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A comprehensive dynamic model for class-1 tensegrity systems based on quaternions

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ABSTRACT

In this paper we propose a new dynamic model, based on quaternions, for tensegrity systems of class-1. Quaternions are used to represent orientations of a rigid body in the 3-dimensional space eliminating the problem of singularities. Moreover, the equations based on quaternions allow to perform more precise calculations and simulations because they do not use trigonometric functions for the representation of angles. We present a thorough introduction of tensegrities and the current state of research. We also introduce the quaternions and provide in the appendix some important details and useful properties. Applying the Euler–Lagrange approach we derive a comprehensive dynamic model, first for a simple rigid bar in the space and, at last, for a class-1 tensegrity system. We present two model forms: a matrix and a vectorial form. The first more compact and easier to write, the latter more suitable to apply the tools and the theory based on vector fields.

1. Introduction

Tensegrities are mechanical systems born in the art community in the early 60s (Snelson, 1965), at the beginning studied by architects and successively widely used in engineering applications. Among several definitions we can say that a tensegrity system is an aggregation of mechanical elements, carrying either compression or tension but not both, for which at least one equilibrium state exists. Mixing together the terms *tensile* and *integrity*, which emphasize the main characteristic of such structures, the name *tensegrity* was coined by Buckminister Fuller in the early 60s (Fuller, 1962). An interesting and quite general definition was given by Pugh (1976): *"A tensegrity system is established when a set of discontinuous compressive components interacts with a set of continuous tensile components to define a stable volume in space"*.

Elements working under compression are called *struts* while the elements working in tension are called *tensors*. Usually the struts are rigid bars and the tensors are cables or springs, but several variations may be allowed. Actuating bars and cables allow to dynamically modify their lengths and, hence, change the configuration of the system.

To build a tensegrity structure, several approaches have been proposed. Between others, one of the most powerful techniques is based on the construction of a base module and of a design pattern. The overall structure is the result of assembling the base module along the design pattern following proper rules to ensure the existence of, at least, one equilibrium point. So, usually, the most famous tensegrity structures have some degree of geometric symmetry in their equilibrium positions.

These systems have a lot of interesting properties. It has been shown, for example, that carefully designing the appropriate net of connections between rigid elements, it is possible to provide to the structure the desired rigidity (within the limits of the material employed). Compared with traditional structures, tensegrities can be much more stiff, much more light and occupy much less space and volume. The main applications in the robotic field, tend to exploit the capability of such systems to be extensible and redundant (in case of one, or more, struts or tensors fail, it is possible to guarantee the functionality of the system by means of other elements). See, for instance, Aldrich (2004), Paul et al. (2006), Masic and Skelton (2004) and Mirats-Tur (2010) for some application concerning manipulators and mobile robotics.

For several years they have been studied only from a static point of view, mainly because there were no possibilities to face with the enormous quantity of computations required to model their dynamics. The static analysis of tensegrity has reached a certain level of maturity, with lots of contributions by different authors and fields of study. See for instance Roth and Whiteley (1981), Connelly (1999) for a mathematic perspective, Calladine and Pellegrino (1992), Motro et al. (1986), Hanaor (1988) for studies about the self-stress states of the structure, or Tarnai (1989), Vassart et al.

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(2000) talking about the prestressability problem. A recent review about tensegrity statics was given in Hernandez and Mirats-Tur (2008) presenting different existing definitions for tensegrity structures, as well as their main properties.

The dynamics were first studied by Motro et al. (1986). Kanchanasaratool and Williamson (2002) studied dynamic particle models while considering the bars to be massless; other studies, as those by Skelton et al. (2001) or Sultan (1999) consider mass on bars. Also non-linear models and their linearizations have been considered by Sultan et al. (2002). A recent survey on the dynamics of tensegrity structures including current open research problems was given in Mirats-Tur and Hernández (2009).

In the following we shall focus on the so called *class-1* tensegrity structures. A definition of class-*k* tensegrity structures can be found in Skelton and Oliveira (2009): *a tensegrity system with k rigid components in contact, with a frictionless ball joint, at a given node*. So a class-1 tensegrity is a structure in which there is only one rigid element per node, or in other words (as captured in Pugh definition): *systems where all tensors constitute a continuous set of elements and no rigid bodies are in contact*. This is the simplest category, yet useful enough to allow the study of general dynamical models and their properties.

To obtain a set of ordinary differential equations of a complex system three different methods are commonly used: Newton–Euler, Lagrange and Hamilton approaches. The Newton–Euler approach is based on the two cardinal equations of the classical mechanics and is particularly well suited for those situations in which it is possible to exploit the recursive structure of the system so that the equations can be written in algorithmic way. It is the case, for example, of robot manipulators. For more details on this approach read Arnold (1978) and Siciliano et al. (2009) for applications to robotics.

In general, for complex systems, Lagrange and Hamilton approaches allow writing the equations of motion in a more straightforward way. Both of them are based on systematic equations which directly lead to the closed form equations of motion once the energy of the system has been determined. These methods are a direct consequence of a variational principle (Hamilton's principle), which states, in a nutshell, that the trajectories of a system can only be solutions when they are stationary points of a special integral function defined by the energy of the system (named the *action integral*). For the systems studied here, these stationary points should be minimums of this function, involving boundary conditions not directly tested with the stationary point of the action integral.

We propose to explore the Lagrange approach for a general system composed of n rigid bodies that are linked between them by means of massless connections. Such connections allow forces to be exchanged between bodies. To represent the dynamics of the system without singularities we will introduce and make use of the quaternions.

This paper is organized as follows: We begin by a section in which an algorithm to associate a quaternion to a bar starting from a vector which locates one endpoint of the bar with respect to the other (the bar vector) is presented. In order to maintain the paper self-contained, further details and properties about quaternions exploited to obtain the dynamic model can be found in the Appendix 1. In Section 3 we apply the Euler–Lagrange equations to a class-1 tensegrity structure using quaternions. Section 3.1 introduces the notation utilized and the framework of reference; in the Sections 3.2 and 3.3 we study, respectively, the kinetic and the potential energy. Finally, in Section 3.4 we deploy the equations of motion, first for a single bar and then for a complete tensegrity structure, differentiating two forms: a vectorial and a matrix form (some calculations and useful properties are collected in Appendix 2). In Section 4 we present some simulation results aimed to show possible applications of the model proposed and to provide an empirical proof of its correctness. Last, Section 5 contains some conclusions and hints about possible future developments.

2. Three-dimensional rotations

In this paragraph we will focus on unit quaternions, with the aim to give an example of how they can be exploited to define the orientation of a rigid body. Please refer to Appendix 1 for further details on quaternions and used notation. A unit quaternion $\underline{\boldsymbol{e}}$ can be written as follows:

$$\underline{\boldsymbol{e}} = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \cdot \boldsymbol{u} \end{pmatrix} \tag{1}$$

where $\boldsymbol{u}$ represents a versor (unit vector). From Eq. (1), it is easy to see that a unit quaternion contains information about a versor and an angle, which may be thought as a rotation angle around that versor ($\theta \in [-\pi, \pi]$). Roughly speaking, a unit quaternion specifies an axis and the rotation around it, hence we can also say that it represents a new reference framework once a base reference frame (with respect to which the vector $\boldsymbol{u}$ is expressed) is fixed. The framework it represents is the result of a rotation of the base frame around the $\boldsymbol{u}$ vector of the θ angle.

Let now take into account a rigid body in the space. Say, for simplicity, and to anticipate the discussions of the next sections concerning tensegrity structures, this body is a simple bar. We want to represent the orientation of a bar in the space by means of a quaternion. We will neglect rotations of the bar around its axis (so we finally have five degrees of freedom in a 5-dimensional coordinate space). To associate a quaternion to a bar means associating it to a relative reference frame: a problem with infinite solutions. Hence, there are infinite ways to associate a quaternion to a bar, while there is only one solution to the reverse problem (a quaternion uniquely identify one orientation).

Let $\underline{\boldsymbol{r}}^0$ be a vector quaternion represented in the inertial reference frame $O - x_0y_0z_0$ (the base frame from now on), and let $\underline{\boldsymbol{r}}^1$ be the representation of the same vector quaternion in another frame $O - x_1y_1z_1$. Let $\underline{\boldsymbol{e}}$ be the quaternion representing the orientation of the frame 1 with respect to the base frame, then, it is known that

$$\underline{\boldsymbol{r}}^0 = \underline{\boldsymbol{e}} \odot \underline{\boldsymbol{r}}^1 \odot \underline{\boldsymbol{e}}^* \tag{2}$$

The quaternion $\underline{\boldsymbol{e}}^*$ represents the orientation of the base frame in the frame $O - x_1y_1z_1$.

Let P_1 and P_2 be the coordinates of the extremes of the bar expressed in the base frame. The bar geometric vector, in the base frame, is defined by:

$$\vec{\boldsymbol{b}} = \vec{\boldsymbol{p}}_2 - \vec{\boldsymbol{p}}_1 \tag{3}$$

where $\vec{\boldsymbol{p}}_1$ and $\vec{\boldsymbol{p}}_2$ are the geometric vectors associated to the points P_1 and P_2. The quaternion associated to the bar in the base frame is:

$$\underline{\boldsymbol{b}}^0 = \begin{pmatrix} 0 \\ \boldsymbol{b} \end{pmatrix} = \begin{pmatrix} 0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{4}$$

where $\boldsymbol{b}$ is the vector associated to the bar meant as element of a vector space (see Fig. 1).

Suppose now to associate a frame to the bar with the origin in the center and one of the main axis directed like the bar; let it be,

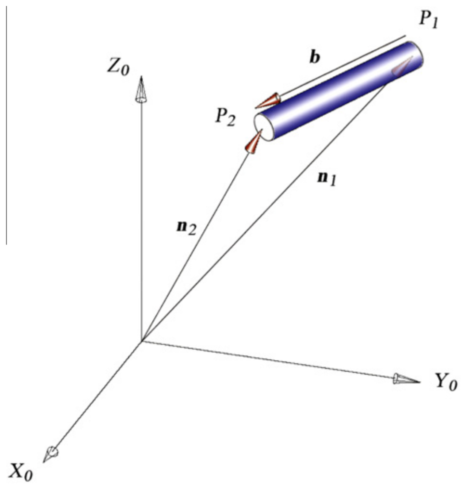


Fig. 1. A single bar in the space.

for example, the x axis. In this frame the vector quaternion associated to the bar is:

$$\underline{\boldsymbol{b}}^1 = \begin{pmatrix} 0 \\ l \\ 0 \\ 0 \end{pmatrix} \tag{5}$$

where l is length of the bar: $\|\boldsymbol{p}_2 - \boldsymbol{p}_1\|$.

From Eq. (2), we can write:

$$\underline{\boldsymbol{b}}^0 = \underline{\boldsymbol{e}} \odot \underline{\boldsymbol{b}}^1 \odot \underline{\boldsymbol{e}}^* = \overset{+}{\mathcal{E}} \overset{+}{\beta}{}^1 \underline{\boldsymbol{e}}^* = \overset{+}{\mathcal{E}} \overline{\mathcal{E}}^T \underline{\boldsymbol{b}}^1 = \overline{\mathcal{E}}^T \overset{+}{\mathcal{E}} \underline{\boldsymbol{b}}^1 \tag{6}$$

The matrices $\overset{+}{\mathcal{E}}$ and $\overline{\mathcal{E}}$ are defined like $\overset{+}{\mathcal{Q}}$ and $\overline{\mathcal{Q}}$ in (A.1).

Eq. (6), in matrix form becomes:

$$\underline{\boldsymbol{b}}^0 = \begin{pmatrix} 0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \quad 0 \quad 0 \\ 0 & \\ 0 & \overset{+}{\boldsymbol{E}} \overline{\boldsymbol{E}}^T \\ 0 & \end{pmatrix} \begin{pmatrix} 0 \\ l \\ 0 \\ 0 \end{pmatrix} \tag{7}$$

where

$$\overset{+}{\boldsymbol{E}} = \left[-\underline{\boldsymbol{e}}_{1:3} \quad (e_0 \boldsymbol{I}_3 + \widetilde{\boldsymbol{E}}) \right] = \begin{pmatrix} -e_1 & e_0 & -e_3 & e_2 \\ -e_2 & e_3 & e_0 & -e_1 \\ -e_3 & -e_2 & e_1 & e_0 \end{pmatrix} \tag{8}$$

$$\overline{\boldsymbol{E}}^T = \left[-\underline{\boldsymbol{e}}_{1:3} \quad (e_0 \boldsymbol{I}_3 - \widetilde{\boldsymbol{E}}) \right]^T = \begin{pmatrix} -e_1 & -e_2 & -e_3 \\ e_0 & -e_3 & e_2 \\ e_3 & e_0 & -e_1 \\ -e_2 & e_1 & e_0 \end{pmatrix} \tag{9}$$

are two special matrices associated to the quaternion that represents the bar orientation. Observe that $\overset{+}{\boldsymbol{E}}{}^T$ and $\overline{\boldsymbol{E}}^T$ can be obtained, respectively, from $\overline{\mathcal{E}}$ and $\overset{+}{\mathcal{E}}$ suppressing the first column:

$$\begin{matrix} \overline{\mathcal{E}} & \overset{+}{\mathcal{E}} \\ \downarrow & \downarrow \\ \overset{+}{\boldsymbol{E}}{}^T & \overline{\boldsymbol{E}}^T \end{matrix} \tag{10}$$

From Eq. (7) we can write

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \overset{+}{\boldsymbol{E}} \overline{\boldsymbol{E}}^T \begin{pmatrix} l \\ 0 \\ 0 \end{pmatrix} \tag{11}$$

which is equivalent to the following non-linear system:

$$\begin{cases} b_1 = l(e_0^2 + e_1^2 - e_2^2 - e_3^2) \\ b_2 = 2l(e_0 e_3 + e_1 e_2) \\ b_3 = 2l(e_1 e_3 - e_0 e_2) \end{cases} \tag{12}$$

System (12) allows to determine the components of $\boldsymbol{b}$ given a quaternion $\underline{\boldsymbol{e}}$. To find a solution of the reverse problem, $\underline{\boldsymbol{e}} = f(\boldsymbol{b})$, we must also add the constraint equation

$$e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1 \tag{13}$$

The resulting system is made of four equations in four unknowns, is non-linear and generally non-invertible, unless some hypothesis are made (for the reasons above discussed the problem has infinite solutions).

A way to find a solution is to put one component of the vector part of $\underline{\boldsymbol{e}}$ to zero. This fact corresponds to fix the vector $\boldsymbol{u}$ of Eq. (1) in a plane defined by two main axis of the base frame. Adding the constraint $u_x = 0$ or $u_y = 0$ or $u_z = 0$, we obtain a system of five equations in four unknowns and it may be proven that it always admits one solution, with the exception of one case.[2] Such case is a singularity of the general solution and corresponds to a special orientation of the bar. There is a different singularity for every choice of the additional constraint on $\boldsymbol{u}$. For all singularities, the unit quaternion is however easily determinable: all components of $\underline{\boldsymbol{e}}$ will be zero, unless one, which will be 1 or -1.

Let us now show the algorithm introduced with an example. Consider the system (12) with the constraint (13). Suppose to add the constraint equation $e_1 = 0$. It corresponds to put $u_x = 0$, i.e., to choose the vector $\boldsymbol{u}$, which defines the direction around which rotate the base frame to obtain the frame associated to the bar, in the yz plane. Furthermore, suppose to fix the origin of the bar frame on the center of the bar and to orient the x axis along the bar itself. The resulting system becomes:

$$\begin{cases} b_1 = l(e_0^2 - e_2^2 - e_3^2) \\ b_2 = 2l e_0 e_3 \\ b_3 = -2l e_0 e_2 \\ 1 = e_0^2 + e_2^2 + e_3^2 \end{cases} \tag{14}$$

Fig. 2 shows the solution for a given orientation of a bar. There are infinite frames with the x axis oriented along the bar and the origin coinciding with one of the extreme points. One of such frames corresponds to a rotation of the base frame around a vector in the yz plane (of the base plane). In the figure, the frame $O_1 - x_1 y_1 z_1$ (in red)[3] is the frame associated to the red versor $\boldsymbol{u}$. The analytical solution can be found as follows.

From the first and the fourth of (14) we have

$$2e_0^2 - 1 = \frac{b_1}{l}$$

which implies

$$\boxed{e_0 = \pm\sqrt{\frac{b_1 + l}{2l}}} \tag{15}$$

From the second we have

$$e_3 = \frac{b_2}{2l e_0}$$

[2] Observe that a non-linear algebraic system, in general, may admit several solutions even if the number of unknowns is less than the number of equations.

[3] For interpretation of color in Figs. 1-9,11,12, the reader is referred to the web version of this article.

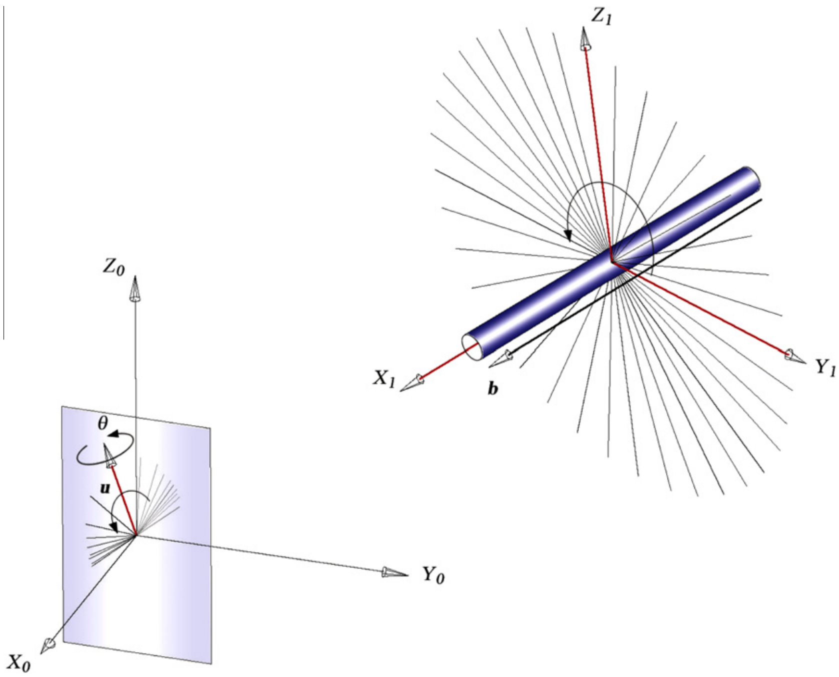


Fig. 2. Correspondence between rotations around the x_1 axis of the frame $O_1 - x_1y_1z_1$ and rotations around the versor $\boldsymbol{u}$ of the frame $O_0 - x_0y_0z_0$.

which, with Eq. (15), gives

$$\boxed{e_3 = \pm\frac{b_2}{\sqrt{2l(b_1+l)}}} \tag{16}$$

Finally, from the third we obtain

$$e_2 = -\frac{b_3}{2le_0}$$

which, together with Eq. (15), gives

$$\boxed{e_2 = \mp\frac{b_3}{\sqrt{2l(b_1+l)}}} \tag{17}$$

Eqs. (15), (17), (16) with

$$\boxed{e_1 = 0} \tag{18}$$

are a solution of the system made of (12) and of Eq. (13): it can be directly proven substituting the solution found into the equations.

The solution found has a singularity for $b_1 = -l$, i.e., for the case in which the bar is oriented in the opposite direction of the x axis of the base frame. In such case, the orientation of the bar frame is represented by the following unit quaternion:

$$\underline{\boldsymbol{e}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \pm 1 \end{pmatrix}$$

There is a singularity in any case, i.e., whichever is the component of the vector part of the unit quaternion that we choose to put equal to zero. In general, if we choose to align the x axis of the bar fixed frame with the bar itself, depending on which component of $\underline{\boldsymbol{e}}$ we put to zero, we have the following cases:

$$e_1 = 0 \quad \Rightarrow \quad \boldsymbol{b} = \begin{pmatrix} -l \\ 0 \\ 0 \end{pmatrix}$$

$$e_2 = 0, \quad e_3 = 0 \quad \Rightarrow \quad \boldsymbol{b} = \begin{pmatrix} l \\ 0 \\ 0 \end{pmatrix}$$

The singularity depends on the way we choose to orient the bar fixed frame with respect to the bar and on which component, of the vector part of $\underline{\boldsymbol{e}}$, we put equal to zero. If we choose a bar fixed frame oriented with the y axis directed along the bar, i.e. $\underline{\boldsymbol{b}}^1 = (0\ 0\ l\ 0)^T$ the system (12) becomes:

$$\begin{cases} b_1 = 2l(e_1e_2 - e_0e_3) \\ b_2 = l(e_0^2 + e_2^2 - e_1^2 - e_3^2) \\ b_3 = 2l(e_0e_1 + e_2e_3) \end{cases} \tag{19}$$

and the singularities occur for:

$$e_1 = 0 \quad \Rightarrow \quad \boldsymbol{b} = \begin{pmatrix} 0 \\ -l \\ 0 \end{pmatrix}$$

$$e_1 = 0, \quad e_3 = 0 \quad \Rightarrow \quad \boldsymbol{b} = \begin{pmatrix} 0 \\ l \\ 0 \end{pmatrix}$$

3. Euler–Lagrange equations for a tensegrity structure

In this section we will focus on the so called *class-1* tensegrity structures. We will deploy a dynamic model making use of quaternions to describe the orientation of the bars.

3.1. The framework and the notations

Consider a class-1 tensegrity system where the struts are rigid bars and the tensors are cables. We admit, for simplicity, that the bars have a radial dimension negligible with respect to their length. On the basis of this hypothesis, we will model the bars as straight lines in the space. For this purpose we will need three independent parameters to describe the position and two independent parameters to describe the orientation.

Before introducing the generalized coordinates and the notation, let us insert here a brief digression concerning terminology and conventions. To avoid confusion and help the reader, we recall some definitions from geometry and algebra.

With the term *vector* we mean an element of a vector space. To avoid confusion, the oriented segment joining two points in $\mathbb{R}^3$ will be called *geometric vector*. The set of all geometric vectors, considered applied at the origin of a reference frame, together with the product of a geometric vector by a scalar and the sum of two geometric vectors is also a vector space. Often, it is easy to get confusion between the Euclidean space and the vector space of geometric vectors. Points of the Euclidean space and geometric vectors representing such points are different entities.

Displacements in $\mathbb{R}^3$ are geometric vectors. Usually, position vectors (and therefore displacements) are considered as applied vectors, while velocities and accelerations (both, angular or linear) are not.

A position vector is said to be *referred to* a frame when it is meant as applied to the origin of such frame.

The transformation applied by a rotation matrix to a vector changes the frame in which it is *expressed*, but not the frame which the vector *refers* to. Consider Fig. 3, the vector $\boldsymbol{p}^1$ is the position vector associated to the point P, referred to the frame *1* and expressed in the same frame. If $\boldsymbol{R}_1^0$ is the rotation matrix representing the orientation of the frame *1* with respect to the frame 0, $\boldsymbol{R}_1^0\boldsymbol{p}^1$ represents the position vector of the point P expressed in the frame 0 but still referring the frame 1. Besides, $\boldsymbol{p}^0 = \boldsymbol{o}_1^0 + \boldsymbol{R}_1^0\boldsymbol{p}^1$ is the vector position associated to the point P expressed in the frame 0 and referred in the same frame. The quantities $\boldsymbol{p}^0$ and $\boldsymbol{p}^1$ are meant as elements of the vector space of all geometric vectors of $\mathbb{R}^3$. This is the reason why we write them without the arrow. The two reference frames, meant coincident in the origin, are the geometric equivalent of a base change in the vector space. In Fig. 3 we associate to the geometric vectors (the arrows) their equivalent vectors meant as elements of a vector space. A superscript is typically used to put in evidence the frame in which a variable is expressed, but sometimes, to simplify the notation, we will omit the superscript to indicate that a variable is expressed in the base frame.

Let us now turn talking about tensegrities. To describe the system we will start introducing the description of a single bar. Let $O - x_0y_0z_0$ be an inertial reference frame (the base frame) and let P_1 and P_2 be the coordinates of the extremes of the bar and $\vec{\boldsymbol{p}}_1$ and $\vec{\boldsymbol{p}}_2$ be the geometric vectors associated to the points P_1 and P_2 (see Fig. 1). Let us now associate a frame to the bar like described in Section 2 (see also Fig. 2): the origin is in the center and the x axis is directed along the bar itself. Let $\underline{\boldsymbol{e}}$ be the unitary quaternion representing the orientation of the bar-fixed frame (and hence also of the bar). The bar may be completely described by means of $\underline{\boldsymbol{e}}$ and by the position of anyone of its points. We choose the center of mass P_c (whose associated vector will be denoted with $\boldsymbol{c}$) which can be easily computed starting from P_1 and P_2:

$$P_c = \frac{1}{2}(P_1 + P_2) \quad \Rightarrow \quad \boldsymbol{c} = \frac{1}{2}(\boldsymbol{p}_1 + \boldsymbol{p}_2)$$

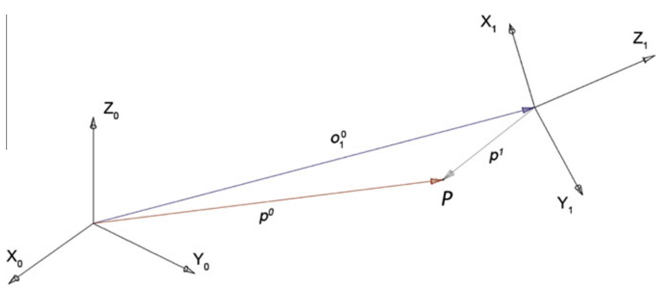


Fig. 3. Reference frames.

On the base of what introduced for a single bar, we define in the following the notation needed to describe a class-1 tensegrity system. We will introduce a double notation, to be able to present two forms of the dynamic model: a matrix form (compact form) and a classical vector form, both based on the Lagrange equations. The former is useful because compact and elegant, the latter, allows to manage the model with vector fields. Therefore, for each quantity we introduce a matrix notation and a vector notation.

For the positions of the nodes and the bars we define

$$\boldsymbol{N} = (\boldsymbol{N}_1 \; \boldsymbol{N}_2) = (\boldsymbol{n}_1 \cdots \boldsymbol{n}_{2n_b})$$

nodes row matrix, $3 \times 2n_b$ where n_b is the number of the bars. Bars are defined as $\boldsymbol{b}_i = \boldsymbol{n}_{i+n_b} - \boldsymbol{n}_i$ where $\boldsymbol{n}_{i+n_b} \in \boldsymbol{N}_2$ and $\boldsymbol{n}_i \in \boldsymbol{N}_1$

$$\boldsymbol{v} = \mathrm{vec}(\boldsymbol{N}) = \begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{n}_1 \\ \vdots \\ \boldsymbol{n}_{2n_b} \end{pmatrix}$$

nodes column matrix, $6n_b \times 1$

$$\boldsymbol{B} = (\boldsymbol{b}_1 \cdots \boldsymbol{b}_{n_b}) = \boldsymbol{N}\begin{pmatrix} -\boldsymbol{I}_{n_b} \\ \boldsymbol{I}_{n_b} \end{pmatrix}$$

bar row matrix, $3 \times n_b$

$$\boldsymbol{\beta} = \mathrm{vec}(\boldsymbol{B}) = (-\boldsymbol{I}_{3n_b} \; \boldsymbol{I}_{3n_b})\boldsymbol{v}$$

bar column matrix, $3n_b \times 1$

In the above definitions, the symbol vec represents the operator which put in one column all the elements of its column arguments (see (A.27)).

To describe the connections made by the cables (also called strings) we define

$$\boldsymbol{C}_{con} = [c_{i,j}]$$

$$c_{i,j} = \begin{cases} 1 & \text{if the string terminates on the node } n_j \\ -1 & \text{if the string originates from the node } n_j \\ 0 & \text{if the string does not concern node } n_j \end{cases}$$

string connectivity matrix, $n_s \times 2n_b$ where n_s is the number of the strings

and, to handle the connections in the vector form, we also define

$$\mathcal{C}_{con} = [c'_{i,j}]$$

$$c'_{i,j} = \begin{cases} \boldsymbol{I}_3 & \text{if the string terminates on the node } n_j \\ -\boldsymbol{I}_3 & \text{if the string originates from the node } n_j \\ \boldsymbol{0}_{3\times 3} & \text{if the string does not concern node } n_j \end{cases}$$

general connectivity matrix, $3n_s \times 6n_b$

The cables are represented by a string vectors:

$$\boldsymbol{S} = (\boldsymbol{s}_1 \cdots \boldsymbol{s}_{n_s}) = \boldsymbol{N}\boldsymbol{C}_{con}^T$$

string matrix, $3 \times n_s$

$$\boldsymbol{\sigma} = \mathrm{vec}(\boldsymbol{S}) = \begin{pmatrix} \boldsymbol{s}_1 \\ \vdots \\ \boldsymbol{s}_{n_s} \end{pmatrix} = \mathcal{C}_{con}\boldsymbol{v}$$

$3n_s \times 1$

To model the effects of undesired interactions with the environment, we consider the presence of some disturbances applied at the extreme of the bars:

$\boldsymbol{D} = [d_{i,j}]$

$d_{i,j} = \begin{cases} 1 & \text{if the disturbance is applied to the node } n_j \\ 0 & \text{otherwise} \end{cases}$ *disturbance connectivity matrix*, $n_w \times 2n_b$ where n_w is the number of external forces applied to the nodes of the structure. Each row only contains one non-zero element.

$\mathcal{D} = \left[d'_{i,j}\right]$

$d'_{i,j} = \begin{cases} \boldsymbol{I}_3 & \text{if disturbance is applied to the node } n_j \\ \boldsymbol{0}_{3\times3} & \text{otherwise} \end{cases}$ *general disturbance connectivity matrix*, $3n_w \times 6n_b$

$\boldsymbol{\Psi} = (\boldsymbol{\psi}_1 \cdots \boldsymbol{\psi}_{n_w})$ *disturbances matrix*, $3 \times n_w$; the set of all external forces acting on the system

$\zeta = \text{vec}(\boldsymbol{\Psi}) = \begin{pmatrix} \boldsymbol{\psi}_1 \\ \vdots \\ \boldsymbol{\psi}_{n_w} \end{pmatrix}$ $3n_w \times 1$

Fig. 4 shows some of the quantities above introduced.

We also define the tension vectors, which represent the forces applied by the cables to the bars. We have a tension vector for each couple of cable-bar connected. Ultimately, we have $2n_s$ tensions applied to the bars. From the concept of 'force coefficient' we define as follows the tension applied by the cable i to the node j:

$$\boldsymbol{t}_{i,j} = -\boldsymbol{C}(i,j)\gamma_i\|\boldsymbol{s}_i\|\frac{\boldsymbol{s}_i}{\|\boldsymbol{s}_i\|} \tag{20}$$

The force coefficients γ_i, for an element in tension, must be always positive, hence, in Eq. (20) it is necessary to take into account the coefficient $\boldsymbol{C}(i,j)$ to coherently modify the sign of the vector. This comes from the fact that, for each cable, the strings have a fixed direction (even if arbitrary): the same for both nodes connected by the tendon. The coefficient $\boldsymbol{C}(i,j)$ may be +1 or -1, depending on the fact that the string associated to the cable, respectively, terminates or originates from the node j and this is why we need to correct the sign of a tension vector with respect to the string vector which it refers to.

Let us define

$\boldsymbol{\Gamma} = \text{diag}\{\gamma_1 \cdots \gamma_{n_s}\}$ *force density matrix*, $n_s \times n_s$; where γ_i are the force density coefficients: forces for length unit

$\boldsymbol{\Gamma}' = \text{diag}\{\gamma_1, \gamma_1, \gamma_1 \ldots \gamma_{n_s}, \gamma_{n_s}, \gamma_{n_s}\}$ *general force density matrix*, $3n_s \times 3n_s$

$\boldsymbol{F} = (\boldsymbol{F}_1\ \boldsymbol{F}_2) = (\boldsymbol{f}_1 \cdots \boldsymbol{f}_{2n_b})$ *forces matrix*, $3 \times 2n_b$; the set of all resultant forces applied to the nodes: include tensions and external forces

$\boldsymbol{\phi} = \text{vec}(\boldsymbol{F}) = \begin{pmatrix} \boldsymbol{\phi}_1 \\ \boldsymbol{\phi}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_1 \\ \vdots \\ \boldsymbol{f}_{2n_b} \end{pmatrix}$ $6n_b \times 1$

It is easy to prove that:

$$\boldsymbol{F} = -\boldsymbol{N}\boldsymbol{C}_{con}^T\boldsymbol{\Gamma}\boldsymbol{C}_{con} + \boldsymbol{\Psi}\boldsymbol{D} \tag{21}$$

$$\boldsymbol{\phi} = -\mathcal{C}_{con}^T\boldsymbol{\Gamma}'\mathcal{C}_{con}\boldsymbol{\nu} + \mathcal{D}^T\zeta \tag{22}$$

Observe the definition of the external forces and of the corresponding matrix $\boldsymbol{D}$: the external forces may be directed anyway, not necessary along the cables.

Adopting the elastic model for the cables, from the equations of a spring, we can also write:

$$\boldsymbol{t}_{i,j} = -k_i\boldsymbol{C}(i,j)(\|\boldsymbol{s}_i\| - \|\boldsymbol{s}_{i0}\|)\frac{\boldsymbol{s}_i}{\|\boldsymbol{s}_i\|} \tag{23}$$

where k_i is the stiffness of the cable, $\|\boldsymbol{s}_i\|$ the actual length and $\|\boldsymbol{s}_{i0}\|$ the rest length.

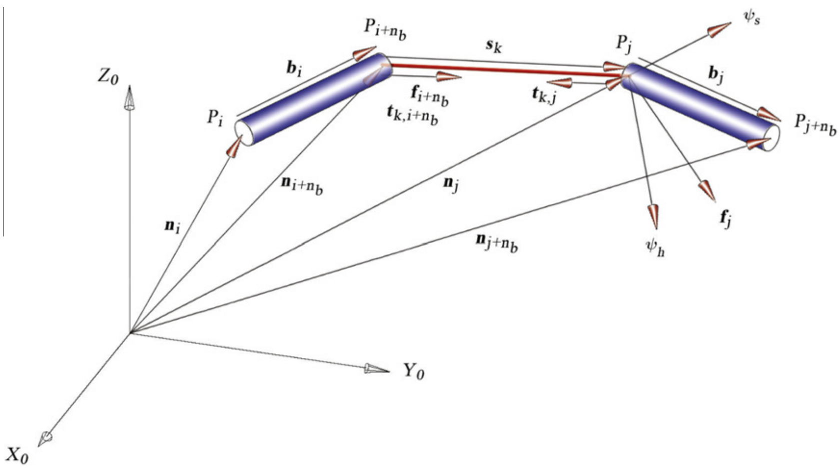


Fig. 4. Definitions of bar vectors, node vectors and forces.

Comparing Eq. (20) with Eq. (23) we obtain:

$$k_i(\|\boldsymbol{s}_i\| - \|\boldsymbol{s}_{i0}\|) = \gamma_i \|\boldsymbol{s}_i\| \tag{24}$$

In (21) the matrix $\boldsymbol{C}_{con}$ is constant and well known (it depends on the mechanical connections). $\boldsymbol{\Psi}$ and $\boldsymbol{D}$ are well known as well, even if they may be time varying. Hence, in (21), the unknown variables are the positions of the bars nodes and $\boldsymbol{\Gamma}$, which is function of the cables length and, therefore, function of $\boldsymbol{N}$. From (24) it results

$$\gamma_i = k_i\left(1 - \frac{\|\boldsymbol{s}_{i0}\|}{\|\boldsymbol{s}_i\|}\right) \tag{25}$$

Finally we define

$\boldsymbol{K} = \mathrm{diag}\{k_i\}$	diagonal matrix of all cables stiffness coefficients, $n_s \times n_s$
$\boldsymbol{S}_0 = \mathrm{diag}\{\|\boldsymbol{s}_{i0}\|\}$	matrix of the cables rest lengths, $n_s \times n_s$

In the following we will treat the forces exerted by the cables like non conservative forces, even if we defined them on the base of the elastic model (they come from a potential function). The reason for this choice is twofold: first, in this way the final equations remain valid even if the model adopted to describe the cables changes, and, second, we will be able to consider the cables (or some of them) as actuated without the need to change anything. In this case the tensions on the cables do not necessary obey more to a model, they are directly imposed, for example, by means of engines and therefore they really are non-conservative.

3.2. Kinetic energy

The kinetic energy of a tensegrity is the sum of the kinetic energy of the single bars. Accordingly, we will write first the kinetic energy for a single bar, which is, itself, sum of two components: the energy due to the translational motion and the component related to a rotational motion.

To write the kinetic energy as a function of the center of mass and the quaternion representing the orientation of the bar, we need to introduce first the *complementary kinetic energy*. Let $\boldsymbol{c}_i$ be the center of mass (expressed in the base frame) and $\boldsymbol{\omega}_i^1$ be the angular velocity vector (expressed in the body-fixed frame) of the *i*th bar. The complementary kinetic energy is defined as (A.26)

$$T_i^* = \frac{1}{2} m_i \dot{\boldsymbol{c}}_i^T \dot{\boldsymbol{c}}_i + \frac{1}{2} \omega_i^{1\,T} \boldsymbol{J}_{3,i}^1 \omega_i^1 \tag{26}$$

$$= \frac{1}{2} m_i \dot{\boldsymbol{c}}_i^T \dot{\boldsymbol{c}}_i + \frac{1}{2} \underline{\boldsymbol{\omega}}_i^{1\,T} \boldsymbol{J}_{4,i}^1 \underline{\boldsymbol{\omega}}_i^1 \tag{27}$$

where m_i is the mass of the bar and $\boldsymbol{J}_{3,i}^1$ is the central tensor of inertia (i.e. the tensor of inertia expressed in a frame positioned in the center of mass). The subscript '3' is used to distinguish this matrix from the version valid with quaternions ($\boldsymbol{J}_{4,i}^1$), which is just a 4 by 4 matrix. Eq. (27) allows to express the complementary kinetic energy making use of quaternions in place of 3-dimensional vectors. The matrix $\boldsymbol{J}_{4,i}^1$ is defined as (Chou, 1992):

$$\boldsymbol{J}_{4,i}^1 = \begin{pmatrix} \frac{1}{2} trace(\boldsymbol{J}_{3,i}^1) & 0 \\ 0 & \boldsymbol{J}_{3,i}^1 \end{pmatrix}$$

where the operator $trace(\boldsymbol{A})$ is the sum of the elements on the main diagonal of the matrix $\boldsymbol{A}$. At last, in the hypothesis that the bar is a solid, regular cylinder (not hollow), we easily compute the central tensor of inertia, considering that all the elements outside the main diagonal (products of inertia) are null and that the elements on the main diagonal simply are the moments of inertia of the bar with respect to the main axis (of the bar-fixed frame):

$$\boldsymbol{J}_{3,i}^1 = \begin{pmatrix} \frac{1}{2} m_i r_i^2 & 0 & 0 \\ 0 & \frac{1}{12} m_i l_i^2 & 0 \\ 0 & 0 & \frac{1}{12} m_i l_i^2 \end{pmatrix} \tag{28}$$

where r_i is the radius of the bar section and l_i is the length of the bar.

Eq. (27) is commonly called *kinetic energy* but strictly speaking it corresponds to the definition of the *complementary kinetic energy* (see Tabarrok and Rimrott, 1994; Rimrott et al., 1993). For mechanical scleronomic systems[4] it happens that the two functions have the same values when computed on the same state, even if this is not true in general. The complementary function may be useful to compute some partial differentiations with respect to variables that do not explicitly appear in the kinetic energy function. For our purposes, it will be useful to write the Lagrange's equations in function of the quaternions $\underline{\boldsymbol{e}}$.

Substituting (A.26) in (26), (A.22) in (27) we obtain $T_i^* = T_i^*(\dot{\boldsymbol{c}}_i, \underline{\boldsymbol{e}}_i, \dot{\underline{\boldsymbol{e}}}_i)$:

$$T_i^* = \frac{1}{2} m_i \dot{\boldsymbol{c}}_i^T \dot{\boldsymbol{c}}_i + 2\dot{\underline{\boldsymbol{e}}}_i^T \overline{\boldsymbol{E}}_i^T J_{3,i}^1 \overline{\boldsymbol{E}}_i \dot{\underline{\boldsymbol{e}}}_i = \frac{1}{2} m_i \dot{\boldsymbol{c}}_i^T \dot{\boldsymbol{c}}_i + 2\underline{\boldsymbol{e}}_i^T \dot{\overline{\boldsymbol{E}}}_i^T J_{3,i}^1 \dot{\overline{\boldsymbol{E}}}_i \underline{\boldsymbol{e}}_i$$

$$= \frac{1}{2} m_i \dot{\boldsymbol{c}}_i^T \dot{\boldsymbol{c}}_i + 2\dot{\underline{\boldsymbol{e}}}_i^T \overset{+}{\mathcal{E}} \boldsymbol{J}_{4,i}^1 \overset{+}{\mathcal{E}}_i^T \dot{\underline{\boldsymbol{e}}}_i = \frac{1}{2} m_1 \dot{\boldsymbol{c}}_i^T \dot{\boldsymbol{c}}_i + 2\underline{\boldsymbol{e}}_i^T \dot{\overset{+}{\mathcal{E}}} \boldsymbol{J}_{4,i}^1 \dot{\overset{+}{\mathcal{E}}}_i^T \underline{\boldsymbol{e}}_i$$

The kinetic energy can be obtained from the complementary function applying the Legendre transformation. Let define the generalized momenta

$$\boldsymbol{p}_i = \left(\frac{\partial T_i^*}{\partial \dot{\boldsymbol{c}}_i}\right)^T = m_i \dot{\boldsymbol{c}}_i \tag{29}$$

$$\underline{\boldsymbol{g}}_i = \left(\frac{\partial T_i^*}{\partial \dot{\underline{\boldsymbol{e}}}_i}\right)^T = 4\overset{+}{\mathcal{E}} \boldsymbol{J}_{4,i}^1 \overset{+}{\mathcal{E}}_i^T \dot{\underline{\boldsymbol{e}}} = 4\overline{\boldsymbol{E}}_i^T J_{3,i}^1 \overline{\boldsymbol{E}}_i \dot{\underline{\boldsymbol{e}}}_i \tag{30}$$

Let $\boldsymbol{q}_i$ be the vector of the generalized coordinates:

$$\boldsymbol{q}_i = \begin{pmatrix} \boldsymbol{c}_i \\ \underline{\boldsymbol{e}}_i \end{pmatrix} \tag{31}$$

It results $T_i^* = T_i^*(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i)$. Consider now the complementary kinetic energy only as function of $\dot{\boldsymbol{q}} : T^* = T^*(\dot{\boldsymbol{q}})$. The definition of the kinetic energy (coming from the application of the Legendre transformation) is (see Arnold, 1978; Shivarama and Fahrenthold, 2004):

$$T = \boldsymbol{x}^T \dot{\boldsymbol{q}} - T^*(\dot{\boldsymbol{q}})$$

where $\boldsymbol{x} = \left(\frac{\partial T^*}{\partial \dot{\boldsymbol{q}}}\right)^T$. Finally, we have

$$T_i = T_i(\boldsymbol{p}_i, \underline{\boldsymbol{e}}_i, \underline{\boldsymbol{g}}_i) = \boldsymbol{p}_i^T \dot{\boldsymbol{c}}_i + \underline{\boldsymbol{g}}_i^T \dot{\underline{\boldsymbol{e}}}_i - T_i^* \tag{32}$$

$$= \frac{1}{2m_i} \boldsymbol{p}_i^T \boldsymbol{p}_i + \frac{1}{8} \underline{\boldsymbol{g}}_i^T \overline{\boldsymbol{E}}_i^T J_{3,1}^{1\;-1} \overline{\boldsymbol{E}}_i \underline{\boldsymbol{g}}_i$$

$$= \frac{1}{2m_i} \boldsymbol{p}_i^T \boldsymbol{p}_i + \frac{1}{8} \underline{\boldsymbol{g}}_i^T \overset{+}{\mathcal{E}} \boldsymbol{J}_{4,i}^{1\;-1} \overset{+}{\mathcal{E}}_i^T \underline{\boldsymbol{g}}_i \tag{33}$$

This last expression, is particularly useful, since it allows to explicit the dependency of the kinetic energy from the quaternion $\underline{\boldsymbol{e}}$. At last, if n_b is the number of the bars of a tensegrity structure, then the total kinetic energy is

$$T = \sum_{i=1}^{n_b} T_i \tag{34}$$

[4] Are said scleronomics the systems for which the transformation equations between cartesian coordinates and generalized coordinates do not explicitly depends on time. A scleronomic constraint is a time-independent constraint, opposite to a rheonomic constraint, which is a time varying constraint.

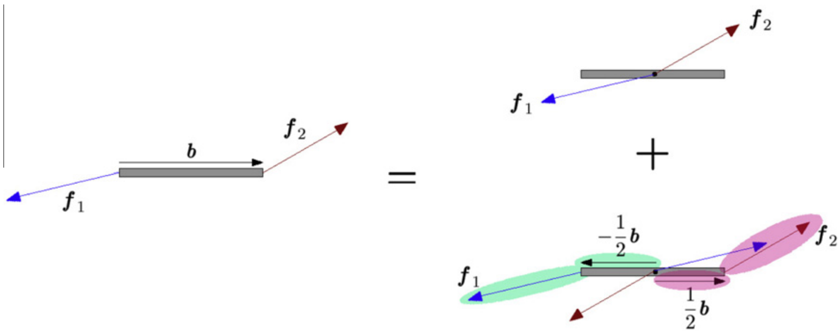


Fig. 5. Forces equivalent frames.

Let us compute now the total differential of both T_i and T_i^*:

$$dT_i^* = \left(\frac{\partial T_i^*}{\partial \dot{\boldsymbol{c}}_i}\right)d\dot{\boldsymbol{c}}_i + \left(\frac{\partial T_i^*}{\partial \underline{\boldsymbol{e}}_i}\right)d\underline{\boldsymbol{e}}_i + \left(\frac{\partial T_i^*}{\partial \dot{\underline{\boldsymbol{e}}}_i}\right)d\dot{\underline{\boldsymbol{e}}}_i$$
$$= \boldsymbol{p}_i^T d\dot{\boldsymbol{c}}_i + \left(\frac{\partial T_i^*}{\partial \underline{\boldsymbol{e}}_i}\right)d\underline{\boldsymbol{e}}_i + \underline{\boldsymbol{g}}_i^T d\dot{\underline{\boldsymbol{e}}}_i \tag{35}$$

differentiating (32) and exploiting Eq. (35) we have

$$dT_i = \boldsymbol{p}_i^T d\dot{\boldsymbol{c}}_i + \dot{\boldsymbol{c}}_i^T d\boldsymbol{p}_i + \underline{\boldsymbol{g}}_i^T d\dot{\underline{\boldsymbol{e}}}_i + \dot{\underline{\boldsymbol{e}}}_i^T d\underline{\boldsymbol{g}}_i - dT_i^*$$
$$= \dot{\boldsymbol{c}}_i^T d\boldsymbol{p}_i + \dot{\underline{\boldsymbol{e}}}_i^T d\underline{\boldsymbol{g}}_i - \left(\frac{\partial T_i^*}{\partial \underline{\boldsymbol{e}}_i}\right)d\underline{\boldsymbol{e}}_i \tag{36}$$

while differentiating (33) we find

$$dT_i = \left(\frac{\partial T_i}{\partial \boldsymbol{p}_i}\right)d\boldsymbol{p}_i + \left(\frac{\partial T_i}{\partial \underline{\boldsymbol{e}}_i}\right)d\underline{\boldsymbol{e}}_i + \left(\frac{\partial T_i}{\partial \underline{\boldsymbol{g}}_i}\right)d\underline{\boldsymbol{g}}_i \tag{37}$$

Finally, comparing (36) with (37) we obtain a very important property, that we will exploit later onto determine the Lagrange's equations:

$$\frac{\partial T_i}{\partial \underline{\boldsymbol{e}}_i} = -\frac{\partial T_i^*}{\partial \underline{\boldsymbol{e}}_i} \tag{38}$$

3.3. Potential energy and non-conservative forces

From now on we will consider the actions exerted by the cables like external non-conservative forces applied to the extremes of the bars (see Subsection 3.1). The potential energy is hence due only to the gravitational field. If the z axis of the base frame is oriented in the vertical direction (upward direction), the potential energy function of a single bar is:

$$V_i = V_i(\boldsymbol{c}_i) = -m_i \boldsymbol{g}_0^T \boldsymbol{c}_i \tag{39}$$

where $\boldsymbol{g}_0^T = (\,0 \quad 0 \quad -g\,)$ and the potential function of the overall system is:

$$V = \sum_{i=1}^{n_b} V_i \tag{40}$$

To be able to apply the Euler–Lagrange's equations we also need to define the non-conservative forces and moments acting on the system. Referring to Fig. 5, in hypothesis that the only external force is the friction, here modeled as viscous friction, the total force applied to the center of mass of a single bar is the vectorial sum of all forces applied at the extremes plus the gravity and the friction:

$$\boldsymbol{f}_i = \boldsymbol{f}_{1,i} + \boldsymbol{f}_{2,i} - \left(\frac{\partial V_i}{\partial \boldsymbol{c}_i}\right)^T - b_{t,i}\dot{\boldsymbol{c}}_i \tag{41}$$

where $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ are the sum of the forces applied, respectively, to the first and to the second endpoint of the bar and $b_{t,i}$ is the coefficient of the viscous friction associated to the translational motion of the ith bar. In term of $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$, the resulting total moment responsible for rotations is:

$$\boldsymbol{\mu}_i = -\frac{1}{2}\boldsymbol{b}_i \times \boldsymbol{f}_{1,i} + \frac{1}{2}\boldsymbol{b}_i \times \boldsymbol{f}_{2,i} - b_{r,i}\boldsymbol{\omega}_i$$
$$= \frac{1}{2}\boldsymbol{b}_i \times (\boldsymbol{f}_{2,i} - \boldsymbol{f}_{1,i}) - b_{r,i}\boldsymbol{\omega}_i \tag{42}$$

where $b_{r,i}$ is the coefficient of the viscous friction associated to the rotational motion and $\boldsymbol{\omega}_i$ is the angular velocity of the bar computed in the base frame. Eqs. (41) and (42) are expressed in the base frame and refers to the cartesian coordinates. Starting from (42) we can determine now the generalized torques acting on quaternions. Let $\mathfrak{q}_i$ be the quasi-coordinates[5,6] associated with the components of the angular velocity vector:

$$\dot{\mathfrak{q}}_i^0 = \boldsymbol{\omega}_i^0$$

The virtual work made by the non-conservative forces is

$$\delta W_i = \left(\boldsymbol{f}_{1,i} + \boldsymbol{f}_{2,i} - b_{t,i}\dot{\boldsymbol{c}}_i\right)^T \delta\boldsymbol{c}_i + \boldsymbol{\mu}_i^T \delta\mathfrak{q}_i^0 \tag{43}$$

where

$$\delta\mathfrak{q}_i^0 = d\mathfrak{q}_i^0 = \boldsymbol{\omega}_i^0 dt = 2\overset{+}{\boldsymbol{E}}_i \delta\underline{\boldsymbol{e}}_i \tag{44}$$

From (43) and (44) we have

$$\delta W_i = \left(\boldsymbol{f}_{1,i} + \boldsymbol{f}_{2,i} - b_{t,i}\dot{\boldsymbol{c}}_i\right)^T \delta\boldsymbol{c}_i + \left(2\overset{+}{\boldsymbol{E}}_i^T \boldsymbol{\mu}_i\right)^T \delta\underline{\boldsymbol{e}}_i \tag{45}$$

hence the generalized torque acting on $\underline{\boldsymbol{e}}_i$ is $2\overset{+}{\boldsymbol{E}}_i^T \boldsymbol{\mu}_i$. Similar calculus can be done using quaternions in place of moment vectors for the rotational part of the kinetic energy. The variables can be referred to the base frame or to the body fixed frame:

$$\delta W_{rot,i} = \underline{\boldsymbol{\mu}}_i^{0\,T} \delta\underline{\mathfrak{q}}_i^0 = \underline{\boldsymbol{\mu}}_i^{1\,T} \delta\underline{\mathfrak{q}}_i^1 \tag{46}$$

where $\underline{\boldsymbol{\mu}}_i$ and $\underline{\mathfrak{q}}_i$ are the quaternions representing, respectively, the vectors $\boldsymbol{\mu}_i$ and $\mathfrak{q}_i$. Eq. (46) can be proven as follows. Considering that

$$\delta\underline{\mathfrak{q}}_i^0 = \underline{\boldsymbol{\omega}}_i^0 dt = 2\overline{\mathcal{E}}_i(\underline{\boldsymbol{e}}_i)^T \delta\underline{\boldsymbol{e}}_i \tag{47}$$

$$\delta\underline{\mathfrak{q}}_i^1 = \underline{\boldsymbol{\omega}}_i^1 dt = 2\overset{+}{\mathcal{E}}_i(\underline{\boldsymbol{e}}_i)^T \delta\underline{\boldsymbol{e}}_i \tag{48}$$

we have

$$\delta W_{rot,i} = \left(2\overline{\mathcal{E}}_i(\underline{\boldsymbol{e}}_i)\underline{\boldsymbol{\mu}}_i^0\right)^T \delta\underline{\boldsymbol{e}}_i = \left(2\overset{+}{\mathcal{E}}_i(\underline{\boldsymbol{e}}_i)\underline{\boldsymbol{\mu}}_i^1\right)^T \delta\underline{\boldsymbol{e}}_i \tag{49}$$

[5] Quasi-coordinates were introduced to derive equations of motion using the Boltzmann–Hamel equations. They are velocity coordinates which are not simply the time derivative of position coordinates (see, for example, Nejmark and Fufaev, 1972 or Papastavridis, 2002 for more details).

[6] Note the differences between $\mathfrak{q}$ and $\boldsymbol{q}$, which is the symbol used in (31) to refer to the set of generalized coordinates of a bar.

Recalling now that $\underline{\boldsymbol{\mu}}_i^1 = \boldsymbol{A}_{4,i}^{-1}\underline{\boldsymbol{\mu}}_i^0 = \overset{+}{\mathcal{E}}_i(\underline{\boldsymbol{e}}_i)^T\overline{\mathcal{E}}_i(\underline{\boldsymbol{e}}_i)\underline{\boldsymbol{\mu}}_i^0$, it can be easily shown that the last two expressions of Eq. (46) are the same. The expression referring to the base frame is, typically, more useful, because it requires to express the torque acting on the bar with respect to the base frame, in which it is naturally deduced from the forces expression. At last, observe that in Eq. (49), $\overline{\mathcal{E}}_i(\underline{\boldsymbol{e}}_i)$ is the square version of $\overset{+}{\boldsymbol{E}}_i^T$ in Eq. (45): the latter can be extracted from the former suppressing the first column (recall property (10)).

3.4. The Lagrange equations

In this section we will obtain the equations of motion for a class-1 tensegrity system starting from the equations of motion of a single bar.

Let $\mathcal{L}_i = T_i - V_i$ be the Lagrangian function of a single bar, the *Euler–Lagrange* equations are

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{c}}_i}\right)^T - \left(\frac{\partial\mathcal{L}}{\partial\boldsymbol{c}_i}\right)^T &= \xi_{\boldsymbol{c},i} \\ \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\underline{\boldsymbol{e}}}_i}\right)^T - \left(\frac{\partial\mathcal{L}}{\partial\underline{\boldsymbol{e}}_i}\right)^T &= \xi_{\boldsymbol{e},i} \end{aligned} \tag{50}$$

where $\xi_{\boldsymbol{c},i}$ and $\xi_{\boldsymbol{e},i}$ are the vectors of the generalized forces acting, respectively, on the state variables $\boldsymbol{c}_i$ and $\underline{\boldsymbol{e}}_i$. To apply Eq. (50), we start computing the following terms:

$$\left(\frac{\partial\mathcal{L}_i}{\partial\boldsymbol{c}_i}\right)^T = \left(-\frac{\partial V_i}{\partial\boldsymbol{c}_i}\right)^T = m_i\boldsymbol{g}_0 \tag{51}$$

$$\left(\frac{\partial\mathcal{L}_i}{\partial\dot{\boldsymbol{c}}_i}\right)^T = \left(\frac{\partial T_i}{\partial\boldsymbol{p}_i}\frac{\partial\boldsymbol{p}_i}{\partial\dot{\boldsymbol{c}}_i}\right)^T = m_i\dot{\boldsymbol{c}}_i \tag{52}$$

$$\left(\frac{\partial\mathcal{L}_i}{\partial\underline{\boldsymbol{e}}_i}\right)^T = \left(\frac{\partial T_i}{\partial\underline{\boldsymbol{e}}_i}\right)^T = \left(-\frac{\partial T_i^*}{\partial\underline{\boldsymbol{e}}}\right)^T = -4\dot{\overset{+}{\mathcal{E}}}\boldsymbol{J}_{4,i}^1\dot{\overset{+}{\mathcal{E}}}_i^T\underline{\boldsymbol{e}}_i \tag{53}$$

$$\left(\frac{\partial\mathcal{L}_i}{\partial\dot{\underline{\boldsymbol{e}}}_i}\right)^T = \left(\frac{\partial T}{\partial\underline{\boldsymbol{g}}_i}\frac{\partial\underline{\boldsymbol{g}}_i}{\partial\dot{\underline{\boldsymbol{e}}}_i}\right)^T = 4\overset{+}{\mathcal{E}}_i\boldsymbol{J}_{4,i}^1\overset{+}{\mathcal{E}}_i^T\dot{\underline{\boldsymbol{e}}}_i \tag{54}$$

where to obtain Eq. (53) the property (38) has been exploited.

Finally, the equations of motion of a single bar are:

$$\ddot{\boldsymbol{c}}_i = \frac{1}{m_i}(\boldsymbol{f}_{1,i} + \boldsymbol{f}_{2,i} + m_i\boldsymbol{g}_0 - b_{t,i}\dot{\boldsymbol{c}}_i + \boldsymbol{f}_{d,i}) \tag{55}$$

$$\frac{d}{dt}\left(4\overset{+}{\mathcal{E}}_i\boldsymbol{J}_{4,i}^1\overset{+}{\mathcal{E}}_i^T\dot{\underline{\boldsymbol{e}}}_i\right) + 4\dot{\overset{+}{\mathcal{E}}}_i\boldsymbol{J}_{4,i}^1\dot{\overset{+}{\mathcal{E}}}_i^T\underline{\boldsymbol{e}}_i = \xi_{\boldsymbol{e},i} \tag{56}$$

where $\boldsymbol{f}_{d,i}$ is a generic disturbance force acting on the motion of the center of mass. To determine $\xi_{\boldsymbol{e},i}$, we can consider the virtual work made by the torques applied to the bar. Recalling Eqs. (42) and (49) we write:

$$\xi_{\boldsymbol{e},i} = 2\overline{\mathcal{E}}_i\begin{bmatrix} 0 \\ \frac{1}{2}(\boldsymbol{b}_i \times (\boldsymbol{f}_{2,i} - \boldsymbol{f}_{1,i})) - b_{r,i}\boldsymbol{\omega}_i^0 + \boldsymbol{\tau}_{d,i}^0 \end{bmatrix} \tag{57}$$

where $\boldsymbol{\tau}_{d,i}^0$ represents a generic disturbance torque vector expressed with respect to the base frame.

Computing the derivative in (56) we obtain:

$$4\dot{\overset{+}{\mathcal{E}}}_i\boldsymbol{J}_{4,i}^1\overset{+}{\mathcal{E}}_i^T\dot{\underline{\boldsymbol{e}}}_i + 4\overset{+}{\mathcal{E}}_i\boldsymbol{J}_{4,i}^1\dot{\overset{+}{\mathcal{E}}}_i^T\dot{\underline{\boldsymbol{e}}}_i + 4\overset{+}{\mathcal{E}}_i\boldsymbol{J}_{4,i}^1\overset{+}{\mathcal{E}}_i^T\ddot{\underline{\boldsymbol{e}}}_i + 4\dot{\overset{+}{\mathcal{E}}}_i\boldsymbol{J}_{4,i}^1\dot{\overset{+}{\mathcal{E}}}_i^T\underline{\boldsymbol{e}}_i = \xi_{\boldsymbol{e},i} \tag{58}$$

which, exploiting (A.13) becomes

$$4\overset{+}{\mathcal{E}}_i\boldsymbol{J}_{4,i}^1\overset{+}{\mathcal{E}}_i^T\ddot{\underline{\boldsymbol{e}}}_i + 4\overset{+}{\mathcal{E}}_i\boldsymbol{J}_{4,i}^1\dot{\overset{+}{\mathcal{E}}}_i^T\dot{\underline{\boldsymbol{e}}}_i = \xi_{\boldsymbol{e},i} \Rightarrow \ddot{\underline{\boldsymbol{e}}}_i = \frac{1}{4}\overset{+}{\mathcal{E}}_i{\boldsymbol{J}_{4,i}^1}^{-1}\overset{+}{\mathcal{E}}_i^T\xi_{\boldsymbol{e},i} - \overset{+}{\mathcal{E}}_i\dot{\overset{+}{\mathcal{E}}}_i^T\dot{\underline{\boldsymbol{e}}}_i \tag{59}$$

and, finally, applying (A.5) we have

$$\ddot{\underline{\boldsymbol{e}}}_i = \frac{1}{4}\overset{+}{\mathcal{E}}_i{\boldsymbol{J}_{4,i}^1}^{-1}\overset{+}{\mathcal{E}}_i^T\xi_{\boldsymbol{e},i} - \overset{+}{\mathcal{E}}_i\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\dot{\underline{\boldsymbol{e}}}_i^T\dot{\underline{\boldsymbol{e}}}_i \Rightarrow \ddot{\underline{\boldsymbol{e}}}_i = \frac{1}{4}\overset{+}{\mathcal{E}}_i{\boldsymbol{J}_{4,i}^1}^{-1}\overset{+}{\mathcal{E}}_i^T\xi_{\boldsymbol{e},i} - \underline{\boldsymbol{e}}_i\dot{\underline{\boldsymbol{e}}}_i^T\dot{\underline{\boldsymbol{e}}}_i \tag{60}$$

The system made of (55) and (56) becomes:

$$\ddot{\boldsymbol{c}}_i = \frac{1}{m_i}(\boldsymbol{f}_{1,i} + \boldsymbol{f}_{2,i} + m_i\boldsymbol{g}_0 - b_{t,i}\dot{\boldsymbol{c}}_i + \boldsymbol{f}_{d,i}) \tag{61}$$

$$\begin{aligned} \ddot{\underline{\boldsymbol{e}}}_i &= \frac{1}{4}\overset{+}{\mathcal{E}}_i{\boldsymbol{J}_{4,i}^1}^{-1}\overset{+}{\mathcal{E}}_i^T\overline{\mathcal{E}}_i\begin{bmatrix} 0 \\ \boldsymbol{b}_i \times (\boldsymbol{f}_{2,i} - \boldsymbol{f}_{1,i}) - b_{r,i}\boldsymbol{\omega}_i + \boldsymbol{\tau}_{d,i} \end{bmatrix} - \underline{\boldsymbol{e}}_i\dot{\underline{\boldsymbol{e}}}_i^T\dot{\underline{\boldsymbol{e}}}_i \\ &= \frac{1}{4}\overset{+}{\mathcal{E}}_i{\boldsymbol{J}_{4,i}^1}^{-1}\overset{+}{\mathcal{E}}_i^T\overline{\mathcal{E}}_i\left[\begin{pmatrix} 0 \\ \boldsymbol{b}_i \times (\boldsymbol{f}_{2,i} - \boldsymbol{f}_{1,i}) + \boldsymbol{\tau}_{d,i} \end{pmatrix} + b_{r,i}2\underline{\boldsymbol{e}} \odot \dot{\underline{\boldsymbol{e}}}^*\right] \\ &\quad - \underline{\boldsymbol{e}}_i\dot{\underline{\boldsymbol{e}}}_i^T\dot{\underline{\boldsymbol{e}}}_i \end{aligned} \tag{62}$$

or,

$$\ddot{\boldsymbol{c}}_i = \frac{1}{m_i}(\boldsymbol{f}_{1,i} + \boldsymbol{f}_{2,i} + m_i\boldsymbol{g}_0 - b_{t,i}\dot{\boldsymbol{c}}_i + \boldsymbol{f}_{d,i}) \tag{63}$$

$$\ddot{\underline{\boldsymbol{e}}}_i = \frac{1}{4}\overset{+}{\mathcal{E}}_i{\boldsymbol{J}_{4,i}^1}^{-1}\begin{bmatrix} 0 \\ \begin{pmatrix} l\\0\\0 \end{pmatrix} \times \boldsymbol{A}_{3,i}^{-1}(\boldsymbol{f}_{2,i} - \boldsymbol{f}_{1,i}) + \boldsymbol{\tau}_{d,i}^1 - 2b_{r,i}\overline{\boldsymbol{E}}\dot{\underline{\boldsymbol{e}}}_i \end{bmatrix} - \underline{\boldsymbol{e}}_i\dot{\underline{\boldsymbol{e}}}_i^T\dot{\underline{\boldsymbol{e}}}_i \tag{64}$$

where $\boldsymbol{\tau}_{d,i}^1$ is the disturbance torque expressed in the body fixed frame. Here the contribution of the friction, referring to the case of viscous friction, has been described in detail, to provide an example of how such kind of forces act on the generalized coordinate chosen. In the following, to simplify the notation, we will consider the friction included in the generic disturbance forces $\boldsymbol{\psi}_i$ introduced in Section 3.1.

We are now ready to write the equations of motion of a class-1 tensegrity system. We will shape two models: a *matrix form* model and a *vector form* model. The former is easier to obtain. It is also more compact and elegant, hence more convenient to write. The latter allows to quickly write the equations of motion in term of the state variables in the classical form $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$. This last form is more suitable for studying properties of a tensegrity by means of the classical analysis tools for non-linear systems based on the *vectors fields*.

We start with the model in matrix form. Define the following set of generalized coordinates for a class-1 tensegrity structure:

$$\boldsymbol{C} = (\boldsymbol{c}_1 \cdots \boldsymbol{c}_{n_b}) \tag{65}$$

$$\boldsymbol{E} = (\underline{\boldsymbol{e}}_1 \cdots \underline{\boldsymbol{e}}_{n_b}) \tag{66}$$

Observe that the total number of variables used to describe the system is $7n_b$, while the number of variables strictly necessary is $5n_b$, considering that the orientation of each bar may be represented only with two coordinates (the rotations around their symmetry axis will be neglected). Therefore (65) and (66) constitute a non-minimum set of coordinates.

In what follows we will make extensive use of the notation introduced in Section 3.1.

To completely explicit the equations of motion in function of the centers of mass and of the unit quaternions, we need to explicit $\boldsymbol{N}$ and $\boldsymbol{\Gamma}$ in Eq. (21). The nodes coordinates can be written as follows:

$$\boldsymbol{N} = (\boldsymbol{N}_1\ \boldsymbol{N}_2) = (\boldsymbol{C}\ \boldsymbol{B})\begin{pmatrix} \boldsymbol{I}_{n_b} & \boldsymbol{I}_{n_b} \\ -\frac{1}{2}\boldsymbol{I}_{n_b} & \frac{1}{2}\boldsymbol{I}_{n_b} \end{pmatrix} = (\boldsymbol{C}\ \boldsymbol{AB}_{diag})\begin{pmatrix} \boldsymbol{I}_{n_b} & \boldsymbol{I}_{n_b} \\ -\frac{1}{2}\boldsymbol{I}_{n_b} & \frac{1}{2}\boldsymbol{I}_{n_b} \end{pmatrix} \tag{67}$$

where

$$\boldsymbol{B} = (\boldsymbol{b}_1 \cdots \boldsymbol{b}_{n_b}) = (\boldsymbol{A}_{3,1} \cdots \boldsymbol{A}_{3,n_b})\begin{pmatrix} \boldsymbol{b}_1^1 & \cdots & \boldsymbol{0}_{3\times1} \\ \vdots & \ddots & \\ \boldsymbol{0}_{3\times1} & & \boldsymbol{b}_{n_b}^1 \end{pmatrix} \triangleq \boldsymbol{AB}_{diag} \tag{68}$$

The matrices $\boldsymbol{A}_{3,i}$ are defined, for the i-th bar, as in (A.19), while the vectors $\boldsymbol{b}_i^1$ represents the bar vectors in a bar relative frame. Choosing, for each bar, a frame join with it as described in Section 2 (see also Fig. 2), we can write $\boldsymbol{b}_i^1 = (l_i\ 0\ 0)^T$.

Note that the matrix $\boldsymbol{A}$ depends on all the quaternions while the vectors $\boldsymbol{b}_i^1$ are constants for fixed length bars and correspond to input variables for variable length bars: $\boldsymbol{b}_i^1 = (u_i\ 0\ 0)^T$.

To separate, in Eq. (67), the state variables from the bar vectors (which, for variable length bars, could be thought as input variables), we observe that

$$(\boldsymbol{C}\ \boldsymbol{AB}_{diag}) = (\boldsymbol{C}\ \boldsymbol{A})\begin{pmatrix} \boldsymbol{I}_{n_b} & \boldsymbol{0}_{n_b\times n_b} \\ \boldsymbol{0}_{3n_b\times n_b} & \boldsymbol{B}_{diag} \end{pmatrix} \tag{69}$$

The matrix $\boldsymbol{\Gamma}$ always depends on the state variables (whether the cables are actuated or not). To explicit this dependence, we can still use the elastic model and observe that

$$\boldsymbol{s}_i = (\boldsymbol{N}_1\ \boldsymbol{N}_2)\left(\boldsymbol{C}_{con}^T\right)_{i-col} \iff \boldsymbol{s}_i^T = (\boldsymbol{C}_{con})_{i-row}\begin{pmatrix} \boldsymbol{N}_1^T \\ \boldsymbol{N}_2^T \end{pmatrix} \tag{70}$$

Let us introduce the matrix

$$\boldsymbol{H} = \begin{pmatrix} (\boldsymbol{C}_{con})_{1-row} & \cdots & \boldsymbol{0}_{1\times2n_b} \\ \vdots & \ddots & \\ \boldsymbol{0}_{1\times2n_b} & & (\boldsymbol{C}_{con})_{n_s-row} \end{pmatrix} \tag{71}$$

it is $[\boldsymbol{H}] = (n_s) \times (2n_bn_s)$. This matrix is constant and made of 0, 1 and -1. Now we can write

$$\begin{aligned} \mathrm{diag}\{\|\boldsymbol{s}_i\|\} &= \left[\begin{pmatrix} \boldsymbol{s}_1^T & \cdots & \boldsymbol{0}_{1\times3} \\ \vdots & \ddots & \\ \boldsymbol{0}_{1\times3} & & \boldsymbol{s}_{n_s}^T \end{pmatrix}\begin{pmatrix} \boldsymbol{s}_1 & \cdots & \boldsymbol{0}_{3\times1} \\ \vdots & \ddots & \\ \boldsymbol{0}_{3\times1} & & \boldsymbol{s}_{n_s} \end{pmatrix}\right]^{\frac{1}{2}} \\ &= \left[\boldsymbol{H}\left(\underbrace{\begin{matrix} \boldsymbol{N}^T & \cdots & \boldsymbol{0}_{2n_b\times3} \\ \vdots & \ddots & \\ \boldsymbol{0}_{2n_b\times3} & & \boldsymbol{N}^T \end{matrix}}_{n_s\text{times}}\right)\begin{pmatrix} \boldsymbol{N} & \cdots & \boldsymbol{0}_{3\times2n_b} \\ \vdots & \ddots & \\ \boldsymbol{0}_{3\times2n_b} & & \boldsymbol{N} \end{pmatrix}\boldsymbol{H}^T\right]^{\frac{1}{2}} \\ &= [\boldsymbol{H}(\boldsymbol{I}_{n_s} \otimes \boldsymbol{N}^T\boldsymbol{N})\boldsymbol{H}^T]^{\frac{1}{2}} \\ &= \left\{\boldsymbol{H}\left[\boldsymbol{I}_{n_s} \otimes \left[\begin{pmatrix} \boldsymbol{I}_{n_b} & -\frac{1}{2}\boldsymbol{I}_{n_b} \\ \boldsymbol{I}_{n_b} & \frac{1}{2}\boldsymbol{I}_{n_b} \end{pmatrix}\begin{pmatrix} \boldsymbol{C}^T \\ (\boldsymbol{AB}_{diag})^T \end{pmatrix}\right.\right.\right. \\ &\quad \left.\left.\left. \times (\boldsymbol{C}\ \boldsymbol{AB}_{diag})\begin{pmatrix} \boldsymbol{I}_{n_b} & \boldsymbol{I}_{n_b} \\ -\frac{1}{2}\boldsymbol{I}_{n_b} & \frac{1}{2}\boldsymbol{I}_{n_b} \end{pmatrix}\right]\right]\boldsymbol{H}^T\right\}^{\frac{1}{2}} \end{aligned} \tag{72}$$

In the above equation, the operator $\otimes$ represents the Kronecker product between matrices (see (A.28)).

Finally, starting from Eq. (25), matrix $\boldsymbol{\Gamma}$ can be now written as follows:

$$\begin{aligned} \boldsymbol{\Gamma} &= \mathrm{diag}\left\{k_i\left(1 - \frac{\|\boldsymbol{s}_{i0}\|}{\|\boldsymbol{s}_i\|}\right)\right\} \\ &= \boldsymbol{K} - \boldsymbol{KS}_0\left\{\boldsymbol{H}\left[\boldsymbol{I}_{n_s} \otimes \left[\begin{pmatrix} \boldsymbol{I}_{n_b} & -\frac{1}{2}\boldsymbol{I}_{n_b} \\ \boldsymbol{I}_{n_b} & \frac{1}{2}\boldsymbol{I}_{n_b} \end{pmatrix}\begin{pmatrix} \boldsymbol{C}^T \\ (\boldsymbol{AB}_{diag})^T \end{pmatrix}\right.\right.\right. \\ &\quad \left.\left.\left. \times (\boldsymbol{C}\ \boldsymbol{AB}_{diag})\begin{pmatrix} \boldsymbol{I}_{n_b} & \boldsymbol{I}_{n_b} \\ -\frac{1}{2}\boldsymbol{I}_{n_b} & \frac{1}{2}\boldsymbol{I}_{n_b} \end{pmatrix}\right]\right]\boldsymbol{H}^T\right\}^{-\frac{1}{2}} \end{aligned} \tag{73}$$

The resulting expression is complicated, and even more complicated is the expression of $\boldsymbol{F}$, where $\boldsymbol{\Gamma}$ should be put (see (21)): it is highly non-linear in the generalized coordinates.

In general it is not possible to sort the nodes in such a way that all the inputs corresponding to the actuated strings appear in the first subset of components of $\boldsymbol{F}$, but it is possible to arrange the nodes numbering so that the actuated bars are the first (or the last) block of components of $\boldsymbol{B}$ (see definition in the table at the beginning of the Section 3.1 or Eq. (68)).

Now, recalling the dynamic equations of a single bar, consider that

$$\begin{bmatrix} 0 \\ (\boldsymbol{b}_i \times (\boldsymbol{f}_{2,i} - \boldsymbol{f}_{1,i})) \end{bmatrix} = \begin{pmatrix} \boldsymbol{0}_{1\times3} \\ \boldsymbol{I}_3 \end{pmatrix}[\boldsymbol{b}_i \times (\boldsymbol{f}_{2,i} - \boldsymbol{f}_{1,i})] \tag{74}$$

and that

$$\begin{aligned} \boldsymbol{b}_i \times (\boldsymbol{f}_{2,i} - \boldsymbol{f}_{1,i}) &= \widetilde{\boldsymbol{B}}_i(\boldsymbol{f}_{2,i} - \boldsymbol{f}_{1,i}) = (\boldsymbol{A}_{3,i}\boldsymbol{b}_i^1) \times (\boldsymbol{f}_{2,i} - \boldsymbol{f}_{1,i}) \\ &= \boldsymbol{A}_{3,i}\widetilde{\boldsymbol{B}}_i^1\boldsymbol{A}_{3,i}^T(\boldsymbol{f}_{2,i} - \boldsymbol{f}_{1,i}) \end{aligned} \tag{75}$$

see (A.29) for more details.

In the Eqs. (74) and (75) the vectors $\boldsymbol{f}_{1,i}$ and $\boldsymbol{f}_{2,i}$ are the vectors $\boldsymbol{f}_i$ and $\boldsymbol{f}_{i+n_b}$ of $\boldsymbol{F}$, hence

$$\begin{aligned} \boldsymbol{f}_{2,i} - \boldsymbol{f}_{1,i} &= i\text{th column of } \boldsymbol{F}\begin{pmatrix} -\boldsymbol{I}_{n_b} \\ \boldsymbol{I}_{n_b} \end{pmatrix} \\ &= \text{row components from } (3i-2) \text{ to } 3i \text{ of } (-\boldsymbol{I}_{3n_b}\ \boldsymbol{I}_{3n_b})\boldsymbol{\phi} \end{aligned} \tag{76}$$

and

$$\begin{aligned} \boldsymbol{f}_{1,i} + \boldsymbol{f}_{2,i} &= i\text{th column of } \boldsymbol{F}\begin{pmatrix} \boldsymbol{I}_{n_b} \\ \boldsymbol{I}_{n_b} \end{pmatrix} \\ &= \text{row components from } (3i-2) \text{ to } 3\ i \text{ of } (\boldsymbol{I}_{3n_b}\ \boldsymbol{I}_{3n_b})\boldsymbol{\phi} \end{aligned} \tag{77}$$

The dynamic model of a tensegrity structure comes now straightforward from Eqs. (61) and (62):

$$\boxed{\ddot{\boldsymbol{C}}} = \boxed{\boldsymbol{F}\begin{pmatrix} \boldsymbol{I}_{n_b} \\ \boldsymbol{I}_{n_b} \end{pmatrix}\begin{pmatrix} m_1 & \cdots & 0 \\ \vdots & \ddots & \\ 0 & & m_{n_b} \end{pmatrix}^{-1} + (\underbrace{1\ \ldots\ 1}_{n_b\ \text{times}}) \otimes \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}} \tag{78}$$

$$\begin{aligned} \boxed{\ddot{\boldsymbol{E}}} = & \frac{1}{4}\left(\overset{+}{\mathcal{E}}_1\boldsymbol{J}_{4,1}^{1^{-1}}\overset{+}{\mathcal{E}}{}_1^T\overline{\mathcal{E}}_1 \cdots \overset{+}{\mathcal{E}}_{n_b}\boldsymbol{J}_{4,n_b}^{1^{-1}}\overset{+}{\mathcal{E}}{}_{n_b}^T\overline{\mathcal{E}}_{n_b}\right)\begin{pmatrix} \begin{matrix}\boldsymbol{0}_{1\times3} \\ \boldsymbol{I}_3\end{matrix} & \cdots & \boldsymbol{0}_{4\times3} \\ \vdots & \ddots & \\ \boldsymbol{0}_{4\times3} & & \begin{matrix}\boldsymbol{0}_{1\times3} \\ \boldsymbol{I}_3\end{matrix} \end{pmatrix} \\ & \cdot \begin{pmatrix} \widetilde{\boldsymbol{B}}_1 & \cdots & \boldsymbol{0}_{3\times3} \\ \vdots & \ddots & \\ \boldsymbol{0}_{3\times3} & & \widetilde{\boldsymbol{B}}_{n_b} \end{pmatrix}\mathrm{diag}\left\{\boldsymbol{F}\begin{pmatrix} -\boldsymbol{I}_{n_b} \\ \boldsymbol{I}_{n_b} \end{pmatrix}\right\} - \boldsymbol{E}Q[\boldsymbol{I}_{n_b} \otimes (\dot{\boldsymbol{E}}^T\dot{\boldsymbol{E}})]\boldsymbol{Q}^T \end{aligned} \tag{79}$$

where (see (A.29))

$$\begin{pmatrix} \widetilde{\boldsymbol{B}}_1 & \dots & \boldsymbol{0}_{3\times3} \\ \vdots & \ddots & \\ \boldsymbol{0}_{3\times3} & & \widetilde{\boldsymbol{B}}_{n_b} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_{3,1} & \dots & \boldsymbol{0}_{3\times3} \\ \vdots & \ddots & \\ \boldsymbol{0}_{3\times3} & & \boldsymbol{A}_{3,n_b} \end{pmatrix} \cdot \begin{pmatrix} \widetilde{\boldsymbol{B}}_1^1 & \dots & \boldsymbol{0}_{3\times3} \\ \vdots & \ddots & \\ \boldsymbol{0}_{3\times3} & & \widetilde{\boldsymbol{B}}_{n_b}^1 \end{pmatrix} \begin{pmatrix} \boldsymbol{A}_{3,1} & \dots & \boldsymbol{0}_{3\times3} \\ \vdots & \ddots & \\ \boldsymbol{0}_{3\times3} & & \boldsymbol{A}_{3,n_b} \end{pmatrix}^T$$

and

$$\boldsymbol{F} = -(\boldsymbol{C} \quad \boldsymbol{A}\boldsymbol{B}_{diag}) \begin{pmatrix} \boldsymbol{I}_{n_b} & \boldsymbol{I}_{n_b} \\ -\frac{1}{2}\boldsymbol{I}_{n_b} & \frac{1}{2}\boldsymbol{I}_{n_b} \end{pmatrix} \boldsymbol{C}_{con}^T \boldsymbol{\Gamma} \boldsymbol{C}_{con} + \boldsymbol{\Psi}\boldsymbol{D}$$

and $\boldsymbol{Q}$ is a constant matrix $n_b \times n_b^2$ (see (A.30)).

Let us introduce now the vectors

$$\boldsymbol{c}_v = \mathrm{vec}(\boldsymbol{C}) \tag{80}$$

$$\boldsymbol{e}_v = \mathrm{vec}(\boldsymbol{E}) \tag{81}$$

The vector form of the model is:

$$\boxed{\ddot{\boldsymbol{c}}_v} = \boxed{\left[\begin{pmatrix} m_1 & \dots & 0 \\ \vdots & \ddots & \\ 0 & & m_{n_b} \end{pmatrix} \otimes \boldsymbol{I}_3\right]^{-1} (\boldsymbol{I}_{3n_b} \ \boldsymbol{I}_{3n_b})\boldsymbol{\phi} + \begin{pmatrix} 1_{first} \\ \vdots \\ 1_{n_b-th} \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}} \tag{82}$$

$$\boxed{\ddot{\boldsymbol{e}}_v} = \frac{1}{4}\boxed{\begin{pmatrix} \overset{+}{\mathcal{E}}_1 \boldsymbol{J}_{4,1}^{1^{-1}} \overset{+}{\mathcal{E}}_1^T \overline{\mathcal{E}}_1 & \dots & \boldsymbol{0}_{4\times4} \\ \vdots & \ddots & \\ \boldsymbol{0}_{4\times4} & & \overset{+}{\mathcal{E}}_{n_b} \boldsymbol{J}_{4,n_b}^{1^{-1}} \overset{+}{\mathcal{E}}_{n_b}^T \overline{\mathcal{E}}_{n_b} \end{pmatrix} \begin{pmatrix} \boldsymbol{0}_{1\times3} & \dots & \boldsymbol{0}_{4\times3} \\ \boldsymbol{I}_3 & & \\ \vdots & \ddots & \\ \boldsymbol{0}_{4\times3} & & \boldsymbol{0}_{1\times3} \\ & & \boldsymbol{I}_3 \end{pmatrix}} \boxed{\cdot \begin{pmatrix} \widetilde{\boldsymbol{B}}_1 & \dots & \boldsymbol{0}_{3\times3} \\ \vdots & \ddots & \\ \boldsymbol{0}_{3\times3} & & \widetilde{\boldsymbol{B}}_{n_b} \end{pmatrix} (-\boldsymbol{I}_{3n_b} \ \boldsymbol{I}_{3n_b})\boldsymbol{\phi} - \left[\left[(\boldsymbol{I}_{n_b} \otimes \dot{\boldsymbol{e}}_v^T)\boldsymbol{Q}'\boldsymbol{Q}'^T(\boldsymbol{I}_{n_b} \otimes \dot{\boldsymbol{e}}_v)\right] \otimes \boldsymbol{I}_4\right]\boldsymbol{e}_v} \tag{83}$$

where $\boldsymbol{Q}'$ is a matrix $4n_b^2 \times 4n_b$ defined in (A.31), and

$$\left[\begin{pmatrix} m_1 & \dots & 0 \\ \vdots & \ddots & \\ 0 & & m_{n_b} \end{pmatrix} \otimes \boldsymbol{I}_3\right]^{-1} = \begin{pmatrix} m_1 & \dots & 0 \\ \vdots & \ddots & \\ 0 & & m_{n_b} \end{pmatrix}^{-1} \otimes \boldsymbol{I}_3$$

$$\boldsymbol{\phi} = -\mathcal{C}_{con}^T \boldsymbol{\Gamma}' \mathcal{C}_{con} \begin{pmatrix} \boldsymbol{I}_{3n_b} & -\frac{1}{2}\boldsymbol{I}_{3n_b} \\ \boldsymbol{I}_{3n_b} & \frac{1}{2}\boldsymbol{I}_{3n_b} \end{pmatrix} \begin{pmatrix} \boldsymbol{c}_v \\ \boldsymbol{\beta} \end{pmatrix} + \mathcal{D}^T \zeta$$

$$\boldsymbol{\beta} = \mathrm{vec}(\boldsymbol{B}) = \begin{pmatrix} \boldsymbol{b}_1 \\ \vdots \\ \boldsymbol{b}_{n_b} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_{3,1} & \dots & \boldsymbol{0}_{3\times3} \\ \vdots & \ddots & \\ \boldsymbol{0}_{3\times3} & & \boldsymbol{A}_{3,n_b} \end{pmatrix} \begin{pmatrix} \boldsymbol{b}_1^1 \\ \vdots \\ \boldsymbol{b}_{n_b}^1 \end{pmatrix}$$

and

$$\boldsymbol{\Gamma}' = \boldsymbol{\Gamma} \otimes \boldsymbol{I}_3 \tag{84}$$

Consider now that

$$\begin{pmatrix} \overbrace{\|\boldsymbol{s}_1\| \quad 0 \quad 0}^{3 \text{ times } \|\boldsymbol{s}_1\|} & & & & \\ 0 \quad \|\boldsymbol{s}_1\| \quad 0 & \dots & & 0 & \\ 0 \quad 0 \quad \|\boldsymbol{s}_1\| & & & & \\ \vdots & \ddots & & & \\ & & \|\boldsymbol{s}_{n_s}\| & 0 & 0 \\ 0 & & 0 & \|\boldsymbol{s}_{n_s}\| & 0 \\ & & 0 & 0 & \|\boldsymbol{s}_{n_s}\| \end{pmatrix} = \mathrm{diag}\{\|\boldsymbol{s}_i\|\} \otimes \boldsymbol{I}_3$$

and that, similarly to (70), we can write

$$\boldsymbol{s}_i = \begin{pmatrix} (\mathcal{C}_{con})_{(3i-2)-row} \\ (\mathcal{C}_{con})_{(3i-1)-row} \\ (\mathcal{C}_{con})_{3i-row} \end{pmatrix} \boldsymbol{v} \tag{85}$$

$$\boldsymbol{s}_i^T = \boldsymbol{v}^T \left[(\mathcal{C}_{con}^T)_{(3i-2)-col} \quad (\mathcal{C}_{con}^T)_{(3i-1)-col} \quad (\mathcal{C}_{con}^T)_{3i-col} \right] \tag{86}$$

Let us now introduce the following matrix, equivalent to (71) but for vector form manipulations:

$$\mathcal{H} = \begin{pmatrix} (\mathcal{C}_{con})_{1-row} & & & \\ (\mathcal{C}_{con})_{2-row} & & & \\ (\mathcal{C}_{con})_{3-row} & & & \\ & (\mathcal{C}_{con})_{1-row} & & \\ & (\mathcal{C}_{con})_{2-row} & \dots & 0 \\ & (\mathcal{C}_{con})_{3-row} & & \\ & & (\mathcal{C}_{con})_{1-row} & \\ & & (\mathcal{C}_{con})_{2-row} & \\ & & (\mathcal{C}_{con})_{3-row} & \\ & \vdots & \ddots & \vdots \\ & & & (\mathcal{C}_{con})_{(3n_s-2)-row} \\ & 0 & \dots & (\mathcal{C}_{con})_{(3n_s-1)-row} \\ & & & (\mathcal{C}_{con})_{3n_s-row} \end{pmatrix} \tag{87}$$

it is $[\mathcal{H}] = 9n_s \times 18n_bn_s$. We can write

$$\mathrm{diag}\{\|\boldsymbol{s}_i\|\} \otimes \boldsymbol{I}_3 = \left[\overbrace{\begin{pmatrix} \boldsymbol{v}^T & \dots & \boldsymbol{0}_{1\times6n_b} \\ \vdots & \ddots & \\ \boldsymbol{0}_{1\times6n_b} & & \boldsymbol{v}^T \end{pmatrix}}^{3n_s \text{ blocks}} \mathcal{H}^T\mathcal{H} \begin{pmatrix} \boldsymbol{v} & \dots & \boldsymbol{0}_{6n_b\times1} \\ \vdots & \ddots & \\ \boldsymbol{0}_{6n_b\times1} & & \boldsymbol{v} \end{pmatrix}\right]^{\frac{1}{2}}$$

$$= \left[(\boldsymbol{I}_{3n_s} \otimes \boldsymbol{v}^T)\mathcal{H}^T\mathcal{H}(\boldsymbol{I}_{3n_s} \otimes \boldsymbol{v})\right]^{\frac{1}{2}} \tag{88}$$

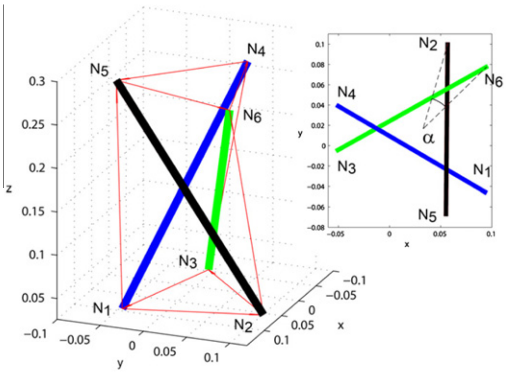
where

$$\boldsymbol{v} = \begin{pmatrix} \boldsymbol{I}_{3n_b} & -\frac{1}{2}\boldsymbol{I}_{3n_b} \\ \boldsymbol{I}_{3n_b} & \frac{1}{2}\boldsymbol{I}_{3n_b} \end{pmatrix} \begin{pmatrix} \boldsymbol{c}_v \\ \boldsymbol{\beta} \end{pmatrix} \tag{89}$$

Finally, it is possible to write $\boldsymbol{\Gamma}'$ as function of $\boldsymbol{c}_v$ and $\boldsymbol{\beta}$:

$$\begin{aligned} \boldsymbol{\Gamma}' &= (\boldsymbol{K} \otimes \boldsymbol{I}_3) - (\boldsymbol{K} \otimes \boldsymbol{I}_3)(\boldsymbol{S}_0 \otimes \boldsymbol{I}_3)(\mathrm{diag}\{\|\boldsymbol{s}_i\|\} \otimes \boldsymbol{I}_3)^{-1} \\ &= (\boldsymbol{K} \otimes \boldsymbol{I}_3) - ((\boldsymbol{K}\boldsymbol{S}_0) \otimes \boldsymbol{I}_3)(\mathrm{diag}\{\|\boldsymbol{s}_i\|\} \otimes \boldsymbol{I}_3)^{-1} \end{aligned} \tag{90}$$

In (90), $\mathrm{diag}\{\|\boldsymbol{s}_i\|\} \otimes \boldsymbol{I}_3$ should be substituted by Eq. (88), using Eq. (89).



$$\begin{aligned} N_1 &= (\ \ 0.095, -0.046, 0.027) \\ N_2 &= (\ \ 0.057, \ \ \ 0.005, 0.027) \\ N_3 &= (-0.053, -0.005, 0.027) \\ N_4 &= (-0.052, \ \ \ 0.040, 0.273) \\ N_5 &= (\ \ 0.055, -0.069, 0.273) \\ N_6 &= (\ \ 0.096, \ \ \ 0.078, 0.273) \end{aligned}$$

Fig. 6. Global and top view of the equilibrium configuration (on the left) and the coordinates of the nodes.

Finally, observe that

$$\begin{aligned} \boldsymbol{\Gamma}' &= (\boldsymbol{K} \otimes \boldsymbol{I}_3) - ((\boldsymbol{K}\boldsymbol{S}_0) \otimes \boldsymbol{I}_3)(\text{diag}\{\|\boldsymbol{s}_i\|\} \otimes \boldsymbol{I}_3)^{-1} \\ &= (\boldsymbol{K} \otimes \boldsymbol{I}_3) - (\boldsymbol{K}\boldsymbol{S}_0 \text{diag}\{\|\boldsymbol{s}_i\|\}^{-1}) \otimes \boldsymbol{I}_3 \\ &= (\boldsymbol{K} - \boldsymbol{K}\boldsymbol{S}_0 \text{diag}\{\|\boldsymbol{s}_i\|\}^{-1}) \otimes \boldsymbol{I}_3 = \boldsymbol{\Gamma} \otimes \boldsymbol{I}_3 \end{aligned}$$

which is a proof of Eq. (84).

4. Simulation results

In this section we present the results of two simulations addressing the problem of form finding. Form finding is a classic problem of tensegrity structures, usually studied from the point of view of statics. It consists in finding the steady states of a tensegrity once the structural properties (the number of bars and the number and type of connections) as well as the physical properties (the mass of the bars, the rest lengths and the stiffness of the cables) have been fixed.

With the first simulation we provide an empirical proof of the correctness of the dynamic model proposed. We have used our model to find an equilibrium of a tensegrity of class-1. To this end, we have chosen a very simple and well known system, in order to compare our results with those of the literature.

The second simulation is aimed to show the effectiveness of our dynamic model in a problem of form finding for a more complex system.

4.1. A tensegrity prism

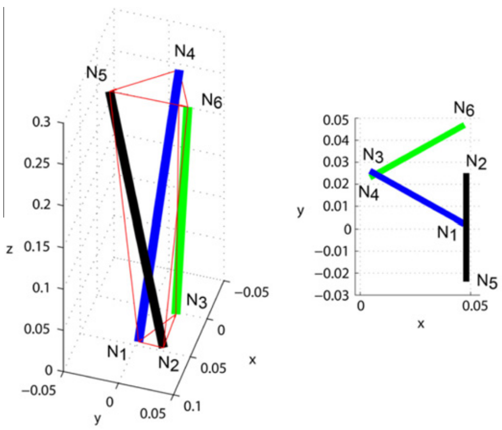
In this paragraph we study the equilibrium state of a minimal and regular tensegrity prism made of 3 bars (Fig. 6) by means of the dynamic model presented above. It is the simplest three-dimensional class-1 tensegrity achievable in the space. For a complete description of tensegrity structures the reader can refer to the literature cited in the introduction of this paper. In particular, in Skelton and Oliveira (2009) the authors provide a detailed description of prismatic tensegrities (in Section 1.4.1) and show that for an unloaded system, only one equilibrium exists. Such equilibrium can be identified by the mutual orientation angle α (see Fig. 6) between the top and the bottom triangles (see also Connelly and Back, 1998) and by the ratio of the force coefficients of the strings. It can therefore be determined without the need to resort to a dynamic model.

In our simulation we computed the equilibrium state as the asymptotic solution to which the system tend in absence of gravity and other external forces with the exception of friction. A term of viscous friction has been added to make the system dissipative and the equilibrium an asymptotic stable one.

We set the length of the bars to 0.3 m and their mass to 0.1 kg. For all cables, we fixed the stiffness to 40 N/m and the rest length to 0.1 m. The viscous coefficients have been fixed to 1.5 N s/m (or N m s/rad for rotations).

Simulations show that whichever is the initial position of the bars, the system always converges to the configuration indicated in Fig. 6. Different initial states can bring to different orientation of such stable configuration, but the mutual position of the bars and the tensions in the cables are always the same.

Consider now the configuration shown in Fig. 7 at rest. Such state is close to the equilibrium but is not an equilibrium. Setting this configuration as the initial state, the system evolves towards the stable state in about 3 s without any contact between its elements.



$$\begin{aligned} N_1 &= (0.047, \ \ \ 0.002, \ \ 0.002) \\ N_2 &= (0.048, \ \ \ 0.025, \ \ 0.002) \\ N_3 &= (0.004, \ \ \ 0.023, \ \ 0.002) \\ N_4 &= (0.004, \ \ \ 0.026, \ \ 0.298) \\ N_5 &= (0.048, -0.024, \ \ 0.298) \\ N_6 &= (0.047, \ \ \ 0.047, \ \ 0.298) \end{aligned}$$

Fig. 7. Global and top view of the initial configuration (on the left) and the coordinates of the nodes.

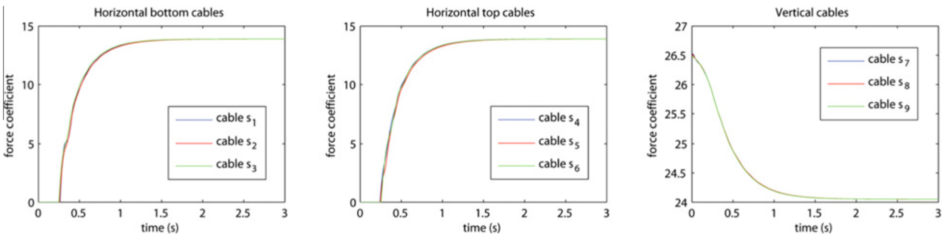


Fig. 8. Force coefficients of the cables.

Starting from the coordinates of the nodes at the equilibrium, it is easy now to compute the relative orientation angle between the top and bottom polygons. It results $\alpha = 30°$, exactly as expected (see for instance, Section 3.6.1 of Skelton and Oliveira, 2009).

Fig. 8 shows the evolution of the force coefficients of the cables. From the graphs it can be observed that the force coefficients of the horizontal bottom cables (s_1, s_2 and s_3) and the force coefficients of the top horizontal cables (s_4, s_5 and s_6) converge to the same value. The same happens for the force coefficients of the set of vertical cables (s_7, s_8 and s_9). Cutting off the decimals representation after the third digit, the values at the equilibrium can be summarized as follows:

$$\begin{pmatrix} \gamma_t \\ \gamma_b \\ \gamma_v \end{pmatrix} = \begin{pmatrix} 13.886 \\ 13.886 \\ 24.051 \end{pmatrix} \tag{91}$$

where γ_t, γ_b and γ_v are, respectively, the force coefficients of the top, bottom and vertical cables. In Skelton and Oliveira (2009, Section 3.6.1), the authors explain that the force coefficients at the equilibrium should satisfy the following relation concerning their ratio:

$$\begin{pmatrix} \gamma_t \\ \gamma_b \\ \gamma_v \end{pmatrix} = \gamma \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 1 \end{pmatrix} \tag{92}$$

Since the values at the equilibrium (91) satisfy Eq. (92) it is empirically proven that our dynamic model successfully describe the dynamics of a tensegrity of class-1.

4.2. A two level structure

Consider a system made of 6 bars and 21 cables, connected and initially disposed as shown in Fig. 9. In the starting state the bars are placed horizontally, in two parallel planes in such a way to form a regular hexagon if observed from the top or from the bottom. Fig. 10 shows the connectivity matrix and the rest length of the cables.

At the beginning, all the cables are in tension. When the actual length of a cable becomes equal to or smaller than its rest length, such cable is no more in tension and the system behaves like if it was cut. The initial state is not an equilibrium and the system evolves (more or less quickly, depending on the friction and the stiffness of the cables) untill a steady state is reached. In the simulation we neglected the contacts between bars and cables and we left the system move like if the rods could penetrate, because our purpose is not to study the dynamic evolution, but only the the final stable equilibrium state. If it exists, it means that it also exists a state close to it which is not an equilibrium, starting from which the system would evolve towards the equilibrium without any contact between bars and cables. Since no contacts with the environment is considered, we also neglected the gravity.

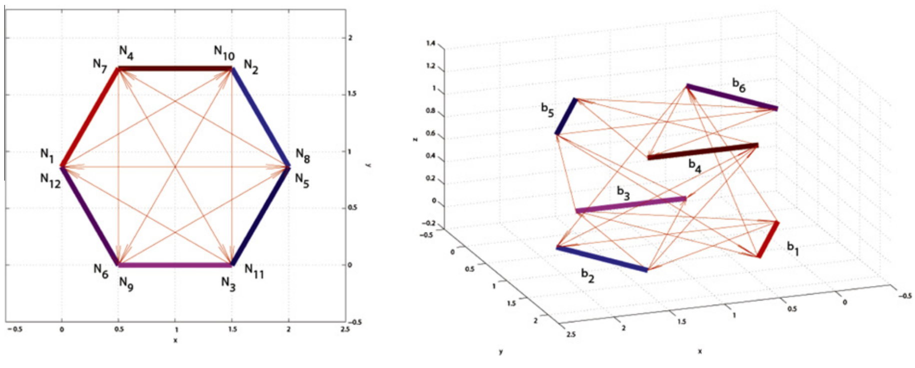


Fig. 9. Initial state: a top view (on the left) and a perspective view.

$$\begin{bmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix}$$

cable	rest length	cable	rest length	cable	rest length
1	0.5	8	0.82	15	0.25
2	0.5	9	0.82	16	0.85
3	0.5	10	0.25	17	0.85
4	0.85	11	0.25	18	0.85
5	0.85	12	0.25	19	0.5
6	0.85	13	0.25	20	0.5
7	0.82	14	0.25	21	0.5

Fig. 10. Connectivity matrix (on the left) and rest length of the cables.

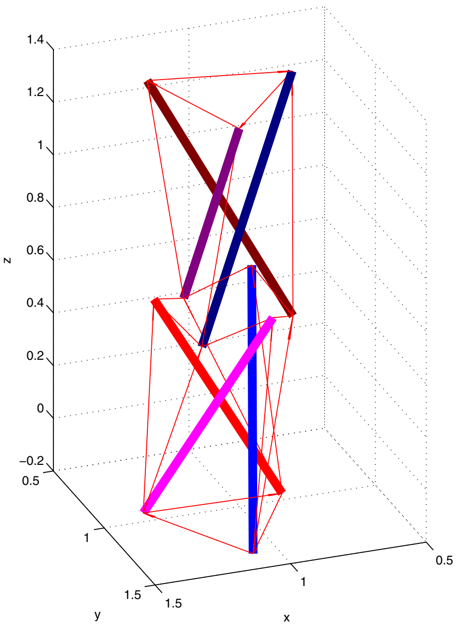


Fig. 11. Stable equilibrium state.

The bars have all unitary length, and mass 0.3 kg. In order to make the system dissipative and to be sure that at least one equilibrium point exists, we considered a viscous friction acting on both the translational and rotational motion of the bars. The viscous coefficients have been fixed all to 1.5 N s/m (or N m s/rad for rotations). Finally, the stiffness coefficients have been set to 40 N/m.

The system stabilizes in about 2 s on the final equilibrium shown in Fig. 11: a two stage prismatic tensegrity. Fig. 12 shows some snapshots of the motion untill the equilibrium. The bars dispose vertically on two levels. The first level is made of the red, the blue and the violet bars (b_1, b_2 and b_3 respectively), while the second level is made of the dark red, the dark blue and the dark violet ones (b_4, b_5 and b_6). Finally Table 1 reports the coordinates of the nodes at the equilibrium.

Table 1
Coordinates of the nodes at the equilibrium.

Node	Coordinates	Node	Coordinates
N_1	$(0.742, 0.728, -0.308)$	N_7	$(1.248, 0.818, 0.549)$
N_2	$(1.010, 1.157, -0.308)$	N_8	$(0.834, 0.674, 0.549)$
N_3	$(1.247, 0.711, -0.308)$	N_9	$(0.917, 1.105, 0.549)$
N_4	$(0.771, 0.912, 0.449)$	N_{10}	$(1.218, 0.671, 1.310)$
N_5	$(1.154, 1.040, 0.449)$	N_{11}	$(0.722, 0.774, 1.310)$
N_6	$(1.073, 0.644, 0.449)$	N_{12}	$(1.059, 1.152, 1.310)$

5. Conclusions

We proposed in this paper a new dynamic model based on quaternions for tensegrity systems of class-1. To our knowledge, it is the first comprehensive model for such kind of mechanical systems based on quaternions. We presented a thorough introduction of quaternions and provided some important details and properties exploited in this work. Despite quaternions are responsible of a growth of the notational complexity, they provide an excellent framework to represent orientations avoiding numerical singularities. Therefore, computations and simulations carried out using equations based on quaternions are more precise.

The model derived is presented in two forms: a matrix and a vector form. The matrix form is more compact and elegant. The latter is the classic form in which we usually write a model making use of vector fields. It is therefore more suitable for the application of the typical tools of the differential geometry.

We proved the effectiveness of the proposed model with some simulations concerning problems of form finding. Comparative results show the correctness of our model and suggest to adopt the quaternion based equations for further applications. For instance,

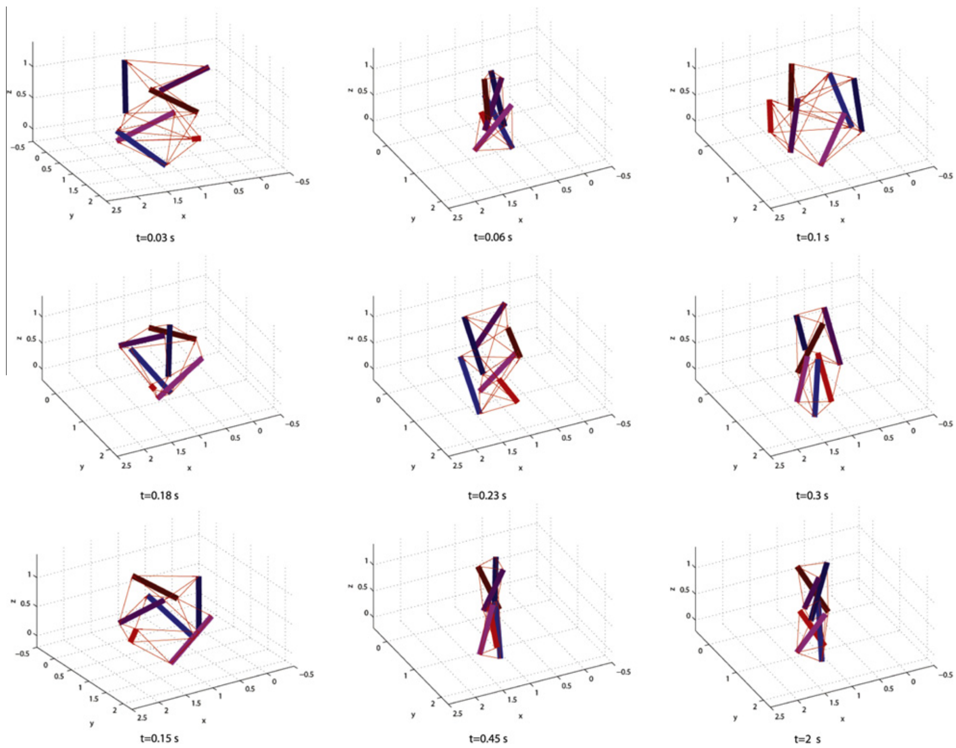


Fig. 12. Screenshots of the motion till the equilibrium.

it is straightforward to modify the equations to account that some cables or bars are actuated. This model could therefore be advantageously exploited to study the control problem of a tensegrity.

As a final remark, mention we can deliver the simulator files (Simulink/Matlab will be needed) for the particular examples given in the paper upon e-mail request to any of the authors.

Acknowledgments

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Appendix A. Quaternions

A.1. Definitions and fundamental properties

Quaternions are a generalization of complex numbers. They were introduced by Hamilton in 1843 (see Hamilton, 1866; Hamilton, 1847) and first applied to mechanical problems in 3-dimensional space. The initial work of Hamilton was intended to generalize complex numbers to a 3-dimensional space, but his attempt failed because the algebra constructed did not have *good* properties, as it was not closed under multiplication. Later on, Hamilton discovered that the appropriate generalization is one in which the scalar (real) axis of the complex plane is left unchanged whereas the vector (imaginary) axis is supplemented by adding two more vector axes. It can be helpful to think of the scalar axis as representing *time* and the three vector axes as representing *space*. The basic algebraic form for a quaternion is,

$$\underline{\boldsymbol{q}} = \mathbf{1}a + \hat{\mathbf{i}}b + \hat{\mathbf{j}}c + \hat{\mathbf{k}}d$$

with $a, b, c, d \in \mathbb{R}$ and $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ literals such that $\hat{\mathbf{i}}^2 = \hat{\mathbf{j}}^2 = \hat{\mathbf{k}}^2 = -1$, $\hat{\mathbf{i}}\hat{\mathbf{j}} = \hat{\mathbf{k}}$ and $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}} = -1$. The 4-dimensional space of quaternions is denoted by $\mathbb{H}$. The above introduced properties imply the results shown in Table 2 for the multiplication of two literals.

Notice that multiplication of a real number by a quaternion is commutative, but not the multiplication of two quaternions ($\hat{\mathbf{i}}\hat{\mathbf{j}} = \hat{\mathbf{k}}$ but $\hat{\mathbf{j}}\hat{\mathbf{i}} = -\hat{\mathbf{k}}$).

The scalar and the vector parts of $\underline{\boldsymbol{q}}$, $s_q = a$ and $\boldsymbol{v}_q = \hat{\mathbf{i}}b + \hat{\mathbf{j}}c + \hat{\mathbf{k}}d$ can be defined separately. The vector part can also be seen as an element of a 3-dimensional vector space on $\mathbb{R} : \boldsymbol{v}_q = (b \quad c \quad d)^T$; then we can write $\underline{\boldsymbol{q}} = [s_q \;\; \boldsymbol{v}_q^T]^T$.

Exploiting the properties of Table 2, for any $\underline{\boldsymbol{q}}, \underline{\boldsymbol{q}}' \in \mathbb{H}$, we introduce the definition of product between quaternions, denoted with a new symbol:

$$\underline{\boldsymbol{q}} \odot \underline{\boldsymbol{q}}' \stackrel{\Delta}{=} (\mathbf{1}a + \hat{\mathbf{i}}b + \hat{\mathbf{j}}c + \hat{\mathbf{k}}d)(\mathbf{1}a' + \hat{\mathbf{i}}b' + \hat{\mathbf{j}}c' + \hat{\mathbf{k}}d') = [s_q \;\; \boldsymbol{v}_q^T][s_{q'} \;\; \boldsymbol{v}_{q'}^T]^T$$
$$= [s_q s_{q'} - \boldsymbol{v}_q^T \boldsymbol{v}_{q'} \;\; s_{q'}\boldsymbol{v}_q + s_q \boldsymbol{v}_{q'} + \boldsymbol{v}_q \times \boldsymbol{v}_{q'}]$$

The operations between quaternions can also be represented as matrix operations. To this end, given a quaternion $\underline{\boldsymbol{q}}$ we define:

$$\overset{+}{\mathcal{Q}} = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}, \quad \overline{\mathcal{Q}} = \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} \tag{A.1}$$

Table 2
Products between the units $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$.

	$\hat{\mathbf{i}}$	$\hat{\mathbf{j}}$	$\hat{\mathbf{k}}$
$\hat{\mathbf{i}}$	-1	$\hat{\mathbf{k}}$	$-\hat{\mathbf{j}}$
$\hat{\mathbf{j}}$	$-\hat{\mathbf{k}}$	-1	$\hat{\mathbf{i}}$
$\hat{\mathbf{k}}$	$\hat{\mathbf{j}}$	$-\hat{\mathbf{i}}$	-1

always denoted with the same letter of the quaternion, but capital and in calligraphic style. Using (A.1) it is easy to show that the quaternion multiplication can be written as: $\underline{\boldsymbol{q}} \odot \underline{\boldsymbol{q}}' = \overset{+}{\mathcal{Q}}\underline{\boldsymbol{q}}' = \overline{\mathcal{Q}}'\underline{\boldsymbol{q}}$.

For any quaternion $\underline{\boldsymbol{q}}$ we also define the *conjugate* $\underline{\boldsymbol{q}}^*$ and the cross product matrix operator $\widetilde{\mathbf{Q}}$:

$$\underline{\boldsymbol{q}}^* = \begin{pmatrix} a \\ -b \\ -c \\ -d \end{pmatrix}, \quad \widetilde{\mathbf{Q}} = \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix} \tag{A.2}$$

With the last definitions, it results:

$$\overset{+}{\mathcal{Q}} = \begin{pmatrix} a & -\underline{\boldsymbol{q}}_{1:3}^T \\ \underline{\boldsymbol{q}}_{1:3}^T & a\boldsymbol{I}_3 + \widetilde{\mathbf{Q}} \end{pmatrix} \overline{\mathcal{Q}} = \begin{pmatrix} a & -\underline{\boldsymbol{q}}_{1:3}^T \\ \underline{\boldsymbol{q}}_{1:3}^T & a\boldsymbol{I}_3 - \widetilde{\mathbf{Q}} \end{pmatrix}$$

where $\boldsymbol{I}_3$ is the identity matrix of dimension 3.

By analogy with complex numbers, we define the norm of $\underline{\boldsymbol{q}}$

$$N_{\underline{\boldsymbol{q}}} = \sqrt{a^2 + b^2 + c^2 + d^2}$$

An angle θ can also be associated with a quaternion $\underline{\boldsymbol{q}}$:

$$\cos\theta = \frac{a}{\sqrt{N_{\underline{\boldsymbol{q}}}}}, \quad \sin\theta = \frac{\sqrt{b^2 + c^2 + d^2}}{\sqrt{N_{\underline{\boldsymbol{q}}}}}$$

and, like for complex numbers, the polar form can be deduced:

$$\underline{\boldsymbol{q}} = N_{\underline{\boldsymbol{q}}}(\mathbf{1}\cos\theta + \hat{\underline{\boldsymbol{q}}}\sin\theta) \quad \text{where } N_{\hat{\underline{\boldsymbol{q}}}} = 1$$

If $N_{\underline{\boldsymbol{q}}} = 1$ the quaternion is called *unit quaternion* or, if seen as a 4-dimensional vector, *Euler parameters set*. The set $\mathbb{H}$ of all quaternions include the set of complex numbers and, moreover, with the operations of sum and product of quaternions, it constitutes a vector space on $\mathbb{R}$. A quaternion with the first component set to zero is called a vector quaternion. All geometric vectors may be represented as vector quaternions having the last three components coincident to those of the geometric vector. This means that the vector space of quaternions contains the vector space of geometric vectors.

Quaternions are often used to simplify representations of rotations and translations of a rigid body in the 3-dimensional space. Typical applications are in robotics and in computer graphics.

A.2. Other important properties

The introduction of quaternions in place of other variables, like Euler angles, to describe the orientation of a rigid body in the space, leads to the advantage of being able to eliminate the problem of singularities. Whichever orientation will assume the body, the quaternion associated to it is always valid. The drawback is the increase of the complexity of notation. To be able to manipulate equations involving quaternions, it is convenient to explore some useful properties.

The following ones are valid for any quaternion $\underline{\boldsymbol{q}}$ and come directly from the definitions:

$$\overline{\mathcal{Q}}(\underline{\boldsymbol{q}}) = \overline{\mathcal{Q}}(\underline{\boldsymbol{q}}^*)^T$$
$$\overset{+}{\mathcal{Q}}(\underline{\boldsymbol{q}}) = \overset{+}{\mathcal{Q}}(\underline{\boldsymbol{q}}^*)^T$$
$$\widetilde{\mathbf{Q}}^* = -\widetilde{\mathbf{Q}}$$
$$\widetilde{\mathbf{Q}}^T = -\widetilde{\mathbf{Q}}$$

Moreover, for any quaternion $\underline{\boldsymbol{q}}$, some properties concerning products can be easily proven by substitution:

$$\overline{\mathbf{Q}}\underline{\boldsymbol{q}} = \boldsymbol{0}_{3\times 1} \tag{A.3}$$

$$\dot{\overline{\mathbf{Q}}}\dot{\underline{\boldsymbol{q}}} = \boldsymbol{0}_{3\times 1} \tag{A.4}$$

$$\dot{\overset{+}{\mathcal{Q}}}{}^T \underline{\dot{\boldsymbol{q}}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \underline{\dot{\boldsymbol{q}}}^T \underline{\dot{\boldsymbol{q}}} \tag{A.5}$$

where $\overline{\boldsymbol{Q}}$ is defined like $\overline{\boldsymbol{E}}$ in Eq. (8) for unit quaternions.

Differentiating once and twice Eq. (A.3) we also obtain, respectively

$$\overline{\boldsymbol{Q}}\underline{\dot{\boldsymbol{q}}} + \dot{\overline{\boldsymbol{Q}}}\underline{\boldsymbol{q}} = \boldsymbol{0}_{3\times 1} \tag{A.6}$$

$$\overline{\boldsymbol{Q}}\underline{\ddot{\boldsymbol{q}}} + \ddot{\overline{\boldsymbol{Q}}}\underline{\boldsymbol{q}} = \boldsymbol{0}_{3\times 1} \tag{A.7}$$

Limiting now the attention to unit quaternions (shown below with the symbol $\underline{\boldsymbol{e}}$ to differentiate them from the generic quaternions), from the definition and the property of unitary norm, we have

$$\overset{+}{\boldsymbol{E}}{}^T \overset{+}{\boldsymbol{E}} \underline{\dot{\boldsymbol{e}}} = (\boldsymbol{I}_4 - \underline{\boldsymbol{e}}\underline{\boldsymbol{e}}^T)\underline{\dot{\boldsymbol{e}}} = \underline{\dot{\boldsymbol{e}}} - \underline{\boldsymbol{e}}(\underline{\dot{\boldsymbol{e}}}^T\underline{\boldsymbol{e}})^T = \underline{\dot{\boldsymbol{e}}}$$

$$\overset{\pm}{\mathcal{E}}{}^T \overset{\pm}{\mathcal{E}} = \overset{\pm}{\mathcal{E}} \overset{\pm}{\mathcal{E}}{}^T = \boldsymbol{I}_4$$

In the last equation, the signs + and − over the matrices, must be used coherently: the property is valid only using the same sign over the two matrices involved in the product.

Differentiating the unitary norm constraint with respect to time we can derive some other useful properties

$$\underline{\boldsymbol{e}}^T\underline{\boldsymbol{e}} = 1 \tag{A.8}$$

$$\underline{\dot{\boldsymbol{e}}}^T\underline{\boldsymbol{e}} = 0 \tag{A.9}$$

$$\underline{\ddot{\boldsymbol{e}}}^T\underline{\boldsymbol{e}} + \underline{\dot{\boldsymbol{e}}}^T\underline{\dot{\boldsymbol{e}}} = 0 \tag{A.10}$$

The unitary norm constraint can also be exploited to write

$$\overset{+}{\mathcal{E}}{}^T\underline{\boldsymbol{e}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{A.11}$$

$$\overline{\boldsymbol{E}}\overline{\boldsymbol{E}}^T = \boldsymbol{I}_3 \tag{A.12}$$

At last, differentiating (A.11) and (A.12), we obtain respectively:

$$\overset{+}{\mathcal{E}}{}^T\underline{\dot{\boldsymbol{e}}} + \dot{\overset{+}{\mathcal{E}}}{}^T\underline{\boldsymbol{e}} = \boldsymbol{0}_{4\times 1} \tag{A.13}$$

$$\overline{\boldsymbol{E}}\dot{\overline{\boldsymbol{E}}}{}^T + \dot{\overline{\boldsymbol{E}}}\overline{\boldsymbol{E}}^T = \boldsymbol{0}_{3\times 1} \tag{A.14}$$

Observe that $\overline{\boldsymbol{E}}^T\overline{\boldsymbol{E}} \neq \boldsymbol{I}_4$ and that $\overline{\boldsymbol{E}}^T\overline{\boldsymbol{E}}$ is not invertible since $\rho(\overline{\boldsymbol{E}}^T\overline{\boldsymbol{E}}) \leqslant \rho(\overline{\boldsymbol{E}}) \leqslant 3$. Besides, called $\boldsymbol{X} = \overline{\boldsymbol{E}}^T\overline{\boldsymbol{E}}$, it is easy to prove that $\overline{\boldsymbol{E}}\boldsymbol{X} = \overline{\boldsymbol{E}}$.

Let us now summarize some properties concerning finite rotations of quaternions. Let $\underline{\boldsymbol{r}}^0$ and $\underline{\boldsymbol{r}}^1$ be, respectively, a vector quaternion before and after the rotation induced by the unit quaternion $\underline{\boldsymbol{e}}$. They can be interpreted as the same quaternion represented in the base frame and in the body fixed frame (whose orientation is precisely identified by $\underline{\boldsymbol{e}}$). The following equations are valid:

$$\underline{\boldsymbol{r}}^0 = \boldsymbol{A}_4\underline{\boldsymbol{r}}^1 \tag{A.15}$$

$$\begin{aligned} \boldsymbol{A}_4 &\triangleq \overset{+}{\mathcal{E}}(\underline{\boldsymbol{e}})\overline{\mathcal{E}}(\underline{\boldsymbol{e}}^*) = \overset{+}{\mathcal{E}}(\underline{\boldsymbol{e}})\overline{\mathcal{E}}(\underline{\boldsymbol{e}})^T = \overline{\mathcal{E}}(\underline{\boldsymbol{e}})^T\overset{+}{\mathcal{E}}(\underline{\boldsymbol{e}}) = \overset{+}{\mathcal{E}}(\underline{\boldsymbol{e}}^*)^T\overline{\mathcal{E}}(\underline{\boldsymbol{e}}^*) \\ &= \begin{pmatrix} 1 & \boldsymbol{0}_3^T \\ \boldsymbol{0}_3 & \overset{+}{\boldsymbol{E}}\overline{\boldsymbol{E}}^T \end{pmatrix} \end{aligned} \tag{A.16}$$

$$\underline{\boldsymbol{r}}^1 = \boldsymbol{A}_4^{-1}\underline{\boldsymbol{r}}^0 \tag{A.17}$$

$$\begin{aligned} \boldsymbol{A}_4^{-1} &= \overset{+}{\mathcal{E}}(\underline{\boldsymbol{e}}^*)\overline{\mathcal{E}}(\underline{\boldsymbol{e}}) = \overset{+}{\mathcal{E}}(\underline{\boldsymbol{e}}^*)\overline{\mathcal{E}}(\underline{\boldsymbol{e}}^*)^T = \overset{+}{\mathcal{E}}(\underline{\boldsymbol{e}})^T\overline{\mathcal{E}}(\underline{\boldsymbol{e}}) \\ &= \overline{\mathcal{E}}(\underline{\boldsymbol{e}})\overset{+}{\mathcal{E}}(\underline{\boldsymbol{e}})^T\begin{pmatrix} 1 & \boldsymbol{0}_3^T \\ \boldsymbol{0}_3 & \overline{\boldsymbol{E}}\overset{+}{\boldsymbol{E}}{}^T \end{pmatrix} \end{aligned} \tag{A.18}$$

From Eqs. (A.16) and (A.18) we also define the 3D rotation matrix and its inverse:

$$\boldsymbol{A}_3 \triangleq \overset{+}{\boldsymbol{E}}\overline{\boldsymbol{E}}^T \tag{A.19}$$

$$\boldsymbol{A}_3^{-1} = \overline{\boldsymbol{E}}\overset{+}{\boldsymbol{E}}{}^T \tag{A.20}$$

Observe that $\boldsymbol{A}_4^{-1} = \boldsymbol{A}_4^T$ as well as $\boldsymbol{A}_3^{-1} = \boldsymbol{A}_3^T$.

Finally let $\underline{\boldsymbol{\omega}}^0$ be the quaternion associated to the angular velocity ω expressed in base frame and $\underline{\boldsymbol{\omega}}^1$ be the same quaternion expressed in the body fixed frame. We have

$$\underline{\boldsymbol{\omega}}^0 = 2\underline{\dot{\boldsymbol{e}}} \odot \underline{\boldsymbol{e}}^* = -2\underline{\boldsymbol{e}} \odot \underline{\dot{\boldsymbol{e}}}^* \tag{A.21}$$

$$\underline{\boldsymbol{\omega}}^1 = -2\underline{\dot{\boldsymbol{e}}}^* \odot \underline{\boldsymbol{e}} = 2\underline{\boldsymbol{e}}^* \odot \underline{\dot{\boldsymbol{e}}} \tag{A.22}$$

$$\underline{\boldsymbol{\omega}}^0 = \boldsymbol{A}_4\underline{\boldsymbol{\omega}}^1 \tag{A.23}$$

$$\underline{\boldsymbol{\omega}}^1 = \boldsymbol{A}_4^T\underline{\boldsymbol{\omega}}^0 \tag{A.24}$$

besides, let $\boldsymbol{\omega}^0$ and $\boldsymbol{\omega}^1$ be the angular velocity vectors expressed, respectively, in the base frame and in the body fixed frame. It can be proven that

$$\boldsymbol{\omega}^0 = 2\overset{+}{\boldsymbol{E}}\underline{\dot{\boldsymbol{e}}} \tag{A.25}$$

$$\boldsymbol{\omega}^1 = 2\overline{\boldsymbol{E}}\underline{\dot{\boldsymbol{e}}} \tag{A.26}$$

Observe also that $\underline{\boldsymbol{\omega}} = (0\ \boldsymbol{\omega}^T)^T$.

For the proofs of the above mentioned properties and more details on quaternions (see Chou, 1992; Shivarama and Fahrenthold, 2004).

Appendix B. Some properties on the matrix calculus

It follows a brief list of definitions and properties of matrix operators recalled in this paper.

1. The operator

   $$\text{vec}(\boldsymbol{v}_1 \cdots \boldsymbol{v}_n)$$

where $\boldsymbol{v}_1 \cdots \boldsymbol{v}_n$ are vectors, defines a new vector made of all the input vectors:

$$\text{vec}(\boldsymbol{v}_1 \cdots \boldsymbol{v}_n) = \begin{pmatrix} \boldsymbol{v}_1 \\ \vdots \\ \boldsymbol{v}_n \end{pmatrix} \tag{A.27}$$

The same operator applied to a matrix is meant as applied to the column vectors of such matrix.

2. The Kronecker product between two matrices is defined as follows:

$$\text{Kronecker}(\boldsymbol{A}, \boldsymbol{B}) = \boldsymbol{A} \otimes \boldsymbol{B} = \begin{pmatrix} a_{1,1}\boldsymbol{B} & \dots & a_{1,n}\boldsymbol{B} \\ \vdots & \ddots & \\ a_{n,1}\boldsymbol{B} & & a_{n,m}\boldsymbol{B} \end{pmatrix} \tag{A.28}$$

If $[A] = m \times n$ and $[B] = r \times s$, then the Kronecker product between A and B is a matrix $m \cdot r \times n \cdot s$.

Some important properties are:

(a) $\boldsymbol{A} \otimes (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \otimes \boldsymbol{B} + \boldsymbol{A} \otimes \boldsymbol{C}$

(b) $(\boldsymbol{A} + \boldsymbol{B}) \otimes \boldsymbol{C} = \boldsymbol{A} \otimes \boldsymbol{C} + \boldsymbol{B} \otimes \boldsymbol{C}$

(c) If $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ are four matrices which allow to compute the products $\boldsymbol{A} \cdot \boldsymbol{C}$ and $\boldsymbol{B} \cdot \boldsymbol{D}$, we have

$$(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{C} \otimes \boldsymbol{D}) = (\boldsymbol{A} \cdot \boldsymbol{C}) \otimes (\boldsymbol{B} \cdot \boldsymbol{D})$$

(d) If $\boldsymbol{A}$ and $\boldsymbol{C}$ are two matrices which allow to compute the products $\boldsymbol{A} \cdot \boldsymbol{C}$ we have:

$$(\boldsymbol{A}\otimes\boldsymbol{B})\boldsymbol{C}=(\boldsymbol{AC})\otimes\boldsymbol{B}$$

and if it possible to compute $\boldsymbol{B}\cdot\boldsymbol{C}$ we can also write:

$$(\boldsymbol{A}\otimes\boldsymbol{B})\boldsymbol{C}=\boldsymbol{A}\otimes(\boldsymbol{BC})$$

(e) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are two invertible matrices, we have

$$(\boldsymbol{A}\otimes\boldsymbol{B})^{-1}=\boldsymbol{A}^{-1}\otimes\boldsymbol{B}^{-1}$$

3. Given two 3-dimensional vectors $\boldsymbol{u}^0$ and $\boldsymbol{v}^0$ represented in the same frame, let $\boldsymbol{u}^1$ be the representation of $\boldsymbol{u}^0$ in a different frame and $\boldsymbol{R}$ be the rotation matrix which define the orientation of the frame 1 with respect to the frame 0: $\boldsymbol{u}^0=\boldsymbol{R}\boldsymbol{u}^1$. Then the following holds:

$$\boldsymbol{u}^0\times\boldsymbol{v}^0=\widetilde{\boldsymbol{U}}^0\boldsymbol{v}^0=\boldsymbol{R}\boldsymbol{u}^1\times\boldsymbol{v}^0=\boldsymbol{R}(\boldsymbol{u}^1\times\boldsymbol{R}^T\boldsymbol{v}^0)=\boldsymbol{R}\widetilde{\boldsymbol{U}}^1\boldsymbol{R}^T\boldsymbol{v}^0$$

from which we have

$$\widetilde{\boldsymbol{U}}^0=\boldsymbol{R}\widetilde{\boldsymbol{U}}^1\boldsymbol{R}^T \tag{A.29}$$

4. Let

$$\boldsymbol{Q}=\begin{pmatrix}\overbrace{1\;0\;\cdots\;0}^{n\text{ components}} & 0\quad\cdots\quad 0 & \cdots & 0\\ 0\quad\cdots\;0 & \underbrace{0\;1\;0\;\cdots\;0}_{n\text{ components}} & & \\ \vdots & & \ddots & \\ 0 & & & 0\;\cdots\;0\;1\end{pmatrix} \tag{A.30}$$

dimensionally it is $[\boldsymbol{Q}]=n\times n^2$.

If $\boldsymbol{e}_i$ are $m\times 1$ vectors, it results:

$$\begin{aligned}\mathrm{diag}(\boldsymbol{e}_1^T\boldsymbol{e}_1\cdots\boldsymbol{e}_n^T\boldsymbol{e}_n)&=\begin{pmatrix}\boldsymbol{e}_1^T\boldsymbol{e}_1 & \ldots & 0\\ \vdots & \ddots & \\ 0 & & \boldsymbol{e}_n^T\boldsymbol{e}_n\end{pmatrix}\\ &=\boldsymbol{Q}(\boldsymbol{I}_n\otimes\boldsymbol{E})^T(\boldsymbol{I}_n\otimes\boldsymbol{E})\boldsymbol{Q}^T=\boldsymbol{Q}(\boldsymbol{I}_n\otimes\boldsymbol{E}^T)(\boldsymbol{I}_n\otimes\boldsymbol{E})\boldsymbol{Q}^T\\ &=\boldsymbol{Q}[\boldsymbol{I}_n\otimes(\boldsymbol{E}^T\boldsymbol{E})]\boldsymbol{Q}^T\end{aligned}$$

where $\boldsymbol{E}=(\boldsymbol{e}_1\cdots\boldsymbol{e}_n)$

5. Let

$$\boldsymbol{E}=(\boldsymbol{e}_1\cdots\boldsymbol{e}_n)$$

and

$$\boldsymbol{e}_v=\mathrm{vec}(\boldsymbol{E})$$

where $\boldsymbol{e}_i$ are $m\times 1$ vectors. We have:

$$\begin{aligned}\boldsymbol{E}&=\Big(\underbrace{\underbrace{\boldsymbol{I}_m\boldsymbol{0}_m\ldots\boldsymbol{0}_m}_{}n\text{blocks}\quad\ldots\quad\boldsymbol{0}_m\ldots\boldsymbol{I}_m\ldots\boldsymbol{0}_m\quad\ldots\quad\boldsymbol{0}_m\ldots\boldsymbol{0}_m\boldsymbol{I}_m}_{n\text{ times}}\Big)\\ &\quad\cdot(\boldsymbol{I}_n\otimes\boldsymbol{e}_v)\\ &=[(\,10\ldots0\;\;\ldots\;\;0\ldots1\ldots0\;\;\ldots\;\;0\ldots01\,)\otimes\boldsymbol{I}_m]\cdot(\boldsymbol{I}_n\otimes\boldsymbol{e}_v)\end{aligned}$$

where $\boldsymbol{0}_m=\boldsymbol{0}_{m\times m}$.

6. The matrix $\mathrm{diag}(\boldsymbol{e}_1^T\boldsymbol{e}_1\cdots\boldsymbol{e}_n^T\boldsymbol{e}_n)$ can also be computed starting from $\boldsymbol{e}_v$ instead of $\boldsymbol{E}$. Consider that

$$\boldsymbol{e}_i=\begin{pmatrix}e_{i,1}\\ \vdots\\ e_{i,m}\end{pmatrix}=(\,\boldsymbol{0}_m\quad\ldots\quad\boldsymbol{I}_m\quad\ldots\quad\boldsymbol{0}_m\,)\boldsymbol{e}_v$$

where $\boldsymbol{I}_m$ is the ith block $m\times m$. Transposing we have

$$\boldsymbol{e}_i^T=\boldsymbol{e}_v^T\begin{pmatrix}\boldsymbol{0}_m\\ \vdots\\ \boldsymbol{I}_m\\ \vdots\\ \boldsymbol{0}_m\end{pmatrix}$$

Finally, we can write

$$\mathrm{diag}(\boldsymbol{e}_1^T\boldsymbol{e}_1\ldots\boldsymbol{e}_n^T\boldsymbol{e}_n)=(\boldsymbol{I}_n\otimes\boldsymbol{e}_v^T)\boldsymbol{Q}'\boldsymbol{Q}'^T(\boldsymbol{I}_n\otimes\boldsymbol{e}_v)$$

where

$$\boldsymbol{Q}'=\begin{pmatrix}\left.\begin{matrix}\boldsymbol{I}_m\\ \boldsymbol{0}_{m\times m}\\ \vdots\\ \boldsymbol{0}_{m\times m}\end{matrix}\right\}n\text{ blocks } m\times m & & & \\ & \begin{matrix}\boldsymbol{0}_{m\times m}\\ \boldsymbol{I}_m\\ \vdots\\ \boldsymbol{0}_{m\times m}\end{matrix} & & \\ & \vdots & \ddots & \\ & & & \begin{matrix}\boldsymbol{0}_{m\times m}\\ \vdots\\ \boldsymbol{0}_{m\times m}\\ \boldsymbol{I}_m\end{matrix}\end{pmatrix} \tag{A.31}$$

dimensionally it is $[\boldsymbol{Q}']=n^2m\times nm$. Observe that $\boldsymbol{Q}'=\boldsymbol{Q}^T\otimes\boldsymbol{I}_m$.

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