Integrable birational maps on the plane: blending dynamics and algebraic geometry.

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Introduction

Some maps arising in mathematical physics are *birational planar* ones with *a rational first integral*. This means that they preserve a foliation of the plane given by *algebraic curves*. As a consequence of Hurwitz theorem if a planar birational map is *not globally periodic* then each irreducible invariant curve is of genus 0 or 1.

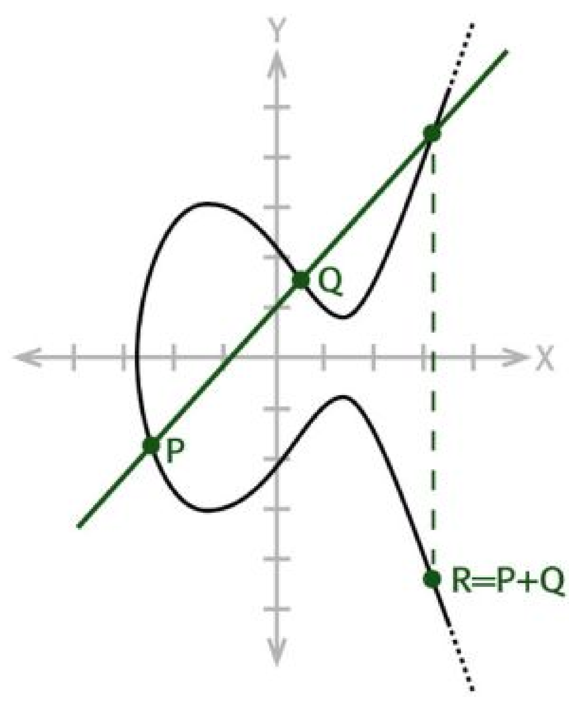
Theorem (Hurwitz)

The group of birational maps of a smooth algebraic curve of genus $g \geq 2$ to itself is finite of order at most $84(g-1)$.

In the case that the invariant foliation is given (generically) by genus 1 curves (*elliptic curves*). *Then the group structure of the elliptic foliation characterizes the dynamics of any birational map preserving it.*

Theorem ([3])

Any birational map F that leaves an elliptic curve $\mathcal{E}$ invariant can be expressed in terms of the group law as either $F : P \mapsto P + Q$ or $F : P \mapsto i(P) + Q$, where i is globally periodic of orders 2,3,4 or 6; and $+$ denotes the inner sum of $\mathcal{E}$.



Recall the group law of an elliptic curve

Observe that $F^n(P) = P + nQ$, so $\mathcal{E}$ is full of p-periodic orbits iff $pQ = 0$, that is, iff Q is in the *torsion* of $(\mathcal{E}, +)$.

Example: The rotation number function of 2-periodic Lyness recurrences, [1]

We consider the 2-periodic Lyness' equations

$$u_{n+2} = \frac{a_n + u_{n+1}}{u_n} \quad \text{where} \quad a_n = \begin{cases} a \text{ for } n = 2\ell + 1, \\ b \text{ for } n = 2\ell. \end{cases}$$

This equation can studied using the *composition map:*

$$F_{b,a}(x,y) := (F_b \circ F_a)(x,y) = \left(\frac{a+y}{x}, \frac{a+bx+y}{xy}\right)$$

where F_a and F_b are the Lyness maps: $F_\alpha(x,y) = \left(y, \frac{\alpha+y}{x}\right)$, since

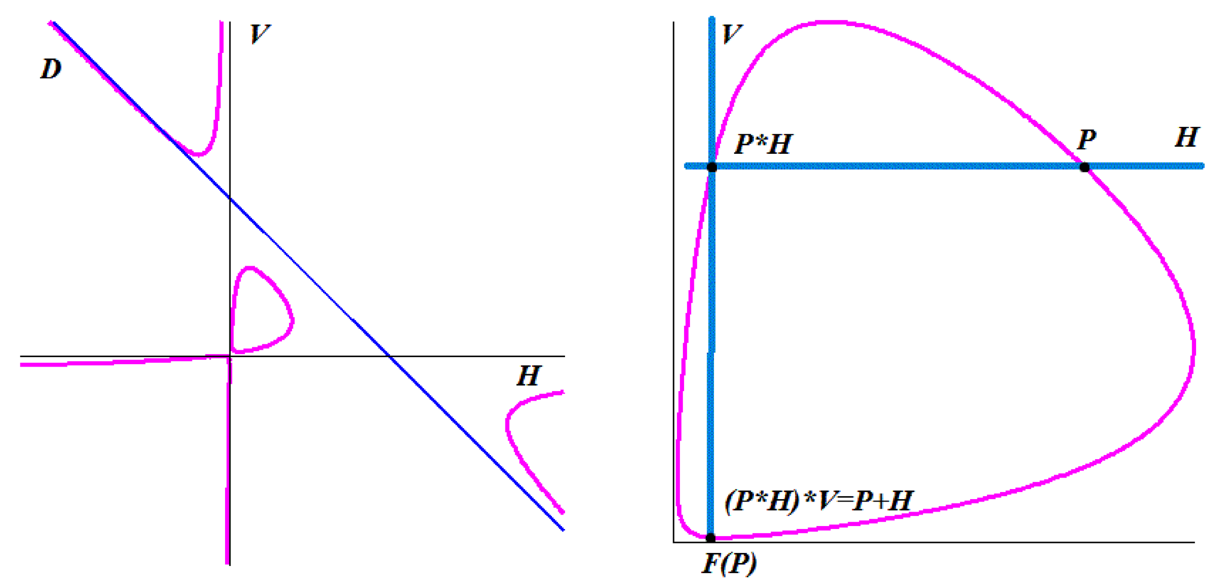
$$(u_1,u_2) \xrightarrow{F_a} (u_2,u_3) \xrightarrow{F_b} (u_3,u_4) \xrightarrow{F_a} (u_4,u_5) \xrightarrow{F_b} (u_5,u_6) \xrightarrow{F_a} \cdots$$

$F_{b,a}$ extends to $\mathbb{C}P^2$ as $\widetilde{F}_{b,a}([x:y:t]) = [ayt + y^2 : at^2 + bxt + yt : xy]$, and it preserves the foliation given by

$$\mathcal{C}_h = \{(bx+at)(ay+bt)(ax+by+abt) - hxyt = 0\}.$$

Some results

- *For each h s.t. $\mathcal{C}_h$ is elliptic, then $\widetilde{F}_{b,a|\mathcal{C}_h}(P) = P + H$, where $+$ is the group law of $\mathcal{C}_h$ taking the infinite point V as the zero element.*
- *If $a > 0$ and $b > 0$, then for all $h > h_c = \min_{\mathcal{Q}^+}(V)$, the curves $\mathcal{C}_h$ are elliptic.*
- *The rotation number function $\theta_{b,a}(h)$ is analytic in $[h_c, +\infty)$*
- *If $(a,b) \neq (1,1)$, then $\exists\, p_0(a,b) \in \mathbb{N}$, generically computable, s.t. for any $p > p_0(a,b)$ $\exists$ at least an oval $\mathcal{C}_h^+$ filled by p–periodic orbits.*



Again $F^n(P) = P + nH$, so $\mathcal{C}_h$ is full of p-periodic orbits iff $pH = V$.

Following the guidelines of Bastien and Rogalski [2], to study the rotation number function we look for a normal form for $\mathcal{C}_h$ instead of $\widetilde{F}$:

$$\begin{array}{ccc} (\mathcal{C}_h, +, V) & \xrightarrow{\cong} & (\mathcal{E}_L, +, V) \\ \widetilde{F}_{|\mathcal{C}_h} : P \mapsto P + H & \longrightarrow & G_{|\mathcal{E}_L} : P \mapsto P + \widehat{H} \end{array}$$

where $\mathcal{E}_L$ is the *Weierstrass Normal Form*:

$$\mathcal{E}_L = \{[x:y:t],\ y^2t = 4\,x^3 - g_2\,xt^2 - g_3\,t^3\},$$

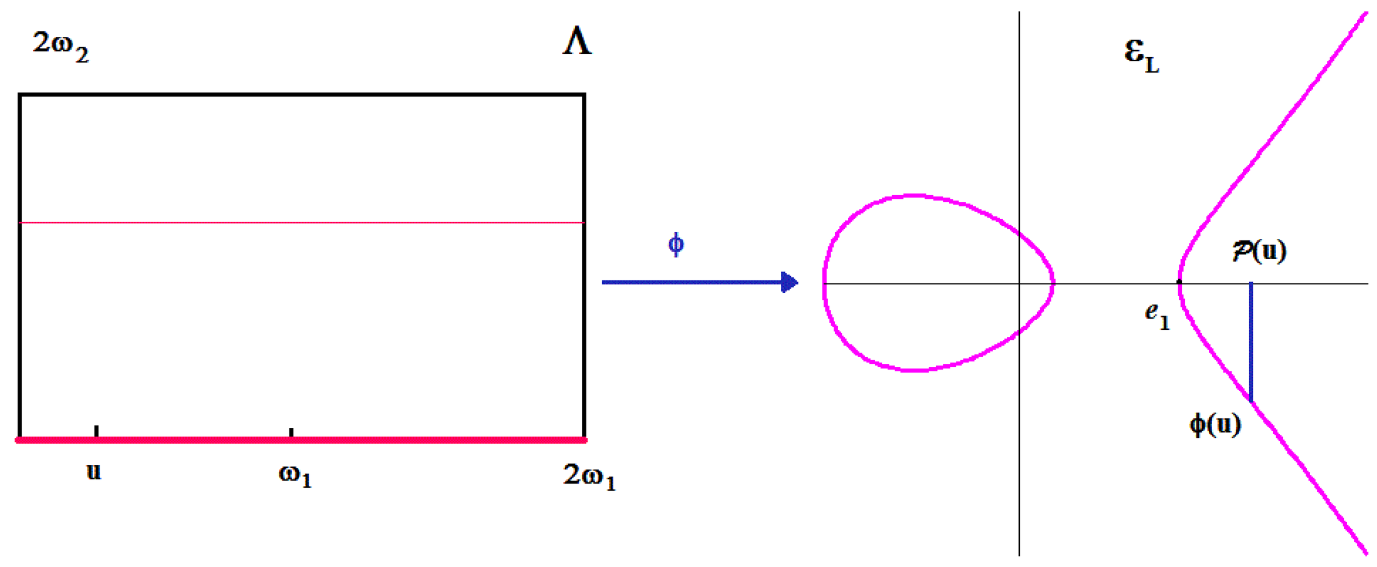
Because $\exists\ \omega_1$ and ω_2 depending on a, b and L and a lattice in $\mathbb{C}$

$$\Lambda = \{2n\omega_1 + 2m\,i\omega_2 \text{ such that } (n,m) \in \mathbb{Z}^2\} \subset \mathbb{C},$$

such that the *Weierstrass $\wp$ function relative to* Λ gives a parametrization of $\mathcal{E}_L$, given by

$$\phi : \mathbb{T}^2 = \mathbb{C}/\Lambda \longrightarrow \mathcal{E}_L$$
$$z \longrightarrow \begin{cases} [\wp(z) : \wp'(z) : 1] \text{ if } & z \notin \Lambda, \\ [0:1:0] = V \text{ if } & z \in \Lambda. \end{cases}$$

Hence $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$, since $y^2 = 4x^3 - g_2x - g_3$.
The oval $\mathcal{C}_h^+$ corresponds with the bounded branch of $\mathcal{E}_L$; and $[0, \omega_1]$ is projected onto the unbounded semi-branch of $\mathcal{E}_L$ with *negative y–coordinates*



Integrating the differential equation $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$ on $[0, u]$:

$$u = \int_{\wp(u)}^{+\infty} \frac{\mathrm{d}s}{\sqrt{4s^3 - g_2 s - g_3}} = \int_{\wp(u)}^{+\infty} \frac{\mathrm{d}s}{\sqrt{4(s-e_1)(s-e_2)(s-e_3)}}.$$

Since

$$G_{|\mathcal{E}_L} : V \mapsto V + \widehat{H} = \widehat{H} \quad \text{is a rotation of angle } \Theta(L) \in \left[0, \frac{1}{2}\right],$$

and $\widehat{H}$ has *negative ordinate* there is a parameter $u = 2\omega_1\Theta(L)$ such that $X(L) := \wp(2\omega_1\Theta(L))$, and using that $e_1 = \wp(\omega_1)$, we obtain

$$2\Theta(L) = \frac{\displaystyle\int_{X(L)}^{+\infty} \frac{\mathrm{d}s}{\sqrt{(s-e_1)(s-e_2)(s-e_3)}}}{\displaystyle\int_{e_1}^{+\infty} \frac{\mathrm{d}s}{\sqrt{(s-e_1)(s-e_2)(s-e_3)}}}.$$

This expression gives the analyticity of the rotation number function and allows to prove that if $(a,b) \neq (1,1)$, then the image of $\theta_{b,a}[h_c, +\infty)$ is an interval, by studying its asymptotic behavior. □

References

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[3] D. Jogia, J.A.G. Roberts, F. Vivaldi, *An algebraic geometric approach to integrable maps of the plane*, Journal of Physics A 39 (2006).

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