Pontryagin duality in the class of precompact Abelian groups and the Baire property

M. Bruguera * M. Tkachenko †

April, 2009; revised version: September, 2010

Abstract

We present a wide class of reflexive, precompact, non-compact, Abelian topological groups G determined by three requirements. They must have the Baire property, satisfy the *open refinement condition*, and contain no infinite compact subsets. This combination of properties guarantees that all compact subsets of the dual group $G^\wedge$ are finite. We also show that many (non-reflexive) precompact Abelian groups are quotients of reflexive precompact Abelian groups. This includes all precompact almost metrizable groups with the Baire property and their products. Finally, given a compact Abelian group G of weight $\geq 2^\omega$, we find proper dense subgroups H_1 and H_2 of G such that H_1 is reflexive and pseudocompact, while H_2 is non-reflexive and almost metrizable.

Keywords: Reflexive; Precompact; Pseudocompact; Baire property; Open refinement condition; h-embedded subgroup; Convergent sequence; Almost metrizable

MSC: primary 43A40, 22D35; secondary 22C05, 54E52, 54C10

The work was done while the authors were visiting CRM in May–July, 2008. Both authors are deeply grateful to the CRM staff for support.

*This author was partially supported by Ministerio de Educación y Ciencia, grant MTM2009-14409-C02-01.

†This author was supported by CONACyT of Mexico, grant 000000000074468.

1 Introduction

The Pontryagin–van Kampen duality theory has been pushed outside the realm of locally compact Abelian groups during the last 60 years. It turned out that many topological Abelian groups were reflexive without being locally compact [4, 5, 22, 26, 27, 32, 35, 39]. Forty years had passed after the emergence of the duality theory until the first reflexive non-complete group was constructed by Y. Komura in [28]. The difficulty of finding such examples was clarified independently by L. Außenhofer and M. J. Chasco in [4] and [8], respectively, where they proved that every metrizable reflexive topological group must be complete.

However, none of the reflexive groups found up to 2008 was a proper dense subgroup of a compact group. Subgroups of compact groups are called *precompact*. Since a precompact group is compact if and only if it is complete, every reflexive precompact metrizable group is compact by Außenhofer–Chasco's theorem. The problem of whether *all* reflexive precompact groups are compact appears explicitly in the article [9] by Chasco and Martín-Peinador.

Very recently, a series of reflexive, *pseudocompact*, non-compact groups appeared in [1] and [23]. We recall that a Tychonoff space is pseudocompact if every continuous real-valued function defined on the space is bounded. Since, by [14, Theorem 1.1], every pseudocompact topological group is precompact, this solved the problem.

Our aim is to continue the study of reflexivity in the class of precompact groups and present an even wider class of precompact, non-compact (hence non-complete) reflexive groups. In fact, one of the main results in [1], the reflexivity of pseudocompact groups without infinite compact subsets (which was also independently proved in [23]), is our Corollary 2.10.

By the Comfort–Ross duality theorem in [13], every precompact Abelian group G becomes reflexive if both the dual group $G^\wedge$ and bidual group $G^{\wedge\wedge}$ carry the topology of pointwise convergence, i.e., $G \cong (G_p^\wedge)_p^\wedge$ (for generalizations, see also [36]). This result is a common basis for applications in [1, 23] and here.

Our strategy is as follows. First, we work with precompact groups G without infinite compact subsets. This implies the coincidence of the compact-open and pointwise convergence topologies on the dual group $G^\wedge$. Then we find conditions under which the dual group $G^\wedge$ contains no infinite compact subsets either. This finishes the job, implying the reflexivity of G by

Corollary 2.2.

To guarantee the absence of infinite compact sets in $G^\wedge$ it suffices, by Theorem 2.7, to require that G have the Baire property and satisfy the *Open Refinement Condition* (ORC, for short) defined in Subsection 1.1 below. Hence every precompact Abelian group with the two properties and without infinite compact subsets is reflexive, which is our Theorem 2.8. Since pseudocompact groups have the Baire property and satisfy ORC, we deduce in Corollary 2.10 that every pseudocompact Abelian group without infinite compact subsets is reflexive (see [1, Theorem 2.2] or [23, Theorem 6.1]). The proof of Theorem 2.8 makes use of the dual characterization of ORC obtained in Lemma 2.5 and saying that a precompact group G satisfies ORC if and only if the closure of every countable set in the dual group $G_p^\wedge$ endowed with the pointwise convergence topology remains countable.

In Section 3 we discuss the relations between the Baire property and ORC in precompact groups. It is shown in Proposition 3.3 and Theorem 3.4 that the class of precompact groups with the Baire property is closed under taking continuous homomorphic images and forming arbitrarily large direct products, respectively. It is worth mentioning that the Baire property fails to be productive even in linear normed spaces [38, 31], so one cannot omit the precompactness requirement in Theorem 3.4. In Lemma 3.5 and Proposition 3.8 we show that the Baire property and ORC are preserved under extensions of precompact groups by pseudocompact groups.

Our main objective in Section 4 is to find out how wide the class of quotients of reflexive precompact groups is. The main result here is Theorem 4.1 which says that every precompact Abelian group with the Baire property that satisfies ORC is a quotient of a reflexive precompact group with respect to a closed pseudocompact subgroup. These quotients include, in particular, all almost metrizable precompact groups with the Baire property and, hence, all precompact second countable groups with the Baire property. These results are complemented in Theorem 4.7 which says that every compact Abelian group G of weight greater than or equal to 2^ω contains a proper dense reflexive pseudocompact subgroup. Finally, in Proposition 4.9 we prove that the product of an arbitrary family of almost metrizable precompact groups with the Baire property is a quotient of a reflexive precompact group. This requires two preliminary steps, Propositions 2.6 and 4.8, where the permanence properties of the class of precompact groups satisfying ORC are established—this class is closed under taking continuous homomorphic images and arbitrarily large products.

Several open problems about the reflexivity and Baire property of precompact groups are posed in Section 5.

1.1 Notation and terminology

We use the additive notation for multiplication in the case of commutative groups, except for the circle group $\mathbb{T}$. All topological groups are assumed to be Hausdorff. A subgroup H of a group G is called *invariant* if $xHx^{-1} = H$, for each $x \in G$. The term 'invariant' replaces 'normal' to avoid ambiguity when considering topological properties of topological groups. The kernel of a group homomorphism $f\colon G \to H$ is $\ker f$.

A topological group G is *ω-narrow* if for every neighborhood U of the neutral element in G, there exists a countable set $C \subseteq G$ such that $UC = G$. A group G is ω-narrow iff it is topologically isomorphic to a subgroup of a product of second countable groups (see [3, Theorem 3.4.23]).

The Raĭkov completion of G is denoted by ϱG. The group ϱG is compact if and only if G is precompact [3, Theorem 3.7.16]. A remarkable property of the Raĭkov completion is that every continuous group homomorphism $f\colon G \to H$ admits an extension to a continuous group homomorphism $\widetilde{f}\colon \varrho G \to \varrho H$ [3, Corollary 3.6.17].

Given a topological Abelian group G, we use $G^\wedge$ to denote the *dual* group of G consisting of all continuous homomorphisms $\chi\colon G \to \mathbb{T}$ to the circle group $\mathbb{T}$. The group $G^\wedge$ is endowed with the pointwise multiplication and carries the compact-open topology. The *bidual group* is $G^{\wedge\wedge} = (G^\wedge)^\wedge$.

The *evaluation homomorphism* $\alpha_G\colon G \to G^{\wedge\wedge}$ is defined by the rule $\alpha_G(x)(\chi) = \chi(x)$, for all $x \in G$ and $\chi \in G^\wedge$. In general, α_G can fail to be continuous, injective or surjective. If however α_G is a topological isomorphism of G onto $G^{\wedge\wedge}$, then the group G is said to be *reflexive*. Pontryagin–van Kampen's duality theorem states that every locally compact topological Abelian group is reflexive.

The same underlying group $G^\wedge$ endowed with the pointwise convergence topology on elements of G will be designated as $G^\wedge_p$. A local base at the neutral element of $G^\wedge_p$ consists of the sets

$$U(g_1, \ldots, g_k) = \{\chi \in G^\wedge : \chi(g_i) \in \mathbb{T}_+ \text{ for each } i \leq k\},$$

where $\mathbb{T}_+ = \{e^{\pi i x} : -1/2 < x < 1/2\}$ and $g_1, \ldots, g_k$ are arbitrary elements of G. Reformulating Corollary 2.2 below, we can say that every precompact Abelian group is reflexive when the dual groups $G^\wedge$ and $G^{\wedge\wedge}$ carry the pointwise convergence topology in place of the compact-open topology.

According to [37, Def. 4.4], a subgroup C of a topological group G is *h-embedded* in G if *every* homomorphism $f\colon C \to K$ to a compact group K admits an extension to a *continuous* homomorphism $\tilde{f}\colon G \to K$. Hence the topology of an h-embedded subgroup C inherited from G is finer than the maximal precompact topology of the (abstract) group C. If the group G is precompact, then every h-embedded subgroup of G carries the finest precompact topological group topology. It was shown by S. Hernández and S. Macario in [24] that, for precompact groups G, the property that all countable subgroups of G are h-embedded is *dual* to pseudocompactness. In other words, if all countable subgroups of G are h-embedded, then the dual group $G^\wedge$ is pseudocompact and, in addition, if G is pseudocompact, then all countable subgroups of $G^\wedge$ are h-embedded.

The following new concept plays an important role here. We say that a topological group G satisfies the *Open Refinement Condition* (abbreviated to *ORC*) if for every continuous homomorphism $f\colon G \to H$ onto a second countable group H, one can find a continuous *open* homomorphism $\pi\colon G \to K$ onto a second countable group K and a continuous homomorphism $g\colon K \to H$ such that $f = g \circ \pi$. It is immediate from the definition that every second countable group satisfies ORC. Since every continuous onto homomorphism of compact groups is open, all compact groups satisfy ORC as well.

A subset Y of a space X is called *meager* or of the *first category* in X provided that Y is the union of a countable family of nowhere dense sets in X. A space X has the *Baire property* if the intersection of any countable family of open dense sets in X is dense in X or, equivalently, every non-empty open subset of X is non-meager. Following [7], we say that a space is Baire if it has the Baire property, and the same applies to topological groups.

A family $\mathcal{N}$ of subsets of a space X is called a *network* for X if for every $x \in X$ and every neighborhood U of x, there exists $N \in \mathcal{N}$ such that $x \in N \subseteq U$. Every base for a space is a network, but not vice versa. It is easy to see that every space with a countable network is separable. For unexplained topological terms, see [17].

2 Reflexive precompact groups

Here we discuss in detail what kind of precompact groups G are reflexive provided that all compact subsets of G are finite. Our approach leans on the Comfort–Ross duality theorem in [13] (a more general form of this duality can be found in [30]). First we introduce a small piece of terminology.

Let G be an abstract Abelian group and $\mathcal{P}_G$ the family of all precompact (not necessarily Hausdorff) topological group topologies on G. Let also $Hom(G, \mathbb{T})$ be the group of homomorphisms of G to the circle group $\mathbb{T}$ and $\mathcal{H}_G$ be the family of subgroups of $Hom(G, \mathbb{T})$. For every $H \in \mathcal{H}_G$, denote by τ_H the coarsest topology on G that makes continuous each $\chi \in H$. Clearly, (G, τ_H) is a precompact topological group, so $\tau_H \in \mathcal{P}_G$ for each $H \in \mathcal{H}_G$. Conversely, given a topology $\tau \in \mathcal{P}_G$, let $H_\tau = (G, \tau)^\wedge \in \mathcal{H}_G$.

Theorem 2.1 (Comfort–Ross duality theorem) *Let G be an abstract Abelian group and $\Phi\colon \mathcal{H}_G \to \mathcal{P}_G$ and $\Psi\colon \mathcal{P}_G \to \mathcal{H}_G$ be mappings defined by $\Phi(H) = \tau_H$ for each $H \in \mathcal{H}_G$ and $\Psi(\tau) = H_\tau$ for each $\tau \in \mathcal{P}_G$. Then both Φ and Ψ are bijections and $\Psi = \Phi^{-1}$.*

In fact, we will only use the following corollary to Theorem 2.1 whose direct proof can be found in [36]:

Corollary 2.2 *Let G be a precompact topological group and $G_p^\wedge$ be the dual group endowed with the pointwise convergence topology. Then the canonical evaluation mapping $\alpha_G\colon G \to (G_p^\wedge)_p^\wedge$ is a topological isomorphism.*

The interplay between the properties of G and $G_p^\wedge$, for precompact groups G, was studied in [24].

By Corollary 2.2, to guarantee the (Pontryagin) reflexivity of a precompact group G, it suffices that all compact subsets of G and of the dual group $G^\wedge$ be finite—in this case the compact-open topology and the pointwise convergence topology coincide.

We do the job in two steps. First, in Corollary 2.4, we describe the groups G with the property that the dual group $G^\wedge$ does not contain non-trivial convergent sequences. Second, imposing additional conditions on G, we deduce that $G^\wedge$ does not contain infinite compact subsets (see Theorem 2.7).

The following important fact is well known (see [18] or [10, Proposition 1.4]):

Proposition 2.3 *Let $\{h_n : n \in \omega\}$ be a family of continuous homomorphisms of a topological group G to a topological group H. If the set of all $x \in G$ such that $h_n(x)$ is a Cauchy sequence in H is non-meager in G, then the family $\{h_n : n \in \omega\}$ is equicontinuous.*

We now use Proposition 2.3 to show that the dual group of a precompact group with the Baire property does not contain non-trivial convergent sequences.

Corollary 2.4 *Let G be a precompact Baire group, and suppose that a sequence $\xi = \{h_n : n \in \omega\} \subseteq G^\wedge$ converges pointwise on each element of G. Then the sequence ξ is eventually constant.*

Proof. Let $K = \varrho G$ be the completion of G. Then K is a compact topological group and G is a non-meager dense subgroup of K. Every h_n admits an extension to a character χ_n of K. It now follows from Proposition 2.3 that the sequence $\xi_K = \{\chi_n : n \in \omega\}$ is equicontinuous on K. Hence Arzelá–Ascoli's theorem implies that ξ_K has compact closure in $K^\wedge$. Since the group $K^\wedge$ is discrete, we conclude that ξ_K and ξ are eventually constant. □

In the lemma below we present a characterization of precompact groups satisfying ORC in dual terms.

Lemma 2.5 *A precompact Abelian group G satisfies the open refinement condition iff for every countable set $S \subseteq G^\wedge$, the closure of S in $G^\wedge_p$ is countable.*

Proof. Necessity. Clearly, $G^\wedge_p$ is a topological subgroup of $\mathbb{T}^G$. Let S be a countable subset of $G^\wedge_p$. Denote by f the diagonal product of the characters of S. Then f is a continuous homomorphism of G to $\mathbb{T}^S$. Since the weight of $\mathbb{T}^S$ is not greater than $|S| \cdot \omega = \omega$ and G satisfies ORC, we can find an open continuous homomorphism $\pi\colon G \to K$ onto a second countable group K and a continuous homomorphism $h\colon K \to \mathbb{T}^S$ such that $f = h \circ \pi$. Note that the group K is precompact as a continuous homomorphic of the precompact group G, so the group ϱK is compact.

Our choice of K, π, and h implies that $S \subseteq \pi^\wedge(K^\wedge_p)$, where $\pi^\wedge\colon K^\wedge_p \to G^\wedge_p$ is the dual homomorphism defined by $\pi^\wedge(\xi) = \xi \circ \pi$, for each $\xi \in K^\wedge_p$. It is easy to see that $\pi^\wedge$ is a topological and isomorphic embedding of $K^\wedge_p$ into $G^\wedge_p$. Since the homomorphism π is open, the image $\pi^\wedge(K^\wedge_p)$ coincides with the annihilator of $\ker \pi$ in $G_p^\wedge$ and hence $\pi^\wedge(K_p^\wedge)$ is closed in $G_p^\wedge$. Therefore, the closure of S in $G_p^\wedge$ is homeomorphic to a subspace of $K_p^\wedge$.

Since K is a dense subgroup of the compact group ϱK, we can identify the abstract groups $(\varrho K)^\wedge$ and $K^\wedge$. Clearly, ϱK has a countable base, so $|K^\wedge| \le \omega$ by [25, Theorem 24.15] or [3, Corollary 9.6.7]. Therefore, the closure of S in $G^\wedge$ is countable.

Sufficiency. Consider a continuous homomorphism $f\colon G \to H$ onto a second countable group H. There exist an open continuous homomorphism $\pi\colon G \to K$ and a continuous isomorphism (not necessarily a homeomorphism) $i\colon K \to H$ such that $f = i \circ \pi$. Clearly, the group K is precompact.

Since H is second countable, the group $H_p^\wedge$ is countable. Let $i^\wedge\colon H_p^\wedge \to K_p^\wedge$ be the homomorphism dual to i. Since i is a bijection, the subgroup $i^\wedge(H_p^\wedge)$ of $K_p^\wedge$ separates points of K. Applying [14, Theorem 1.9], we conclude that $i^\wedge(H_p^\wedge)$ is dense in $K_p^\wedge$.

It follows from the continuity of the homomorphism $\pi^\wedge\colon K_p^\wedge \to G_p^\wedge$ that the countable group $\pi^\wedge(i^\wedge(H_p^\wedge)) = f^\wedge(H_p^\wedge)$ is dense in $\pi^\wedge(K_p^\wedge)$. By our assumption, the closure of $f^\wedge(H_p^\wedge)$ in $G_p^\wedge$ is countable. Hence $\pi^\wedge(K_p^\wedge)$ is also countable. But $\pi^\wedge$ is injective, so $K_p^\wedge$ is countable as well. Therefore, the group K is second countable. Since the homomorphism $\pi\colon G \to K$ is continuous and open, it follows from $f = i \circ \pi$ that G satisfies ORC. □

Here is an application of Lemma 2.5 to the study of permanence properties of precompact groups satisfying ORC:

Proposition 2.6 *The class of precompact topological groups satisfying the open refinement condition is closed with respect to taking continuous homomorphic images.*

Proof. Let $f\colon G \to H$ be a continuous group homomorphism of G onto H, where G is precompact and satisfies ORC. The dual homomorphism $f^\wedge\colon H_p^\wedge \to G_p^\wedge$ is a topological isomorphism of $H_p^\wedge$ onto a subgroup of $G_p^\wedge$. Since G satisfies ORC, Lemma 2.5 implies that the closure of every countable set in $G_p^\wedge$ is countable. Hence the groups $f^\wedge(H_p^\wedge)$ and $H_p^\wedge$ have the same property. Applying Lemma 2.5 once again, we conclude that H satisfies ORC. □

In the next theorem we present conditions on a precompact group G under which the dual group $G^\wedge$ does not contain infinite compact subsets.

Theorem 2.7 *Let G be a precompact Baire group satisfying the open refinement condition. Then the dual group $G_p^\wedge$ does not contain infinite compact subsets.*

Proof. Suppose to the contrary that $G_p^\wedge$ contains an infinite compact subset F. Let S be a countable infinite subset of F. By Lemma 2.5, the closure of S in $G_p^\wedge$, say, C is a countable infinite compact set. Hence C is a non-discrete metrizable space which must contain non-trivial convergent sequences. The latter contradicts Corollary 2.4. □

Here is one of the main results of the article.

Theorem 2.8 *Suppose that G is a precompact Abelian group satisfying the open refinement condition. If G has the Baire property and contains no infinite compact subsets, then G is reflexive.*

Proof. It follows from Theorem 2.7 that $G_p^\wedge$ has no infinite compact subsets. Hence the groups $G^\wedge$ and $G^{\wedge\wedge}$ carry the topology of pointwise convergence on elements of G and $G^\wedge$, respectively. The conclusion now follows from Corollary 2.2. □

Our next aim is to show that the class of groups satisfying the conditions of Theorem 2.8 is fairly wide. For example, it contains all pseudocompact Abelian groups without infinite compact subsets (see Corollary 2.10 below). To see how wide the latter class is it suffices to refer to the following weaker version of [23, Theorem 5.8]: Under the Singular Cardinal Hypothesis, SCH, every pseudocompact Abelian group admits another pseudocompact Hausdorff topological group topology with no infinite compact subsets. We recall that SCH is the statement consistent with ZFC and saying that for every singular cardinal κ, if $2^{cf(\kappa)} < \kappa$, then $\kappa^{cf(\kappa)} = \kappa^+$. In particular, SCH implies that if $\kappa > 2^\omega$ is a cardinal of countable cofinality, then $\kappa^\omega = \kappa^+$.

In fact, the class of groups satisfying the conditions of Theorem 2.8 contains many precompact non-pseudocompact groups as well—this follows from Theorem 4.1 or Proposition 4.9 given below. It is worth noting that the first examples of reflexive precompact non-pseudocompact groups were presented in [1, Theorem 3.3].

Since every continuous homomorphism of a pseudocompact group onto a topological group with a countable base is open [14], the following fact is immediate:

Proposition 2.9 *Every pseudocompact group satisfies the open refinement condition.*

We can now give an alternative proof of [1, Theorem 2.2] (see also [23, Theorem 6.1]):

Corollary 2.10 *Every pseudocompact Abelian group without infinite compact subsets is reflexive.*

Proof. Every pseudocompact space X has the Baire property since it meets each non-empty G_δ-set in the Stone–Čech compactification βX of X (see [17, 3.10.F]). By Proposition 2.9, every pseudocompact topological group satisfies ORC. Therefore, Theorem 2.8 implies the required conclusion. □

3 The Baire property and the open refinement condition

To present a series of reflexive precompact groups in Section 4, we need several facts about precompact groups with the Baire property and/or satisfying ORC. In the lemma below we establish that the Baire property is preserved under extensions of topological groups by second countable groups. This result complements the fact that the product $X \times Y$ of two Baire spaces is also Baire provided one of the factors is second countable (see [33]).

Lemma 3.1 *Let K be a closed second countable subgroup of a topological group G (not necessarily Abelian). If the coset space G/K and the group K are Baire, so is G.*

Proof. Denote by π the natural projection of G onto the left coset space $H = G/K$. Suppose that $\{F_k : k \in \omega\}$ is a sequence of closed nowhere dense subsets of G. We have to show that the complement $G \setminus F$ is dense in G, where $F = \bigcup_{k\in\omega} F_k$.

Let $\{U_n : n \in \omega\}$ be a countable local base at the neutral element of K. Given integers $n, k \in \omega$, we put

$$P_{n,k} = \{h \in H : xU_n \subseteq F_k \cap \pi^{-1}(h), \text{ for some } x \in \pi^{-1}(h)\}.$$

Let us show that each $P_{n,k}$ is nowhere dense in H. Since the group K is second countable, we can find, for every $n \in \omega$, a countable set $C_n \subseteq K$ such that $K = U_n C_n$. We claim that $\pi^{-1}(P_{n,k}) \subseteq F_k C_n$, for all $n, k \in \omega$. Indeed, if $h \in P_{n,k}$ then $xU_n \subseteq F_k \cap \pi^{-1}(h)$, for some $x \in \pi^{-1}(h)$. Then $\pi(x) = h$ and we have that

$$F_k C_n \supseteq xU_n C_n = xK = \pi^{-1}(h).$$

This implies the inclusion $\pi^{-1}(P_{n,k}) \subseteq F_k C_n$. Since $F_k C_n$ is a first category set in G, we conclude that so are the set $\pi^{-1}(P_{n,k})$ in G and its image $P_{n,k}$ in H, respectively.

Let U be a non-empty open subset of G. Then $V = \pi(U)$ is open in H and, since $P = \bigcup_{n,k\in\omega} P_{n,k}$ is a first category set in the Baire space H, there exists a point $y \in V \setminus P$. Choose $x \in U$ with $\pi(x) = y$. Then $U \cap xK$ is a non-empty open subset of xK. It follows from our definition of the sets $P_{n,k}$ and P and the choice of the elements $y \in V$ and $x \in U$ that $xK \cap F_k$ is a nowhere dense subset of xK, for each $k \in \omega$. Indeed, otherwise one can find $x' \in xK$ and $k, n \in \omega$ such that $x'U_n \subseteq F_k \cap xK$. Then $y = \pi(x) = \pi(x') \in P_{k,n}$, thus contradicting our choice of y.

We conclude, therefore, that $xK \cap F$ is a first category set in xK. Since K is Baire and $xK \cong K$, the complement $(xK \cap U) \setminus F$ is non-empty. Thus, $U \setminus F \neq \emptyset$, which finishes the proof of the lemma. □

Lemma 3.2 *The class of precompact Baire groups is closed with respect to taking continuous homomorphic images.*

Proof. Suppose that $f\colon G \to H$ is a continuous surjective homomorphism of precompact topological groups, where G is Baire. Denote by g an extension of f to a continuous homomorphism of ϱG to ϱH. Since the groups ϱG and ϱH are compact, g is an open surjection. Suppose to the contrary that H can be covered by countably many nowhere dense sets, say, $H = \bigcup_{n\in\omega} C_n$. Clearly, $G \subseteq \bigcup_{n\in\omega} F_n$, where $F_n = g^{-1}(C_n)$ for each $n \in \omega$. Since the homomorphism g is open, the sets F_n are nowhere dense in ϱG. However, the density of G in ϱG implies that each $F_n \cap G$ is nowhere dense in G, which contradicts the Baire property of G. □

The following result is a kind of a *reflection principle* for the Baire property in the class of precompact groups. It reduces the problem of whether a given (non-metrizable) precompact group is Baire to the verification of whether all second countable continuous homomorphic images of the group are Baire.

Proposition 3.3 *The following conditions are equivalent for a precompact topological group G (not necessarily Abelian):*

a) *G is Baire;*

b) *G is non-meager in its completion ϱG;*

c) *every second countable continuous homomorphic image H of G is Baire.*

Proof. The equivalence a) $\Leftrightarrow$ b) is almost immediate. Indeed, suppose that G can be covered by a countable family $\{F_n : n \in \omega\}$ of nowhere dense subsets of the group ϱG. We can assume without loss of generality that each F_n is closed in ϱG. Then $U_n = \varrho G \setminus F_n$ is a dense open subset of ϱG and, since G is dense in ϱG, $V_n = G \cap U_n$ is a dense open subset of G, for each $n \in \omega$. It follows from our definition of the sets V_n that $\bigcap_{n\in\omega} V_n = \emptyset$, so G fails to be Baire.

Conversely, suppose that G is non-meager in ϱG, and consider a sequence $\{V_n : n \in \omega\}$ of open dense subsets of G. For every $n \in \omega$, choose an open set U_n in ϱG such that $U_n \cap G = V_n$. Then $F_n = \varrho G \setminus U_n$ is a closed nowhere dense subset of ϱG, so the set $P = G \setminus \bigcup_{n\in\omega} F_n$ is non-empty. We claim that P is dense in G. If not, there exists a non-empty open set W in G disjoint from P. Since G is precompact, we can find a finite set $C \subseteq G$ such that $G = WC$. It follows from our choice of W that $W \subseteq \bigcup_{n\in\omega} F_n$, i.e., W is of the first category in ϱG. Hence $G = WC$ is also of the first category in ϱG, thus contradicting our assumption about G.

The implication b) $\Rightarrow$ c) follows from Lemma 3.2 and the equivalence of a) and b).

It remains to verify that c) implies a). To this end, it suffices to prove that for every first category set S in G, there exists a continuous homomorphism $f\colon G \to H$ onto a second countable group H such that $f(S)$ is of the first category in H.

We call an open subset U of G *standard* if one can find a continuous homomorphism $p\colon G \to H$ onto a second countable group H and an open set $V \subseteq H$ such that $U = p^{-1}(V)$. Since G is precompact, the standard open sets constitute a base for G. It is almost immediate from the definition that the family of standard open sets in G is closed with respect to taking countable unions.

Let $\{F_n : n \in \omega\}$ be a sequence of nowhere dense subsets of G. For every $n \in \omega$, let γ_n be the family of standard open sets U in G such that

$U \cap F_n = \emptyset$. Then $U_n = \bigcup \gamma_n$ is a dense open subset of G disjoint from F_n. Clearly, G is a dense subgroup of the compact group ϱG, whence it follows that G has countable cellularity. Hence, for every $n \in \omega$, the family γ_n contains a countable subfamily λ_n such that $\bigcup \lambda_n$ is dense in U_n. Then $W_n = \bigcup \lambda_n$ is a standard open set in G, for each $n \in \omega$. Let a continuous homomorphism $f_n \colon G \to H_n$ onto a second countable group H_n witness that the set W_n is standard, where $n \in \omega$. Taking the diagonal product of the homomorphisms f_n, we obtain a continuous homomorphism $f \colon G \to H$ onto a second countable group H and a family $\{O_n : n \in \omega\}$ of open sets in H such that $W_n = f^{-1}(O_n)$, for each $n \in \omega$. Since each W_n is dense in G and $O_n = f(W_n)$, we conclude that the sets O_n are dense H. It also follows from the definition of W_n and the choice of O_n that $f(F_n) \cap O_n = \emptyset$, i.e., $f(F_n)$ is nowhere dense in H. Hence $f(\bigcup_{n\in\omega} F_n)$ is a first category set in H, as required. □

It is well known that the product of two Baire spaces can fail to be Baire (see [11, 19]). Even the product of two linear normed spaces with the Baire property need not have it, as M. Valdivia showed in [38] (see also [31]). It turns out that in the class of precompact groups, the Baire property becomes productive.

Theorem 3.4 *The class of precompact Baire groups is closed under formation of arbitrary direct products.*

Proof. Let $G = \prod_{i\in I} G_i$ be a product of precompact Baire groups. By Proposition 3.3, it suffices to verify that for every continuous homomorphism $f \colon G \to H$ onto a second countable group H, the image H is Baire.

Extend f to a continuous homomorphism $\varrho f \colon \prod_{i\in I} \varrho G_i \to \varrho H$ and denote by N the kernel of ϱf. Since the groups H and ϱH are first countable, N is a G_δ-set in the compact group $\varrho G = \prod_{i\in I} \varrho G_i$. Hence we can find, for every $i \in I$, a closed invariant subgroup $N_i \subseteq \varrho G_i$ of type G_δ in ϱG_i such that $\prod_{i\in I} N_i \subseteq N$. Let $\pi_i \colon \varrho G_i \to \varrho G_i/N_i$ be the quotient homomorphism, $i \in I$. Then the quotient group $\varrho G_i/N_i$ is compact and metrizable, so the subgroup $K_i = \pi_i(G_i)$ of $\varrho G_i/N_i$ has a countable base. It follows from Lemma 3.2 that the group K_i is Baire.

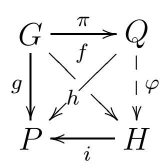
Let $\pi = \prod_{i\in I} \pi_i$ be the product of the homomorphisms π_i's. Clearly, the homomorphism $\pi \colon \varrho G \to \prod_{i\in I} \varrho G_i/N_i$ is continuous and surjective, while $\prod_{i\in I} N_i$ is the kernel of π. Since ϱG is a compact group, the homomorphism π is open. It follows from the inclusion $\ker \pi \subseteq N = \ker \varrho f$ that there exists a homomorphism $\varphi\colon \prod_{i\in I} \varrho G_i/N_i \to \varrho H$ such that $\varrho f = \varphi\circ\pi$. Since π is open, the homomorphism φ is continuous. It is also clear that $\pi(G) = \prod_{i\in I} K_i$.

To finish the proof, it suffices to note that the product $K = \prod_{i\in I} K_i$ of second-countable Baire spaces is Baire, by [33, Theorem 3]. Finally, by Lemma 3.2, the image $H = \varphi(K)$ is Baire as well. $\square$

In some cases, Lemma 3.1 remains valid for a non-metrizable subgroup K of a precompact group G. Again, we impose no commutativity conditions on the groups that appear in the lemma below.

Lemma 3.5 *Let K be a closed invariant pseudocompact subgroup of a precompact topological group G. If the quotient group G/K is Baire, so is G.*

Proof. By Proposition 3.3, it suffices to verify that every second countable continuous homomorphic image H of G is Baire. Let $f\colon G \to H$ be a continuous onto homomorphism. Denote by π the canonical projection of G onto G/K. Then the diagonal product of π and f, say p is a continuous homomorphism of G to the product group $G/K \times H$. Put $M = p(G)$. Then there exist continuous homomorphisms $\varphi\colon M \to G/K$ and $g\colon M \to H$ satisfying $\pi = \varphi \circ p$ and $f = g \circ p$.



Clearly, φ and g are restrictions to M of the projections of $G/K \times H$ to the first and second factor, respectively.

It is easy to see that the kernel of φ is a compact metrizable group. Indeed, let e be the neutral element of G/K. Then $\ker\varphi = \varphi^{-1}(e) = p(\pi^{-1}(e)) = p(K)$ is a pseudocompact subgroup of M. Since φ is the restriction to M of the projection of the product $G/K \times H$ to the first factor, its kernel is contained in $\{e\} \times H \cong H$, which is metrizable. Thus $\ker\varphi$ is compact metrizable as a pseudocompact subspace of a metrizable space.

Notice that the homomorphism φ is open. Indeed, if U is open in M, then $\varphi(U) = \pi(p^{-1}(U))$. The latter set in open in G/K since p is continuous and π is open. Therefore, G/K is a quotient group of M with respect to a compact metrizable subgroup.

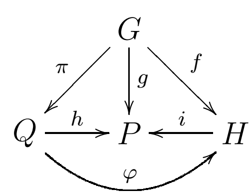
Since, by the assumptions of the lemma, the group G/K is Baire, it follows from Lemma 3.1 that so is M. Therefore, the continuous homomorphic image

H of M is also Baire. It remains to apply Proposition 3.3 to conclude that G is Baire as well. □

We recall that a topological group H has *countable pseudocharacter* if the neutral element of H is a G_δ-set in H.

Lemma 3.6 *Let $f\colon G \to H$ be a continuous homomorphism of an ω-narrow topological group G satisfying the open refinement condition onto a group H of countable pseudocharacter. Then H has a countable network.*

Proof. The group H is ω-narrow as a continuous homomorphic image of the ω-narrow group G. Since, by our assumptions, the neutral element of H is a G_δ-set, it follows from [3, Corollary 5.2.12] that there exists a continuous isomorphism $i\colon H \to P$ onto a second countable topological group P. Then $g = i \circ f$ is a continuous homomorphism of G onto P. Hence we can find a continuous open homomorphism $\pi\colon G \to Q$ onto a second countable group Q and a continuous homomorphism $h\colon Q \to P$ such that $g = h \circ \pi$.



Then $\varphi = i^{-1} \circ h$ is a homomorphism of Q onto H. Since the homomorphism π is open, it follows from the equality $\varphi \circ \pi = f$ that φ is continuous. Therefore, the images under φ of the elements of a countable base for Q form a countable network for H. □

Here we give some additional information on Baire groups. It will be used in the proof of Proposition 3.8.

Lemma 3.7 *Let G be a Baire topological group. If G has a countable network, then G is separable metrizable.*

Proof. Let $\mathcal{N}$ be a countable network for G. Since G is regular, the closures of the elements of $\mathcal{N}$ also form a countable network for G. Hence we can assume that each element of $\mathcal{N}$ is closed in G. For every $F \in \mathcal{N}$, let $F^* = F \setminus \operatorname{Int} F$. Then F^* is a closed nowhere dense subset of G and, since G is Baire, the set

$P = G \setminus \bigcup\{F^* : F \in \mathcal{N}\}$ is not empty. Take a point $x \in P$. We claim that the family

$$\mathcal{B}(x) = \{\operatorname{Int} F : x \in F \in \mathcal{N}\}$$

is a local base for G at x. Indeed, take an arbitrary neighbourhood O of x in G. Since $\mathcal{N}$ is a network for G, there exists $F \in \mathcal{N}$ such that $x \in F \subseteq O$. It follows from $x \in P$ that $x \in \operatorname{Int} F$. Clearly, $\operatorname{Int} F \in \mathcal{B}(x)$ and $\operatorname{Int} F \subseteq F \subseteq O$, whence our claim follows.

Since topological groups are homogeneous spaces, we conclude that G is first countable. Finally, every first countable topological group is metrizable by the Birkhoff–Kakutani theorem, while metrizable spaces with a countable network are separable and, hence, second countable. □

We need one more fact about extensions of topological groups:

Proposition 3.8 *Let K be a closed invariant pseudocompact subgroup of an ω-narrow group G such that quotient group G/K is Baire and satisfies the open refinement condition. Then G satisfies the open refinement condition (and is Baire).*

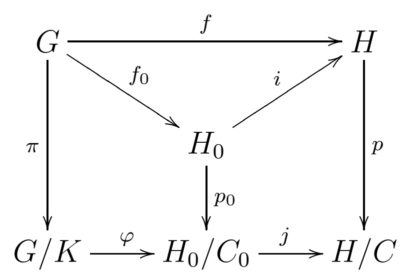
Proof. That G is Baire follows directly from Lemma 3.5. Denote by π the canonical projection of G onto the quotient group G/K and consider an arbitrary continuous homomorphism $f\colon G \to H$ onto a second countable group H. Let H_0 be the same underlying group H that carries the quotient topology with respect to the homomorphism f; then the homomorphism f considered as a mapping of G to H_0 is denoted by f_0. Clearly, f_0 is open and continuous. Let i be the identity homomorphism of H_0 onto H. Then i is continuous and satisfies $f = i \circ f_0$. To finish the proof, it suffices to verify that the group H_0 has a countable base.

Put $C = f(K)$ and $C_0 = f_0(K)$. Then C and C_0 are pseudocompact invariant subgroups of H and H_0, respectively. Since the group H is second countable, C is compact. The group C_0 is also compact. Indeed, the neutral element of H_0 is a G_δ-set since $i\colon H_0 \to H$ is a continuous isomorphism onto a second countable group. Therefore, every pseudocompact subspace of H_0 is compact and has a countable base [2, Proposition 3.4]. In particular, C_0 is a compact metrizable subgroup of H_0. It is clear that $i(C_0) = C$ and the restriction of i to C_0 is a topological isomorphism of C_0 onto C.

Let $p\colon H \to H/C$ and $p_0\colon H_0 \to H_0/C_0$ be canonical projections. Then there exists a continuous isomorphism $j\colon H_0/C_0 \to H/C$ satisfying $j \circ p_0 =$

$p \circ i$. The group H/C is second countable as the image of the second countable group H under the open continuous homomorphism p. Since j is one-to-one and continuous, the group G_0/H_0 has countable pseudocharacter.

Since the kernel K of the homomorphism π is contained in the kernel of the homomorphism $p_0 \circ f_0$, there exists a homomorphism $\varphi\colon G/K \to H_0/C_0$ satisfying $\varphi \circ \pi = p_0 \circ f_0$.



Since π is open and the composition $p_0 \circ f_0$ is continuous, we conclude that φ is also continuous. Further, since p_0 and f_0 are open, the homomorphisms $p_0 \circ f_0$ and φ are open as well. Therefore the group H_0/C_0 is the Baire as an open continuous image of the Baire group G/K.

Clearly, G/K is ω-narrow as a continuous homomorphic image of the ω-narrow group G. By the assumptions of the proposition, the group G/K satisfies ORC, while the neutral element of the group H_0/C_0 is a G_δ-set. Hence, by Lemma 3.6, H_0/C_0 has a countable network. Then, according to Lemma 3.7, H_0/C_0 has a countable base. Finally, since $C_0 = \ker p_0$ has a countable base, Vilenkin's theorem (see [40] or [3, Corollary 1.5.21]) implies that the group H_0 has a countable base as well. □

4 Quotients of reflexive groups

We are finally in a position to show that the class of reflexive groups satisfying the conditions of Theorem 2.8 contains many precompact non-pseudocompact groups. Further, we will also show that taking quotients destroys reflexivity quite easily (see Corollaries 4.2 and 4.4, as well as Theorem 4.6).

Theorem 4.1 *Let H be a precompact Abelian group with the Baire property that satisfies ORC. Then there is a precompact group G such that $H = G/N$, where N is a closed pseudocompact subgroup of G, and*

(i) *G is Baire;*

(ii) *G satisfies ORC;*

(iii) *G contains no infinite compact subsets;*

(iv) *G is reflexive;*

(v) *all countable subgroups of G are h-embedded.*

Proof. Denote by ϱH the completion of H. Then ϱH is a compact group. According to [16, Theorem 5.5], one can find a pseudocompact Abelian group P all countable subgroups of which are h-embedded and a continuous open homomorphism π of P onto ϱH such that the kernel N of π is a pseudocompact subgroup of P. [The fact that all countable subgroups of P are h-embedded was explicitly verified in the proof of [16, Theorem 5.5] though it was not given in the body of the theorem.] Hence we can apply [1, Proposition 2.1] to conclude that all compact subsets of P are finite.

Let φ be the restriction of π to the subgroup $G = \pi^{-1}(H)$ of P. Then G is precompact as a subgroup of the pseudocompact (hence precompact) group P and φ is a continuous open homomorphism of G onto H. It follows that $H \cong G/N$. It is also clear that $\ker\varphi = N = \ker\pi$ is a closed pseudocompact subgroup of G, so Lemma 3.5 implies that the group G is Baire, i.e., G satisfies (i). Since G is a subgroup of P, all compact subsets of G are finite and all countable subgroups of G are h-embedded. This implies (iii) and (v). By Proposition 3.8, G satisfies ORC, which gives (ii). Therefore, item (iv) of the theorem, the reflexivity of G, follows from Theorem 2.8. □

Proposition 2.6 shows that the open refinement condition imposed upon the group H in the above theorem appeared not accidentally.

Since every second countable group satisfies ORC, the next result is immediate from Theorem 4.1:

Corollary 4.2 *Every second countable precompact Abelian group H with the Baire property is a quotient group of a reflexive precompact group with respect to a closed pseudocompact subgroup.*

Recall that a topological group G is called *almost metrizable* or *feathered* if G contains a compact subgroup K such that the quotient space G/K is metrizable (see [34] or [3, Section 4.3]). Clearly, all compact groups and all metrizable groups are almost metrizable.

Remark 4.3 It is easy to see that for a precompact almost metrizable group G, one can always find a compact *invariant* subgroup K of G such that the quotient group G/K has a countable base. We assume for simplicity that G is Abelian (even if the conclusion is valid in general). Suppose we have chosen a compact subgroup K of a precompact Abelian group G such that the quotient group G/K is metrizable. Since the group G/K is also precompact (hence ω-narrow), G/K has a countable base by [3, Proposition 3.4.5].

Here is a more general version of Corollary 4.2.

Corollary 4.4 *Every precompact almost metrizable Abelian group H with the Baire property is a quotient group of a reflexive precompact group with respect to a closed pseudocompact subgroup.*

Proof. Let K be a compact subgroup of H such that the quotient group H/K has a countable base. Then Proposition 3.8 implies that H satisfies ORC. Hence the required conclusion follows from Theorem 4.1. □

Many groups H satisfying the conditions of Corollary 4.2 or 4.4 need not be reflexive. One can take, for example, any proper subgroup of $\mathbb{T}$ of countable index, where $\mathbb{T}$ is the circle group with the usual topology. A considerably wider class of such groups is presented below.

We say, following [12], that a dense subgroup H of a topological Abelian group G *determines* G if the dual groups $G^\wedge$ and $H^\wedge$ are topologically isomorphic (under the natural restriction mapping). The theorem below refines Corollary 2.10 of [15], where the authors just claim the existence of a proper dense subgroup $H \subseteq G$ that determines G. First, one simple lemma is in order:

Lemma 4.5 *If H is a subgroup of a Baire topological group G and the index of H in G is countable, then H is Baire.*

Proof. Suppose to the contrary that there exists a meager non-empty open set $U \subseteq H$. Since translations in H are homeomorphisms, the family $\mathcal{B}$ of the sets xV, where $x \in H$ and V is a non-empty open set in H with $V \subseteq U$, constitutes a base of H. It is clear that every element of $\mathcal{B}$ is a meager subset of G. Let $\mathcal{D}$ be a maximal disjoint subfamily of $\mathcal{B}$. Then the open set $D = \bigcup \mathcal{D}$ is dense in H, so $F = H \setminus D$ is a closed nowhere dense set in H. In particular, F is nowhere dense in G.

We claim that D is meager in G. Indeed, let $\mathcal{D} = \{D_i : i \in I\}$. Since every D_i is meager in G, we can find a countable family $\{M_{i,n} : n \in \omega\}$ of nowhere dense sets in G such that $D_i = \bigcup_{n\in\omega} M_{i,n}$. Then the set $M_n = \bigcup_{i\in I} M_{i,n}$ is nowhere dense in G (we use the fact that $\mathcal{D}$ is disjoint) and $D = \bigcup_{n\in\omega} M_n$ is meager in G. Therefore, $H = D \cup F$ is also meager in G.

Finally, the group G is covered by countably many cosets of H, so G is meager in itself. This contradicts the Baire property of G. ☐

Theorem 4.6 *Every infinite compact Abelian group G contains a proper dense almost metrizable Baire subgroup H which determines G and satisfies the open refinement condition. Therefore, H fails to be reflexive.*

Proof. Our first step is to prove the existence of such a subgroup H in the case G is metrizable. Then G has countable weight, $|G| = 2^\omega = \mathfrak{c}$, and there exists a countable dense subgroup S of G.

Denote by $D(G)$ the divisible hull of G containing G as an essential subgroup (see [20]). Then $D(G)$ is the direct sum of countable subgroups, say, $D(G) = \oplus_{\alpha<\mathfrak{c}} C_\alpha$. Since $S \subseteq G$ is countable, there exists a countable set $A \subseteq \mathfrak{c}$ such that $S \subseteq \oplus_{\alpha\in A} C_\alpha$. Take any countable infinite set $B \subseteq \mathfrak{c}$ disjoint from A and put $D = \mathfrak{c} \setminus B$. Then $A \subseteq D$ and, hence, S is a subgroup of the group $H = G \cap \oplus_{\alpha\in D} C_\alpha$. Since G is an essential subgroup of $D(G)$, the intersection $G \cap \oplus_{\alpha\in B} C_\alpha$ is non-trivial, which in its turn implies that H is a proper subgroup of G. It is also clear that H has countable index in G. Since $S \subseteq H$, we conclude that H is dense in G. By Lemma 4.5, H is Baire.

The metrizable group H is clearly almost metrizable. Since, in addition, H is dense in the compact metrizable group G, the dual group $H^\wedge$ coincides with the discrete group $G^\wedge$ (see [8, Theorem 2] or [4, Proposition 4.11]). It follows that H determines G and that the second dual $H^{\wedge\wedge} \cong G^{\wedge\wedge} \cong G$ is compact, while H is not. Therefore, H is not reflexive.

Suppose that G is not metrizable. Then there exists a continuous homomorphism $f\colon G \to K$ onto an infinite compact metrizable group K. We have just proved that K contains a proper dense Baire subgroup, say, H_0. Let $H = f^{-1}(H_0)$. Then H is a proper subgroup of G and, since the homomorphism f is open, H is dense in G. Let φ be the restriction of f to H. Then φ is open and $\ker f = \ker \varphi$, i.e., the homomorphism φ has the compact kernel N. Hence the group H is almost metrizable, while Lemma 3.5 implies that H is Baire. Since N is compact and the quotient group $H_0 \cong H/N$ is not reflexive, neither is H (see [6, Theorem 2.6]). Note that H satisfies ORC according to Proposition 2.9.

Finally, since $H_0^\wedge$ is discrete and N is compact, a direct verification shows the dual group $H^\wedge$ is discrete as well. The density of H in G implies that the restriction mapping $\chi \mapsto \chi\restriction_H$, with $\chi \in G^\wedge$, is an isomorphism of the (abstract) group $G^\wedge$ onto $H^\wedge$. Since both dual groups are discrete, this mapping is a topological isomorphism. Hence H determines G. □

Since every precompact group is a dense subgroup of a compact group, it is natural to ask which compact groups G contain proper dense reflexive subgroups. By [8, Theorem 2], the dual group of every precompact metrizable Abelian group is discrete, so the answer is in the negative for every compact metrizable group G. Therefore, under the Continuum Hypothesis, the following result gives a complete answer to the question.

Theorem 4.7 *Every compact Abelian group G of weight greater than or equal to 2^ω contains a proper dense reflexive pseudocompact subgroup.*

Proof. By [25, Theorem 24.15], we have $|G^\wedge| = w(G) \geq \mathfrak{c} = 2^\omega$, where $w(G)$ is the weight of G, and a standard argument (along with the fact that the cofinality of $\mathfrak{c}$ is uncountable) shows that $G^\wedge$ contains a subgroup S isomorphic to the direct sum of $\mathfrak{c}$ many non-trivial cyclic subgroups, say, $S = \oplus_{\alpha<\mathfrak{c}} C_\alpha$. Since the group G is topologically isomorphic to $G^{\wedge\wedge}$, it admits a continuous homomorphism onto the product group $\prod_{\alpha<\mathfrak{c}} C_\alpha^\wedge$, where each $C_\alpha^\wedge$ is either a finite group or the circle group $\mathbb{T}$. To see this, it suffices to consider the restriction to S of every character χ defined on $G^\wedge$. Again, since $cf(\mathfrak{c}) > \omega$, there are either $\mathfrak{c}$ summands C_α with $C_\alpha \cong \mathbb{Z}$ or $\mathfrak{c}$ summands $C_\alpha \cong \mathbb{Z}(n)$, for some integer $n > 1$. Hence G admits a continuous homomorphism either onto $\mathbb{T}^\mathfrak{c}$ or onto $\mathbb{Z}(n)^\mathfrak{c}$, for some $n > 1$.

According to [29], each of the groups $\mathbb{T}^\mathfrak{c}$ and $\mathbb{Z}(n)^\mathfrak{c}$ contains a proper dense pseudocompact subgroup without non-trivial convergent sequences. In fact, the argument from [16, Theorem 5.5] shows that the corresponding subgroup P can be chosen so that all countable subgroups of P are h-embedded. Therefore, by [1, Proposition 2.1], all compact subsets of P are finite. It now follows from Corollary 2.10 that the group P is reflexive.

Let $f\colon G \to K$ be a continuous homomorphism onto K, where K is either $\mathbb{T}^\mathfrak{c}$ or $\mathbb{Z}(n)^\mathfrak{c}$. Since the homomorphism f is open, $H = f^{-1}(P)$ is a proper dense pseudocompact subgroup of G. The restriction $\varphi = f\restriction_H$ is an open continuous homomorphism with a compact kernel N. Since the image

$P = \varphi(H) \cong H/N$ is a reflexive group, we conclude that so is H (see [6, Theorem 2.6]). □

It was shown in [21, Corollary 2.10] that every connected Abelian group G of weight $\kappa = \kappa^\omega$ contains a proper dense pseudocompact subgroup without non-trivial convergent sequences. According to Corollary 2.10, this implies that such a group G contains a proper dense reflexive pseudocompact subgroup. We have just proved in Theorem 4.7 that G does contain a proper dense reflexive pseudocompact subgroup without the assumption that G is connected.

In Proposition 4.8 below we complement Proposition 2.6 and clarify the permanence properties of the class of topological groups satisfying ORC, with the aim to generalize Corollary 4.4.

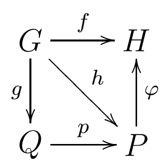
Proposition 4.8 *The class of precompact topological groups satisfying the open refinement condition is closed under the formation of arbitrary direct products.*

Proof. Consider a product $G = \prod_{i\in I} G_i$, where each group G_i is precompact and satisfies ORC. Suppose that $f\colon G \to H$ is a continuous homomorphism onto a second countable group H. We can apply [3, Lemma 8.5.4] to find a countable set $J \subseteq I$ and a continuous homomorphism $q\colon \prod_{i\in J} G_i \to H$ such that $f = q \circ \pi_J$, where π_J is the projection of G onto the subproduct $G_J = \prod_{i\in J} G_i$. Taking into account that the projection π_J is an open homomorphism, we can assume without loss of generality that the index set I is countable.

Extend f to a continuous homomorphism $\varrho f\colon \varrho G \to \varrho H$. In what follows we identify the compact groups ϱG and $\prod_{i\in I} \varrho G_i$. Arguing as in the proof of Theorem 3.4, we can find, for each $i \in I$, a continuous homomorphism $h_i\colon \varrho G_i \to P_i$ onto a second countable group P_i and a continuous homomorphism $\varphi\colon \prod_{i\in I} P_i \to \varrho H$ such that $\varrho f = \varphi \circ h$, where $h\colon \prod_{i\in I} \varrho G_i \to \prod_{i\in I} P_i$ is the topological product of the homomorphisms h_i's.

The subgroup $h_i(G_i)$ of P_i has a countable base and, since G_i satisfies ORC, we can find an open continuous homomorphism $g_i\colon G_i \to Q_i$ onto a second countable group Q_i and a continuous homomorphism $p_i\colon Q_i \to P_i$ satisfying $h_i\restriction_{G_i} = p_i \circ g_i$. The topological product g of the homomorphisms g_i's is an open continuous homomorphism of G onto the second countable group $Q = \prod_{i\in I} Q_i$. Also we have the equality $h\restriction_G = p \circ g$, where $p\colon Q \to$

$P = \prod_{i \in I} P_i$ is the product of the homomorphisms p_i's.



It follows from the definition of the homomorphisms φ, g, p that $f = (\varphi \circ p) \circ g$. Finally, since $g\colon G \to Q$ is an open continuous homomorphism and the group Q is second countable, we conclude that G satisfies ORC. □

We finish with a generalization of Corollary 4.4 to topological products.

Proposition 4.9 *Let H be the product of an arbitrary family of precompact almost metrizable Abelian groups with the Baire property. Then H is a quotient group of a reflexive precompact group G with respect to a closed pseudocompact subgroup, where all countable subgroups of G are h-embedded.*

Proof. Suppose that $H = \prod_{i \in I} H_i$, where each H_i is a precompact almost metrizable Abelian group with the Baire property. Clearly, H is precompact. It follows from Theorem 3.4 that H is Baire, while Proposition 3.8 and Remark 4.3 together imply that each factor H_i satisfies ORC. Hence, by Proposition 4.8, H satisfies ORC as well. To finish the proof, it suffices to apply Theorem 4.1. □

Corollary 4.10 *Let H be the product of an arbitrary family of precompact metrizable Abelian groups with the Baire property. Then H is a quotient group of a reflexive precompact group with respect to a closed pseudocompact subgroup.*

5 Open problems

Theorem 2.8 on the reflexivity of some precompact Abelian groups G is based on the fact that both groups G and its dual $G^\wedge$ have no infinite compact subsets. However, the product $G \times K$ of a reflexive group G with any compact Abelian group K remains reflexive. Therefore, one can try to extend Theorem 4.1 to all precompact Abelian groups:

Problem 5.1 *Is every precompact Abelian group (with the Baire property) a quotient of a reflexive precompact group (with the Baire property)?*

Here is a problem of a similar nature:

Problem 5.2 *Is every precompact Abelian group a continuous homomorphic image of a reflexive precompact group G?*

The methods of the article do not work to settle the next problem since infinite non-discrete Baire groups are uncountable:

Problem 5.3 *Do there exist countable infinite reflexive precompact groups?*

It is also interesting, after Lemma 3.5, to find out whether the Baire property is stable under extensions of groups with respect to a compact invariant subgroup:

Problem 5.4 *Let K be a compact invariant subgroup of a topological group G such that the quotient group G/K is Baire. Is G Baire?*

Acknowledgements. The authors are indebted to M. J. Chasco for helpful discussions regarding the contents of the article and to the referee for careful reading of the manuscript and a number of valuable comments and suggestions, including the sufficiency part of Lemma 2.5.

References

[1] S. Ardanza-Trevijano, M. J. Chasco, X. Domínguez, M. G. Tkachenko, Precompact non-compact reflexive Abelian groups, *Forum Math.*, to appear. Doi: 10.1515/form.2011.061.

[2] A. V. Arhangel'skii, Relations among the invariants of topological groups and their subspaces, *Russian Math. Surveys* **35** (1980), 1–23. Russian original in: *Uspekhy Mat. Nauk* **35** (1980), 3–22.

[3] A. V. Arhangel'skii and M. G. Tkachenko, *Topological Groups and Related Structures*, Atlantis Studies in Mathematics, Vol. I, Atlantis Press/World Scientific, 2008.

[4] L. Außenhofer, Contributions to the duality theory of abelian topological groups and to the theory of nuclear groups, *Dissertationes Math.* (Rozprawy Mat.) **384**:113, 1999.

[5] W. Banaszczyk, Additive subgroups of topological vector spaces, *Lecture Notes in Math.* vol. 1466, Springer Verlag, Berlin–Heidelberg, 1991.

[6] W. Banaszczyk, M. J. Chasco, and E. Martín-Peinador, Open subgroups and Pontryagin duality, *Math. Z.* **215** (1994), no. 2, 195–204.

[7] N. Bourbaki, Elements of Mathematics. General Topology. Part I, Paris 1966.

[8] M. J. Chasco, Pontryagin duality for metrizable groups, *Arch. Math.* **70** (1998), no. 1, 22–28.

[9] M. J. Chasco and E. Martín-Peinador, An approach to duality on Abelian precompact groups, *J. Group Theory* **11** (2008), 635–643.

[10] M. J. Chasco, E. Martín-Peinador, and V. Tarieladze, On Mackey topology for groups, *Studia Math.* **132** (3) (1999), 257–262.

[11] P. E. Cohen, Products of Baire spaces, *Proc. Amer. Math. Soc.* **55** (1976), 119–124.

[12] W. W. Comfort, S. U. Raczkowski, F. J. Trigos-Arrieta, The dual group of a dense subgroup, *Czech. Math. J.* **54** (2004), no. 2, 509–533.

[13] W. W. Comfort and K. A. Ross, Topologies induced by groups of character, *Fund. Math.* **55** (1964), 283–291.

[14] W. W. Comfort and K. A. Ross, Pseudocompactness and uniform continuity in topological groups, *Pacific J. Math.* **16** (1966), no. 3, 483–496.

[15] D. Dikranjan and D. B. Shakhmatov, Quasi-convex density and determining subgroups of compact abelian groups, *J. Math. Anal. Appl.* **363** (2010), no. 1, 42–48.

[16] D. Dikranjan and M. G. Tkachenko, Sequential completeness of quotient groups, *Bull. Austral. Math. Soc.* **61** (2000), 129–151.

[17] R. Engelking, *General Topology,* Heldermann Verlag, Berlin 1989.

[18] I. Fleischer and T. Traynor, Continuity of homomorphisms on a Baire group, *Proc. Amer. Math. Soc.* **93** (1985), no. 2, 367–368.

[19] W. Fleissner and K. Kunen, Barely Baire spaces, *Fund. Math.* **101** (1978), no. 3, 229–240.

[20] L. Fuchs, *Infinite Abelian Groups I*, Pure and Applied Mathematics 36, Academic Press 1970.

[21] J. Galindo and S. García-Ferreira, Compact groups containing dense pseudocompact subgroups without non-trivial convergent sequences, *Topol. Appl.* **154** (2007), no. 2, 476–490.

[22] J. Galindo and S. Hernández, Pontryagin–van Kampen reflexivity for free Abelian topological groups, *Forum Math.* **11** (1999), no. 4, 399–415.

[23] J. Galindo and S. Macario, *Pseudocompact group topologies with no infinite compact subsets*, Preprint, available at `http://arxiv.org/abs/0812.5033`, 2008.

[24] S. Hernández and S. Macario, Dual properties in totally bounded abelian groups, *Arch. Math.* (Basel) **80** (2003), no. 3, 271–283.

[25] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis, Volume I*, Springer-Verlag, Berlin-Göttingen-Heidelberg 1979.

[26] S. Kaplan, Extension of the Pontryagin duality I: Infinite products, *Duke Math. J.* **15** (1948), 649–658.

[27] S. Kaplan, Extensions of the Pontryagin duality. II. Direct and inverse sequences, *Duke Math. J.* **17** (1950), 419–435.

[28] Y. Komura, Some examples on linear topological spaces, *Math. Ann.* **153** (1984), 150–162.

[29] V. Malykhin and L. Shapiro, Pseudocompact groups without convergent sequences, *Math. Notes* **37** (1985), 59–62.

[30] C. Menini and A. Orsatti, Dualities between categories of topological modules, *Comm. Algebra* **11** (1983), no. 1, 21–66.

[31] J. van Mill and R. Pol, The Baire category theorem in products of linear spaces and topological groups, *Topol. Appl.* **22** (1986), 267–282.

[32] N. Noble, k-groups and duality, *Trans. Amer. Math. Soc.* **151** (1970), 551–561.

[33] J. Oxtoby, Cartesian products of Baire spaces, *Fund. Math.* **49** (1961), 157–161.

[34] B. A. Pasynkov, Almost metrizable topological groups, *Dokl. AN SSSR* **161** (1965), 281–284 (*in Russian*).

[35] V. G. Pestov, Free abelian topological groups and the Pontryagin–van Kampen duality, *Bull. Austral. Math. Soc.* **52** (1995), 297–311.

[36] S. U. Raczkowski and F. J. Trigos-Arrieta, Duality of totally bounded Abelian groups, *Bol. Soc. Mat. Mex.* III, Ser. 7 (2001), no. 1, 1–12.

[37] M. G. Tkachenko, Compactness type properties in topological groups, *Czech. Math. J.* **38** (1988), no. 2, 324–341.

[38] M. Valdivia, Products of Baire topological vector spaces, *Fund. Math.* **125** (1985), no. 1, 71–80.

[39] R. Venkataraman, Extensions of Pontryagin duality, *Math. Z.* **143** (1975), no. 2, 105–112.

[40] N. Ya. Vilenkin, Theory of weakly separable topological groups, *Mat. Sb.* **22** (1948), 135–177 (in Russian).

M. Bruguera
Dept. de Matemática Aplicada I
Universidad Politécnica de Cataluña
C/ Gregorio Marañón 44-50, 08028 Barcelona, España
e-mail: m.montserrat.bruguera@upc.edu

M. Tkachenko
Departamento de Matemáticas
Universidad Autónoma Metropolitana
Av. San Rafael Atlixco # 186, Col. Vicentina, Iztapalapa,
C.P. 09340, Mexico D. F., Mexico
e-mail: mich@xanum.uam.mx