BETWEENNESS-SELFCENTRIC GRAPHS

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Abstract. The betweenness centrality of a vertex of a graph is the portion of shortest paths between all pairs of vertices passing through that vertex. In this paper, we study properties and constructions of graphs whose vertices have the same value of betweenness centrality.

Keywords: betweenness centrality, betweenness-selfcentric graph

MSC 2010: 05C12

1. Introduction

In applied graph theory, a lot of attention is paid on the analysis of complex networks that represent relations between various objects. One of typical task of such an analysis is the identification of objects which play a key role within the network. The measure of importance of these objects is usually expressed by the values of centrality indices. Given a graph G, a *centrality index* is a function $c\colon V(G) \to \mathbb{R}$ which is invariant under graph isomorphism. The most frequently used centrality indices are vertex degree, eccentricity or sum of all distances from a vertex. Another widely used centrality is the *betweenness centrality*, which measures the amount of information that flows through a network node. It is roughly defined as the portion of shortest paths in a graph which pass through a selected vertex. More precisely, if $\sigma_{u,v}$ is the number of shortest paths with the start vertex u and end vertex v, and $\sigma_{u,v}(x)$ is the number of these u-v-paths passing through the vertex $x \neq u, v$, then

the betweenness of x is $B(x) = \sum\limits_{(u,v) \in V^2} \dfrac{\sigma_{u,v}(x)}{\sigma_{u,v}}$; the symbol $\overline{B}(G)$ will stand for the average of values of betweenness of the vertices of a graph G. Besides practical applications, an attention is recently paid also on graph-theoretical properties of betweenness centrality (see the papers [4], [12], [5], [8]) as the average betweenness centrality of an n-vertex graph G directly depends on the mean distance $\overline{\ell}(G)$ by the formula $\overline{B}(G) = (n-1)(\overline{\ell}(G) - 1)$ (see [7], [2]). The related variant of betweenness, the *edge betweenness centrality* (defined as the sum $\sum\limits_{(u,v) \in V^2} \dfrac{\sigma_{u,v}(e)}{\sigma_{u,v}}$ where $\sigma_{u,v}(e)$ is the number of shortest u-v-paths that pass through the edge e was also studied mainly in the connection with detection of subgroups in social networks, see the paper [10].

According to a selected centrality index, one may consider two kinds of extremal networks: heterogeneous networks with vertices having different centralities and uniform networks with all vertices having the same centrality. For instance, if the centrality index is vertex degree, the extremal networks would correspond to antiregular graphs (that is, graphs with just two vertices of same degree) and regular graphs.

In this paper, we explore *betweenness-selfcentric* graphs: the graphs with vertices having the same betweenness. These graphs seem to be rare – using the graph databases provided by Brendan McKay [1] and Wolfram Mathematica 8 code for betweenness testing, we have found out that, among all connected graphs on at most 10 vertices, only 75 are betweenness–selfcentric (see Figure 3 with list of all such graphs on 4–10 vertices). Yet these graphs constitute a relatively dense family – we show that, for sufficiently large integer n, there are superpolynomially many betweenness-selfcentric graphs on n vertices.

It is easy to see that any vertex-transitive graph is betweenness-selfcentric; on the contrary, we show that there are many betweenness-selfcentric graphs which are not vertex-transitive. Namely, we show that this property holds for the wide class of distance-regular graphs (and its subset of strongly regular graphs) which yields to betweenness-selfcentric graphs with arbitrarily chosen automorphism groups.

As both vertex-transitive and distance-regular graphs are regular, we are interested in constructions of betweenness-selfcentric graphs which are nonregular. Particularly, we ask the following: which degree sequences may betweenness-selfcentric graphs possess? We present several constructions based on graph cloning and local join operation; in addition, we estimate the maximum number of distinct degrees that a betweenness-selfcentric graph may have, and determine the structure of betweenness-selfcentric graphs with an universal vertex.

Throughout this paper, we consider connected graphs without loops or multiple edges; we use the standard graph terminology as used in [3]. Given a graph

$G = (V, E)$, the order and the size of G is the number of vertices and edges of G, respectively. We denote by $\Delta(G)$ the maximum degree of G, and by $N(x)$ the set of neighbours of a vertex x of G. The distance between two vertices u and v is the lenght of the shortest path between both vertices and it is denoted by $d(u, v)$; the maximum distance between the vertices of the graph $\max\limits_{u,v \in V} d(u, v)$ is called the *diameter* of G, diam(G), and we also denote by $N_i(u) = \{v \in V : d(u, v) = i\}$. The complete graph on n vertices is denoted by K_n. A real-valued function f is called *superpolynomial* if it grows faster than any polynomial (that is, if $\lim\limits_{x \to \infty} \dfrac{f(x)}{x^n} = +\infty$ for each positive integer n).

2. General properties

In this section, we present several results on general properties of betweenness-selfcentric graphs.

Theorem 2.1. *Each betweenness-selfcentric graph is 2-connected.*

Proof. By contradiction. Let G be a betweenness-selfcentric graph which is not 2-connected, and let H_1 be an endblock of G (that is, a block which contains only one cutvertex) with the minimum number $h_1 + 1$ of vertices. Let x be the cut-vertex of G which belongs to H_1, let $H_2, \ldots, H_k$ be components of $G - x$ which are different from H_1, let $h_i = |V(H_i)|$ for $i = 2, \ldots, k$, and let y be the vertex with the maximum distance from x in H_1. It is easy to see that $h_1 \geq 2$. Now $B(x) \geq 2 \sum\limits_{\substack{i,j=1 \\ i<j}}^{k} h_i h_j$ (since any pair of vertices from distinct components H_i, H_j contributes at least 1 to the sum for $B(x)$). On the other hand, if $u \in V(H_i), v \in V(H_j), H_i \neq H_j \neq H_1$ or $u \in V(H_i), v \in V(H_1), i \neq 1$, then the pair (u, v) does not contribute to the sum for $B(y)$. We conclude that the only pairs contributing to the sum for $B(y)$ are those ones consisting of vertices of H_1 (excluding the vertex y). Therefore, using the fact that the betweenness of a vertex in an n-vertex graph is at most $(n-1)(n-2)$ (see [8]), we obtain

$$B(y) \leq h_1(h_1 - 1) < h_1^2 < 2 \sum_{\substack{i,j=1 \\ i<j}}^{k} h_i h_j \leq B(x),$$

a contradiction. $\square$

Lemma 2.1. *Let G be a graph of diameter 2. Then, for every vertex $x \in V(G)$,*
$$B(x) = \sum_{\substack{(u,v) \in N^2(x) \\ uv \notin E(G)}} \frac{1}{\sigma_{u,v}}.$$

Proof. By definition,

$$B(x) = \sum_{\substack{(u,v)\in V^2(G) \\ u,v\neq x}} \frac{\sigma_{u,v}(x)}{\sigma_{u,v}} = \sum_{(u,v)\in N^2(x)} \frac{\sigma_{u,v}(x)}{\sigma_{u,v}} +$$

$$(\star) \qquad \sum_{(u,v)\in (V(G)\setminus N(x))^2} \frac{\sigma_{u,v}(x)}{\sigma_{u,v}} + \sum_{\substack{(u,v):u\in N(x), \\ v\notin N(x)}} \frac{\sigma_{u,v}(x)}{\sigma_{u,v}}.$$

Now, if both $u,v \notin N(x)$, then each u-v-path containing x has length 4; since $d(u,v) \leq 2$, we get $\sigma_{u,v}(x) = 0$. If $u \in N(x), v \notin N(x)$ and $uv \in E(G)$, then trivially $\sigma_{u,v}(x) = 0$; if $uv \notin E(G)$, then $d(u,v) = 2$ but each u-v-path passing through x has length 3, so again $\sigma_{u,v}(x) = 0$. Hence, in $(\star)$, the second and third sums are equal to zero. Moreover, since $\sigma_{u,v}(x) = 1$ for nonadjacent $u,v \in N(x)$ and 0 for adjacent ones, the result holds. $\square$

Lemma 2.2. *Let G be betweenness-selfcentric graph of diameter 2, order n and size e. Then for any $x \in V(G)$, $B(x) = n - 1 - \dfrac{2e}{n}$.*

Proof. By the previous Lemma, $B(x) = \sum_{\substack{(u,v)\in N^2(x) \\ uv\notin E(G)}} \dfrac{1}{\sigma_{u,v}}$. Then, in the sum $\sum_{x\in V(G)} B(x)$, each pair (u,v) of nonadjacent vertices contributes $\dfrac{1}{\sigma_{u,v}}$ to each their common neighbour; for this pair, there are $\sigma_{u,v}$ common neighbours, hence, we obtain

$$\sum_{x\in V(G)} B(x) = \sum_{\substack{(u,v)\in V^2(G) \\ uv\notin E(G)}} \sigma_{u,v} \cdot \frac{1}{\sigma_{u,v}} = \sum_{\substack{(u,v)\in V^2(G) \\ uv\notin E(G)}} 1 = n(n-1) - 2e$$

thus, for any $x \in V(G)$, $B(x) = \dfrac{1}{n} \sum_{y\in V(G)} B(y) = n - 1 - \dfrac{2e}{n}$. $\square$

Now we turn our attention to degree sequences of betweenness-selfcentric graphs and the possible values of betweenness that the vertices of betweenness-selfcentric graphs may reach. Our first result shows that a non-complete betweenness-selfcentric graph does not contain an universal vertex (that is, the vertex adjacent to every other vertex):

Theorem 2.2. *Let G be an n-vertex betweenness-selfcentric graph with $\Delta(G) = n - 1$. Then $G \cong K_n$.*

Proof. Let x be vertex of G with $d(x) = n - 1$ and let $H = G - x, V_H = V(H)$. Let $S = \{(u,v) \in V_H^2 : d_H(u,v) \geq 3\}, T = \{(u,v) \in V_H^2 : d_H(u,v) = 2\}$ and $\sigma'_{u,v}$ be the number of shortest u-v-paths in H. Observe that if $(u,v) \in S$, then there is unique shortest u-v-path in G of length 2, and this path contains x; further, if $(u,v) \in T$, then, among all u-v-paths of length 2 in G, only one contains x.

We obtain

$$B(x) = \sum_{(u,v) \in S} \frac{\sigma_{u,v}(x)}{\sigma_{u,v}} + \sum_{(u,v) \in T} \frac{\sigma_{u,v}(x)}{\sigma_{u,v}} = \sum_{(u,v) \in S} 1 + \sum_{(u,v) \in T} \frac{1}{\sigma'_{u,v} + 1},$$

$$B(y) = \sum_{(u,v) \in S} \frac{\sigma_{u,v}(y)}{\sigma_{u,v}} + \sum_{(u,v) \in T} \frac{\sigma_{u,v}(y)}{\sigma_{u,v}} = \sum_{(u,v) \in S} 0 + \sum_{\substack{(u,v) \in T \\ uy, vy \in E(H)}} \frac{1}{\sigma'_{u,v} + 1}.$$

Since G is betweenness-selfcentric, $B(x) = B(y)$, thus,

$$\sum_{(u,v) \in S} 1 + \sum_{(u,v) \in T} \frac{1}{\sigma'_{u,v} + 1} = \sum_{\substack{(u,v) \in T \\ uy, vy \in E(H)}} \frac{1}{\sigma'_{u,v} + 1}.$$

This implies that $S = \emptyset$ and every vertex of H is a common neighbour of each two non-adjacent vertices in H. Now, if there exists a vertex $y \neq x$ which is of degree $< n - 1$, then we obtain

$$B(x) = \sum_{(u,v) \in T} \frac{1}{\sigma'_{u,v} + 1} = \sum_{(u,v) \in T} \frac{1}{(n-3) + 1} = \frac{1}{n-2} \left(|2E(\overline{H})| \right)$$

$$B(y) = \sum_{\substack{(u,v) \in T \\ uy, vy \in E(H)}} \frac{1}{\sigma'_{u,v} + 1} = \sum_{\substack{(u,v) \in T \\ uy, vy \in E(H)}} \frac{1}{(n-3) + 1} < \frac{1}{n-2} \left(|2E(\overline{H})| \right)$$

(since y is non-adjacent to at least one vertex). Hence $B(x) \neq B(y)$, a contradiction.

Thus, in H, each vertex has degree $n - 1$ and we obtain that $G \cong K_n$. $\qquad\square$

The results in Section 3 show that there exist a wide variety of non-isomorphic betweenness-selfcentric graphs with sub-universal vertex (that is, a vertex which is adjacent to all but one vertex); nevertheless, we can show

Theorem 2.3. *Let G be an n-vertex betweenness-selfcentric graph with $\Delta(G) = n - 2$. Then $\mathrm{diam}(G) = 2$.*

Proof. Let u be a vertex of G of degree $n-2$, $N(u) = \{v_1, \ldots, v_{n-2}\}$ and let w be a vertex of G which is not adjacent to u. Since G is connected, there exists $s \in \{1, \ldots, n-2\}$ such that $v_s w \in E(G)$; this implies that $\mathrm{diam}(G) \leq 3$.

Now,

$$B(u) = \sum_{\substack{v_i, v_j \in N(u) \\ v_i v_j \notin E(G)}} \frac{1}{\sigma_{v_i, v_j}} + \sum_{\substack{v_i \in N(u) \\ d(v_i, w) = 3}} \frac{\sigma_{v_i, w}(u)}{\sigma_{v_i, w}} \geq \sum_{\substack{v_i, v_j \in N(u) \cap N(w) \\ v_i v_j \notin E(G)}} \frac{1}{\sigma_{v_i, v_j}} = B(w)$$

Since G is betweenness-selfcentric and $N(u) \cap N(w) \subset N(u)$, we obtain that $\sum_{\substack{v_i \in N(u) \\ d(v_i, w) = 3}} \frac{\sigma_{v_i, w}(u)}{\sigma_{v_i, w}} = 0$ and $\sum_{\substack{v_i, v_j \in N(u) \\ v_i v_j \notin E(G)}} \frac{1}{\sigma_{v_i, v_j}} = \sum_{\substack{v_i, v_j \in N(u) \cap N(w) \\ v_i v_j \notin E(G)}} \frac{1}{\sigma_{v_i, v_j}}$. Thus, if $v_i v_j \notin E(G)$, then $v_i, v_j \in N(w)$.

Consider a vertex $v_k \in N(u)$. If $v_k \in N(w)$, then $d(v_k, w) = 1$; otherwise, either, for each $i \in \{1, \ldots, n-2\} \setminus \{k\}$, $v_k v_i \in E(G)$ (this, however, gives $v_k v_s \in E(G)$ and, subsequently, $d(v_k, w) = 2$), or there is $v_j \in N(u)$ such that $v_k v_j \notin E(G)$ (but this gives $v_k \in N(w)$). In any case, we obtain $\mathrm{diam}(G) = 2$. $\qquad\square$

Now we turn our attention to values of betweenness that the vertices of betweenness-selfcentric graphs may possess; for a betweenness-selfcentric graph G, let $B(G)$ denote the common value of the betweenness of its vertices.

Lemma 2.3. *For any betweenness-selfcentric graph G, $B(G) = 0$ or $B(G) \geq 1$.*

Proof. Fix the number n of vertices of a betweenness-selfcentric graph G. Then the maximum possible number of edges of G is $\binom{n}{2}$ for $G \cong K_n$; in this case, $B(G) = 0$.

Let H be an n-vertex betweenness-selfcentric graph with the second highest number of edges. By Theorem 2.2, $\Delta(H) \leq n-2$ which implies that $|E(H)| \leq \binom{n}{2} - \frac{n}{2}$ and the equality is attained if and only if $n = 2k$ and H is isomorphic to the graph $K_{2k} - kK_2$ of k-dimensional hyperoctahedron. Further, we use Theorem 2 from [8]: if a graph G_2 is the factor subgraph of a graph G_1, then $\overline{B}(G_1) \leq \overline{B}(G_2) - \frac{2(|E(G_1)| - |E(G_2)|)}{n}$.

We distinguish two cases:

(1) Let $n = 2k$. From the above findings, it follows that each n-vertex betweenness-selfcentric graph different from K_n has at least $\frac{n}{2}$ edges less than K_n, hence

$$B(G) = \overline{B}(G) \geq \overline{B}(K_n) + \frac{2(|E(K_n)| - |E(G)|)}{n} \geq 0 + \frac{2\frac{n}{2}}{n} = 1.$$

(2) Let $n = 2k + 1$. Then

$$B(G) = \overline{B}(G) \geq \overline{B}(K_n) + \frac{2(|E(K_n)| - |E(G)|)}{n} \geq 0 + \frac{2\frac{n+1}{2}}{n} = 1 + \frac{1}{n} > 1.$$

$\square$

We do not know an example of betweenness-selfcentric graph with betweenness belonging to $\left(1, \frac{3}{2}\right)$; we conjecture that, for any rational number $\alpha \in \left(\frac{3}{2}, \infty\right)$, there exists a betweenness-selfcentric graph with betweenness centrality equal to α.

In the next, we determine the betweenness of vertices of distance-regular graphs. Recall that a graph G is *distance-regular* if, for any two vertices u and v at distance i, the number of vertices at distance j from u and at distance $i - j$ from v is the same; in other words, the number $p_{ij} = |N_i(u) \cap N_j(v)|$, $i, j = 1, \ldots, D = \text{diam}(G)$, does not depend on u and v. For a distance-regular graph G, one also defines a matrix (called *intersection array* of G)

$$i(G) = \left\{ \begin{array}{ccccc} - & c_1 & \ldots & c_{D-1} & c_D \\ a_0 & a_1 & \ldots & a_{D-1} & a_D \\ b_0 & b_1 & \ldots & b_{D-1} & - \end{array} \right\},$$

where $c_k = |N_{k-1}(u) \cap N(v)|$, $a_k = |N_k(u) \cap N(v)|$ and $b_k = |N_{k+1}(u) \cap N(v)|$, for any two vertices u, v of G, $0 \leq k \leq D$.

Theorem 2.4. *Every distance regular graph is betweenness-selfcentric.*

Proof. Let G be a distance-regular graph and w be arbitrary vertex of G lying on a shortest path between two vertices u, v with the distance $d(u, v) = k$. Suppose that $d(u, w) = l$. By induction on k, it is easy to prove that $\sigma_{uv} = \prod_{i=1}^{k} c_i$. Thus

$$b_{u,v}(w) = \frac{\sigma_{u,v}(w)}{\sigma_{u,v}} = \frac{\sigma_{u,w}\sigma_{w,v}}{\sigma_{u,v}} = \frac{\prod_{i=1}^{l} c_i \prod_{i=1}^{k-l} c_i}{\prod_{i=1}^{k} c_i} = \frac{\prod_{i=1}^{l} c_i}{\prod_{i=k-l}^{k} c_i}.$$

This means that the betweenness does not depend on the choice of vertices u, v, just on the distance between them. Moreover, $B(w) = \sum_{u,v \neq w} b_{u,v}(w)$ does not depend on w, hence, it is equal for all vertices of G. $\square$

In particular, every strongly regular graph is distance-regular, and therefore betweenness-selfcentric. This provides the existence of betweenness-selfcentric non-transitive graphs, according to theorem of Phelps:
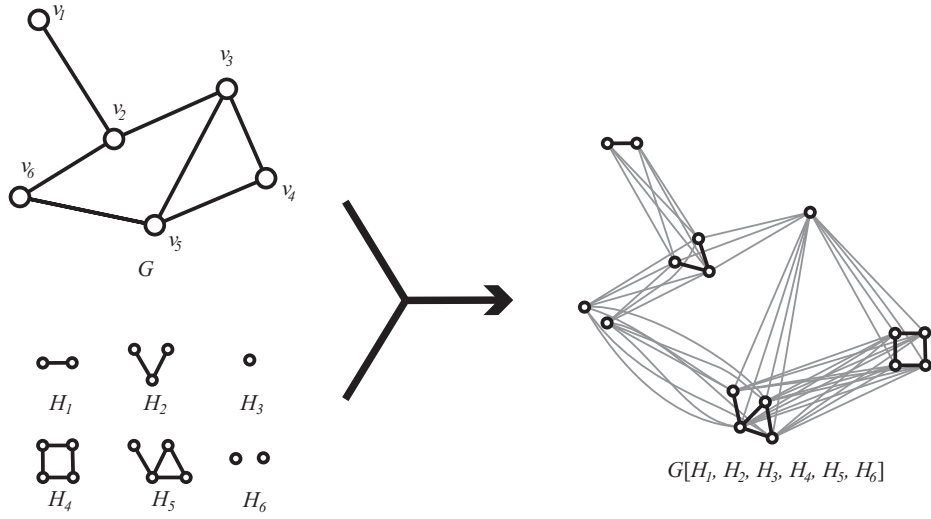
Figure 1: An example of graph cloning

Theorem 2.5 ([11]). *For each finite group A, there exists a strongly regular graph with the automorphism group isomorphic to A.*

Corollary 2.1. *For each finite group A, there exists a betweenness-selfcentric graph whose automorphism group is isomorphic to A.*

3. Constructions of betweenness-selfcentric graphs

In this section, we study betweenness-selfcentric nonregular graphs; we present constructions which yield to betweenness-selfcentric graphs having vertices of several different degrees by using the operation of graph cloning (see, for example, [9]).

Let G be a graph with the vertex set $\{v_1, \ldots, v_n\}$ and $H_1, \ldots, H_n$ be other graphs. The graph $G[H_1, \ldots, H_n]$ is obtained from G by replacing each vertex v_i in G by H_i and taking a join on H_i and H_j whenever $v_i v_j$ is an edge of G (see Figure 1); for $H_1 = \cdots = H_n$, we write $G[H]$ for short. The subgraph H_i in $G[H_1, \ldots, H_n]$ resulted from the above described replacement will be called *substitute* for v_i.

Theorem 3.1. *Let G be betweenness-selfcentric graph and m be an integer. Then $G[K_m]$ is betweenness-selfcentric.*

Proof. Let x be an arbitrary vertex of $G[K_m]$ and x_t be the vertex of G that corresponds to the substitute $K^t \cong K_m$ in $G[K_m]$. Let $u \in K^i, v \in K^j$ be two distinct vertices of $G[K_m]$ (different from x_t) and let u_i, v_j be vertices in G that correspond to substitutes K^i, K^j. Now, $d_{G[K_m]}(u, v) = d_G(u_i, v_j) = d$, $\sigma_{u,v} = \sigma_{u_i,v_j} m^{d-1}$ (each fixed shortest u_i-v_j-path of length d in G results in m^{d-1} shortest u-v-paths in

$G[K_m]$) and if $d_G(u_i, x_l) = h, d_G(x_l, v_j) = k$, then $\sigma_{u,v}(x) = m^{h-1}m^{k-1}\sigma_{u_i,v_j}(x_t) = \sigma_{u_i,v_j}(x_t)m^{h+k-2} = \sigma_{u_i,v_j}(x_t)m^{d-2}$ (any shortest u-v-path passing through x corresponds to some shortest u_i-v_j-path passing through x_t). Hence, we obtain

$$B_{G[K_m]}(x) = \sum_{(u,v)\in V^2(G[K_m])} \frac{\sigma_{u,v}(x)}{\sigma_{u,v}} = \sum_{(u,v)\in V^2(G[K_m])} \frac{m^{d-2}\sigma_{u_i,v_j}(x_t)}{m^{d-1}\sigma_{u_i,v_j}} =$$

$$\frac{1}{m} \sum_{(u,v)\in V^2(G[K_m])} \frac{\sigma_{u_i,v_j}(x_t)}{\sigma_{u_i,v_j}} = \frac{1}{m} \sum_{\substack{(u,v)\in V^2(G[K_m]) \\ u\in K^i, v\in K^j}} \frac{\sigma_{u_i,v_j}(x_t)}{\sigma_{u_i,v_j}} =$$

$$\frac{1}{m} \sum_{(u_i,v_j)\in V^2(G)} m\cdot m\cdot \frac{\sigma_{u_i,v_j}(x_t)}{\sigma_{u_i,v_j}} = m \sum_{(u_i,v_j)\in V^2(G)} \frac{\sigma_{u_i,v_j}(x_t)}{\sigma_{u_i,v_j}} = mB_G(x_t)$$

which is independent of x due to the fact that G is betweenness-selfcentric. $\qquad\square$

Theorem 3.2. *Let G be betweenness-selfcentric k-regular graph and let $H = \bigcup_{i=1}^{\ell} n_i K_{r_i}$ be a disjoint union of n_i complete graphs of orders $r_i, i = 1\ldots,\ell$. Then $G[H]$ is betweenness-selfcentric.*

Proof. Put $m = \sum_{i=1}^{\ell} n_i r_i$. Let x be an arbitrary vertex of $G[H]$ and x_t be the vertex of G that corresponds to a substitute $H^t \cong H$ in $G[H]$. Let $u \in H^i, v \in H^j$ be two distinct nonadjacent vertices of $G[H]$ (different from x) and let u_i, v_j be vertices in G that correspond to substitutes H^i, H^j. If $i \neq j$ then, like in the previous proof, $d_{G[H]}(u,v) = d_G(u_i,v_j) = d$, $\sigma_{u,v} = \sigma_{u_i,v_j}m^{d-1}$ and if $d_G(u_i,x_l) = h, d_G(x_l,v_j) = p$, then $\sigma_{u,v}(x) = m^{h-1}m^{p-1}\sigma_{u_i,v_j}(x_t) = \sigma_{u_i,v_j}(x_t)m^{h+p-2} = \sigma_{u_i,v_j}(x_t)m^{d-2}$. If $i = j$ then $\sigma_{u,v} = km$ and $\sigma_{u,v}(x) = 1$. Hence, we obtain

$$B_{G[H]}(x) = \sum_{\substack{(u,v)\in V^2(G[H]) \\ }} \frac{\sigma_{u,v}(x)}{\sigma_{u,v}} = \sum_{\substack{(u,v)\in V^2(G[H]) \\ u\in H^i, v\in H^j, i\neq j}} \frac{m^{d-2}\sigma_{u_i,v_j}(x_t)}{m^{d-1}\sigma_{u_i,v_j}} +$$

$$\sum_{\substack{(u,v)\in V^2(G[H]) \\ u,v\in H^i}} \frac{\sigma_{u,v}(x)}{\sigma_{u,v}} = \frac{1}{m} \sum_{\substack{(u,v)\in V^2(G[H]) \\ u\in H^i, v\in H^j, i\neq j}} \frac{\sigma_{u_i,v_j}(x_t)}{\sigma_{u_i,v_j}} +$$

$$k\cdot 2\left(\binom{m}{2} - \sum_{i=1}^{\ell} n_i\binom{r_i}{2}\right)\cdot\frac{1}{km} = \frac{1}{m} \sum_{(u_i,v_j)\in V^2(G)} m\cdot m\cdot \frac{\sigma_{u_i,v_j}(x_t)}{\sigma_{u_i,v_j}} +$$

$$\frac{1}{m}(m(m-1) - \sum_{i=1}^{\ell} n_i r_i(r_i-1)) = mB_G(x) + m - 1 + \frac{\sum_{i=1}^{\ell} n_i r_i(r_i-1)}{m}$$

Since G is betweenness-selfcentric, this expression does not depend on x. $\qquad\square$

This result provides a lower bound of order $\Omega(\sqrt{n})$ for the maximum possible number of distinct degrees (the graph *diversity*) of an n-vertex betweenness-selfcentric graph: setting $G \cong K_2$ and $H \cong \bigcup_{i=1}^{t} K_i$, one obtains the betweenness-selfcentric graph $G[H]$ with $2 \sum_{i=1}^{t} i = t(t+1)$ vertices and t distinct degrees $\frac{t(t+1)}{2} + i, i = 0, \ldots, t-1$; with $t(t+1) = n$, we have $t = \frac{\sqrt{4n+1}-1}{2}$. For $n \geq 4$, an easy upper bound $n-3$ can be deduced from Theorem 2.2 and the fact that an n-vertex graph with the maximum degree $n-2$ and $n-2$ distinct degrees contains a vertex of degree 1, but its betweenness is zero. It is an open question whether the upper bound for the diversity of betweenness-selfcentric graph is a sublinear function in terms of number of vertices.

Theorem 3.2 can be also used to show the following

Corollary 3.1. *There exists an integer N such that, for each even integer $n \geq N$, there exist superpolynomially many nonisomorphic nontransitive betweenness-selfcentric graphs on n vertices.*

Proof. Put $n = 2k$ and consider an integer partition $k = k_1 + \cdots + k_l$. Let H be a disjoint union of complete graphs of orders $k_i, i = 1 \ldots, l$ and let $G \cong K_2$. By previous Theorem, $G[H]$ is betweenness-selfcentric. Also, it is easy to see that, using this construction, different number partitions yield nonisomorphic resulting graphs. Thus, the number of n-vertex nonisomorphic betweenness-selfcentric graphs is at least the number of integer partitions $p(k)$ of the number $k = \frac{n}{2}$, which is superpolynomial in k by Hardy-Ramanujan formula $p(k) \sim \frac{1}{4k\sqrt{3}} e^{\pi \sqrt{\frac{2k}{3}}}$. Note also that, for fixed k, the number of integer partitions of k with equal parts is sublinear in k, and any partition of k with at least two distinct parts yields a betweenness-selfcentric graph $G[H]$ which is not vertex transitive; hence, the number of nontransitive graphs arisen from the above construction is also superpolynomial in k. $\qquad \square$

The constructions of Theorems 3.1 and 3.2 are based on cloning betweenness-selfcentric graphs with complete graphs or disjoint unions of complete graphs. However, betweenness-selfcentric graphs can be also constructed by using clonings of any kind of graph, by replacing vertices with different complete graphs, as it can be seen in the next

Theorem 3.3. *Let s, m, n be integers and let G be the graph obtained from two disjoint copies K^1, K^2 of K_s on vertices $u_1, \ldots, u_s$ and $v_1, \ldots, v_s$, respectively, by adding edges $u_i v_i$ for each $i = 1, \ldots, s$. Then the graph $G' = G[K_m, \ldots, K_m,$*
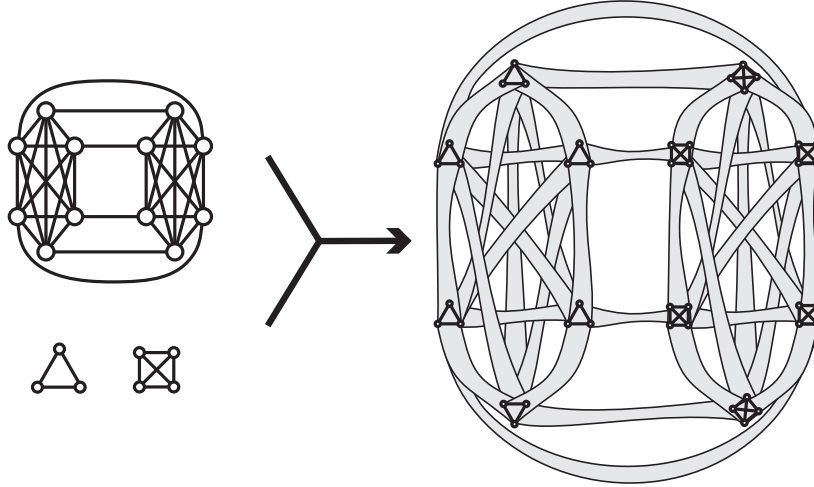
Figure 2: The construction from Theorem 3.2 (grey strips correspond to bunches of edges in local joins)

$K_n, \ldots, K_n$] (where the replacement by K_m is applied on $u_1, \ldots, u_s$) is betweenness-selfcentric.

Proof. It is easy to see that the diameter of G is 2. Denote by $(A_1, \ldots, A_s)$ and $(B_1, \ldots, B_s)$ the partitions of $V(K^1)$ and $V(K^2)$ such that A_i and B_j are vertex sets of substitutes isomorphic to K_m and K_n, respectively, in G'.

Let x be a vertex of G'; without loss of generality, let $x \in A_i$ (the case $x \in V(K^2)$ is similar). Then $N(x) = V(K^1) \cup B_i$. For a pair (u, v) of nonadjacent vertices of G', $\sigma_{u,v} = m + n$. Count now the pairs (u, v) of nonadjacent vertices $u, v \in N(x)$: we obtain that one of u, v is from $V(K^1) \setminus A_i$ and another one is from B_i. Thus, there are $2(sm - m)n = 2mn(s - 1)$ such pairs.

Thus, by Lemma 2.1,

$$B(x) = \sum_{\substack{(u,v) \in N^2(x) \\ uv \notin E(G')}} \frac{1}{\sigma_{u,v}} = \sum_{\substack{(u,v) \in N^2(x) \\ uv \notin E(G')}} \frac{1}{m + n} = \frac{2mn(s - 1)}{m + n}$$

which is independent of the choice of x. □

Thus, we see that the operation of cloning provides, under appropriate choice of basic graph and the substitutes, a wide variety of betweenness-selfcentric graphs. We believe that the underlying global structure of cloned betweenness-selfcentric graphs can be chosen in an arbitrary way:

Conjecture 3.1. *For any graph G on vertex set $\{v_1, \ldots, v_n\}$, there exist graphs $H_1, \ldots, H_n$ such that $G[H_1, \ldots, H_n]$ is betweenness-selfcentric.*

To provide a partial support for this conjecture, we examine betweenness-selfcentric graphs obtained by cloning from graphs from selected graph families.

Theorem 3.4. *The graph $C_4[K_a, K_b, K_c, K_d]$ is betweenness-selfcentric if and only if $bd(b+d) = ac(a+c)$.*

Proof. It is easy to calculate that, for x being a vertex of substitute K_a or K_c in $C_4[K_a, K_b, K_c, K_d]$, $B(x) = 2bd \cdot \dfrac{1}{a+c}$, and if x is a vertex of substitute K_b or K_d, then $B(x) = 2ac \cdot \dfrac{1}{b+d}$; the theorem follows. $\qquad\square$

By numerical calculations, one can check that there exist many quadruples of four distinct integers a, b, c, d satisfying the above condition.

Theorem 3.5. *The graph $C_5[K_a, K_b, K_c, K_d, K_e]$ is betweenness-selfcentric if and only if $a = b = c = d = e$.*

Proof. By straightforward calculation, we obtain that the betweenness of a vertex x which is a vertex of substitute of K_a, K_b, K_c, K_d and K_e in $C_5[K_a, K_b, K_c, K_d, K_e]$ are $\dfrac{2be}{a}, \dfrac{2ac}{b}, \dfrac{2bd}{c}, \dfrac{2ce}{d}$ and $\dfrac{2ad}{e}$, respectively; putting these values equal and forming the corresponding four equations, it can be easily verified that the only solution is $a = b = c = d = e$. $\qquad\square$

The conditions obtained for the existence of betweenness-selfcentric clones from longer cycles are less simple. For example, in $C_6[K_a, K_b, K_c, K_d, K_e, K_f]$, the betweenness of vertices from the substitute K_a is equal to $\dfrac{bf}{a} + \dfrac{b}{ab+de} + \dfrac{f}{af+cd}$ (the other betweenness values can be obtained by cyclic shifting of variables $a, \ldots, f$ in this expression). It is an open question whether there exists a nontrivial solution of the system of five equations formed from comparison of these betweenness values; by numerical calculations, we verified that the only solutions with all six parameters $a, \ldots, f$ being less than 20 are only those ones with all parameters equal.

Theorem 3.6. *Let r, s be integers and $K_{r,s} = (\{x_1, \ldots, x_r\}, \{y_1, \ldots, y_s\}, E)$ be a complete bipartite graph. Then $K_{r,s}[K_{a_1}, \ldots, K_{a_r}; K_{b_1}, \ldots, K_{b_s}]$ is betweenness-selfcentric if and only if*

$$\left(\sum_{i=1}^{r} a_i \right) \left(\sum_{\substack{i,j=1 \\ i<j}}^{r} a_i a_j \right) = \left(\sum_{i=1}^{s} b \right) \left(\sum_{\substack{i,j=1 \\ i<j}}^{s} b_i b_j \right).$$

Proof. Observe that, in the resulting clone, the neighbourhood of any vertex x from a substitute K_{a_t} (for fixed $t = 1, \ldots, r$) consists of $\sum\limits_{j=1}^{s} b_j$ vertices belonging to substitutes $K_{b_1}, \ldots, K_{b_s}$ and of $a_t - 1$ vertices from K_{a_t}; in this neighbourhood, the only missing edges are those ones between two vertices from distinct substitutes K_{b_p}, K_{b_q}, and there are $\sum\limits_{i=1}^{r} a_i$ shortest paths between any pair of such vertices. Thus, by Lemma 2.1, $B(x) = 2\left(\sum\limits_{\substack{i,j=1 \\ i<j}}^{s} b_i b_j\right) \dfrac{1}{\sum\limits_{i=1}^{r} a_i}$. By the symmetry, we obtain that, for any vertex y from a substitute K_{b_t} (for fixed $t = 1, \ldots, s$), $B(x) = 2\left(\sum\limits_{\substack{i,j=1 \\ i<j}}^{r} a_i a_j\right) \dfrac{1}{\sum\limits_{i=1}^{s} b_i}$. Setting $B(x) = B(y)$, the result follows. $\qquad\square$

By applying this result we obtain, for example, that the graphs $K_{2,3}[K_1, K_4; K_1, K_1, K_2]$ and $K_{3,7}[K_1, K_1, K_4; K_1, K_1, K_1, K_1, K_1, K_1, K_2]$ are betweenness-selfcentric. However, it is not clear whether the numbers $a_1, \ldots, a_r$, $b_1, \ldots, b_s$ can be always found for arbitrary complete bipartite graphs.

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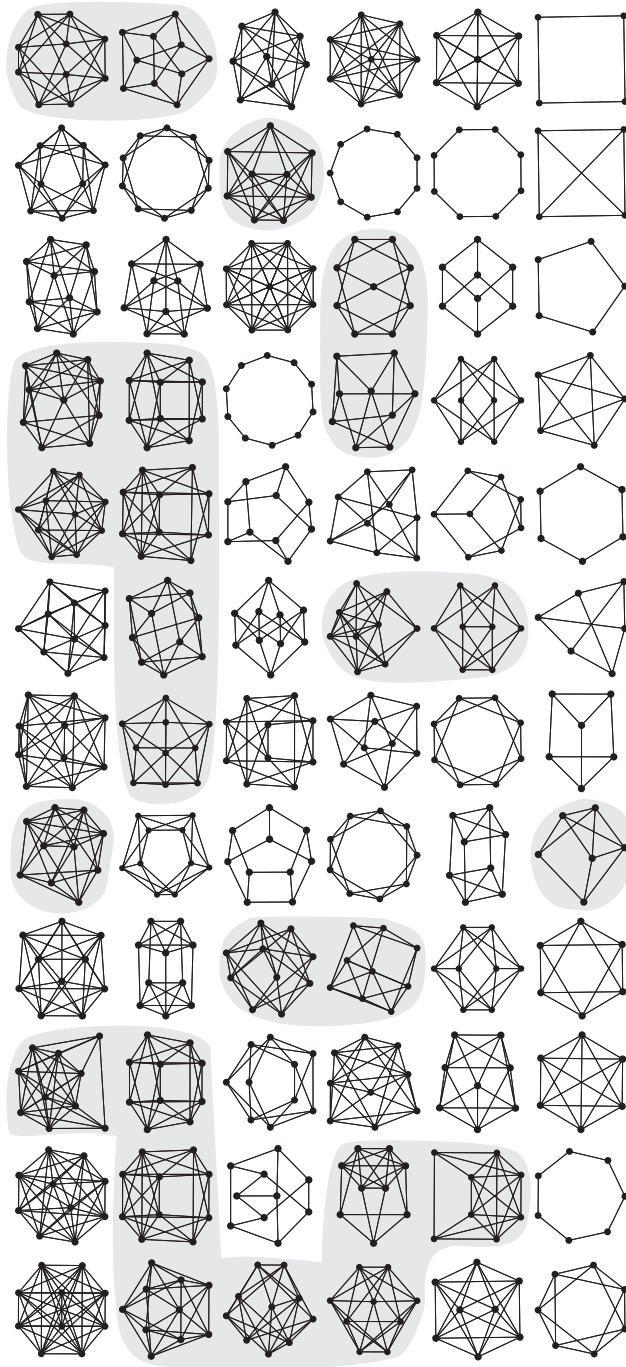*Silvia Gago*, Dept. Matemàtica Aplicada III, EUETIB, Universitat Politècnica de Catalunya,

Figure 3: All betweenness-selfcentric graphs on 4–10 vertices
(the grey graphs are not vertex transitive)

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