_\Sigma := \frac{1}{\eta_\Sigma} : \Sigma \times \Sigma \times \Sigma \longrightarrow [0, 1],$$

is also a quality measure, in the sense of [3]. It is important to point out that this quality measure holds the same properties of the corresponding original planar quality measure q.

Finally, we use the expression of the distortion η_Σ, Equation (12), to define the distortion measure for triangles on parametric coordinates as:

$$\eta_{\mathcal{U}} := \eta_\Sigma \circ \tilde{\varphi} = \eta \circ \tilde{\mathbf{T}} \circ \tilde{\varphi} : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \longrightarrow [1, \infty). \tag{13}$$

Accordingly, the quality measure for triangles on parametric coordinates is:

$$q_{\mathcal{U}} := \frac{1}{\eta_{\mathcal{U}}} : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \longrightarrow [0, 1]. \tag{14}$$

2.3 Extension to quadrilaterals on parametric coordinates

According to [4], the distortion measure for a planar quadrilateral is evaluated through the decomposition of the quadrilateral into four triangles, see Figure 4. In this work, we also compute the distortion measure of a quadrilateral element on a parameterized surface as the mean value of the distortion measure of the four corner triangles. To this end, let $(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ be the vertices of a quadrilateral element of a mesh with the nodes on a parameterized surface, and let $(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ be their parametric coordinates. The *distortion measure for quadrilaterals on parametric coordinates* is:

$$\eta_{\mathcal{U}}(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) := \frac{\eta_{\mathcal{U}}(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2) + \eta_{\mathcal{U}}(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_3) + \eta_{\mathcal{U}}(\mathbf{u}_0, \mathbf{u}_2, \mathbf{u}_3) + \eta_{\mathcal{U}}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)}{4}, \tag{15}$$

where $\eta_{\mathcal{U}}(\mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_k)$ is the distortion on parametric coordinates for the triangle $(\mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_k)$, see Equation 13.

Accordingly, the *quality measure for quadrilaterals on parametric coordinates* is:

$$q_{\mathcal{U}}(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) := \frac{1}{\eta_{\mathcal{U}}(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)}. \tag{16}$$

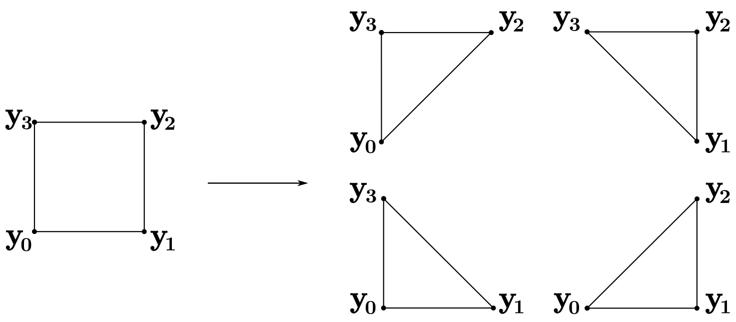

Figure 4: Decomposition of a planar quadrilateral into four triangles.

3 OPTIMIZATION OF SURFACE MESH QUALITY

The main goal of a simultaneous smoothing and untangling method is to obtain high-quality meshes composed by valid (non-inverted) elements. Note that the best possible result, can be characterized in terms of the distortion measure. That is, given a distortion measure $\eta_{\mathcal{U}}$ and a mesh $\mathcal{M}$ on a parameterized surface composed by n_N nodes and n_E elements, the node location is ideal if

$$\eta_{\mathcal{U}}(t^j_{\mathcal{U}}) = 1 \qquad j = 1, \ldots, n_E, \tag{17}$$

where $t^j_{\mathcal{U}} = (\mathbf{u}_{j_1}, \mathbf{u}_{j_2}, \mathbf{u}_{j_3})$ is the jth element expressed on parametric coordinates. However, for a fixed mesh topology the node location that leads to an ideal mesh distortion is not in general achievable. That is, the constraints in Equation (17) cannot be imposed strongly and therefore, we just enforce the ideal mesh distortion in the least-squares sense.

For a given mesh topology and a set of fixed nodes (nodes on the surface boundary), we formulate the least-squares problem in terms of the parametric coordinates of a set of free nodes (inner nodes on the surface). To this end, we reorder the parametric coordinates of the nodes, $\mathbf{u}_i$, in such a way that $i = 1, \ldots, n_F$ are the indices corresponding to the free nodes, and $i = n_F + 1, \ldots, n_N$ correspond to the fixed nodes. Thus, we can formulate the mesh optimization problem as

$$\min_{\mathbf{u}_1,\ldots,\mathbf{u}_{n_F}} f(\mathbf{u}_1, \ldots, \mathbf{u}_{n_F}; \mathbf{u}_{n_F+1}, \ldots, \mathbf{u}_{n_N}), \tag{18}$$

where

$$f(\mathbf{u}_1, \ldots, \mathbf{u}_{n_F}; \mathbf{u}_{n_F+1}, \ldots, \mathbf{u}_{n_N}) := \frac{1}{2}\sum_{j=1}^{n_E}(\eta_{\mathcal{U}}(t^j_{\mathcal{U}}) - 1)^2$$

denotes the objective function.

Finally, the optimal configuration is found between the candidates for the minimization of (18). The candidates are the critical parametric coordinates $(\mathbf{u}_1, \ldots, \mathbf{u}_{n_F})$ of f. They are characterized by ensuring, for $i = 1, \ldots, n_F$,

$$\frac{\partial f}{\partial \mathbf{u}_i}(\mathbf{u}_1, \ldots, \mathbf{u}_{n_F}; \mathbf{u}_{n_F+1}, \ldots, \mathbf{u}_{n_N}) = \sum_{j=1}^{n_E}(\eta_{\mathcal{U}}(t^j_{\mathcal{U}}) - 1)\,\frac{\partial \eta_{\mathcal{U}}}{\partial \mathbf{u}_i}(t^j_{\mathcal{U}}) = 0. \tag{19}$$

To solve the optimization problem in Equation (18), we have to find the optimum between the candidate configurations. These configurations are characterized by the global non-linear constraints in Equation (19). To solve these constraints, we choose a non-linear iterative method that: exploits the locality of the problem, avoids solving large linear systems, and is well suited

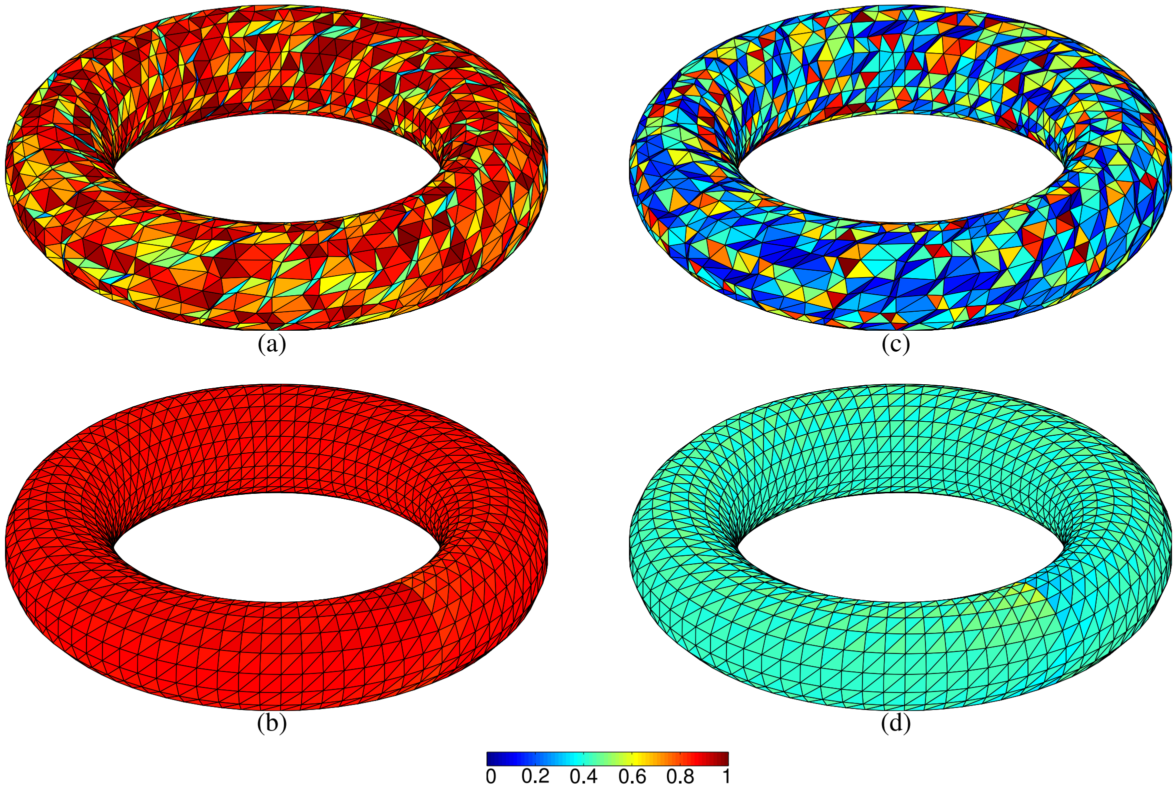

Figure 5: Meshes for a torus. Meshes colored according to the shape quality measure: (a) initial mesh, and (b) smoothed and untangled mesh. Meshes colored according to the Oddy quality measure: (c) initial mesh, and (d) smoothed and untangled mesh.

for parallelization (by coloring the mesh nodes). Specifically, we use a non-linear iterative Gauss-Seidel method determined by the iteration

$$\mathbf{u}_i^{k+1} = \mathbf{u}_i^k - \alpha_i^k \, [\nabla^2_{ii} f(\mathbf{w}_i^k)]^{-1} \, \boldsymbol{\nabla}_i f(\mathbf{w}_i^k) \qquad i = 1, \ldots, n_F, \tag{20}$$

where α_i^k is the step length, and

$$\mathbf{w}_i^k = (\mathbf{u}_1^{k+1}, \ldots, \mathbf{u}_{i-1}^{k+1}, \mathbf{u}_i^k, \mathbf{u}_{i+1}^k, \ldots, \mathbf{u}_{n_F}^k; \mathbf{u}_{n_F+1}^0, \ldots, \mathbf{u}_{n_N}^0)$$

is the vector of updated node locations for the $i-1$ first nodes. Note that $\boldsymbol{\nabla}_i$ and ∇^2_{ii} denote the gradient and the Hessian with respect to the parametric coordinates $\mathbf{u}_i$ of node i.

4 NUMERICAL EXAMPLES

In this section, we present several examples in order to illustrate the behavior of the proposed method. To this end we present several examples and we analyze the minimum, the maximum, the mean, the standard deviation of the quality of the elements, and the number of tangled elements. We highlight that in all cases, the smoothed mesh increases the minimum and mean values of the mesh quality and decreases its standard deviation. All algorithms have been implemented in C++ in the meshing environment EZ4U [16, 17, 18].

The goal of the first example is to show that any planar distortion measure can be extended to parameterized surfaces. First, we generate a triangular mesh on a torus composed by 1600 nodes and 3002 elements. In Figures 5(a) and 5(c) we show the initial mesh, coloring the elements with respect to the two different selected measures. Note that the mesh contains 73 inverted elements. Then, in Figures 5(b) and 5(d) we present the two resulting optimized meshes.

Table 1: Shape and Oddy quality statistics of the meshes on the torus.

Measure	Mesh	Figure	Min. Q.	Max. Q.	Mean Q.	Std. Dev.	Tang. el.
Shape	Tangled	5(a)	0.00	1.00	0.72	0.24	73
	Smoothed	5(a)	0.79	0.92	0.86	0.02	0
Oddy	Tangled	5(c)	0.00	1.00	0.34	0.26	73
	Smoothed	5(d)	0.31	0.59	0.42	0.03	0

Srf.	Mesh	Fig.	Min.Q.	Max.Q.	Mean Q.	Std.Dev.	Tang.
Revol.	Initial	6(a)	0.44	0.88	0.79	0.10	0
	Tangled	6(b)	0.00	0.99	0.30	0.32	664
	Smoothed	6(c)	0.65	1.00	0.83	0.03	0
Tubular	Initial	6(d)	0.37	1.00	0.81	0.19	0
	Tangled	6(e)	0.00	0.97	0.15	0.26	786
	Smoothed	6(f)	0.52	1.00	0.84	0.08	0

Table 2: Shape quality statistics of the meshes on the revolution and tubular surfaces.

Table 1 summarizes the quality statistics of the meshes presented in Figure 5. Note that the proposed algorithm untangles an input mesh with inverted elements. In addition, for both cases, the proposed method improves the quality of the initial surface meshes. Note that the Oddy measure is more restrictive. That is, Oddy measure quantifies as low quality the rectangular triangles (the ideal triangle is the equilateral). Nevertheless, both measures properly detect the degenerated and the valid elements.

The goal of the second example is to illustrate the robustness of the developed smoothing and untangling method. To this end, we use the shape distortion measure, Equation (4), and we consider two NURBS surfaces. The first one is meshed using triangular elements, and the second one is meshed with quadrilateral elements (see Figure 6). For each surface, three figures are presented. First, we display an initial mesh generated on the NURBS surface. Second, we show a mesh with the same topology than the initial one, but with a large number of tangled elements. This tangled mesh is the input of the smoothing and untangling algorithm. Third, we present the optimized mesh.

Figure 6(a) presents a triangular mesh generated on a revolution surface. This mesh is composed by 800 nodes and 1482 elements. Figure 6(b) shows a mesh with 664 tangled elements, obtained by a random perturbation of the initial mesh. Figure 6(c) presents the optimized mesh obtained using the proposed method. Analogously, Figures 6(d), 6(e) and 6(f), present the same scheme for a quadrilateral mesh on a tubular surface. The mesh is composed by 1200 nodes and 1121 elements, and the perturbed configuration has 786 tangled elements.

Table 2 summarizes the shape quality statistics of the meshes presented in Figure 6. Note that the proposed algorithm untangles an input mesh composed by a large number of tangled elements. In addition, for both cases, the proposed method improves the quality of the initial surface meshes.

In the third example we apply the smoothing and untangling procedure using the shape distortion measure, Equation (4), to two CAD models composed by multiple patches: a knob and a crank arm. Figure 7(a) shows the initial mesh on the knob. It is composed by 15137 nodes and 14521 quadrilateral elements. The initial mesh has intentionally been generated with 497 tangled elements. Figure 7(b) presents the smoothed mesh, where all the inverted and degenerated elements have been untangled. Then, Figure 8(a) presents the initial mesh on the crank arm.

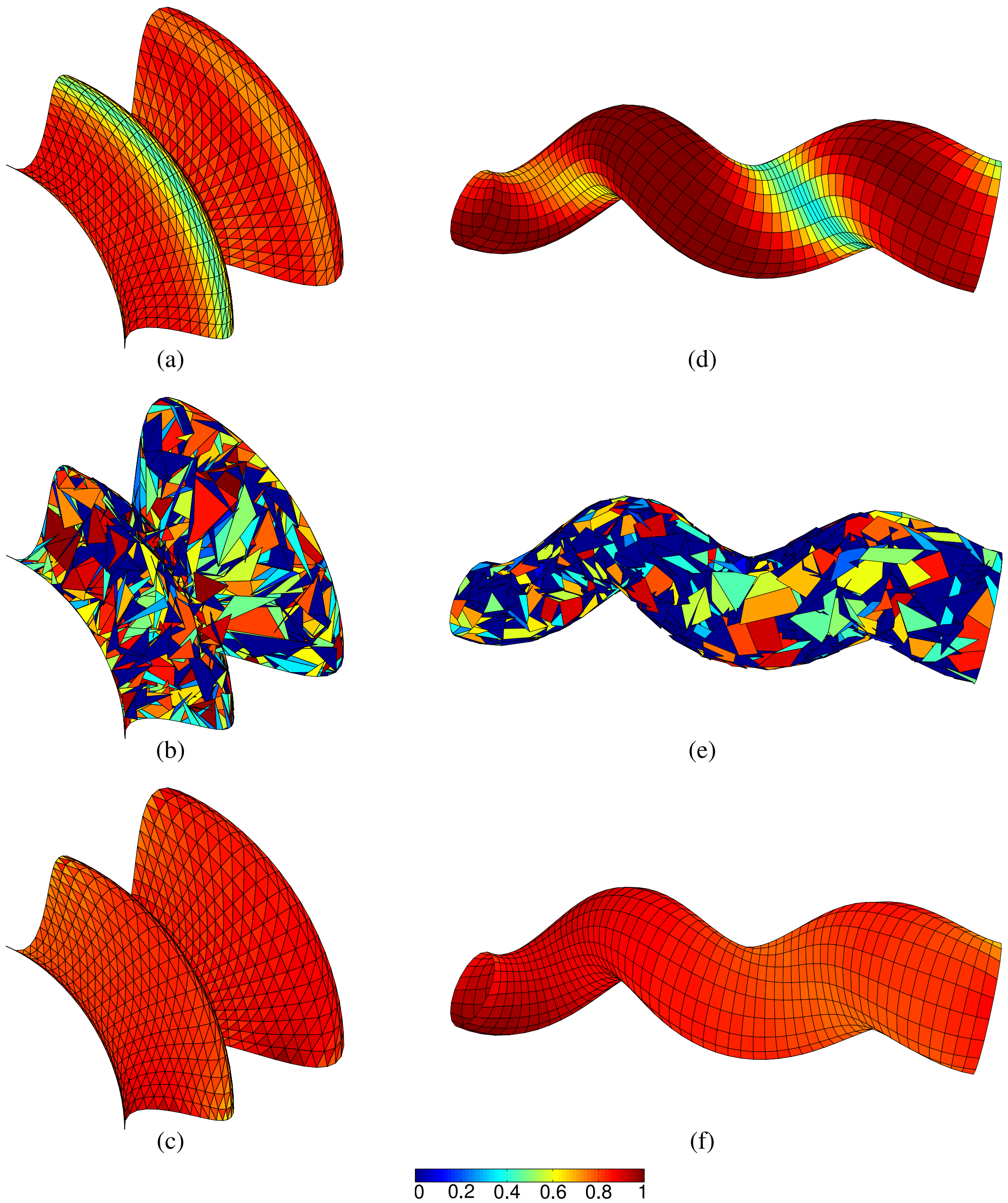

Figure 6: Meshes colored according to the shape quality measure for a revolution surface: (a) initial mesh, (b) tangled mesh, and (c) smoothed and untangled mesh. Meshes colored according to the shape quality measure for a tubular surface, (d) initial mesh, (e) tangled mesh, and (f) smoothed and untangled mesh.

It is composed by 2966 nodes and 4600 triangular elements. Figure 8(b) presents the resulting mesh from the smoothing procedure.

Table 3 details the shape quality statistics of the presented meshes. Note that the smoothing procedure properly improves the quality of the surface mesh in both cases. Moreover, it increases the minimum and the mean value of the quality of the mesh.

5 CONCLUDING REMARKS

In this work, we have proposed a new and robust technique to smooth and untangle meshes on parameterized surfaces. To this end, we first have developed an analytical procedure to

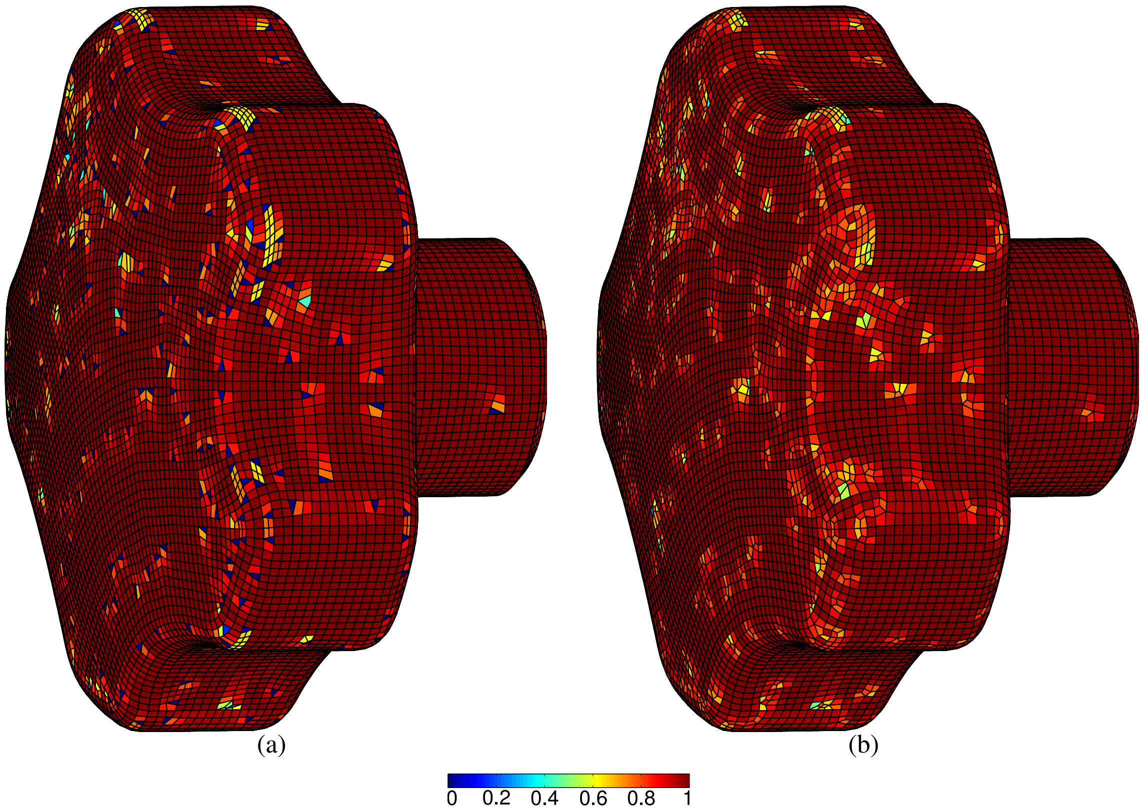

Figure 7: Quadrilateral meshes colored according to the shape quality measure on a knob: (a) initial mesh, and (b) smoothed mesh.

Table 3: Shape quality statistics of the meshes on the knob and a crank arm.

Surface	Mesh	Figure	Min. Q.	Max. Q.	Mean Q.	Std. Dev.	Tang. el.
Knob	Initial	7(a)	0.00	1.00	0.92	0.20	497
	Smoothed	7(b)	0.21	1.00	0.94	0.10	0
Crank arm	Initial	8(a)	0.00	1.00	0.82	0.12	11
	Smoothed	8(b)	0.31	1.00	0.88	0.09	0

extend any Jacobian-based quality measure for planar elements (triangles or quadrilaterals) to parameterized surfaces. Then, using the proposed quality measure, we have presented a new technique to optimize (smooth and untangle) meshes on parameterized surfaces. This optimization process is performed by minimizing the mesh distortion measure (inverse of the quality) expressed on parametric coordinates. In addition, we use the surface parameterization to obtain an analytical expression of the first and second derivatives of the proposed quality measure and to ensure that the nodes move on the surface (via the parameterization mapping). We have presented several examples to illustrate the capabilities of the proposed method. In particular, have applied our method using two planar quality measures, and we have untangled highly and randomly tangled meshes to test the robustness of the implementation. Finally, we have optimized two multi-patch CAD models to show practical applications of the method.

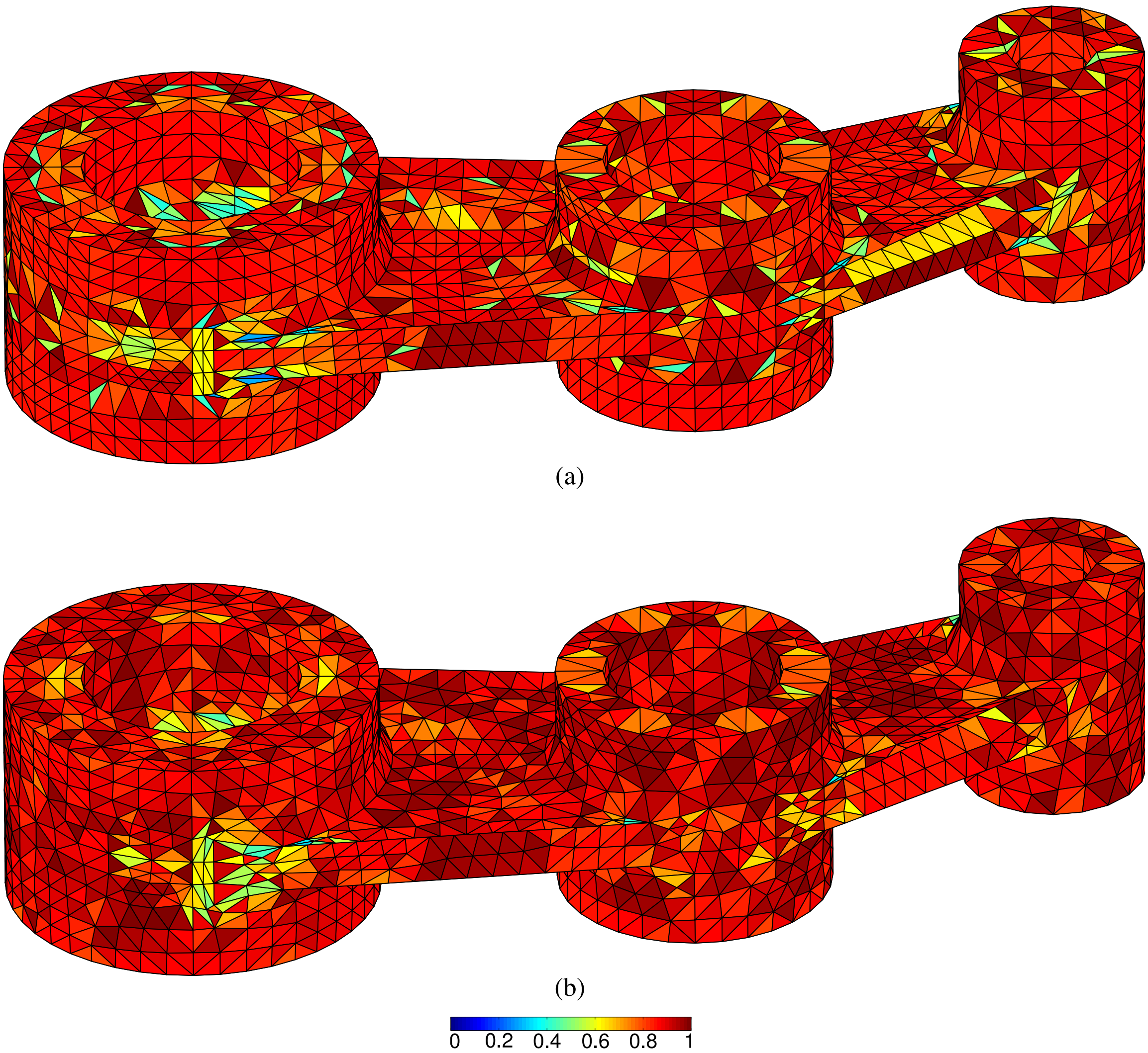

Figure 8: Triangular meshes for a crank arm colored according to the shape quality measure: (a) initial mesh, and (b) smoothed mesh.

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