Global periodicity conditions for maps and recurrences via Normal Forms

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Abstract

We face the problem of characterizing the periodic cases in parametric families of rational diffeomorphisms of $\mathbb{K}^k$, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, having a fixed point. Our approach relies on the Normal Form Theory, to obtain necessary conditions for the existence of a formal linearization of the map, and on the introduction of a suitable rational parametrization of the parameters of the family. Using these tools we can find a finite set of values p for which the map can be p-periodic, reducing the problem of finding the parameters for which the periodic cases appear to simple computations. We apply our results to several two and three dimensional classes of polynomial or rational maps. In particular we find the global periodic cases for several Lyness type recurrences.

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1 Introduction

A map F such that $F^p(x) \equiv x$, for some $p \in \mathbb{N}$ and for all x for which F^p is well defined, will be called a *periodic map*. If p is the smallest positive integer with this property, then F is called *p-periodic*. In this paper we treat the problem of characterizing the p-periodic cases in parametric families of rational maps of $\mathbb{K}^k$, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, having a fixed point.

When F is a p-periodic differentiable map having a fixed point, $\mathbf{x}_0$, it is well-known that $(DF(\mathbf{x}_0))^p = \mathrm{Id}$. In fact this is a simple consequence of the chain rule. As we will see in Proposition 10, $m = p$ is the smallest positive integer number such that $(DF(\mathbf{x}_0))^m = \mathrm{Id}$. This simple result allows to treat in a easy way the periodicity problem when a value p such that $(DF(\mathbf{x}_0))^p = \mathrm{Id}$ is known. For instance if F has a fixed point $\mathbf{x}_0$ such that $(DF(\mathbf{x}_0))^2 = \mathrm{Id}$ then if F is p-periodic then p must be 2, and not $p = 2m$, $m \in \mathbb{N}$ as we could think in principle, and then we simply have to check whether $F^2 = \mathrm{Id}$ or not.

In general, given a parametric family of maps $F_{\mathbf{a}}$, $\mathbf{a} \in \mathbb{K}^m$, the most difficult problem for finding the periodic maps is to determine which are the possible values p such that there exists some $\mathbf{a}$ such that $F_{\mathbf{a}}$ is p-periodic. The tools that we will introduce in this paper will allow to find a finite set of possible values of p for which the map can be p-periodic, converting the problem of finding these values of $\mathbf{a}$ into a computational problem.

Proposition 10 as well as our approach to the characterization of p-periodic maps via Normal Form Theory are based on the Montgomery-Bochner Theorem, see [23]. It will be recalled and proved in Section 2. In a few words it says that any p-periodic, $\mathcal{C}^1$-map with a fixed point is locally conjugated with the linear map $L(\mathbf{x}) = DF(\mathbf{x}_0)\mathbf{x}$, and so *locally linearizable*. Notice that the differentiability condition is necessary since it is well known that there are periodic involutions (i.e. $F^2 = \mathrm{Id}$) given by homemorphisms with fixed points which are not linearizable, see [8].

Hence any p-periodic case in a given family with fixed points can be locally linearized. Thus, the application of a suitable Normal Form algorithm, will give necessary conditions for the existence of the linearization. As we will see, these conditions are sometimes also sufficient.

We remark that this approach does not cover the problem in its full generality, because there are periodic diffeomorphisms without fixed points in $\mathbb{R}^k$ with $k \geq 7$, see [18, 20].

It is well-known that the Normal Form algorithms often lead to very complicated expressions which are difficult to handle when dealing with the given parameters of the map. Sometimes, these obstructions can be significatively softened by introducing new parameters *rationally depending* on the *old* ones, and such that the coordinates of the fixed points as well as the eigenvalues of the jacobian matrix at these fixed points, depend rationally on these *new* parameters. This is the second main characteristic of our approach, when dealing with concrete applications.

The Normal Form Theory is briefly recalled in Section 3. In Section 4 we obtain some results for planar maps in the case that the linear part of F at the fixed point is given by a matrix $\mathrm{diag}(\alpha,\beta)$ with $\alpha\beta=1$, or $\mathrm{diag}(\alpha,1)$. As first applications of the method, we get:

Theorem 1. *Consider a smooth complex map of the form*

$$F(x,y)=\left(\alpha x+\sum_{i+j\geq 2} f_{i,j}x^iy^j,\frac{1}{\alpha}y+\sum_{i+j\geq 2} g_{i,j}x^iy^j\right), \tag{1}$$

where α is a primitive p-root of unity, $p\geq 5$. Then the conditions $\mathcal{P}_1(F)=\mathcal{P}_2(F)=\mathcal{P}_3(F)=0$ are necessary for F to be p-periodic, where

$$\begin{aligned}\mathcal{P}_1(F):=&\left(f_{2,1}+f_{1,1}g_{1,1}\right)\alpha^4-f_{1,1}\left(2f_{2,0}-g_{1,1}\right)\alpha^3+\left(2g_{2,0}f_{0,2}-f_{1,1}f_{2,0}+f_{1,1}g_{1,1}\right)\alpha^2\\&-\left(f_{2,1}+f_{1,1}f_{2,0}\right)\alpha+f_{1,1}f_{2,0},\\ \mathcal{P}_2(F):=&g_{0,2}g_{1,1}\alpha^4-\left(g_{1,2}+g_{0,2}g_{1,1}\right)\alpha^3+\left(f_{1,1}g_{1,1}+2g_{2,0}f_{0,2}-g_{0,2}g_{1,1}\right)\alpha^2\\&+g_{1,1}\left(-2g_{0,2}+f_{1,1}\right)\alpha+f_{1,1}g_{1,1}+g_{1,2},\end{aligned}$$

and $\mathcal{P}_3(F)$ is given in Appendix A.

In fact, conditions $\mathcal{P}_1(F)=0$ and $\mathcal{P}_2(F)=0$ also work for $p=4$.

Theorem 2. *Consider a smooth complex map of the form*

$$F(x,y)=\left(\alpha x+\sum_{i+j\geq 2} f_{i,j}x^iy^j,y+\sum_{i+j\geq 2} g_{i,j}x^iy^j\right), \tag{2}$$

where α is a primitive p-root of unity. Then the following are necessary conditions for F to be p-periodic:

$$\begin{aligned}\mathcal{P}_1(F):=&f_{1,1}=0,\\ \mathcal{P}_2(F):=&g_{0,2}=0,\\ \mathcal{P}_3(F):=&f_{1,2}\alpha-2f_{2,0}f_{0,2}+2f_{0,2}g_{1,1}-f_{1,2}=0,\\ \mathcal{P}_4(F):=&g_{0,3}\alpha-g_{0,3}-f_{0,2}g_{1,1}=0,\\ \mathcal{P}_5(F):=&f_{1,3}\alpha^2+\left(-2f_{2,0}f_{0,3}+3f_{0,3}g_{1,1}+2g_{1,2}f_{0,2}-2f_{2,1}f_{0,2}-2f_{1,3}\right)\alpha+f_{1,3}+2f_{2,0}f_{0,3}\\&+2f_{2,1}f_{0,2}-4g_{2,0}f_{0,2}^2-2g_{1,2}f_{0,2}-3f_{0,3}g_{1,1}=0,\\ \mathcal{P}_6(F):=&g_{0,4}\alpha^2-\left(f_{0,3}g_{1,1}+2g_{0,4}+g_{1,2}f_{0,2}\right)\alpha+g_{2,0}f_{0,2}^2+f_{0,3}g_{1,1}+g_{0,4}+g_{1,2}f_{0,2}=0.\end{aligned}$$

In this last case, and in contrast with the one treated in Theorem 1, it is not difficult to obtain additional periodicity conditions. Two more periodicity conditions are given in Appendix B.

The above results are applied in several contexts. The first application is for polynomial maps. Periodic *polynomial* maps are notorious examples of invertible polynomial ones, which, in turn, are the focus of many deep open problems like the Jacobian conjecture, or the linearization conjecture. This second conjecture says that if $F : \mathbb{C}^n \to \mathbb{C}^n$ is a p-periodic polynomial map, then there exists a polynomial automorphism φ (i.e. an invertible polynomial map with polynomial inverse) such that $\varphi \circ F \circ \varphi^{-1}$ is a linear map. This conjecture is true for $n = 2$ and as far as we know it is open for $n \geq 3$, see [15, Chaps. 8 and 9] and [21].

In Section 5 we characterize the p-periodic maps in a family of triangular maps, see Theorem 15, and we give a simple and self-contained proof of the linearization conjecture for this case. As an application of this result and Theorem 1 we prove:

Proposition 3. *Consider a complex polynomial map*

$$F(x, y) = \left(\alpha x + \sum_{i+j=2}^{3} f_{i,j} x^i y^j, y/\alpha + \sum_{i+j=2}^{3} g_{i,j} x^i y^j \right), \tag{3}$$

The map is p-periodic if and only if α is a primitive p-root of the unity, and it holds one of the following conditions

(i) $p = 1$ *and* $F(x, y) = (x, y)$*;*

(ii) $p = 2, 4$ *and* $F(x, y) = (\alpha x + f_{0,2} y^2, y/\alpha)$ *or* $F(x, y) = (\alpha x, y/\alpha + g_{2,0} x^2)$*;*

(iii) $p = 3,$ *and* $F(x, y) = (\alpha x + f_{0,3} y^3, y/\alpha)$ *or* $F(x, y) = (\alpha x, y/\alpha + g_{3,0} x^3)$*;*

(iv) $p \geq 5$ *and* $F(x, y) = (\alpha x + f_{0,2} y^2 + f_{0,3} y^3, y/\alpha)$ *or* $F(x, y) = (\alpha x, y/\alpha + g_{2,0} x^2 + g_{3,0} x^3)$*.*

Similarly, as an application of Theorem 2, we prove:

Proposition 4. *The only p-periodic cases in the family of complex maps*

$$F(x, y) = \left(\frac{\alpha x + bx^2 + cxy + dy^2}{1 + m\,(x^2 + y^2)}, \frac{y + rx^2 + sxy + ty^2}{1 + m\,(x^2 + y^2)} \right),$$

are, either $F(x, y) = (x, y)$ when $\alpha = 1$, or the ones given the polynomial maps $F(x, y) = (\alpha x + dy^2, y)$ or $F(x, y) = (\alpha x, y + rx^2)$ when α a primitive p-root of the unity with $p > 1$ and d and r arbitrary complex numbers.

In all the rest of examples, given in Sections 6 and 7, the maps are the ones associated to some recurrences. Recall that given a recurrence, autonomous or not, it is said that it is *globally p-periodic* if for all initial conditions for which the sequence is well-defined it gives rise to a p-periodic sequence and p is the smallest positive integer number with this property. We will face this question studying an associated map F. With this point of view, the recurrence will be globally periodic if and only if the map F is periodic.

The study of the global periodicity in difference equations is nowadays the subject of an active research, see for instance [1, 2, 4, 5, 6, 7, 9, 10, 11, 12, 13, 16, 22, 25, 26], and references therein and several techniques have been used to approach the problem. To the best of our knowledge, this is the first time that the Normal Form Theory is used in this setting. As a second application of Theorem 1, we classify the globally periodic second order Lyness recurrences, reobtaining the results in [13] for this case:

Proposition 5. *The only globally periodic Lyness recurrences* $x_{n+2} = \dfrac{a + x_{n+1}}{x_n}$ *with* $a \in \mathbb{C}$*, are the* 5*-periodic case with* $a = 1$*; and the* 6*-periodic case with* $a = 0$.

Also as a direct consequence of Theorem 1 we get next result for some Gumovski-Mira-type recurrences [17],

Proposition 6. *There are no globally periodic cases in the family of Gumovski-Mira recurrences*

$$x_{n+2} = -x_n + \frac{x_{n+1}}{b + x_{n+1}^2}, \quad b \in \mathbb{C}.$$

One of the main applications in this setting concerns the 2-periodic Lyness recurrence

$$x_{n+2} = \frac{a_n + x_{n+1}}{x_n}, \qquad \text{where} \qquad a_n = \begin{cases} a & \text{for} \quad n = 2\ell + 1, \\ b & \text{for} \quad n = 2\ell, \end{cases} \tag{4}$$

and $a, b \in \mathbb{C}$. In Section 6.3 we solve the global periodicity problem for it by studying the family of maps

$$F_{b,a}(x, y) = \left(\frac{a + y}{x}, \frac{a + bx + y}{xy} \right),$$

which as we will see describes the behavior of (4).

Theorem 7. *The only globally periodic recurrences in* (4) *are:*

(i) The cases $a = b = 0$ *(*6*-periodic) and* $a = b = 1$ *(*5*-periodic).*

(ii) The cases $a = (-1 \pm i\sqrt{3})/2$ *and* $b = \overline{a} = 1/a$*,* 10*-periodic.*

Notice that the cases given in (i) correspond to the well-known autonomous globally periodic Lyness recurrences also appearing in Proposition 5.

Finally, to show an application in $\mathbb{K}^3$ we find the globally periodic third order Lyness recurrences, reobtaining again the result in [13]:

Proposition 8. *The only globally periodic third-order Lyness recurrence*

$$x_{n+3} = \frac{a + x_{n+1} + x_{n+2}}{x_n}, \quad a \in \mathbb{C},$$

corresponds to $a = 1$ *and is* 8*-periodic.*

2 Some consequences of the Montgomery-Bochner Theorem

The next version of Montgomery-Bochner Theorem is a simplified one, adapted to our interests. The general one applies in a much more general context, see [23].

Theorem 9 (**Montgomery-Bochner**). *Let $F:\mathcal{U}\to\mathcal{U}$ be a p-periodic $\mathcal{C}^1$-diffeomorphism, where U is an open set of $\mathbb{K}^k$. Let $\mathbf{x}_0\in\mathcal{U}$ be a fixed point of F. Then, there exists a neighbourhood of $\mathbf{x}_0$ where F is conjugated with the linear map $L(\mathbf{x})=DF(\mathbf{x}_0)\mathbf{x}$. Moreover the linearization is given by the local diffeomorphism*

$$\psi(\mathbf{x})=\frac{1}{p}\sum_{i=0}^{p-1}(DF(\mathbf{x}_0))^{-i}\left(F^i(\mathbf{x})\right).$$

Proof. Since F is p-periodic $(DF(\mathbf{x}_0))^p=\mathrm{Id}$. So $(\det(DF(\mathbf{x}_0))^p=1$ and $DF(\mathbf{x}_0)$ is invertible. Consider ψ as in the statement. By the inverse function theorem it is clear that the map ψ is a local diffeomophism because $D\psi(\mathbf{x}_0)=\mathrm{Id}$. Moreover, using again the p-periodicity of F we get that $\psi(F(\mathbf{x}))=L(\psi(\mathbf{x}))$, as we wanted to prove. ∎

As we have seen in the proof of the above theorem, if F is a p-periodic differentiable map with a fixed point $\mathbf{x}_0$, then $(DF(\mathbf{x}_0))^p=\mathrm{Id}$. Next result relates p with the minimum positive m such that $(DF(\mathbf{x}_0))^m=\mathrm{Id}$.

Proposition 10. *Let F be a differentiable map having a fixed point $\mathbf{x}_0$. Assume that F is p-periodic and let m be the minimum positive m such that $(DF(\mathbf{x}_0))^m=\mathrm{Id}$. Then $p=m$.*

Proof. By using the Montgomery-Bochner Theorem we know that F is $\mathcal{C}^1$-conjugated to $L(\mathbf{x})=DF(\mathbf{x}_0)\mathbf{x}$ in a neighborhood of $\mathbf{x}_0$. Thus $F=\psi^{-1}\circ L\circ\psi$, for some $\mathcal{C}^1$ diffeomorphism ψ. Since $L^m=\mathrm{Id}$ if and only if $F^m=\psi^{-1}\circ L^m\circ\psi=\psi^{-1}\circ\psi=\mathrm{Id}$, the result follows. ∎

Corollary 11. *Let $F_{\mathbf{a}}(\mathbf{x})=L\mathbf{x}+G(\mathbf{x},\mathbf{a})$, with $\mathbf{x}\in\mathcal{U}\subset\mathbb{K}^n$ and $\mathbf{a}\in\mathbb{K}^m$, a smooth family of maps such that $G(\mathbf{0},\mathbf{a})\equiv D_{\mathbf{x}}G(\mathbf{0},\mathbf{a})\equiv 0$ for all $\mathbf{a}\in\mathbb{K}^m$. Assume that p is the minimum positive integer number such that $L^p=\mathrm{Id}$. Then if $F_{\mathbf{a}}$ is periodic for some $\mathbf{a}\in\mathbb{K}$ then it is p-periodic, i.e. $F_{\mathbf{a}}^p=\mathrm{Id}$.*

In particular note that if $L=\mathrm{Id}$ then the only periodic case is $F_{\mathbf{a}}(\mathbf{x})=\mathbf{x}$ and when $L^2=\mathrm{Id}$ the periodicity conditions are given by $F_{\mathbf{a}}(F_{\mathbf{a}}(\mathbf{x}))\equiv\mathbf{x}$. For example the fact proved in [25, Ex. 2], that the only periodic map of the form $F(x_1,x_2)=(x_2+ax_1^2,x_1+bx_1x_2)$ corresponds to the linear case $a=b=0$, follows easily using this approach.

Notice that using Montgomery-Bochner Theorem a necessary condition for a map of the form $F_{\mathbf{a}}(\mathbf{x})=L\mathbf{x}+G(\mathbf{x},\mathbf{a})$, to be periodic is that $F_{\mathbf{a}}$ is linearizable in a neighbourhood of $\mathbf{0}$. The linearizable cases can be detected by following the well-know Normal Form Theory, which, as far as we know, has not been used for this purpose. Some results useful for applying it will be recalled in the next section.

3 Periodicity conditions via Normal Form Theory

We start introducing some well-known issues of Normal Form Theory, while referring the reader to [3, Sec. 2.5], for further details.

Let $F := F^{(1)} : \mathbb{K}^k \to \mathbb{K}^k$, be a family of smooth maps depending on some parameters and satisfying $F^{(1)}(0) = 0$. Let

$$F^{(1)}(\mathbf{x}) = F_1^{(1)}(\mathbf{x}) + F_2^{(1)}(\mathbf{x}) + \cdots + F_k^{(1)}(\mathbf{x}) + O(|\mathbf{x}|^{k+1}) \tag{5}$$

be the Taylor expansion of F at $\mathbf{0}$, where $F_r^{(1)} \in \mathcal{H}_r$, the real vector space of maps whose components are homogeneous polynomials of degree r.

The aim of the Normal Form Theory is to construct a sequence of transformations Φ_n, starting from $n = 2$, such that at each step, Φ_n simplifies, as much as possible, the terms of the corresponding homogeneous part of degree n. To this end, let $F_1^{(1)}(\mathbf{x}) = DF^{(1)}(\mathbf{0})\,\mathbf{x} =: L\mathbf{x}$ and suppose that

$$F^{(n-1)}(\mathbf{x}) = L\,\mathbf{x} + F_n^{(n-1)}(\mathbf{x}) + O(|\mathbf{x}|^{n+1})\,,\; n \geq 2.$$

Consider a transformation

$$\mathbf{x} = \Phi_n(\mathbf{y}) := \mathbf{y} + \phi_n(\mathbf{y}),$$

with $\phi_n \in \mathcal{H}_n$, such that it conjugates the map $F^{(n-1)}$ with a new map $F^{(n)}$, via the conjugation

$$F^{(n-1)}(\Phi_n) = \Phi_n(F^{(n)}).$$

From the above equation, it can be easily seen that

$$F^{(n)}(\mathbf{y}) = L\,\mathbf{y} + L\,\phi_n(\mathbf{y}) - \phi_n(L\mathbf{y}) + F_n^{(n-1)}(\mathbf{y}) + O(|\mathbf{y}|^{n+1}).$$

Clearly, if $\phi_n(\mathbf{y})$ can be chosen in such a way that

$$M_L\left(\phi_n(\mathbf{y})\right) := L\,\phi_n(\mathbf{y}) - \phi_n(L\mathbf{y}) = -F_n^{(n-1)}(\mathbf{y}), \tag{6}$$

then $F^{(n-1)}$ is transformed into

$$F^{(n)}(\mathbf{y}) = L\,\mathbf{y} + F_{n+1}^{(n)}(\mathbf{y}) + O(|\mathbf{y}|^{n+2}) = L\,\mathbf{y} + O(|\mathbf{y}|^{n+1}).$$

The vectorial equation (6) is the well-known *homological equation* associated with $L = DF^{(1)}(\mathbf{0})$, and the existence of solutions of it is the necessary and sufficient condition to be able to remove the homogeneous terms of degree n.

From now, one we will assume that the linear map is diagonalizable, and so that it is $L = \mathrm{diag}(\lambda_i)_{i=1}^k$. In this case, the linear operator $M_L := \mathcal{H}_n \to \mathcal{H}_n$, given in (6), has the eigenvectors $\mathbf{x}^{\mathbf{m}}\, e_i$, $i = 1, 2, \ldots, k$, with $\mathbf{m} = (m_1, m_2, \ldots, m_k) \in \mathcal{M}_n^k := \{\mathbf{m} \in \mathbb{N}^k$ satisfying $\sum_{i=1}^k m_i = n\}$; $\mathbf{x}^{\mathbf{m}} = x_1^{m_1}\, x_2^{m_2} \cdots\, x_n^{m_k}$ where $\mathbf{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{K}^k$; and where e_i is the i-th member of the natural basis for $\mathbb{K}^k$. Hence

$$M_L\,(\mathbf{x}^{\mathbf{m}}\, e_i) = (\lambda_i - \lambda^{\mathbf{m}})\,\mathbf{x}^{\mathbf{m}}\, e_i, \tag{7}$$

where $\lambda^{\mathbf{m}} = \lambda_1^{m_1}\,\lambda_2^{m_2} \cdots\, \lambda_n^{m_k}$.

Set

$$F_n^{(n-1)}(\mathbf{x}) = \left(\sum_{\mathbf{m}} f_{1;\mathbf{m}}^{(n-1)}\mathbf{x}^{\mathbf{m}}, \sum_{\mathbf{m}} f_{2;\mathbf{m}}^{(n-1)}\mathbf{x}^{\mathbf{m}}, \ldots, \sum_{\mathbf{m}} f_{k;\mathbf{m}}^{(n-1)}\mathbf{x}^{\mathbf{m}}\right),$$

and

$$\phi_n(\mathbf{x}) = \left(\sum_{\mathbf{m}} a_{1;\mathbf{m}}\mathbf{x}^{\mathbf{m}}, \sum_{\mathbf{m}} a_{2;\mathbf{m}}\mathbf{x}^{\mathbf{m}}, \ldots, \sum_{\mathbf{m}} a_{k;\mathbf{m}}\mathbf{x}^{\mathbf{m}}\right),$$

where $\mathbf{m} \in \mathcal{M}_n^k$.

When $\lambda_i - \lambda^{\mathbf{m}} \neq 0$ for all the suitable values of $\mathbf{m} \in \mathbb{N}^k$ and for all $i = 1, 2, \ldots, k$, it is said that there are no resonances. In this case the operator M_L is invertible, the homological equation always has solution and so the linearization process can continue. On the contrary, if $\lambda_i - \lambda^{\mathbf{m}} = 0$ for some $\mathbf{m} \in \mathcal{M}_n^k$ and some $i \in \{1, 2, \ldots, k\}$, then the vector $\lambda = (\lambda_1, \lambda_1, \ldots, \lambda_k)$ is said to be *resonant of order* n. In this case, by simple inspection of the homological equation, and using (7), we obtain that the *the* n*th order obstruction equation associated to the resonance* is given by

$$(\lambda_i - \lambda^{\mathbf{m}})\, a_{i;\mathbf{m}} \; = \; -f_{i;\mathbf{m}}^{(n-1)}.$$

However, there are some maps having this resonance for which the process can continue. This happens if the right-hand side of this scalar equations vanish, namely $f_{i;\mathbf{m}}^{(n-1)} = 0$, and these cases are the ones candidate to be linearized. Hence, we have obtained the following result

Proposition 12. *If* $L := \mathrm{diag}(\lambda_i)_{i=1}^k$, *then a necessary condition for the map (5) to be periodic is given by the* n*th order periodicity condition associated to the resonance condition,* $\lambda_i - \lambda^{\mathbf{m}} = 0$, *given by* $f_{i;\mathbf{m}}^{(n-1)} = 0$.

Remark 13. *Notice that* p*-periodic maps with* L *diagonal are such that* $\lambda_i^p = 1$, *for all* i. *Therefore for these maps many resonances* $\lambda_i - \lambda^{\mathbf{m}} = 0$ *appear.*

By following the Normal Form Algorithm, it is straightforward (and well known) to see that the numerator of $f_{i;\mathbf{m}}^{(n-1)}$ is a polynomial in the coefficients of $F^{(1)}$. Thus, for each particular case, the above equations give periodicity conditions, which are algebraic in terms of the initial parameters of the map, once expressed in form (5).

To fix the ideas we give a simple example. Suppose that $k = 2$. Assume that $L = \mathrm{diag}(\alpha, \beta)$. Set $\Phi_2(\mathbf{y}) := \mathbf{y} + \phi_2(\mathbf{y})$, where

$$\phi_2(x,y) := \begin{pmatrix} a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ b_{20}x^2 + b_{11}xy + b_{02}y^2 \end{pmatrix}.$$

Consider the map $F^{(1)}(\mathbf{x}) = L\mathbf{x} + F_2^{(1)}(\mathbf{x}) + O(|\mathbf{y}|^3)$ with

$$F_2^{(1)}(x,y) = \begin{pmatrix} f_{20}x^2 + f_{11}xy + f_{02}y^2 \\ g_{20}x^2 + g_{11}xy + g_{02}y^2 \end{pmatrix},$$

where to simplify the notation, and from now on, if there is no possibility of confusion, we will drop the superscript (1) of the coefficients of $F^{(1)}$.

The homological equation at order 2 is $L\,\phi_2(\mathbf{y}) - \phi_2(L\mathbf{y}) = -F_2^{(1)}(\mathbf{y})$, and gives the following six scalar equations:

$$\begin{cases} (\alpha-\alpha^2)a_{20} = -f_{20}, & (\beta-\alpha^2)b_{20} = -g_{20}, \\ (\alpha-\alpha\beta)a_{11} = -f_{11}, & (\beta-\alpha\beta)b_{11} = -g_{11}, \\ (\alpha-\beta^2)a_{02} = -f_{02}, & (\beta-\beta^2)b_{02} = -g_{02}. \end{cases}$$

If no one of the six 2*nd order resonance conditions*: $\alpha^2-\alpha$, $\alpha\beta-\alpha$, $\beta^2-\alpha$, $\beta^2-\beta$, $\alpha\beta-\beta$, and $\alpha^2-\beta$, vanish, there is no obstruction to remove the second order terms of $F^{(1)}$ using the conjugation Φ_2.

Suppose now, that the map $F^{(1)}$ is such that the *resonance* $\beta-\alpha^2 = 0$ occurs. Then the scalar equation $(\beta-\alpha^2)b_{20} = -g_{20}$ is an *obstruction equation.* But this obstruction to the linearization process disappears if g_{20} vanishes. In summary, if $\beta = \alpha^2$, then $g_{20} = 0$ is a *periodicity condition.*

4 Proof of Theorems 1 and 2

We keep the notation introduced in the above section, i.e., $F^{(k)}$ is the map obtained after $k-1$ steps of the normal form procedure, $F^{(k)}(\mathbf{x}) = L\,\mathbf{x} + F_{k+1}^{(k)}(\mathbf{x}) + O(|\mathbf{x}|^{k+2})$, and its coefficients are $f_{i,j}^{(k)}$ and $g_{i,j}^{(k)}$. First consider the case treated in Theorem 1:

$$F^{(k)}(x,y) = \left(\alpha x + \sum_{i+j\geq k+1} f_{i,j}^{(k)} x^i y^j, \frac{1}{\alpha}\,y + \sum_{i+j\geq k+1} g_{i,j}^{(k)} x^i y^j\right).$$

It is easy to check that the scalar equations associated to equation (6) are

$$\begin{cases} \alpha(1-\alpha^{n-2i-1})\,a_{n-i,i} = -f_{n-i,i}^{(n-1)}, \\ \alpha^{-1}(1-\alpha^{n-2i+1})\,b_{n-i,i} = -g_{n-i,i}^{(n-1)}, \end{cases}$$

for $i=0,\ldots,n$. Hence, for any α, and any odd n we obtain the periodicity conditions

$$f^{(n-1)}_{\frac{n+1}{2},\frac{n-1}{2}}=0 \quad \text{and} \quad g^{(n-1)}_{\frac{n-1}{2},\frac{n+1}{2}}=0. \tag{8}$$

We remark that for a given α, primitive m-root of the unity, other conditions can be added. For instance when $m=3$,

$$f^{(1)}_{0,2}=0,\ g^{(1)}_{2,0}=0,\ f^{(3)}_{4,0}=0,\ f^{(3)}_{1,3}=0,\ g^{(3)}_{3,1}=0 \text{ and } g^{(3)}_{0,4}=0,$$

are also periodicity conditions. Also, when $m=5$, $f^{(3)}_{0,4}=0$ and $g^{(3)}_{4,0}=0$ are periodicity conditions.

Returning to the general case, we want to obtain explicitly the first conditions given in (8). The first four are given by $f^{(2)}_{2,1}=g^{(2)}_{1,2}=0$; and $f^{(4)}_{3,2}=g^{(4)}_{2,3}=0$. Some straightforward computations applying the Normal Form Algorithm explained in Section 3 show that

$$f^{(2)}_{2,1}=\frac{\mathcal{P}_1(F)}{\alpha(\alpha^3-1)},\quad g^{(2)}_{1,2}=\frac{\mathcal{P}_2(F)}{\alpha^3-1}\quad \text{and} \quad f^{(4)}_{3,2}=\frac{\mathcal{P}_3(F)}{\alpha^3(\alpha^3-1)^3\,(\alpha^2+1)\,(\alpha+1)},$$

giving the desired result. Recall that $\alpha^n-1\neq 0, n\leq 4$. We do not give the periodicity condition associated to $g^{(4)}_{2,3}$ for the sake of brevity and because we will not use in the specific examples.

Now we consider the maps that appear in Theorem 2. When $\alpha=1$ the result follows trivially by Corollary 11. When $\alpha\neq 1$, the scalar equations associated to equation (6) are

$$\begin{cases} \alpha(1-\alpha^{n-i-1})\,a_{n-i,i}=-f^{(n-1)}_{n-i,i}\,, \\ (1-\alpha^{n-i})\,b_{n-i,i}=-g^{(n-1)}_{n-i,i}\,, \end{cases}$$

for $i=0,\ldots,n$. Hence, for any $\alpha\neq 1$, we obtain the periodicity conditions

$$f^{(n-1)}_{1,n-1}=0 \quad \text{and} \quad g^{(n-1)}_{0,n}=0.$$

Thus the first periodicity conditions are given by $f_{1,1}=g_{0,2}=0$; $f^{(2)}_{1,2}=g^{(2)}_{0,3}=0$; and $f^{(3)}_{1,3}=g^{(3)}_{0,4}=0$. Applying the Normal Form Algorithm we get

$$f^{(2)}_{1,2}=\frac{\mathcal{P}_3(F)}{\alpha-1},\quad g^{(2)}_{0,3}=\frac{\mathcal{P}_4(F)}{\alpha-1},\quad f^{(3)}_{1,3}=\frac{\mathcal{P}_5(F)}{\alpha-1}\quad \text{and} \quad g^{(3)}_{0,4}=\frac{\mathcal{P}_6(F)}{\alpha-1}.$$

5 On some polynomial and rational maps

In this section, first we study the periodicity problem for a family of triangular maps and then we apply this result and Theorem 1 to characterize the periodic maps of the family (3), proving Proposition 3. Finally we prove Proposition 4.

5.1 Preliminary results and a triangular family

First, recall that a polynomial automorphism is a bijective polynomial map with polynomial inverse. Also recall the following well-known result, where as usual $\mathbb{C}^*=\mathbb{C}\setminus\{0\}$.

Lemma 14. *Let $F:\mathbb{C}^n\to\mathbb{C}^n$ be polynomial map.*

(a) If F is an automorphism then $\det(DF(\mathbf{x}))\in\mathbb{C}^*$.

(b) If F is p-periodic, $p\geq 1$ then F is an automorphism and $\det(DF(\mathbf{x}))\in\mathbb{C}^*$.

Proof. (a) Since $F\circ F^{-1}=\mathrm{Id}$, we have that $\det\left(DF(F^{-1}(\mathbf{x}))\right)\cdot\det\left(DF^{-1}(\mathbf{x})\right)\equiv 1$. Since the only non-vanishing complex polynomials are the constant ones, the result follows.

(b) If F is p-periodic, then $F^{-1}=F^{p-1}$, hence F is a polynomial automorphism. ∎

In fact, the reciprocal of item (a) above is precisely the celebrated Jacobian Conjecture. Next results characterize the periodic cases in a family of triangular maps.

Theorem 15. *Let α be a primitive p-root of unity, and consider the $\mathcal{C}^1$-map $F:\mathbb{K}^2\to\mathbb{K}^2$,*

$$F(x,y)=(\alpha x+f(y),y/\alpha)\,,$$

with $f(0)=f'(0)=0$. Then:

(i) F is periodic if and only if

$$\sum_{j=0}^{p-1}\alpha^j f(\alpha^j y)\equiv 0$$

and then it is p-periodic.

(ii) If F is p-periodic, the linearization given in the Montgomery-Bochner Theorem,

$$\psi(\mathbf{x})=\frac{1}{p}\sum_{j=0}^{p-1}(DF(0,0))^{-j}(F^j)(\mathbf{x}),$$

is a global linearization.

Proof. (i) By Corollary 11, if F is periodic then it is p-periodic. It is not difficult to prove that

$$F^p(x,y)=\left(\alpha^p x+\sum_{j=1}^{p}\alpha^{p-j}f\left(\frac{y}{\alpha^{j-1}}\right),\frac{y}{\alpha^p}\right).$$

Therefore, using that $\alpha^p=1$, the condition of being p-periodic writes as

$$\sum_{j=1}^{p}\alpha^{p-j}f\left(\frac{y}{\alpha^{j-1}}\right)=\sum_{j=1}^{p}\alpha^{p-j}f\left(\alpha^{p+1-j}y\right)=\sum_{n=1}^{p}\alpha^{n-1}f\left(\alpha^n y\right)=0,$$

for all $y \in \mathbb{K}$. Multiplying the last expression by α we obtain condition (i).

(ii) Note that in the expression of the local diffeomorphism ψ given in the statement,

$$(DF(0,0))^{-j}(F^j)(\mathbf{x}) = \begin{pmatrix} \alpha^{-j} & 0 \\ 0 & \alpha^j \end{pmatrix} \begin{pmatrix} \alpha^j x + g_j(y) \\ y/\alpha^j \end{pmatrix} = \begin{pmatrix} x + \alpha^{-j} g_j(y) \\ y \end{pmatrix},$$

for some given map $g_j(y)$. Therefore $\psi(x,y) = (x + g(y), y)$, for some map smooth map g. Since clearly, ψ is a diffeomorphism, the result follows. ∎

Corollary 16. *Let α be a primitive p-root of unity. A map $F : \mathbb{K}^2 \to \mathbb{K}^2$,*

$$F(x,y) = \left(\alpha x + \sum_{k=2}^{\infty} f_k y^k, \frac{y}{\alpha}\right),$$

is p-periodic if and only if $f_k = 0$ for all $k = mp - 1, m \in \mathbb{N}$. In particular, it is always periodic if f is a polynomial and $p > \deg(f) + 1$.

Proof. Let us write condition (i) in this setting. We obtain

$$\begin{aligned} 0 =& \sum_{k\geq 2} \left(1 + \alpha^{k+1} + (\alpha^{k+1})^2 + \cdots + (\alpha^{k+1})^{p-1}\right) f_k y^k \\ =& \sum_{k\geq 2,\, k\neq mp-1} \frac{1-(\alpha^{k+1})^p}{1-\alpha^{k+1}} f_k y^k + \sum_{k\geq 2,\, k=mp-1} p f_k y^k = p\left(\sum_{k\geq 2,\, k=mp-1} f_k y^k\right). \end{aligned}$$

Therefore, the characterization holds. ∎

5.2 Proof of Proposition 3

First, we apply Lemma 14 (b), taking into account that in this case $\det(DF(\mathbf{x})) \equiv 1$, obtaining that a necessary condition for a map F in the family of maps (3) to be periodic, is to belong to one of the following cases I,II, III and IV considered below. The cases $p \leq 4$ follow easily using Corollary 11. So, from now one we also assume that $p \geq 5$.

Case I: It holds that $g_{1,2} \neq 0$, $g_{0,3} \neq 0$ and

$$\begin{cases} f_{2,0} = -\dfrac{\alpha^2 g_{1,1}}{2}, f_{1,1} = -\dfrac{3\alpha^2 g_{1,1} g_{0,3}}{g_{1,2}}, f_{0,2} = -\dfrac{9\alpha^2 g_{0,3}^2 g_{1,1}}{2{g_{1,2}}^2}, \\ f_{3,0} = -\dfrac{\alpha^2 g_{1,2}^2}{9 g_{0,3}}, f_{2,1} = -\alpha^2 g_{1,2}, f_{1,2} = -3\alpha^2 g_{0,3}, f_{0,3} = -\dfrac{3\alpha^2 g_{0,3}^2}{g_{1,2}}, \\ g_{2,0} = \dfrac{g_{1,1} g_{1,2}}{6 g_{0,3}}, g_{0,2} = \dfrac{3 g_{1,1} g_{0,3}}{2 g_{1,2}}, g_{3,0} = \dfrac{g_{1,2}^3}{27 g_{0,3}^2}, g_{2,1} = \dfrac{g_{1,2}^2}{3 g_{0,3}}. \end{cases}$$

When $g_{1,1} \neq 0$, both conditions $\mathcal{P}_1(F) = \mathcal{P}_2(F) = 0$ in Theorem 1 give

$$g_{0,3} = -\frac{2(\alpha-1)\left(\alpha^2+\alpha+1\right) g_{1,2}^2}{3\alpha\,(\alpha+1)\,(2\alpha^2+\alpha+2)\, g_{1,1}^2}.$$

Observe that $2\alpha^2+\alpha+2\neq 0$ because α is a root of unity. Applying again Theorem 1, the condition $\mathcal{P}_3(F)=0$ writes as

$$\frac{\alpha^5\,(\alpha-1)^2\,(\alpha+1)^2\,\left(\alpha^2+\alpha+1\right)^2 P_8(\alpha)\,g_{1,2}^2}{3\,(2\,\alpha^2+\alpha+2)^2}=0,$$

where

$$P_8(\alpha)=30\alpha^8+36\alpha^7+133\alpha^6+114\alpha^5+214\alpha^4+114\alpha^3+133\alpha^2+36\alpha+30.$$

Since $g_{1,2}\neq 0$, $\alpha\neq -1, \alpha^3\neq 1$, and the roots of P_8 have modulus different from 1 we obtain that the above equality never holds and there are no periodic maps in this subfamily when $g_{1,1}\neq 0$. When $g_{1,1}=0$, then $\mathcal{P}_1(F)\neq 0$ and the same result holds.

Case II:

$$\begin{cases} f_{2,0}=f_{1,1}=f_{3,0}=f_{2,1}=f_{1,2}=0,\\ g_{2,0}=g_{1,1}=g_{0,2}=g_{3,0}=g_{2,1}=g_{1,2}=g_{0,3}=0,\end{cases}$$

being $f_{0,2}$ and $f_{0,3}$ free parameters. Hence, in this case F has the form

$$F(x,y)=(\alpha x+f_{0,2}y^2+f_{0,3}y^3, y/\alpha) \tag{9}$$

and the result follows from Corollary 16.

Case III: It holds that $f_{0,2}\neq 0$,

$$\begin{cases} f_{2,0}=\dfrac{\alpha^4 {g_{0,2}}^2}{f_{0,2}}, f_{1,1}=-2\alpha^2 g_{0,2}, f_{3,0}=f_{2,1}=f_{1,2}=f_{0,3}=0,\\ g_{2,0}=\dfrac{\alpha^4 {g_{0,2}}^3}{{f_{0,2}}^2}, g_{1,1}=-\dfrac{\alpha^2 2{g_{0,2}}^2}{f_{0,2}}, g_{0,3}=g_{2,1}=g_{1,2}=g_{3,0}=0,\end{cases}$$

being $g_{0,2}$ a free parameter. By using again condition $\mathcal{P}_1(F)=0$ in Theorem 1 we obtain that a necessary condition for periodicity is

$$\frac{2\alpha^6\,(\alpha+1)\left(2\alpha^2+\alpha+2\right)g_{0,2}^3}{f_{0,2}}=0.$$

So $g_{0,2}=0$ and we are in a subcase of (9). Hence we are done.

Case IV:

$$\begin{cases} f_{2,0}=f_{1,1}=f_{0,2}=f_{3,0}=f_{2,1}=f_{1,2}=f_{0,3}=0,\\ g_{1,1}=g_{0,2}=g_{2,1}=g_{1,2}=g_{0,3}=0,\end{cases}$$

being $g_{2,0}$ and $g_{3,0}$ free parameters. This case, is symmetric with respect the Case II and can be treated analogously.

5.3 Proof of Proposition 4

The proof when $p \leq 3$ follows easily from Corollary 11. So, from now on we can assume that $p \geq 4$. From Theorem 2 we have $\mathcal{P}_1(F) = c$ and $\mathcal{P}_2(F) = t$. Hence, to obtain a periodic map, these coefficients must vanish. When $c = t = 0$, $\mathcal{P}_4(F) = -\alpha m + m - ds$, so another necessary periodicity condition is $m = ds/(1-\alpha)$. Taking into account the above relation, one gets $\mathcal{P}_3(F) = (\alpha s - 2b + 2s)\, d$.

Let us assume first, that $d \neq 0$. In this case, imposing that $\mathcal{P}_3(F)$ vanishes we get $b = (\alpha+2)s/2$. In this case $\mathcal{P}_5(F) = -4\mathcal{P}_6(F) = -4rd^2$, hence $r = 0$ is another necessary periodicity condition. Assuming that r vanishes and using the expressions of $\mathcal{P}_7(F)$ and $\mathcal{P}_8(F)$ given in Appendix B we get

$$\mathcal{P}_7(F) = -d^2 s\,(\alpha - 2)\,(\alpha s - 2d)\,/2 \text{ and } \mathcal{P}_8(F) = -d^2 s\,(\alpha s - 2\,d)\,/2.$$

If $s = 0$, the map is then given by $F(x, y) = (\alpha x + dy^2, y)$ which is p-periodic if and only if α is a primitive p-root of the unity, because

$$F^p(x,y) = \left(\alpha^p x + dy^2 \sum_{i=0}^{p-1} \alpha^i, y\right) = \left(\alpha^p x + dy^2 \left(\frac{1-\alpha^p}{1-\alpha}\right), y\right) = (x, y)\,.$$

If $s \neq 0$, then from the above expressions of $\mathcal{P}_7(F)$ and $\mathcal{P}_8(F)$ we have that $d = \alpha s/2$. In this case we will see that the map is not periodic. Indeed, if $c = r = t = 0$, $b = (\alpha+2)s/2$, $d = \alpha s/2$ and $m = \alpha s^2/(2(1-\alpha))$, then map has a continuum of fixed points containing the origin, as well as an isolated fixed point at $\mathbf{x}_0 = ((1-\alpha)/s, 0)$.

Notice that if F were p-periodic then $\alpha^p = 1$ and $(DF(\mathbf{x}_0))^p = \mathrm{Id}$. Hence $|\det(DF(\mathbf{x}_0))| = 1$. On the other hand $|\det(DF(\mathbf{x}_0)| = 4/|\alpha+1|^2 \neq 1$, because $|\alpha| = 1$ and recall that $\alpha \neq 1$. Therefore the map is not periodic.

Finally, when $d = 0$, since $m = ds/(1-\alpha)$, the map becomes $F(x, y) = (\alpha x + bx^2, y + rx^2 + sxy)$. From Lemma 14 its determinant must be constant, which trivially gives $b = s = 0$. Then $F(x, y) = (\alpha x, y + rx^2)$. Following a similar argument as above it is easy to see that the map is p-periodic if and only α is a primitive p-root of the unity.

6 On some second order rational difference equations

6.1 Global periodicity in the Lyness recurrence

Proof of Proposition 5. It is well known that the dynamics of the Lyness recurrence can be studied using the dynamical system generated by its associated Lyness map

$$G_a(x, y) = \left(y, \frac{a+x}{y}\right). \tag{10}$$

It is easy to see that $G_a^p \neq \mathrm{Id}$ for $p \leq 4$. So from now one we search for p-periodic maps with $p \geq 5$.

In order to significatively simplify the computations to apply Theorem 1 we will introduce a new parameter λ being one of the eigenvalues of the Jacobian matrix of G_a at some fixed point.

Indeed, G_a always has some fixed point (x_0, x_0) with $x_0^2 - x_0 - a = 0$ and $x_0 \neq 0$. The eigenvalues λ of DG_a at this fixed point satisfy $\lambda^2 - \lambda/x_0 + 1 = 0$. Using both equations it is natural to introduce the following rational parametrization for a,

$$a = -\frac{\lambda(\lambda^2 - \lambda + 1)}{(1 + \lambda^2)^2}, \quad \lambda^2 + 1 \neq 0,\ \lambda \neq 0, \quad \lambda \in \mathbb{C}. \tag{11}$$

Note that this parametrization covers all values of $a \in \mathbb{C}$. Moreover using it, a fixed point is $x_0 = \lambda/(1 + \lambda^2)$ and its associated eigenvalues are λ and $1/\lambda$.

After the translation $(x, y) \to (x - x_0, y - x_0)$, that brings the fixed point to the origin, the map G_a conjugates, using again x and y as variables, with

$$g_\lambda(x, y) := \left(y, -\frac{\lambda x - (1 + \lambda^2)y}{\lambda + x(1 + \lambda^2)}\right) \quad \text{with linear part} \quad L_\lambda(x, y) = \left(y, -x + \frac{1 + \lambda^2}{\lambda} y\right).$$

The linear change of variables $\Psi(x, y) = (x - \lambda y, x - y/\lambda)$ gives a conjugation between L_λ and its diagonal form $L(x, y) := (\lambda x, y/\lambda)$. Using this conjugation we consider the map $F_\lambda := \Psi \circ g_\lambda \circ \Psi^{-1}$, which, for the sake of brevity, we do not explicitly write. We will apply Theorem 1 to F_λ which is conjugated to the Lyness map G_a.

Imposing that $\mathcal{P}_1(F_\lambda) = 0$ we get that the a necessary condition for F_λ to be periodic is

$$\left(\lambda^2 - \lambda + 1\right)\left(\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1\right)\left(\lambda^2 + 1\right)^3 = 0.$$

The roots of the factor $\lambda^2 - \lambda + 1$, which are primitive 6-th roots of the unity, correspond to the 6-periodic case, $a = 0$. Finally, using again (11), we get that all the roots of the polynomial $\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1$, which are primitive 5-th roots of the unity, correspond to the 5-periodic case, $a = 1$. ∎

6.2 Non global periodicity in a Gumovski-Mira recurrence

Proof of Proposition 6. Proceeding as in the Lyness case, we consider the map associated with the Gumovski-Mira recurrence,

$$G_b(x, y) = \left(y, -x + \frac{y}{b + y^2}\right).$$

It is easy to prove that $G_b^p \neq \mathrm{Id}$ for $p \leq 4$. So from now one we look for p-periodic maps with $p \geq 5$.

We consider separately the case $b = 0$. In this situation $\mathbf{x}_0 := (\sqrt{2}/2, \sqrt{2}/2)$ is a fixed point of G_0. It is easy to see that $(DG_0(\mathbf{x}_0)^p \neq \mathrm{Id}$, for any positive integer p, because the matrix is not diagonalizable. So G_0 is not a periodic map.

When $b \neq 0$ we introduce a new parameter λ, and write $b = \lambda/(1+\lambda^2)$ with $\lambda^2+1 \neq 0$ and $\lambda \neq 0$. Notice that this parametrization covers all values of b in $\mathbb{C}\setminus\{0\}$. We rename the new map corresponding to G_b as g_λ. The eigenvalues of its Jacobian matrix at the origin, which is always a fixed point, are λ and $1/\lambda$. The linear map $\Psi(x, y) = (x - \lambda y, x - y/\lambda)$ is a conjugation between $Dg_\lambda(\mathbf{0})$ and its diagonal form $L(x, y) := (\lambda x, y/\lambda)$. Using this conjugation we consider the map $F_\lambda := \Psi \circ g_\lambda \circ \Psi^{-1}$. Using Theorem 1 we impose that $\mathcal{P}_2(F_\lambda) = 0$. We get that a necessary condition for F_λ to be periodic is

$$\left(\lambda^2+1\right)^2\left(\lambda^2+\lambda+1\right) = 0.$$

If λ is a root of $\lambda^2+\lambda+1$, then it is a primitive 3rd-root of the unity. Then by Corollary 11, F_λ should be globally 3-periodic. But we have already discarded this possibility. So the result follows. ∎

6.3 Global periodicity in the 2-periodic non-autonomous Lyness recurrence

In this section we study the problem of the global periodicity of the the sequence generated by the 2-periodic Lyness recurrence (4). The sequence $\{x_n\}$ given by this recurrence can be reobtained as

$$(x_1, x_2) \xrightarrow{G_a} (x_2, x_3) \xrightarrow{G_b} (x_3, x_4) \xrightarrow{G_a} (x_4, x_5) \xrightarrow{G_b} (x_5, x_6) \xrightarrow{G_a} \cdots$$

where $G_\alpha(x, y)$, with $\alpha \in \{a, b\}$, is the Lyness map given in (10). So the behavior of (4) is given by the dynamical system generated by the map:

$$G_{b,a}(x, y) := G_b \circ G_a(x, y) = \left(\frac{a+y}{x}, \frac{a+bx+y}{xy}\right). \tag{12}$$

Proof of Theorem 7. As we have seen it suffices to study the perodicity problem for the map (12). It is easy to see that $G_a^p \neq \mathrm{Id}$ for $p = 1, 2, 4$. Moreover it is 3-periodic if and only if $a = b = 0$. Notice that this case corresponds to the globally 6-periodic recurrence. We continue searching p-periodic maps with $p \geq 5$.

Following similar ideas that in the previous subsections we introduce a more suitable rational parametrization of a and b. We consider

$$\begin{aligned} a &= \frac{B^3\left(\lambda^2+1\right)+\lambda\left(2\,B^3-1\right)}{B\left(\lambda+1\right)^2}, \\ b &= -B+(B^2-a)^2, \quad \text{with} \quad B(\lambda+1)\neq 0 \quad \text{and} \quad \lambda \neq 0. \end{aligned} \tag{13}$$

Using these new parameters we cover all the values of a and b in $\mathbb{C}$. Moreover the fixed point is $(B, B^2 - a)$, where a is given in (13), and the eigenvalues of $G_{b,a}$ at this point are λ and $1/\lambda$. After a translation $(x, y) \to (x - B, y - (B^2 - a))$, which brings the fixed point to the origin, the map $G_{b,a}$ conjugates, using again x and y as variables, with

$$g_{B,\lambda}(x,y) = \left(\frac{y - Bx}{x + B}, -\frac{B^2 (\lambda+1)^2 x - B\left(\lambda^2 + \lambda + 1\right) y + \lambda\, xy}{\left(B(\lambda+1)^2 y + \lambda\right)(x + B)} \right),$$

with linear part.

$$L_{B,\lambda}(x,y) = \left(-x + \frac{y}{B}, -\frac{B(\lambda+1)^2 x}{\lambda} + \frac{\left(\lambda^2 + \lambda + 1\right) y}{\lambda} \right).$$

The linear change of variables $\Psi(x,y) = \left(x + y, (\lambda+1)\, Bx + \left(1 + \frac{1}{\lambda}\right) By\right)$ gives a conjugation between $L_{B,\lambda}$ and its diagonal form $L(x,y) := (\lambda x, y/\lambda)$. Using this conjugation we consider the map

$$F_{B,\lambda}(x,y) := \Psi \circ g_{B,\lambda}(x,y) \circ \Psi^{-1}(x,y),$$

which satisfies $DF_{B,\lambda}(0,0) = \operatorname{diag}(\lambda, 1/\lambda)$. For simplicity, we omit its explicit expression. Recall that $\lambda^p - 1 \neq 0$ for $p = 1, 2, 3$.

By Theorem 1, when $p \geq 5$, from both conditions $\mathcal{P}_i(F_{B,\lambda}) = 0, i = 1, 2$, we obtain the same periodicity condition $C_1(B, \lambda) = 0$, where

$$\begin{aligned} C_1(B,\lambda) := \quad & B^6\lambda^{10} + 9B^6\lambda^9 + 35B^6\lambda^8 + 80B^6\lambda^7 + 124B^6\lambda^6 + 2B^3\lambda^9 + 142B^6\lambda^5 + 8B^3\lambda^8 \\ & + 124B^6\lambda^4 + 18B^3\lambda^7 + 80B^6\lambda^3 + 32B^3\lambda^6 + 35B^6\lambda^2 + 40B^3\lambda^5 + 9B^6\lambda \\ & + 32B^3\lambda^4 + \lambda^7 + B^6 + 18B^3\lambda^3 + 3\lambda^6 + 8B^3\lambda^2 + 2\lambda^5 + 2B^3\lambda + 3\lambda^4 + \lambda^3. \end{aligned}$$

Using again Theorem 1, we obtain another polynomial restriction $C_2(B, \lambda) := \mathcal{P}_3(F_{B,\lambda}) = 0$. The expression of $C_2(B, \lambda)$ is given in Appendix C. To study the periodicity of $F_{B,\lambda}$ it suffices to deal with the two conditions

$$C_1(B,\lambda) = 0, \quad C_2(B,\lambda) = 0.$$

Computing $R(\lambda) := \operatorname{Res}(C_1(B, \lambda), C_2(B, \lambda); B)$ we get

$$R(\lambda) = \lambda^{36} (\lambda - 1)^{24} (\lambda + 1)^{72} \left(\lambda^2 + 1\right)^6 \left(\lambda^2 + \lambda + 1\right)^{24} S^6(\lambda)\, T^6(\lambda),$$

where $S(\lambda) = \lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1$ and $T(\lambda) = 3\lambda^4 + 15\lambda^3 + 20\lambda^2 + 15\lambda + 3$. Then, a necessary condition for $F_{B,\lambda}$ to be p-periodic with $p \geq 4$ is that λ is a primitive p-th of the unity and that either $S(\lambda) = 0$ or $T(\lambda) = 0$. Let us discard the former possibility.

It turns out that T has two real roots and two complex roots of modulus one. We have to prove that they are not roots of the unity. This can be seen, for instance, proving that T

is not divisible by any cyclotomic polynomial. This holds because if it had a cyclotomic polynomial divisor, its degree should be at most 4. The cyclotomic polynomials of degree at most 4 correspond to $p \in \{1,2,3,4,5,6,8,10,12\} := \mathcal{D}_4$. This is because these are the cases which correspond to cyclotomic polynomials of degree $\varphi(p) \leq 4$, being φ the Euler's function, see for instance [24]. Since

$$\mathrm{Res}(T(\lambda), \lambda^p - 1; \lambda) \neq 0, \quad \text{for} \quad p \in \mathcal{D}_4,$$

the result follows.

Finally, when $S(\lambda) = 0$ notice that λ is a primitive 5-th root of the unity. So, by Corollary 11 if $F_{B,\lambda}$ is p-periodic it should be 5-periodic. Therefore it suffices to study whether $F^5_{B,\lambda} = \mathrm{Id}$ or not, or equivalently whether $G^5_{b,a} = \mathrm{Id}$. Computing the numerator of the first component of $G^5_{b,a}(x,y) - (x,y)$ we get that it writes as $a^4 b(1-ab)x + O(2)$, where as usual $O(m)$ denotes terms of degree at least m in x and y. Hence only three possibilities for $G_{b,a}$ to be 5-periodic appear: either $a = 0$ or $b = 0$ or $ab = 1$.

The first two cases can easily discarded. It holds that $G^5_{0,a} \neq \mathrm{Id}$ and $G^5_{b,0} \neq \mathrm{Id}$. On the other hand, when $b = 1/a, a \neq 0$ the numerator of the first component of $G^5_{1/a,a}(x,y) - (x,y)$ writes as $-a(a-1)^2(a^2+a+1)^2x^2y + O(3)$. Since this last function has to vanish we get three candidates to be 5-periodic: $a = 1$ and $a = (-1 \pm i\sqrt{3})/2$ with $b = 1/a = \overline{a}$. It is easy see that all them give rise to 5-periodic maps $G_{b,a}$. The last two correspond to the globally 10-periodic recurrence. ∎

Remark 17. *The characterization of the globally periodic difference equations treated in this section can also be obtained following the approach developed in [27] that gives all the periodic QRT-maps. This result also appears in [14, p. 165] and [19].*

7 The third order Lyness recurrence

We start proving a general result which will useful for solving the periodicity problem for the Lyness recurrence.

Proposition 18. *Consider the smooth family of maps*

$$F(x,y,z) = \left(\alpha x + \sum_{\mathbf{m}} f_{\mathbf{m}} \mathbf{x}^{\mathbf{m}}, \beta y + \sum_{\mathbf{m}} g_{\mathbf{m}} \mathbf{x}^{\mathbf{m}}, \gamma z + \sum_{\mathbf{m}} h_{\mathbf{m}} \mathbf{x}^{\mathbf{m}} \right),$$

with $\mathbf{m} \in \{(i,j,k)$ such that $i+j+k \geq 2\}$, and where $\mathbf{x}^{\mathbf{m}} = x^i y^j z^k$. When $\alpha = \pm 1$, $\beta\gamma = 1$, and $\beta \neq 1, \gamma \neq 1$, some necessary conditions for it to be periodic are

$$f^{(2)}_{3,0,0} = f^{(2)}_{1,1,1} = g^{(2)}_{2,1,0} = g^{(2)}_{0,2,1} = h^{(2)}_{2,0,1} = h^{(2)}_{0,1,2} = 0,$$

where $f^{(2)}_{i,j,k}$ and $g^{(2)}_{i,j,k}$ are the expressions given in the second step of the normal form procedure described in Section 3.

Proof. By inspection of the 3*rd order resonance conditions*, we observe that when $\alpha = \pm 1$ and $\beta\gamma = 1$ there appear the resonances $\alpha^3 - \alpha$, $\alpha\beta\gamma - \alpha$,$\alpha^2\beta - \beta$,$\beta^2\gamma - \beta$,$\alpha^2\gamma - \gamma$ and $\beta^2\gamma - \gamma$ which are associated to the coefficients $f^{(2)}_{3,0,0}$, $f^{(2)}_{1,1,1}$, $g^{(2)}_{2,1,0}$, $g^{(2)}_{0,2,1}$, $h^{(2)}_{2,0,1}$ and $h^{(2)}_{0,2,1}$ respectively. So all them must vanish to have a periodic map. ∎

Proof of Proposition 8. The dynamics of the third-order Lyness' equation can be studied through the Lyness maps

$$G_a(x,y,z) = \left(y, z, \frac{a+y+z}{x}\right).$$

It is easy to see that $G_a^p \neq \mathrm{Id}$ for $p = 1, 2$. We continue searching p-periodic maps with $p \geq 3$. It has always some fixed point (x_0, x_0, x_0) with $x_0^2 - 2x_0 - a = 0$ and $x_0 \neq 0$. Moreover the eigenvalues λ of the Jacobian matrix at this points are given by the zeroes of $-(\lambda+1)(\lambda^2 - (1 + 1/x_0)\lambda + 1) = 0$. These two equations suggest us to introduce the rational parametrization of a as

$$a = -\frac{\lambda\left(2\,\lambda^2 - 3\,\lambda + 2\right)}{\left(\lambda^2 - \lambda + 1\right)^2}, \quad \text{with} \quad \lambda^2 - \lambda + 1 \neq 0 \quad \text{and} \quad \lambda \neq 0,$$

which covers all values of $a \in \mathbb{C}$. Then the fixed point is (x_0, x_0, x_0) with $x_0 = \lambda/(\lambda^2 - \lambda + 1)$ and the eigenvalues of DG_a at this point are $-1, \lambda, 1/\lambda$. Notice that since $p \geq 3$, we can assume $\lambda \neq 1$. To apply Proposition 18 we perform the translation $(x,y,z) \to (x - x_0, y - x_0, z - x_0)$, which brings the fixed point to the origin, obtaining

$$g_\lambda(x,y,z) := \left(y, z, \frac{-\lambda x + \left(\lambda^2 - \lambda + 1\right) y + \left(\lambda^2 - \lambda + 1\right) z}{\left(\lambda^2 - \lambda + 1\right) x + \lambda}\right),$$

with linear part

$$L_\lambda(x,y) = \left(y, z, -x + \frac{\left(\lambda^2 - \lambda + 1\right) y}{\lambda} + \frac{\left(\lambda^2 - \lambda + 1\right) z}{\lambda}\right).$$

The linear change of variables $\Psi(x,y,z) = (x + y + \lambda^2 z, -x + \lambda\, y + \lambda\, z, x + \lambda^2 y + z)$ gives a conjugation between L_λ and its diagonal form $L(x,y,z) := (-x, \lambda y, z/\lambda)$. Using the conjugation Ψ, we finally obtain a map with diagonal linear part $F_\lambda := \Psi \circ g_\lambda \circ \Psi^{-1}$, which is under the assumptions of Proposition 18.

Applying this proposition and the Normal Form Algorithm to F_λ we can compute $g^{(2)}_{2,1,0}$. From the equation $g^{(2)}_{2,1,0} = 0$ we obtain that

$$\left(\lambda^2 - \lambda + 1\right)^3 \left(\lambda^4 + 1\right) = 0.$$

Thus λ has to be a primitive 8-th root of the unity. All these values of λ correspond to the same value $a = 1$, which gives a globally 8-periodic recurrence. So the result follows ∎

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Appendix A. Expression of $\mathcal{P}_3(F)$ when $\alpha\beta = 1$

Consider the map (1), applying the Normal Form Algorithm one gets that the periodicity condition associated to $f_{3,2}^{(4)}$ is given by

$$
\begin{aligned}
&\mathcal{P}_3(F) := f_{1,1}g_{0,2}g_{1,1}^2\alpha^{17} + (2g_{1,1}^2f_{1,1}g_{0,2} - f_{3,1}g_{0,2} - f_{1,1}g_{1,1}g_{1,2} - f_{1,1}g_{0,2}g_{2,1})\alpha^{16}\\
&+ (f_{3,2} + 3g_{1,1}^2f_{1,1}g_{0,2} + 3f_{1,1}g_{0,2}f_{3,0} - 3f_{1,1}g_{0,2}g_{2,1} + 2f_{1,1}f_{2,0}g_{0,2}g_{1,1} + 2f_{1,1}^2g_{0,2}g_{2,0}\\
&- 2f_{3,1}g_{0,2} + 2f_{2,1}f_{2,0}g_{0,2} + g_{1,1}^2f_{1,2} + 2f_{2,2}g_{1,1} + f_{1,1}g_{2,2} + 2f_{0,2}f_{1,1}g_{1,1}g_{2,0} - f_{1,1}g_{1,1}g_{1,2}\\
&+ g_{1,1}^2f_{1,1}^2)\alpha^{15} + (3g_{1,1}^2f_{1,1}g_{0,2} + 6f_{1,1}g_{0,2}f_{3,0} - 6f_{0,2}f_{3,0}g_{1,1} - 4f_{1,2}f_{2,0}g_{1,1} - 3f_{1,1}g_{1,1}f_{2,1}\\
&+ 2f_{1,2}g_{2,1} - 3f_{1,2}f_{3,0} + 5g_{1,1}^2f_{1,2} + 2g_{1,1}^2f_{1,1}^2 + 2f_{0,2}f_{1,1}g_{1,1}g_{2,0} - 2f_{2,2}f_{2,0} - 2f_{1,1}^2g_{2,1}\\
&- 3f_{3,1}g_{0,2} - 2f_{1,1}g_{0,2}^2g_{2,0} + 3f_{1,1}f_{2,0}g_{0,2}g_{1,1} + 4f_{2,1}f_{2,0}g_{0,2} - 3f_{0,2}f_{1,1}g_{3,0} - 2f_{1,2}g_{0,2}g_{2,0}\\
&+ 4f_{2,2}g_{1,1} + f_{3,2} + 2g_{1,1}^3f_{0,2} + 2f_{0,2}g_{1,1}g_{2,1} - 4f_{1,1}f_{2,0}^2g_{0,2} - 4f_{1,1}g_{0,2}g_{2,1} + 6f_{1,1}^2g_{0,2}g_{2,0}\\
&- 3f_{1,1}f_{3,1} - 2f_{1,1}f_{2,0}g_{1,2} - 2f_{1,1}f_{1,2}g_{2,0} - 4f_{4,0}f_{0,2} - 2g_{1,1}^2f_{0,2}f_{2,0} + 2f_{1,1}g_{2,2})\alpha^{14}\\
&+ (4f_{1,1}f_{2,0}f_{2,1} + 4g_{1,1}^2f_{1,1}g_{0,2} + 11f_{1,1}g_{0,2}f_{3,0} - 12f_{0,2}f_{3,0}g_{1,1} + 6f_{1,1}^2f_{2,0}g_{1,1}\\
&- 10f_{1,2}f_{2,0}g_{1,1} - 8f_{1,1}g_{1,1}f_{2,1} + 2f_{1,2}g_{2,1} - 3f_{1,2}f_{3,0} + 4f_{1,2}f_{2,0}^2 + 3f_{1,1}^3g_{2,0} + 10g_{1,1}^2f_{1,2}\\
&+ g_{1,1}^2f_{1,1}^2 + 6f_{1,1}^2f_{3,0} - 4f_{0,2}f_{1,1}g_{1,1}g_{2,0} - 2f_{2,2}f_{2,0} - 7f_{1,1}^2g_{2,1} + 12f_{3,0}f_{2,0}f_{0,2}\\
&+ 4f_{0,2}f_{2,0}g_{0,2}g_{2,0} - f_{3,1}g_{0,2} - 2f_{0,2}g_{0,2}g_{3,0} - 6f_{1,1}g_{0,2}^2g_{2,0} + 8f_{0,2}f_{1,1}f_{2,0}g_{2,0} + 6f_{2,1}f_{2,0}g_{0,2}\\
&- 6f_{0,2}f_{1,1}g_{3,0} - 4f_{1,2}g_{0,2}g_{2,0} + 6f_{2,2}g_{1,1} + f_{3,2} + 6g_{1,1}^3f_{0,2} + 3f_{1,1}g_{0,3}g_{2,0} + 6f_{0,2}g_{1,1}g_{2,1}\\
&+ 6f_{0,3}g_{1,1}g_{2,0} - 8f_{1,1}f_{2,0}^2g_{0,2} - 3f_{1,1}g_{0,2}g_{2,1} + 5f_{1,1}g_{1,1}g_{1,2} + 8f_{0,2}f_{2,0}^2g_{1,1} + 11f_{1,1}^2g_{0,2}g_{2,0}\\
&- 5f_{1,1}f_{3,1} - 5f_{1,1}f_{2,0}g_{1,2} + 3f_{1,3}g_{2,0} - 8f_{1,1}f_{1,2}g_{2,0} - 4f_{4,0}f_{0,2} - 12g_{1,1}^2f_{0,2}f_{2,0} + 2f_{0,2}g_{3,1}\\
&- 4f_{0,2}f_{2,0}g_{2,1} + 3f_{1,1}g_{2,2})\alpha^{13} + (6f_{1,1}f_{2,0}f_{2,1} + 9g_{1,1}^2f_{1,1}g_{0,2} + 7f_{1,1}g_{0,2}f_{3,0} - 24f_{0,2}f_{3,0}g_{1,1}\\
&+ 19f_{1,1}^2f_{2,0}g_{1,1} - 16f_{1,2}f_{2,0}g_{1,1} - 15f_{1,1}g_{1,1}f_{2,1} + 4f_{1,2}g_{2,1} - 5f_{1,2}f_{3,0} - 8f_{0,2}f_{2,0}^3 + 4f_{1,2}f_{2,0}^2\\
&+ 12f_{1,1}^3g_{2,0} + 15g_{1,1}^2f_{1,2} - 5g_{1,1}^2f_{1,1}^2 + 12f_{1,1}^2f_{3,0} + 3f_{0,3}g_{3,0} - 33f_{0,2}f_{1,1}g_{1,1}g_{2,0} - 2f_{2,2}f_{2,0}\\
&- 13f_{1,1}^2g_{2,1} + 12f_{3,0}f_{2,0}f_{0,2} + 12f_{0,2}f_{2,0}g_{0,2}g_{2,0} + 2f_{3,1}g_{0,2} - 4f_{0,2}g_{0,2}g_{3,0} - 8f_{1,1}g_{0,2}^2g_{2,0}\\
&- 6f_{0,2}f_{2,1}g_{2,0} - 8f_{1,1}g_{0,2}^2g_{2,0} - 6f_{0,2}f_{2,1}g_{2,0} + 26f_{0,2}f_{1,1}f_{2,0}g_{2,0} - 14f_{1,1}f_{2,0}g_{0,2}g_{1,1}\\
&+ 2f_{2,1}f_{2,0}g_{0,2} - 15f_{0,2}f_{1,1}g_{3,0} - 2f_{1,2}g_{0,2}g_{2,0} - 8f_{1,1}g_{0,2}^2g_{2,0} - 6f_{0,2}f_{2,1}g_{2,0} + 2f_{2,1}f_{2,0}g_{0,2}\\
&+ 26f_{0,2}f_{1,1}f_{2,0}g_{2,0} - 14f_{1,1}f_{2,0}g_{0,2}g_{1,1} - 15f_{0,2}f_{1,1}g_{3,0} - 2f_{1,2}g_{0,2}g_{2,0} + 2f_{2,2}g_{1,1} - 2f_{3,2}\\
&+ 16g_{1,1}^3f_{0,2} + 6f_{1,1}g_{0,3}g_{2,0} + 14f_{0,2}g_{1,1}g_{2,1} + 12f_{0,3}g_{1,1}g_{2,0} - 14f_{1,1}f_{2,0}^2g_{0,2} + 3f_{1,1}g_{0,2}g_{2,1}
\end{aligned}
$$

$$+ 9f_{1,1}g_{1,1}g_{1,2} + 20f_{0,2}f_{2,0}^2g_{1,1} - 6f_{1,1}^2f_{2,0}^2 + 10f_{1,1}^2g_{0,2}g_{2,0} - 7f_{1,1}f_{3,1} - 9f_{1,1}f_{2,0}g_{1,2}$$
$$+ 3f_{1,3}g_{2,0} - 19f_{1,1}f_{1,2}g_{2,0} - 6f_{0,3}f_{2,0}g_{2,0} - 4f_{4,0}f_{0,2} - 30g_{1,1}^2f_{0,2}f_{2,0} + 2f_{0,2}g_{3,1}$$
$$- 8f_{0,2}f_{2,0}g_{2,1} + 4f_{0,2}g_{0,2}g_{1,1}g_{2,0} + f_{1,1}g_{2,2})\alpha^{12} + (8f_{1,1}f_{2,0}f_{2,1} + 15g_{1,1}^2f_{1,1}g_{0,2}$$
$$+ 2f_{1,1}g_{0,2}f_{3,0} - 18f_{0,2}f_{3,0}g_{1,1} + 40f_{1,1}^2f_{2,0}g_{1,1} - 12f_{1,2}f_{2,0}g_{1,1} - 15f_{1,1}g_{1,1}f_{2,1} - 2f_{1,2}g_{2,1}$$
$$+ 4f_{1,2}f_{3,0} - 8f_{0,2}f_{2,0}^3 + 6f_{1,2}f_{2,0}^2 + 29f_{1,1}^3g_{2,0} + 11g_{1,1}^2f_{1,2} - 16g_{1,1}^2f_{1,1}^2 + 21f_{1,1}^2f_{3,0}$$
$$+ 3f_{0,3}g_{3,0} - 74f_{0,2}f_{1,1}g_{1,1}g_{2,0} + 4f_{2,2}f_{2,0} - 16f_{1,1}^2g_{2,1} + 16f_{3,0}f_{2,0}f_{0,2} + 12f_{0,2}f_{2,0}g_{0,2}g_{2,0}$$
$$+ 5f_{3,1}g_{0,2} - 6f_{0,2}g_{0,2}g_{3,0} - 8f_{1,1}g_{0,2}^2g_{2,0} - 6f_{0,2}f_{2,1}g_{2,0} + 70f_{0,2}f_{1,1}f_{2,0}g_{2,0}$$
$$- 35f_{1,1}f_{2,0}g_{0,2}g_{1,1} - 4f_{2,1}f_{2,0}g_{0,2} - 19f_{0,2}f_{1,1}g_{3,0} - 4f_{2,2}g_{1,1} + 4g_{1,2}f_{0,2}g_{2,0} - 3f_{3,2}$$
$$+ 24g_{1,1}^3f_{0,2} + 9f_{1,1}g_{0,3}g_{2,0} + 14f_{0,2}g_{1,1}g_{2,1} + 18f_{0,3}g_{1,1}g_{2,0} - 8f_{1,1}f_{2,0}^2g_{0,2} + 6f_{1,1}g_{0,2}g_{2,1}$$
$$+ 10f_{1,1}g_{1,1}g_{1,2} + 40f_{0,2}f_{2,0}^2g_{1,1} - 10f_{1,1}^2f_{2,0}^2 - 3f_{1,1}^2g_{0,2}g_{2,0} - 8f_{1,1}f_{2,0}g_{1,2} + 3f_{1,3}g_{2,0}$$
$$- 25f_{1,1}f_{1,2}g_{2,0} - 12f_{0,3}f_{2,0}g_{2,0} + 4f_{4,0}f_{0,2} - 54g_{1,1}^2f_{0,2}f_{2,0} + 2f_{0,2}g_{3,1} - 12f_{0,2}f_{2,0}g_{2,1}$$
$$+ 10f_{0,2}g_{0,2}g_{1,1}g_{2,0} - 2f_{1,1}g_{2,2})\alpha^{11} + (-2f_{1,1}f_{2,0}f_{2,1} + 22g_{1,1}^2f_{1,1}g_{0,2} - 9f_{1,1}g_{0,2}f_{3,0}$$
$$- 10f_{0,2}f_{3,0}g_{1,1} + 54f_{1,1}^2f_{2,0}g_{1,1} + 2f_{1,2}f_{2,0}g_{1,1} - 8f_{1,1}g_{1,1}f_{2,1} - 2f_{1,2}g_{2,1} + 5f_{1,2}f_{3,0}$$
$$- 12f_{0,2}f_{2,0}^3 - 6f_{1,2}f_{2,0}^2 + 46f_{1,1}^3g_{2,0} + 4g_{1,1}^2f_{1,2} - 25g_{1,1}^2f_{1,1}^2 + 13f_{1,1}^2f_{3,0} + 3f_{0,3}g_{3,0}$$
$$- 115f_{0,2}f_{1,1}g_{1,1}g_{2,0} + 6f_{2,2}f_{2,0} - 9f_{1,1}^2g_{2,1} - 8f_{3,0}f_{2,0}f_{0,2} + 4f_{0,2}f_{2,0}g_{0,2}g_{2,0} + 5f_{3,1}g_{0,2}$$
$$+ 2f_{0,2}g_{0,2}g_{3,0} - 6f_{0,2}f_{2,1}g_{2,0} + 98f_{0,2}f_{1,1}f_{2,0}g_{2,0} - 52f_{1,1}f_{2,0}g_{0,2}g_{1,1} - 10f_{2,1}f_{2,0}g_{0,2}$$
$$- 17f_{0,2}f_{1,1}g_{3,0} + 8f_{1,2}g_{0,2}g_{2,0} - 10f_{2,2}g_{1,1} + 4g_{1,2}f_{0,2}g_{2,0} - 3f_{3,2} + 32g_{1,1}^3f_{0,2}$$
$$+ 6f_{1,1}g_{0,3}g_{2,0} + 12f_{0,2}g_{1,1}g_{2,1} - 8f_{0,2}^2g_{2,0}^2 + 15f_{0,3}g_{1,1}g_{2,0} + 9f_{1,1}g_{0,2}g_{2,1}$$
$$+ 5f_{1,1}g_{1,1}g_{1,2} + 40f_{0,2}f_{2,0}^2g_{1,1} - 16f_{1,1}^2f_{2,0}^2 - 20f_{1,1}^2g_{0,2}g_{2,0} + 7f_{1,1}f_{3,1} - 3f_{1,1}f_{2,0}g_{1,2}$$
$$- 3f_{1,3}g_{2,0} - 19f_{1,1}f_{1,2}g_{2,0} - 12f_{0,3}f_{2,0}g_{2,0} + 8f_{4,0}f_{0,2} - 62g_{1,1}^2f_{0,2}f_{2,0} - 2f_{0,2}g_{3,1}$$
$$- 4f_{0,2}f_{2,0}g_{2,1} + 24f_{0,2}g_{0,2}g_{1,1}g_{2,0} - 5f_{1,1}g_{2,2})\alpha^{10} + (-10f_{1,1}f_{2,0}f_{2,1} + 21g_{1,1}^2f_{1,1}g_{0,2}$$
$$- 11f_{1,1}g_{0,2}f_{3,0} + 14f_{0,2}f_{3,0}g_{1,1} + 53f_{1,1}^2f_{2,0}g_{1,1} + 16f_{1,2}f_{2,0}g_{1,1} + 5f_{1,1}g_{1,1}f_{2,1}$$
$$- 8f_{1,2}g_{2,1} + 11f_{1,2}f_{3,0} + 4f_{0,2}f_{2,0}^3 - 8f_{1,2}f_{2,0}^2 + 53f_{1,1}^3g_{2,0} - 9g_{1,1}^2f_{1,2} - 29g_{1,1}^2f_{1,1}^2$$
$$+ 2f_{1,1}^2f_{3,0} - 6f_{0,3}g_{3,0} - 130f_{0,2}f_{1,1}g_{1,1}g_{2,0} + 6f_{2,2}f_{2,0} + f_{1,1}^2g_{2,1} - 16f_{3,0}f_{2,0}f_{0,2}$$
$$- 24f_{0,2}f_{2,0}g_{0,2}g_{2,0} + 2f_{3,1}g_{0,2} + 6f_{0,2}g_{0,2}g_{3,0} + 4f_{1,1}g_{0,2}^2g_{2,0} + 98f_{0,2}f_{1,1}f_{2,0}g_{2,0}$$
$$- 56f_{1,1}f_{2,0}g_{0,2}g_{1,1} - 10f_{2,1}f_{2,0}g_{0,2} - 3f_{0,2}f_{1,1}g_{3,0} + 4f_{1,2}g_{0,2}g_{2,0} - 10f_{2,2}g_{1,1}$$
$$+ 4g_{1,2}f_{0,2}g_{2,0} + 28g_{1,1}^3f_{0,2} - 4f_{0,2}g_{1,1}g_{2,1} - 8f_{0,2}^2g_{2,0}^2 + 3f_{0,3}g_{1,1}g_{2,0} + 14f_{1,1}f_{2,0}^2g_{0,2}$$
$$+ 3f_{1,1}g_{0,2}g_{2,1} - 3f_{1,1}g_{1,1}g_{1,2} + 28f_{0,2}f_{2,0}^2g_{1,1} - 4f_{1,1}^2f_{2,0}^2 - 38f_{1,1}^2g_{0,2}g_{2,0} + 13f_{1,1}f_{3,1}$$
$$+ 5f_{1,1}f_{2,0}g_{1,2} - 6f_{1,3}g_{2,0} - f_{1,1}f_{1,2}g_{2,0} + 8f_{4,0}f_{0,2} - 56g_{1,1}^2f_{0,2}f_{2,0} - 4f_{0,2}g_{3,1}$$
$$+ 4f_{0,2}f_{2,0}g_{2,1} + 26f_{0,2}g_{0,2}g_{1,1}g_{2,0} - 5f_{1,1}g_{2,2})\alpha^9 + (-16f_{1,1}f_{2,0}f_{2,1} + 17g_{1,1}^2f_{1,1}g_{0,2}$$
$$- 10f_{1,1}g_{0,2}f_{3,0} + 22f_{0,2}f_{3,0}g_{1,1} + 36f_{1,1}^2f_{2,0}g_{1,1} + 24f_{1,2}f_{2,0}g_{1,1} + 15f_{1,1}g_{1,1}f_{2,1}$$
$$- 2f_{1,2}g_{2,1} + 2f_{1,2}f_{3,0} + 8f_{0,2}f_{2,0}^3 - 14f_{1,2}f_{2,0}^2 + 45f_{1,1}^3g_{2,0} - 13g_{1,1}^2f_{1,2} - 22g_{1,1}^2f_{1,1}^2$$
$$- 17f_{1,1}^2f_{3,0} - 6f_{0,3}g_{3,0} - 103f_{0,2}f_{1,1}g_{1,1}g_{2,0} + 14f_{1,1}^2g_{2,1} - 24f_{3,0}f_{2,0}f_{0,2} - f_{3,1}g_{0,2}$$
$$- 28f_{0,2}f_{2,0}g_{0,2}g_{2,0} + 10f_{0,2}g_{0,2}g_{3,0} + 10f_{1,1}g_{0,2}^2g_{2,0} + 6f_{0,2}f_{2,1}g_{2,0} + 58f_{0,2}f_{1,1}f_{2,0}g_{2,0}$$
$$- 41f_{1,1}f_{2,0}g_{0,2}g_{1,1} - 4f_{2,1}f_{2,0}g_{0,2} + 16f_{0,2}f_{1,1}g_{3,0} + 6f_{1,2}g_{0,2}g_{2,0} - 4f_{2,2}g_{1,1} + 3f_{3,2}$$
$$+ 22g_{1,1}^3f_{0,2} - 6f_{1,1}g_{0,3}g_{2,0} - 10f_{0,2}g_{1,1}g_{2,1} - 8f_{0,2}^2g_{2,0}^2 - 9f_{0,3}g_{1,1}g_{2,0} + 16f_{1,1}f_{2,0}^2g_{0,2}$$

$$
\begin{aligned}
&- 8f_{1,1}g_{1,1}g_{1,2} - 4f_{0,2}f_{2,0}^2g_{1,1} + 6f_{1,1}^2f_{2,0}^2 - 36f_{1,1}^2g_{0,2}g_{2,0} + 10f_{1,1}f_{3,1} + 10f_{1,1}f_{2,0}g_{1,2}\\
&- 6f_{1,3}g_{2,0} + 20f_{1,1}f_{1,2}g_{2,0} + 18f_{0,3}f_{2,0}g_{2,0} + 4f_{4,0}f_{0,2} - 26g_{1,1}^2f_{0,2}f_{2,0} - 4f_{0,2}g_{3,1}\\
&+ 16f_{0,2}f_{2,0}g_{2,1} + 36f_{0,2}g_{0,2}g_{1,1}g_{2,0} - 2f_{1,1}g_{2,2})\alpha^8 + (-10f_{1,1}f_{2,0}f_{2,1} + 8g_{1,1}^2f_{1,1}g_{0,2}\\
&- 3f_{1,1}g_{0,2}f_{3,0} + 24f_{0,2}f_{3,0}g_{1,1} + 12f_{1,1}^2f_{2,0}g_{1,1} + 14f_{1,2}f_{2,0}g_{1,1} + 16f_{1,1}g_{1,1}f_{2,1}\\
&- 2f_{1,2}g_{2,1} - f_{1,2}f_{3,0} + 16f_{0,2}f_{2,0}^3 - 2f_{1,2}f_{2,0}^2 + 25f_{1,1}^3g_{2,0} - 14g_{1,1}^2f_{1,2} - 13g_{1,1}^2f_{1,1}^2\\
&- 21f_{1,1}^2f_{3,0} - 6f_{0,3}g_{3,0} - 62f_{0,2}f_{1,1}g_{1,1}g_{2,0} - 6f_{2,2}f_{2,0} + 15f_{1,1}^2g_{2,1} - 12f_{3,0}f_{2,0}f_{0,2}\\
&- 28f_{0,2}f_{2,0}g_{0,2}g_{2,0} - 3f_{3,1}g_{0,2} + 6f_{1,1}g_{0,2}^2g_{2,0} + 6f_{0,2}f_{2,1}g_{2,0} - 22f_{1,1}f_{2,0}g_{0,2}g_{1,1}\\
&+ 2f_{2,1}f_{2,0}g_{0,2} + 21f_{0,2}f_{1,1}g_{3,0} - 4f_{1,2}g_{0,2}g_{2,0} + 2f_{2,2}g_{1,1} - 4g_{1,2}f_{0,2}g_{2,0} + 3f_{3,2}\\
&+ 10g_{1,1}^3f_{0,2} - 9f_{1,1}g_{0,3}g_{2,0} - 18f_{0,2}g_{1,1}g_{2,1} - 8f_{0,2}^2g_{2,0}^2 - 21f_{0,3}g_{1,1}g_{2,0} + 12f_{1,1}f_{2,0}^2g_{0,2}\\
&- 5f_{1,1}g_{0,2}g_{2,1} - 9f_{1,1}g_{1,1}g_{1,2} - 24f_{0,2}f_{2,0}^2g_{1,1} + 20{f_{1,1}}^2f_{2,0}^2 - 29f_{1,1}^2g_{0,2}g_{2,0} + f_{1,1}f_{3,1}\\
&+ 9f_{1,1}f_{2,0}g_{1,2} - 3f_{1,3}g_{2,0} + 26f_{1,1}f_{1,2}g_{2,0} + 18f_{0,3}f_{2,0}g_{2,0} - 4f_{4,0}f_{0,2} - 4g_{1,1}^2f_{0,2}f_{2,0}\\
&- 2f_{0,2}g_{3,1} + 12f_{0,2}f_{2,0}g_{2,1} + 22f_{0,2}g_{0,2}g_{1,1}g_{2,0} + f_{1,1}g_{2,2})\alpha^7 + (2f_{1,1}f_{2,0}f_{2,1}\\
&+ 3g_{1,1}^2f_{1,1}g_{0,2} + f_{1,1}g_{0,2}f_{3,0} + 12f_{0,2}f_{3,0}g_{1,1} - f_{1,1}^2f_{2,0}g_{1,1} + 4f_{1,2}f_{2,0}g_{1,1}\\
&+ 11f_{1,1}g_{1,1}f_{2,1} + 4f_{1,2}g_{2,1} - 7f_{1,2}f_{3,0} + 8f_{0,2}f_{2,0}^3 + 4f_{1,2}f_{2,0}^2 + 8f_{1,1}^3g_{2,0} - 7g_{1,1}^2f_{1,2}\\
&- 3g_{1,1}^2f_{1,1}^2 - 16f_{1,1}^2f_{3,0} + 3f_{0,3}g_{3,0} - 14f_{0,2}f_{1,1}g_{1,1}g_{2,0} - 6f_{2,2}f_{2,0} + 13f_{1,1}^2g_{2,1}\\
&+ 4f_{3,0}f_{2,0}f_{0,2} - 2f_{3,1}g_{0,2} - 2f_{0,2}g_{0,2}g_{3,0} + 4f_{1,1}g_{0,2}^2g_{2,0} + 6f_{0,2}f_{2,1}g_{2,0}\\
&- 22f_{0,2}f_{1,1}f_{2,0}g_{2,0} + 6f_{2,1}f_{2,0}g_{0,2} + 20f_{0,2}f_{1,1}g_{3,0} - 2f_{1,2}g_{0,2}g_{2,0} + 6f_{2,2}g_{1,1}\\
&+ 2f_{3,2} + 4g_{1,1}^3f_{0,2} - 6f_{1,1}g_{0,3}g_{2,0} - 10f_{0,2}g_{1,1}g_{2,1} - 15f_{0,3}g_{1,1}g_{2,0} + 2f_{1,1}f_{2,0}^2g_{0,2}\\
&- 4f_{1,1}f_{2,0}g_{0,2}g_{1,1} - 4g_{1,2}f_{0,2}g_{2,0} - 3f_{1,1}g_{0,2}g_{2,1} - 5f_{1,1}g_{1,1}g_{1,2} - 28f_{0,2}f_{2,0}^2g_{1,1}\\
&+ 16f_{1,1}^2f_{2,0}^2 - 10f_{1,1}^2g_{0,2}g_{2,0} - 5f_{1,1}f_{3,1} + 5f_{1,1}f_{2,0}g_{1,2} + 3f_{1,3}g_{2,0} + 23f_{1,1}f_{1,2}g_{2,0}\\
&+ 12f_{0,3}f_{2,0}g_{2,0} - 4f_{4,0}f_{0,2} + 14g_{1,1}^2f_{0,2}f_{2,0} + 2f_{0,2}g_{3,1} + 8f_{0,2}f_{2,0}g_{2,1} + 3f_{1,1}g_{2,2}\\
&+ 16f_{0,2}g_{0,2}g_{1,1}g_{2,0})\alpha^6 + (8f_{1,1}f_{2,0}f_{2,1} + 2f_{1,1}g_{0,2}f_{3,0} + 2f_{0,2}f_{3,0}g_{1,1} - 6f_{1,1}^2f_{2,0}g_{1,1}\\
&- 8f_{1,2}f_{2,0}g_{1,1} + 3f_{1,1}g_{1,1}f_{2,1} + 2f_{1,2}g_{2,1} - 4f_{1,2}f_{3,0} + 10f_{1,2}f_{2,0}^2 - 3f_{1,1}^3g_{2,0}\\
&- 3g_{1,1}^2f_{1,2} - 5f_{1,1}^2f_{3,0} + 3f_{0,3}g_{3,0} - 4f_{2,2}f_{2,0} + 4f_{1,1}^2g_{2,1} + 8f_{3,0}f_{2,0}f_{0,2} - f_{3,1}g_{0,2}\\
&+ 4f_{0,2}f_{2,0}g_{0,2}g_{2,0} - 4f_{0,2}g_{0,2}g_{3,0} - 30f_{0,2}f_{1,1}f_{2,0}g_{2,0} + f_{1,1}f_{2,0}g_{0,2}g_{1,1} + 4f_{2,1}f_{2,0}g_{0,2}\\
&+ 6f_{0,2}f_{1,1}g_{3,0} - 4f_{1,2}g_{0,2}g_{2,0} + 4f_{2,2}g_{1,1} - 4g_{1,2}f_{0,2}g_{2,0} - f_{3,2} - 3f_{1,1}g_{0,3}g_{2,0}\\
&- 6f_{0,2}g_{1,1}g_{2,1} - 9f_{0,3}g_{1,1}g_{2,0} - 4f_{1,1}f_{2,0}^2g_{0,2} - 2f_{1,1}g_{0,2}g_{2,1} - 2f_{1,1}g_{1,1}g_{1,2} + 6f_{1,1}^2f_{2,0}^2\\
&- 16f_{0,2}f_{2,0}^2g_{1,1} - 3f_{1,1}^2g_{0,2}g_{2,0} - 8f_{1,1}f_{3,1} + 3f_{1,3}g_{2,0} + 7f_{1,1}f_{1,2}g_{2,0} - 6f_{0,3}f_{2,0}g_{2,0}\\
&- 4f_{4,0}f_{0,2} + 10g_{1,1}^2f_{0,2}f_{2,0} + 2f_{0,2}g_{3,1} - 4f_{0,2}f_{2,0}g_{2,1} + 2f_{1,1}g_{2,2})\alpha^5 + (10f_{1,1}f_{2,0}f_{2,1}\\
&+ f_{1,1}g_{0,2}f_{3,0} - 2f_{0,2}f_{3,0}g_{1,1} - 6f_{1,2}f_{2,0}g_{1,1} + 2f_{1,2}g_{2,1} - f_{1,2}f_{3,0} - 4f_{0,2}f_{2,0}^3\\
&+ 6f_{1,2}f_{2,0}^2 - 2f_{1,1}^3g_{2,0} + g_{1,1}^2f_{1,1}^2 + 3f_{1,1}^2f_{3,0} + 3f_{0,3}g_{3,0} + 6f_{0,2}f_{1,1}g_{1,1}g_{2,0} + 2f_{2,2}f_{2,0}\\
&+ f_{1,1}^2g_{2,1} + 8f_{3,0}f_{2,0}f_{0,2} + 8f_{0,2}f_{2,0}g_{0,2}g_{2,0} - 4f_{0,2}f_{1,1}f_{2,0}g_{2,0} + 2f_{1,1}f_{2,0}g_{0,2}g_{1,1}\\
&+ 2f_{2,1}f_{2,0}g_{0,2} + 2f_{0,2}f_{1,1}g_{3,0} + 2f_{2,2}g_{1,1} - f_{3,2} - 4f_{1,1}f_{2,0}^2g_{0,2} - 4f_{1,1}^2f_{2,0}^2 + 2f_{1,1}^2g_{0,2}g_{2,0}\\
&- 3f_{1,1}f_{3,1} - f_{1,1}f_{2,0}g_{1,2} + 3f_{1,3}g_{2,0} + f_{1,1}f_{1,2}g_{2,0} - 6f_{0,3}f_{2,0}g_{2,0} + 6g_{1,1}^2f_{0,2}f_{2,0}\\
&+ 2f_{0,2}g_{3,1} - 4f_{0,2}f_{2,0}g_{2,1} + f_{1,1}g_{2,2})\alpha^4 + (-f_{1,1}^3g_{2,0} + 4f_{0,2}f_{2,0}^2g_{1,1} - f_{1,1}f_{2,0}g_{1,2}\\
&- 8f_{1,1}^2f_{2,0}^2 - 2f_{1,1}f_{2,0}^2g_{0,2} - 2f_{0,2}f_{1,1}g_{3,0} + f_{1,2}f_{3,0} - 4f_{1,2}f_{2,0}g_{1,1} + 2f_{2,2}f_{2,0}
\end{aligned}
$$

$$- f_{1,1}g_{1,1}f_{2,1} + 2f_{1,1}^2 f_{3,0} + f_{1,1}^2 f_{2,0}g_{1,1} - f_{3,2} - f_{1,1}^2 g_{2,1} - 6f_{0,3}f_{2,0}g_{2,0} - 4f_{0,2}f_{2,0}g_{2,1}$$
$$- 3f_{1,1}f_{1,2}g_{2,0} - 2f_{0,2}f_{3,0}g_{1,1} - 4f_{0,2}f_{2,0}^3 - f_{1,1}f_{3,1} + 2f_{1,1}f_{2,0}f_{2,1})\alpha^3 + (-2f_{1,1}^2 f_{2,0}^2$$
$$+ 4f_{0,2}f_{2,0}^2 g_{1,1} + f_{1,2}f_{3,0} + f_{1,1}^3 g_{2,0} + f_{1,1}^2 f_{3,0} + 4f_{0,2}f_{1,1}f_{2,0}g_{2,0} + 2f_{2,2}f_{2,0} + f_{1,1}f_{3,1}$$
$$- 2f_{1,2}f_{2,0}^2 + 2f_{1,1}^2 f_{2,0}g_{1,1})\alpha^2 + (-2f_{1,2}f_{2,0}^2 - 2f_{1,1}f_{2,0}f_{2,1} - f_{1,1}^2 f_{3,0})\alpha + 2f_{1,1}^2 f_{2,0}^2.$$

Appendix B. Expression of $\mathcal{P}_7(F)$ and $\mathcal{P}_8(F)$ when $\beta = 1$

Applying the Normal Form Algorithm to the map (2) we get that, when $\alpha^3 \neq 1$, then $f_{1,4}^{(4)} = \mathcal{P}_7(F)/(\alpha^3 - 1)$ and $g_{0,5}^{(4)} = \mathcal{P}_8(F)/(\alpha^3 - 1)$ where

$$\mathcal{P}_7(F) := f_{1,4}\alpha^3 + (3f_{0,3}g_{1,2} - 3f_{1,4} - 2f_{0,3}f_{2,1} - 2f_{2,0}f_{0,4} + 2g_{1,3}f_{0,2} - 2f_{2,2}f_{0,2}$$
$$+ 4f_{0,4}g_{1,1})\alpha^2 + (-4g_{1,3}f_{0,2} + 4f_{0,3}f_{2,1} - 8f_{0,4}g_{1,1} - 6f_{0,3}g_{1,2} - 4g_{2,1}f_{0,2}^2$$
$$- 10f_{0,3}f_{0,2}g_{2,0} + 3f_{1,4} + 3f_{3,0}f_{0,2}^2 + 4f_{2,0}f_{0,4} + 4f_{2,2}f_{0,2})\alpha - f_{1,4} - 2f_{2,2}f_{0,2}$$
$$+ 2g_{1,3}f_{0,2} + 4g_{2,1}f_{0,2}^2 - 2f_{2,0}f_{0,4} + 2{f_{2,0}}^2{f_{0,2}}^2 - 3f_{3,0}f_{0,2}^2 - 8g_{1,1}f_{2,0}f_{0,2}^2$$
$$+ 8f_{0,2}^2 g_{1,1}^2 + 4f_{0,4}g_{1,1} + 3f_{0,3}g_{1,2} + 10f_{0,3}f_{0,2}g_{2,0} - 2f_{0,3}f_{2,1},$$
$$\mathcal{P}_8(F) := g_{0,5}\alpha^3 + (-3g_{0,5} - f_{0,4}g_{1,1} - g_{1,3}f_{0,2} - f_{0,3}g_{1,2})\alpha^2 + (2f_{0,3}g_{1,2} + 2f_{0,4}g_{1,1}$$
$$+ g_{2,1}f_{0,2}^2 + 2f_{0,3}f_{0,2}g_{2,0} + 2g_{1,3}f_{0,2} + 3g_{0,5})\alpha - f_{0,3}g_{1,2} - g_{0,5} - 2f_{0,3}f_{0,2}g_{2,0}$$
$$- f_{0,4}g_{1,1} + g_{1,1}f_{2,0}f_{0,2}^2 - g_{1,3}f_{0,2} - g_{2,1}f_{0,2}^2 - 2f_{0,2}^2 g_{1,1}^2.$$

Appendix C. Expression of $C_2(B, \lambda)$ in the proof of Theorem 7

$$C_2(B,\lambda) := 8B^{12}\lambda^{25} + 125B^{12}\lambda^{24} + 912B^{12}\lambda^{23} + 4140B^{12}\lambda^{22} + 13091B^{12}\lambda^{21} + 23B^9\lambda^{24}$$
$$+30388B^{12}\lambda^{20} + 264B^9\lambda^{23} + 52493B^{12}\lambda^{19} + 1457B^9\lambda^{22} + 64792B^{12}\lambda^{18}$$
$$+5130B^9\lambda^{21} + 44963B^{12}\lambda^{17} + 12792B^9\lambda^{20} + 17B^6\lambda^{23} - 22114B^{12}\lambda^{16}$$
$$+23399B^9\lambda^{19} + 152B^6\lambda^{22} - 126694B^{12}\lambda^{15} + 30518B^9\lambda^{18} + 685B^6\lambda^{21}$$
$$-230443B^{12}\lambda^{14} + 23012B^9\lambda^{17} + 2027B^6\lambda^{20} - 285544B^{12}\lambda^{13} - 7945B^9\lambda^{16}$$
$$+4241B^6\lambda^{19} - 265465B^{12}\lambda^{12} - 59005B^9\lambda^{15} + 6222B^6\lambda^{18} + 23B^3\lambda^{21}$$
$$-182980B^{12}\lambda^{11} - 111409B^9\lambda^{14} + 5530B^6\lambda^{17} + 126B^3\lambda^{20} - 80299B^{12}\lambda^{10}$$
$$-140407B^9\lambda^{13} - 138B^6\lambda^{16} + 356B^3\lambda^{19} - 280B^{12}\lambda^9 - 131599B^9\lambda^{12}$$
$$-10552B^6\lambda^{15} + 644B^3\lambda^{18} + 37544B^{12}\lambda^8 - 90967B^9\lambda^{11} - 21809B^6\lambda^{14}$$
$$+723B^3\lambda^{17} + 40086B^{12}\lambda^7 - 40111B^9\lambda^{10} - 28180B^6\lambda^{13} + 253B^3\lambda^{16} + 8\lambda^{19}$$
$$+26571B^{12}\lambda^6 - 883B^9\lambda^9 - 26229B^6\lambda^{12} - 844B^3\lambda^{15} + 29\lambda^{18} + 12701B^{12}\lambda^5$$
$$+17318B^9\lambda^8 - 17474B^6\lambda^{11} - 2101B^3\lambda^{14} + 36\lambda^{17} + 4481B^{12}\lambda^4 + 18449B^9\lambda^7$$
$$-6870B^6\lambda^{10} - 2804B^3\lambda^{13} + 34\lambda^{16} + 1143B^{12}\lambda^3 + 12036B^9\lambda^6 + 902B^6\lambda^9$$
$$-2563B^3\lambda^{12} - 33\lambda^{15} + 199B^{12}\lambda^2 + 5561B^9\lambda^5 + 4012B^6\lambda^8 - 1532B^3\lambda^{11} - 71\lambda^{14}$$
$$+21B^{12}\lambda + 1827B^9\lambda^4 + 3635B^6\lambda^7 - 364B^3\lambda^{10} - 137\lambda^{13} + B^{12} + 404B^9\lambda^3$$
$$+2079B^6\lambda^6 + 335B^3\lambda^9 - 92\lambda^{12} + 53B^9\lambda^2 + 839B^6\lambda^5 + 493B^3\lambda^8 - 56\lambda^{11}$$
$$+3B^9\lambda + 232B^6\lambda^4 + 348B^3\lambda^7 + 8\lambda^{10} + 37B^6\lambda^3 + 149B^3\lambda^6 + 29\lambda^9 + 2B^6\lambda^2$$
$$+35B^3\lambda^5 + 25\lambda^8 + 3B^3\lambda^4 + 9\lambda^7 + \lambda^6.$$

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