

# DARBOUX INTEGRALS FOR SCHRÖDINGER PLANAR VECTOR FIELDS VIA DARBOUX TRANSFORMATIONS

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**ABSTRACT.** In this paper we study the Darboux transformations of planar vector fields of Schrödinger type. Using the isogaloisian property of Darboux transformation we prove the “invariance” of the elements of the “Darboux Theory of Integrability”. In particular, we also show how the shape invariance property of the potential is important in order to preserve the structure of the transformed vector field. Free particle, square well, harmonic oscillator, three dimensional harmonic oscillator and Coulomb potential, are presented as natural examples coming from supersymmetric quantum mechanics.

## INTRODUCTION

In this paper we connect different results of Darboux: We deal with some generalization of the Darboux theory of integrability for planar vector fields and the Darboux transformation of the associated equation.

Darboux published in 1882 the paper [15] in where he presents as a proposition in a general way, which in particular case the history proved to be a notable theorem today known as *Darboux transformation*, see Theorem 12 and Corollary 13. Darboux had shown that whenever one knows to integrate the equation

$$\frac{d^2y}{dx^2} = (\Phi(x) + m)y,$$

for all the values of the constant  $m$ , one can obtain an infinite set of equations, displaying the variable parameter in the same way, which are integrable for any value of the parameter. This proposition also can be found in his book [16, p. 210]. Curiously Darboux transformation was forgotten for a long time until that was recovered as an exercise in 1926 by Ince (see exercises 5, 6 and 7 [20, p. 132]) following closely the formulation of Darboux given in [15, 16]. In 1930, P. Dirac publishes *The Principles of Quantum Mechanics*, in where he gave a mathematically rigorous formulation of quantum

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2000 *Mathematics Subject Classification.* Primary 12H05, Secondary 34A30, 34C14, 81Q60, 32S65.

*Key words and phrases.* Darboux theory of integrability, Darboux transformations, Differential Galois theory, Schrödinger equation, Supersymmetric Quantum Mechanics.

The authors are partial supported by the MICIIN/FEDER grant number MTM2009-06973. CP is additionally partially supported by the MICIIN/FEDER grant MTM2008-03437 and by the Generalitat de Catalunya grant number 2009SGR859.

mechanics. In 1938, J. Delsarte wrote introduced the notion of transformation (transmutation) operator, today know as *intertwining operator* which is closely related with Darboux transformation and ladder operators. In 1941, E. Schrödinger factorized in several ways the hypergeometric equation. This was a byproduct of his *factorization method* originating an approach that can be traced back to Dirac's raising and lowering operators for the harmonic oscillator. Ten years later, in 1951, another factorization method was presented. L. Infeld and T. E. Hull in where they gave the classification of their factorizations of linear second order differential equations for eigenvalue problems of wave mechanics. In 1955, M.M. Crum inspired in the Liouville's work about Sturm-Liouville systems developed one kind of iterative generalization of Darboux transformation. Crum surprisingly did not mention to Darboux. In 1971, G.A. Natanzon studied a general form of the transformation that converts the hypergeometric equation to the Schrödinger equation writing down the most general *solvable potential*, potential for which the Schrödinger equation can be reduced to hypergeometric or confluent hypergeometric form, concept introduced by himself.

Almost one hundred years later of the Darboux's proposition, in 1981, Edward Witten in his renowned paper [39] gave birth to the *Supersymmetric Quantum Mechanics*, discussing general conditions for dynamical supersymmetry breaking. Since the work of Witten, thousands of papers, about supersymmetric quantum mechanics, has been written. The most relevant of these papers was written by L. É. Gendenshtein in 1983, where he presented the *Shape invariance* condition, i.e. preserving the shape under Darboux transformation, and used to find the complete spectra for a broad class of problems including all known exactly solvable problems of quantum mechanics (bound state and reflectionless potentials). Today this kind of exactly solvable potentials satisfying the shape invariance condition are called *Shape invariant potentials*, see [17].

In 2009, in the PhD thesis of the first author which was published as book (see [1] and see also [2]), was presented a Galoisian point of view to Supersymmetric Quantum Mechanics, in particular to Darboux Transformations and shape invariance condition. This paper agree with this point of view.

From the other hand, Darboux in 1878 presented a simple way to construct first integrals and integrating factors for planar polynomial vector fields, see [14]. The key point of his method are the invariant algebraic curves of such vector fields. His approach has been related with problems concernig limit cycles, centers and bifurcation problems, see for instance [21, 28, 35]. Moreover, the geometric scenario of the algebraic curves determines the structure of the vector fields, see [9, 10, 32]. Nowadays Darboux's method has been improved for polynomial vector fields basically taking into

account the exponential factors [8] and the multiplicity of the invariant algebraic curves see for instance [12, 9]. Moreover, the existence of a rational first integral and Darboux's method are related by Jouanolou's results, see [22]. Recently, Jouanolou's work has been reproved using simple arguments of linear algebra, see [29]. Preme and Singer [33] and later Singer in [34] give the relation between elementary/Liouvillian first integrals and integrating factors that are constructed by Darboux's method. Additionally, Darboux's ideas have been extended to a particular class of non-autonomous vector fields, see [27].

In this paper we use the results of [3]. For a given polynomial vector field we obtain its associated foliation being of Riccati type. Then, this Riccati vector field is analyzed throughout differential Galois theory to connect the existence of integrating factors/first integrals of the vector field with the Galoisian structure of the Riccati equation.

Our main aim in this paper is to relate Darboux's theory of integrability for planar vector fields with Schrödinger equation and Darboux transformations of the same equation. More concrete, for a given polynomial vector field, whose foliation is of Riccati type, we obtain an analogous of Darboux transformation to vector fields of Riccati type, that is, we consider polynomial vector fields of the form

$$\begin{aligned}\dot{v} = \frac{dv}{dt} &= S_0(x) + S_1(x)v + S_2(x)v^2, \\ \dot{x} = \frac{dx}{dt} &= N(x),\end{aligned}$$

with  $S_0, S_1, S_2 \in \mathbb{C}[x]$ . In particular for non-relativistic quantum mechanics we have  $S_2(x) = -N(x), S_1(x) = 0$  and  $S_0(x) = N(x)(V(x) - \lambda)$  with  $V(x) = T(x)/N(x)$ ,  $N, T \in \mathbb{C}[x]$  and  $\lambda$  a constant.

Hence, we deal with systems of the form

$$\begin{aligned}\frac{dv}{dt} &= N(x) (V(x) - \lambda - v^2), \\ \frac{dx}{dt} &= N(x),\end{aligned}\tag{1}$$

or equivalently we have the polynomial vector field

$$\begin{aligned}\frac{dv}{dt} &= T(x) - N(x)\lambda - N(x)v^2, \\ \frac{dx}{dt} &= N(x),\end{aligned}\tag{2}$$

of degree  $m = \max\{\deg T(x), \deg N(x) + 2\}$ . The associated foliation of system (1) is

$$v' = \frac{dv}{dx} = V(x) - \lambda - v^2, \quad (3)$$

with  $V \in \mathbb{C}(x)$ .

The structure of the paper is the following: In Section 1 we present the basic concepts of differential Galois theory, Schrödinger equation, Darboux transformation and Darboux theory of integrability of planar polynomial vector fields. In Section 2 we present our main results. More concrete, in Theorem 21 we show that the strong isogalossian property of the Darboux Transformation and the shape invariance property of the potential are necessary in order to preserve the rational structure of the transformed vector field. Moreover, in Proposition 27 we present a condition for the non-existence of a rational first integral of a Riccati equation. At the end, in Section 3 we give several examples as applications of our results: we consider the case of free particle, square well potential, harmonic oscillator, three dimensional harmonic oscillator and the Coulomb potential.

## 1. THEORETICAL BACKGROUND

In this section we establish the theoretical background that we use in this work.

**1.1. Differential Galois Theory.** We start regarding an algebraic model for functions and the corresponding Galois theory, see [37], which is known as *differential Galois theory*. The structure of the following preliminaries also can be found in [1, 2].

**Definition 1** (Differential Fields). *Let  $F$  be a commutative field of characteristic zero. A derivation of  $F$  is a map  $\frac{d}{dx} : F \rightarrow F$  satisfying*

$$\frac{d}{dx}(a + b) = \frac{da}{dx} + \frac{db}{dx}, \quad \frac{d}{dx}(a \cdot b) = \frac{da}{dx} \cdot b + a \cdot \frac{db}{dx},$$

for all  $a, b \in F$ . We then say that  $(F, \frac{d}{dx})$  (or just  $F$ , when there is no ambiguity) is a differential field with the derivation  $\frac{d}{dx}$ .

We assume that  $F$  contains an element  $x$  such that  $\frac{d}{dx}(x) = 1$ . Let  $\mathcal{C}$  denote the field of constants of  $F$ :

$$\mathcal{C} = \left\{ c \in F \mid \frac{dc}{dx} = 0 \right\}.$$

It is also of characteristic zero and will be assumed to be algebraically closed.

Throughout this paper, the *coefficient field* for a differential equation will be defined as the smallest differential field containing all the coefficients of the equation.

In particular we deal with second order linear homogeneous differential equations, i.e., equations of the form

$$\frac{d^2y}{dx^2} + \alpha \frac{dy}{dx} + \beta y = 0, \quad \alpha, \beta \in F. \quad (4)$$

So the rest of the theory will be considered in this context, although there is naturally extended for higher dimension.

**Definition 2** (Picard-Vessiot Extension). *Consider the differential equation (4). Let  $L$  be a differential field containing  $F$  (a differential extension of  $F$ ). We say that  $L$  is a Picard-Vessiot extension of  $F$  for the differential equation (4) if there exist two linearly independent  $y_1, y_2 \in L$  solutions of the differential equation (4) such that  $L = F\langle y_1, y_2 \rangle$  (i.e  $L = F(y_1, y_2, dy_1/dx, dy_2/dx)$ ) and  $L$  and  $F$  has the same field of constants  $\mathcal{C}$ .*

In what follows, we choose a Picard-Vessiot extension and the term “solution of (4)” will mean “solution of (4) in  $L$ ”. So any solution of the differential equation (4) is a linear combination (over  $\mathcal{C}$ ) of  $y_1$  and  $y_2$ .

**Definition 3** (Differential Galois Groups). *An  $F$ -automorphism  $\sigma$  of the Picard-Vessiot extension  $L$  is called a differential automorphism if it leaves  $F$  fixed and commutes with the derivation. This means that*

$$\sigma \left( \frac{da}{dx} \right) = \frac{d}{dx}(\sigma(a)) \quad \text{for all } a \in L \quad \text{and} \quad \forall a \in F, \quad \sigma(a) = a.$$

*The group of all differential automorphisms of  $L$  over  $F$  is called the differential Galois group of  $L$  over  $F$  and is denoted by  $\text{DGal}(L/F)$ .*

Given  $\sigma \in \text{DGal}(L/F)$ , we see that  $\{\sigma y_1, \sigma y_2\}$  are also solutions of the equation (4). Hence there exists a matrix

$$A_\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{C}),$$

such that

$$\sigma \begin{pmatrix} y_1 & y_2 \end{pmatrix} = \begin{pmatrix} \sigma(y_1) & \sigma(y_2) \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix} A_\sigma.$$

As  $\sigma$  commutes with the derivation, this extends naturally to an action on a fundamental solution matrix of the companion first order system associated with the equation (4).

$$\sigma \left( \begin{pmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{pmatrix} \right) = \begin{pmatrix} \sigma(y_1) & \sigma(y_2) \\ \sigma \left( \frac{dy_1}{dx} \right) & \sigma \left( \frac{dy_2}{dx} \right) \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{pmatrix} A_\sigma.$$

This defines a faithful representation  $\text{DGal}(L/K) \rightarrow \text{GL}(2, \mathbb{C})$  and it is possible to consider  $\text{DGal}(L/K)$  as a subgroup of  $\text{GL}(2, \mathbb{C})$ . It depends on the choice of the fundamental system  $\{y_1, y_2\}$ , but only up to conjugacy.

Recall that an algebraic group  $G$  is an algebraic manifold endowed with a group structure. Let  $\mathrm{GL}(2, \mathbb{C})$  denote, as usual, the set of invertible  $2 \times 2$  matrices with entries in  $\mathbb{C}$  (and  $\mathrm{SL}(2, \mathbb{C})$  be the set of matrices with determinant equal to 1). A linear algebraic group will be a subgroup of  $\mathrm{GL}(2, \mathbb{C})$  equipped with a structure of algebraic group. One of the fundamental results of the Picard-Vessiot theory is the following theorem (see [23, 24]).

**Theorem 4.** *The differential Galois group  $\mathrm{DGal}(L/F)$  is an algebraic subgroup of  $\mathrm{GL}(2, \mathbb{C})$ .*

In fact, the differential Galois group measures the algebraic relations between the solutions (and their derivatives) of the differential equation (4). It is sometimes viewed as the object which should tell “what algebra sees of the dynamics of the solutions”.

In an algebraic group  $G$ , the largest connected algebraic subgroup of  $G$  containing the identity, noted  $G^\circ$ , is a normal subgroup of finite index. It is often called the *connected component of the identity*. If  $G = G^\circ$  then  $G$  is a *connected group*.

When  $G^\circ$  satisfies some property, we say that  $G$  virtually satisfies this property. For example, *virtually solvability* of  $G$  means solvability of  $G^\circ$  (see [38]).

**Theorem 5** (Lie-Kolchin). *Let  $G \subseteq \mathrm{GL}(2, \mathbb{C})$  be a virtually solvable group. Then  $G^\circ$  is triangularizable, i.e it is conjugate to a subgroup of upper triangular matrices.*

The following are some examples of subgroups of  $\mathrm{SL}(2, \mathbb{C})$ .

**Reducible subgroups** These are the groups which leave a non-trivial subspace of  $V$  invariant. They are classified in two categories.

*Diagonal groups:* the identity group:

$$\{e\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

the  $n$ -roots group(cyclic):

$$\mathbb{G}^{[n]} = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}, \quad c^n = 1 \right\},$$

the multiplicative group:

$$\mathbb{G}_m = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}, \quad c \in \mathbb{C}^* \right\}.$$

*Triangular groups:* the additive group:

$$\mathbb{G}_a = \left\{ \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, \quad d \in \mathbb{C} \right\},$$

the  $n$ -quasi-roots group:

$$\mathbb{G}^{\{n\}} = \left\{ \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}, \quad c^n = 1, \quad d \in \mathbb{C} \right\},$$

the Borel group:

$$\mathbb{B} = \mathbb{C}^* \ltimes \mathbb{C} = \left\{ \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}, \quad c \in \mathbb{C}^*, \quad d \in \mathbb{C} \right\}.$$

**Irreducible subgroups** The infinite dihedral group (also called meta-abelian group):

$$\mathbb{D}_\infty = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}, \quad c \in \mathbb{C}^* \right\} \cup \left\{ \begin{pmatrix} 0 & d \\ -d^{-1} & 0 \end{pmatrix}, \quad d \in \mathbb{C}^* \right\}$$

and its finite subgroups  $\mathbb{D}_{2n}$  (where  $c$  and  $d$  spans the  $n$ -th roots of unity).

There are also three other finite irreducible (primitive) groups: the tetrahedral group  $A_4^{\text{SL}_2}$  of order 24, the octahedral group  $S_4^{\text{SL}_2}$  of order 48, and the icosahedral group  $A_5^{\text{SL}_2}$  of order 120.

Throughout this work we will use the following definition.

**Definition 6** (Liouvillian Integrability). *We say that the linear differential equation (4) is (Liouville) integrable if the Picard-Vessiot extension  $L \supset F$  is obtained as a tower of differential fields  $F = L_0 \subset L_1 \subset \dots \subset L_m = L$  such that  $L_i = L_{i-1}(\eta)$  for  $i = 1, \dots, m$ , where either*

- (1)  $\eta$  is algebraic over  $L_{i-1}$ , that is  $\eta$  satisfies a polynomial equation with coefficients in  $L_{i-1}$ .
- (2)  $\eta$  is primitive over  $L_{i-1}$ , that is  $\frac{d\eta}{dx} \in L_{i-1}$ .
- (3)  $\eta$  is exponential over  $L_{i-1}$ , that is  $\left(\frac{d\eta}{dx}\right)/\eta \in L_{i-1}$ .

We remark that in the usual terminology of differential algebra for integrable equations the corresponding Picard-Vessiot extensions are called *Liouvillian*. From now on we say that an equation is integrable whether it is integrable in the sense of differential Galois theory according to Definition 6.

The following theorem is due to Kolchin.

**Theorem 7.** *The equation (4) is integrable if and only if  $\text{DGal}(L/F)$  is virtually solvable.*

There is an algorithm due to Kovacic [25] that decides about the integrability of the equation (4) in the case where  $F = \mathbb{C}(x)$ . In practice, Kovacic's algorithm deals with the reduced form of equation (4), namely with the form  $y'' = ry$ , where  $r$  is a rational function. Kovacic used the fact that  $\text{DGal}(L/F) \subseteq \text{SL}(2, \mathbb{C})$ , in order to separate his algorithm in three cases for the integrability of the equation (4).

**Case 1.:**  $\text{DGal}(L/F)$  is reducible,

**Case 2.:**  $\text{DGal}(L/F)$  is irreducible ( $\subseteq \mathbb{D}_\infty$ ),

**Case 3.:**  $\text{DGal}(L/F)$  is finite primitive ( $A_4^{\text{SL}_2}, S_4^{\text{SL}_2}, A_5^{\text{SL}_2}$ ).

It is well known that the case 3 in Kovacic's algorithm is very complicated and there are a few examples of differential equations solved by Kovacic's that fall in case 3.

Sometimes, Kovacic's algorithm can be avoided, this is the case for the *Whittaker equation*

$$\frac{d^2y}{dx^2} = \left( \frac{1}{4} - \frac{\kappa}{x} + \frac{4\mu^2 - 1}{4x^2} \right) y, \quad (5)$$

where its integrability was analyzed in [30] and summarized as follow:

**Theorem 8** (Martinet-Ramis,[30]). *The Whittaker differential equation (5) is integrable if and only if either,  $\kappa + \mu \in \frac{1}{2} + \mathbb{N}$ , or  $\kappa - \mu \in \frac{1}{2} + \mathbb{N}$ , or  $-\kappa + \mu \in \frac{1}{2} + \mathbb{N}$ , or  $-\kappa - \mu \in \frac{1}{2} + \mathbb{N}$ .*

One equation that can be transformed into a Whittaker equation is the *Bessel equation*

$$\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \frac{x^2 - \mu^2}{x^2} y = 0. \quad (6)$$

In this way, Martinet-Ramis theorem can be applied directly over the Bessel equation and we recover the ancient and well known result:

**Corollary 9.** *The Bessel differential equation (6) is integrable if and only if  $\mu \in \frac{1}{2} + \mathbb{Z}$ .*

**1.2. The Schrödinger Equation.** Here we first introduce the Schrödinger equation and then we present the preliminaries about Schrödinger equation from a Galoisian point of view, see [1, 2].

In classical mechanics for a particle of mass  $m$  moving under the action of a potential  $U$  the Hamiltonian is given by

$$H = \frac{\|\vec{p}\|^2}{2m} + U(\vec{x}), \quad \vec{p} = (p_1, \dots, p_n), \quad \vec{x} = (x_1, \dots, x_n),$$

and corresponds to the energy (kinetic plus potential). From the other hand in quantum mechanics the momentum  $\vec{p}$  is given by  $\vec{p} = -i\hbar\nabla$ , where  $\hbar$  is the Planck constant and  $\nabla$  is the Laplacian operator. In this case the Hamiltonian operator  $H$  is the Schrödinger (non-relativistic, stationary) operator which is given by

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{x}),$$

where  $\vec{x}$  is the *coordinate* and  $V(\vec{x})$  is the *potential or potential energy*. The Schrödinger equation is given by  $H\Psi = \lambda\Psi$ , where the eigenfunction  $\Psi$  is the *wave function* and the eigenvalue  $\lambda \in \mathbb{R}_+$  is the *energy level*. The solutions  $\Psi$  of the Schrödinger equation are the *states* of the particle. In [36] it can be found the details about the mathematical foundations of quantum mechanics for the Schrödinger equation.

According to [13, 39], a *supersymmetric quantum mechanical system* is one in which there are operators  $Q_i$  that commute with the Hamiltonian  $\mathcal{H}$  and satisfying

$$\begin{aligned} [Q_i, \mathcal{H}] &= Q_i H - H Q_i = 0, \\ \{Q_i, Q_j\} &= Q_i Q_j + Q_j Q_i = \delta_{ij} \mathcal{H} \quad \text{and} \quad \delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases} \end{aligned}$$

For  $n = 2$ , we obtain the simplest example of a supersymmetric quantum mechanical system. In this case we have that  $x \in \mathbb{R}$ . The physical interpretation involves a spin one half particle moving on the line. In this case the *supercharges*  $Q_i$  are defined as

$$Q_{\pm} = \frac{\sigma_1 p \pm \sigma_2 W(x)}{2}, \quad Q_+ = Q_1, \quad Q_- = Q_2,$$

where  $p = -i\hbar \frac{d}{dx}$ ,  $W : \mathbb{R} \rightarrow \mathbb{R}$  is the *superpotential* and  $\sigma_i$  are the *Pauli spin matrices* defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that the supercharges define a system of two ordinary differential equations.

The operator  $\mathcal{H}$ , satisfying  $Q_i \mathcal{H} = \mathcal{H} Q_i$  and  $2Q_i^2 = \mathcal{H}$ , is

$$\mathcal{H} = \frac{I_2 p^2 + I_2 W^2(x) + \hbar \sigma_3 \frac{d}{dx} W(x)}{2} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \quad \text{with} \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and  $H_-$  and  $H_+$  are the *supersymmetric partner Hamiltonians* and are given by

$$H_{\pm} = -\frac{1}{2} \frac{d^2}{dx^2} + V_{\pm}, \quad V_{\pm} = \left( \frac{W}{\sqrt{2}} \right)^2 \pm \frac{1}{\sqrt{2}} \frac{d}{dx} \left( \frac{W}{\sqrt{2}} \right). \quad (7)$$

The potentials  $V_{\pm}$  are called *supersymmetric partner potentials*.

Now we follow [1, 2] in order to present the Schrödinger equation in the context of differential Galois theory. Thus, the Schrödinger equation (stationary and unidimensional) now is written as

$$H\Psi = \lambda\Psi, \quad H = -\frac{d^2}{dx^2} + V(x), \quad V \in F, \quad (8)$$

where  $F$  is a differential field (with  $\mathcal{C} = \mathbb{C}$  as field of constants). We will deal with the integrability of equation (8) in agreement with our definition of *integrability*, i.e., in the sense of differential Galois theory, see Definition 6.

We introduce the following notations.

- Denote by  $\Lambda \subseteq \mathbb{C}$  the set of eigenvalues  $\lambda$  such that equation (8) is integrable (Definition 6).
- Denote by  $\Lambda_+$  the set  $\{\lambda \in \Lambda \cap \mathbb{R} : \lambda \geq 0\}$  and by  $\Lambda_-$  the set  $\{\lambda \in \Lambda \cap \mathbb{R} : \lambda \leq 0\}$ .
- Denote by  $L_\lambda$  the Picard-Vessiot extension of equation (8). Thus, the differential Galois group of (8) is denoted by  $\text{DGal}(L_\lambda/K)$ .

The set  $\Lambda$  will be called *the algebraic spectrum* (or alternatively *the Liouvilian spectral set*) of  $H$ . We remark that  $\Lambda$  can be  $\emptyset$ , i.e.,  $\text{DGal}(L_\lambda/F) = \text{SL}(2, \mathbb{C}) \forall \lambda \in \mathbb{C}$ . On the other hand, by Theorem 7, if  $\lambda_0 \in \Lambda$  then  $(\text{DGal}(L_{\lambda_0}/F))^0 \subseteq \mathbb{B}$ .

**Definition 10** (Algebraically Solvable and Quasi-Solvable Potentials). *We say that the potential  $V(x) \in F$  is:*

- *an algebraically solvable potential when  $\Lambda$  is an infinite set, or*
- *an algebraically quasi-solvable potential when  $\Lambda$  is a non-empty finite set, or*
- *an algebraically non-solvable potential when  $\Lambda = \emptyset$ .*

*When  $\text{Card}(\Lambda) = 1$ , we say that  $V(x) \in F$  is a trivial algebraically quasi-solvable potential.*

The following theorem proved in [1], see also [2], shows that if there exist more than one eigenvalue in the algebraic spectrum of the Schrödinger operator with  $F = \mathbb{C}(x)$ , then we can't fall in case 3 of Kovacic's algorithm.

**Theorem 11.** *Consider the Schrödinger equation (8) with  $F = \mathbb{C}(x)$  and Picard-Vessiot extension  $L_\lambda$ . If  $\text{DGal}(L_0/F)$  is finite primitive, then  $\text{DGal}(L_\lambda/F)$  is not finite primitive for all  $\lambda \in \Lambda \setminus \{0\}$ .*

From [1, 2] note that the known cases of rational potentials in quantum mechanics leads to Schrödinger equations falling in case 1 of Kovacic's algorithm. Additionally, if  $\text{Card}(\Lambda) > 1$  then any algebraic solution of the Riccati equation associated to the Schrödinger equation (8) is a root of a polynomial of degree at most two.

**1.3. Darboux Transformation.** Darboux gave in [15] a transformation that allow us to transform some type of differential equations into other differential equations preserving the type. Here we present the Darboux's result taken faithfully from [15]. Note that  $P, Q, R$  according to [15] are arbitrary functions of  $x$ .

**Theorem 12** (General Darboux Transformation, [15]). *Suppose that we know how to integrate, for any value of the constant  $m$ , the following equation*

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + (Q - mR)y = 0. \quad (9)$$

*If  $\theta \neq 0$  is an integral of the equation*

$$\frac{d^2\theta}{dx^2} + P\frac{d\theta}{dx} + (Q - mR)\theta = 0,$$

*then the function*

$$u = \frac{\frac{dy}{dx} - \frac{1}{\theta}\frac{d\theta}{dx}y}{\sqrt{R}}, \quad (10)$$

*will be an integral of the equation*

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + \left( \theta\sqrt{R}\frac{d}{dx}\left(\frac{P}{\theta\sqrt{R}}\right) - \theta\sqrt{R}\frac{d^2}{dx^2}\left(\frac{1}{\theta\sqrt{R}}\right) - mR \right) u = 0, \quad (11)$$

*for  $m \neq 0$ .*

Darboux in [15, 16] presented the particular case for  $R = 1$ ,  $P = 0$  and  $\Phi = -Q$  which today is known as *Darboux transformation*, but really is a corollary of the *general Darboux transformation* given in Theorem 12.

**Corollary 13** (Darboux Transformation, [15, 16]). *Suppose that we know how to integrate*

$$\frac{d^2\xi}{dx^2} = (\Phi(x) + m)\xi, \quad (12)$$

*for any value of  $m$ . If  $\theta$  satisfies the equation*

$$\frac{d^2\theta}{dx^2} = (\Phi(x) + m_1)\theta,$$

*then the function*

$$u = \frac{d\xi}{dx} - \frac{1}{\theta}\frac{d\theta}{dx}\xi,$$

*will be an integral of the equation*

$$\frac{d^2u}{dx^2} = \left( \theta\frac{d^2}{dx^2}\left(\frac{1}{\theta}\right) - m_1 + m \right) u, \quad (13)$$

*for  $m \neq m_1$ . Furthermore,*

$$\theta\frac{d^2}{dx^2}\left(\frac{1}{\theta}\right) - m_1 = \Phi(x) - 2\frac{d}{dx}\left(\frac{1}{\theta}\frac{d\theta}{dx}\right) = 2\left(\frac{1}{\theta}\frac{d\theta}{dx}\right)^2 - \Phi(x) - 2m_1.$$

In practice, we need at least two values of  $m$  to apply the Darboux's results.

In quantum mechanics, particularly in Supersymmetric Quantum Mechanics, the function  $\Phi$  corresponds to the potential  $V$  and the parameter

$m$  corresponds to  $-\lambda$ . We also denote by  $DT$  the Darboux Transformation.

The following results corresponds to the Darboux transformation in the Galoisian and quantum mechanic formalism, see [1, 2].

**Theorem 14** (Galoisian version of DT). *Assume  $H_{\pm} = -\frac{d^2}{dx^2} + V_{\pm}(x)$  and  $\Lambda \neq \emptyset$ . Consider the Schrödinger equation  $H_- \Psi^{(-)} = \lambda \Psi^{(-)}$  with  $V_-(x) \in F$ . Let DT be the transformation such that  $V_- \mapsto V_+$ ,  $\Psi^{(-)} \mapsto \Psi^{(+)}$ ,  $F \mapsto \tilde{F}$ . Then for the Schrödinger equation  $H_+ \Psi^{(+)} = \lambda \Psi^{(+)}$  with  $V_+(x) \in \tilde{F}$  the following statements holds.*

- i)  $DT(V_-) = V_+ = \Psi_{\lambda_1}^{(-)} \frac{d^2}{dx^2} \left( \frac{1}{\Psi_{\lambda_1}^{(-)}} \right) + \lambda_1 = V_- - 2 \frac{d^2}{dx^2} (\ln \Psi_{\lambda_1}^{(-)})$ ,  
 $DT(\Psi_{\lambda_1}^{(-)}) = \Psi_{\lambda_1}^{(+)} = \frac{1}{\Psi_{\lambda_1}^{(-)}}$ , where  
 $\Psi_{\lambda_1}^{(-)}$  is a particular solution of  $H_- \Psi^{(-)} = \lambda_1 \Psi^{(-)}$ ,  $\lambda_1 \in \Lambda$ .
- ii)  $DT(\Psi_{\lambda}^{(-)}) = \Psi_{\lambda}^{(+)} = \frac{d}{dx} \Psi_{\lambda}^{(-)} - \frac{d}{dx} (\ln \Psi_{\lambda_1}^{(-)}) \Psi_{\lambda}^{(-)}$ ,  $\lambda \neq \lambda_1$ , where  
 $\Psi_{\lambda}^{(-)}$  is the general solution of  $H_- \Psi^{(-)} = \lambda \Psi^{(-)}$  for  $\lambda \in \Lambda \setminus \{\lambda_1\}$   
and  
 $\Psi_{\lambda}^{(+)}$  is the general solution of  $H_+ \Psi^{(+)} = \lambda \Psi^{(+)}$  also for  $\lambda \in \Lambda \setminus \{\lambda_1\}$ .

**Remark 15.** *According to [1, 2] we have that a transformation is called isogaloisian whether it preserves the differential Galois group: the initial equation and the transformed equation have the same differential Galois group. Furthermore, when the differential field and the Picard-Vessiot extension are preserved, then the transformation is called strong isogaloisian.*

In agreement with Theorem 14 and Remark 15, we obtain the following results, see [1, 2] for complete statements and proofs.

**Proposition 16.** *In general, DT is isogaloisian and virtually strong isogaloisian. Furthermore, if  $\partial_x (\ln \Psi_{\lambda_1}^{(-)}) \in F$ , then DT is strong isogaloisian.*

**Proposition 17.** *The supersymmetric partner potentials  $V_{\pm}$  are rational functions if and only if the superpotential  $W$  is a rational function.*

In this work, and in particular in Section 2 we will use the following result, (see also [1, 2]).

**Corollary 18.** *The superpotential  $W \in \mathbb{C}(x)$  if and only if DT is strong isogaloisian.*

**Remark 19.** *The examples presented in Section 2 are in agreement with the previous results, since  $\tilde{F} = F$  due to the fact that the superpotential  $W$  belongs to  $F = \mathbb{C}(x)$ . The following definition is a partial Galoisian adaptation of the original definition given in [17] ( $F = \mathbb{C}(x)$ ). The complete Galoisian adaptation is given when  $F$  is any differential field, see [1, 2].*

**Definition 20** (Rational Shape Invariant Potentials, [1, 2]). *Assume  $V_{\pm}(x; \mu) \in \mathbb{C}(x; \mu)$ , where  $\mu$  is a family of parameters. The potential  $V = V_- \in \mathbb{C}(x)$  is said to be rational shape invariant potential with respect to  $\mu$  and  $\lambda = \lambda_n$  being  $n \in \mathbb{Z}_+$ , if there exists a function  $f$  such that*

$$V_+(x; a_0) = V_-(x; a_1) + R(a_1), \quad a_1 = f(a_0), \quad \lambda_n = \sum_{k=2}^{n+1} R(a_k), \quad \lambda_0 = 0.$$

Hence the form of the potentials  $V_{\pm}$  are preserved up to parameters. By Theorem 14 and Propositions 16 and 17 we have the following result, see [1, 2].

**Theorem 21.** *Consider  $H\Psi^{(-)} = \lambda_n\Psi^{(-)}$  with Picard-Vessiot extension  $L_n$ , where  $n \in \mathbb{Z}_+$ . If  $V = V_- \in \mathbb{C}(x)$  is a shape invariant potential with respect to  $\lambda = \lambda_n$ , then*

$$\text{DGal}(L_{n+1}/\mathbb{C}(x)) = \text{DGal}(L_n/\mathbb{C}(x)), \quad n > 0.$$

Hence for rational shape invariant potentials Galois group is preserved due to Picard Vessiot extension and differential field.

**1.4. Darboux's theory of integrability for planar polynomial vector fields.** In this subsection we present the basic ideas of Darboux's method for planar polynomial vector fields. Our aim is not to give an extensive presentation of this theory but we only present the basic results that we use in Section 2.

We consider the *polynomial (differential) system* in  $\mathbb{C}^2$  defined by

$$\frac{dx}{dt} = \dot{x} = P(x, y), \quad \frac{dy}{dt} = \dot{y} = Q(x, y), \quad (14)$$

where  $P$  and  $Q$  are polynomials in the variables  $x$  and  $y$ . The independent variable  $t$  can be real or complex. Associated to the polynomial differential system (14) in  $\mathbb{C}^2$  there is the *polynomial vector field*

$$X = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}, \quad (15)$$

in  $\mathbb{C}^2$  and its associated foliation is given by

$$Pdy - Qdx = 0.$$

An algebraic curve  $f(x, y) = 0$  in  $\mathbb{C}^2$  with  $f \in \mathbb{C}[x, y]$  is an *invariant algebraic curve* of a polynomial system (14) if

$$X(\log(f)) = K, \quad (16)$$

for some polynomial  $K \in \mathbb{C}[x, y]$  called the *cofactor* of the invariant algebraic curve  $f = 0$ . Note that due to the definition (16) we have that the degree of the cofactor  $K$  is less than the degree of the polynomial vector field (14). Moreover, the curve  $f = 0$  is formed by trajectories of the vector field  $X$ .

For a given system (14) of degree  $m$  the calculation of all the invariant algebraic curves is a very hard problem because in general we don't know

about the maximum degree of such curves. However, imposing additionally conditions either for the structure of the system or for the nature of the curves we can have an evidence of a such a bound [11, 6, 7].

Let  $h, g \in \mathbb{C}[x, y]$  be relatively prime in the ring  $\mathbb{C}[x, y]$ . The function  $\exp(g/h)$  is called an *exponential factor* of the polynomial system (14) if there is a polynomial  $L \in \mathbb{C}[x, y]$  (also called cofactor) that satisfies the equation

$$X\left(\frac{g}{h}\right) = L. \quad (17)$$

Additionally, is assumed that the degree of the cofactor  $L$  is less than the degree of the polynomial vector field  $X$ . It turns out that if  $h$  is not a constant polynomial, then  $h = 0$  is one of the invariant algebraic curve of (14).

The following theorem is a short version of Darboux theory of integrability for planar polynomial differential systems. For more details and also for higher dimension see [26].

**Theorem 22.** *Suppose that a polynomial system (14) of degree  $m$  admits*

- $p$  irreducible invariant algebraic curves  $f_i = 0$  with cofactors  $K_i$  for  $i = 1, \dots, p$ .
- $q$  exponential factors  $F_j$  with cofactors  $L_j$  for  $j = 1, \dots, q$ .

*Then the following statements hold.*

(a) *The function*

$$f_1^{\lambda_1} \dots f_p^{\lambda_p} F_1^{\mu_1} \dots F_q^{\mu_q}, \quad (18)$$

*is a first integral of (14) if and only if*

$$\sum_{i=1}^p \lambda_i K_i + \sum_{i=1}^q \mu_i L_i = 0. \quad (19)$$

(b) *The function (18) is an (Darboux) integrating factor of the vector field (15) if and only if there is a linear combination of the cofactors and of the divergence of the system (14),*

$$\sum_{i=1}^p \lambda_i K_i + \sum_{i=1}^q \mu_i L_i + (P_x + Q_y) = 0.$$

Integrating factors for planar systems can be thought as reparameterization of the independent variable (time) that transforms the original system into a divergence free system.

As we show in Theorem 22 for polynomial vector fields the invariant algebraic curves (and because of their multiplicity, see [12]) the exponential factors are the basic elements in order to construct first integrals/integrating factors. As we will see the method of Theorem 22 also works for more general expressions of vector fields, curves, exponential factors and cofactors, see also [18, 19].

**Definition 23** (Generalized exponential factor). Consider  $S(x) \in \mathbb{C}(x)$ . We define a generalized exponential factor of a polynomial vector field  $X$  any expression of the form  $F = e^{\int S(x)}$  which satisfies  $X(F) = LF$  with  $L$  an appropriate function called generalized cofactor.

**Definition 24** (Generalized Darboux function). We define the Generalized Darboux function any expression of the form

$$\begin{aligned} & (y - S_1(x))^{\lambda_1} \cdots (y - S_p(x))^{\lambda_p} e^{\int S(x)} \\ & = (y - S_1(x))^{\lambda_1} \cdots (y - S_p(x))^{\lambda_p} e^{\tilde{S}(x)} \prod (x - x_i)^{b_i}, \end{aligned} \quad (20)$$

with  $S_i, S, \tilde{S}, g \in \mathbb{C}(x), x_i, b_i, \lambda_i \in \mathbb{C}$  and  $p \in \mathbb{Z}_+$ , see also [40].

An integrating factor of the form (20) will be called *generalized Darboux integrating factor* and similarly a first integral of the form (20) will be called *generalized Darboux first integral*.

**Remark 25.** We consider the Schrödinger equation (8) with  $F = \mathbb{C}(x)$  and the associated Ricatti equation (3). We also consider  $v_1(x), v_2(x), v_3(x)$  particular solutions of equation (3) with  $V \in \mathbb{C}(x)$  and we write  $V(x) = T(x)/N(x)$  with  $T, N \in \mathbb{C}[x]$ . Note that an associated polynomial vector field to equation (3) can be written into the form (2).

(a) Following [3, 25, 40] we can distinguish the following cases about the type of the first integrals of equation (3) or equivalently of the polynomial vector field (2).

Case-1 (i) If only  $v_1(x) \in \mathbb{C}(x)$  then the vector field (2) has a first integral of Darboux–Schwarz–Christoffel Type, namely a first integral of the form

$$I(v, x) = \frac{1}{y - v_1(x)} e^{g(x)} \prod (x - x_i)^{a_i} + \int e^{g(u)} \prod (u - x_i)^{a_i - m_i} P(u) du, \quad (21)$$

with  $P \in \mathbb{C}[x], g \in \mathbb{C}(x), x_i, a_i \in \mathbb{C}$  and  $m_i \in \mathbb{Z}_+$ , see also [40].

(ii) If both  $v_1(x), v_2(x) \in \mathbb{C}(x)$  then the vector field (2) has a generalized Darboux first integral of the form

$$I(v, x) = \frac{-v + v_2(x)}{-v + v_1(x)} e^{\int (v_2(x) - v_1(x)) dx}. \quad (22)$$

Furthermore if the Galois group of the Schrödinger equation (8) is a cyclic group of order  $k$  then the  $k$  power of the first integral (22) is rational, see also [3].

Case 2 If  $v_1$  is a solution of a quadratic polynomial then the vector field (2) has a first integral of hyperelliptic type.

Case 3 *If all  $v_1, v_2, v_3$  are algebraic over  $\mathbb{C}(x)$  then the vector field (2) has a rational first integral of the form*

$$I(v, x) = \frac{(v_2 - v_1)(v_1 - v)}{(v_3 - v_1)(v_2 - v)}.$$

(b) *In general, knowing one algebraic solution  $v_1(x)$  of equation (3) we can obtain a second solution of (3), namely  $v_2(x) = v_1 + \frac{e^{-2 \int v_1 dx}}{\int e^{-2 \int v_1 dx} dx}$ . Then, the vector field (2) has always a first integral of the form (22) and can be rewritten either as Darboux–Schwarz–Christoffel Type or as a generalized Darboux function or as a first integral of hyperelliptic type or as a rational first integral.*

*Hence, the  $\text{Gal}(L/F)$  of the Schrödinger equation (8) is virtually solvable if and only if the first integral of the vector field (2) can be written in one of the forms appearing in statement (a).*

Additionally in [3] appears the following result about the existence of a rational first integral of a polynomial vector field whose foliation is of Riccati type.

**Theorem 26.** *The Galois group of the equation  $y'' = r(x)y$  with  $r \in \mathbb{C}(x)$  is finite if and only if there exist a rational first integral for the associated polynomial vector field of the corresponding Riccati equation.*

Note that the Schrödinger equation (8) can be always written into the form  $y'' = r(x)y$  and so we can always apply Theorem 26.

## 2. MAIN RESULTS

From now on we consider the Schrödinger equation (8) with potential  $V = V_-$ :

$$\Psi''^{(-)} = (V_-(x) - \lambda)\Psi^{(-)}, V_- = \frac{T}{N}, T, N \in \mathbb{C}[x], \lambda \in \Lambda, \text{Car}(\Lambda) > 1, F = \mathbb{C}(x). \quad (23)$$

After the change of coordinates  $\zeta^{(-)} = \Psi^{(-)'} / \Psi^{(-)}$ , equation (8) can be written as

$$\zeta^{(-)'} = V_- - \lambda - \zeta^{(-)2}, \quad (24)$$

and we associate to equation (24) the polynomial vector field

$$X_\lambda^{(-)} = (T - \lambda N - N\zeta^{(-)2}) \frac{\partial}{\partial \zeta^{(-)}} + N \frac{\partial}{\partial x}, \quad (25)$$

of degree  $m = \max\{\deg T(x), \deg N(x) + 2\}$ . By Propositions 16, 17 and Corollary 18, we have that for given  $V_- \in F$  and  $W = \zeta_0^{(-)} = \Psi_0^{(-)'} / \Psi_0^{(-)} \in F$  then we obtain that  $V_+ \in F$ , i.e., the Darboux Transformation DT is strong isogalossian. Moreover, the superpotential  $W$  is rational. The applications considered in this work correspond to this case.

In the next proposition we present a result about the non-existence of a rational first integral.

**Proposition 27.** *Consider the Schrödinger equation (23). If  $D\text{Gal}(L_\lambda/F)$  is not cyclic then the associated foliation of the Schrödinger equation (23) has not rational first integrals.*

*Proof.* It follows directly from Theorems 11 and 26.  $\square$

**Lemma 28.** *Consider  $\Psi_\lambda^{(-)}(x)$  a solution of the Schrödinger equation (23) and we denote by*

$$\zeta_\lambda^{(-)}(x) = \frac{\left(\Psi_\lambda^{(-)}(x)\right)'}{\Psi_\lambda^{(-)}(x)} = \left(\ln\left(\Psi_\lambda^{(-)}(x)\right)\right)', \quad \lambda \in \Lambda. \quad (26)$$

Then for all  $\lambda \in \Lambda$  the vector field (25) admits the following.

- Invariant curve  $f_\lambda^{(-)}(\zeta^{(-)}, x) = -\zeta^{(-)} + \zeta_\lambda^{(-)}(x)$  with generalized co-factor

$$K_\lambda^{(-)}(\zeta^{(-)}, x) = -N(x) \left(\zeta^{(-)} + \zeta_\lambda^{(-)}(x)\right).$$

- Generalized exponential factor  $F_\lambda^{(-)}(\zeta^{(-)}, x) = e^{\int \left(\frac{1}{2} \frac{N'(x)}{N(x)} + \zeta_\lambda^{(-)}(x)\right) dx}$  with generalized cofactor  $L_\lambda^{(-)}(\zeta^{(-)}, x) = N'(x)/2 + N(x)\zeta_\lambda^{(-)}(x)$ .
- An integrating factor of the form

$$R_\lambda^{(-)}(\zeta^{(-)}, x) = \frac{e^{\int \left(-\frac{N'(x)}{N(x)} - 2\zeta_\lambda^{(-)}(x)\right) dx}}{\left(-\zeta^{(-)} + \zeta_\lambda^{(-)}(x)\right)^2}.$$

- A first integral of the form

$$I_\lambda^{(-)}(\zeta^{(-)}, x) = \frac{-\zeta^{(-)} + \zeta_{(\lambda,2)}^{(-)}}{-\zeta^{(-)} + \zeta_{(\lambda,1)}^{(-)}} e^{\int \left(\zeta_{(\lambda,2)}^{(-)} - \zeta_{(\lambda,1)}^{(-)}\right) dx},$$

$$\text{with } \zeta_{(\lambda,1)} \text{ as in (26) and } \zeta_{(\lambda,2)}^{(-)} = \zeta_{(\lambda,1)}^{(-)} + \frac{e^{-2 \int \zeta_{(\lambda,1)}^{(-)}}}{\int e^{-2 \int \zeta_{(\lambda,1)}^{(-)}}}.$$

*Proof.* Note that  $\zeta_\lambda^{(-)}(x)$  is a solution of the associated equation  $\zeta^{(-)'} = V_-(x) - \lambda - \zeta^{(-)2}$  for all  $\lambda \in \Lambda$ . Using the expression of the vector field (25)

we have

$$\begin{aligned} X_\lambda^{(-)} \left( f_\lambda^{(-)}(\zeta^{(-)}, x) \right) &= (T - \lambda N - N\zeta^{(-)2})(-1) + N\zeta_\lambda^{(-)'} = N(\zeta^{(-)2} - \zeta_\lambda^{(-)2}) \\ &= K_\lambda^{(-)}(\zeta^{(-)}, x) \cdot f_\lambda^{(-)}(\zeta^{(-)}, x). \end{aligned}$$

Note that the curve  $f_\lambda^{(-)}$  is polynomial in the variable  $\zeta^{(-)}$  but it could be not polynomial in the variable  $x$ . Direct computations shows that system (24) admits the generalized exponential factor  $F_\lambda^{(-)}(\zeta^{(-)}, x) = e^{\int \left( \frac{1}{2} \frac{N'(x)}{N(x)} + \zeta_\lambda^{(-)}(x) \right) dx}$  with generalized cofactor  $L_\lambda^{(-)}(\zeta^{(-)}, x) = N'(x)/2 + \zeta_\lambda^{(-)}(x)$ . Note that vector field (25) has divergence  $\text{div}(X_\lambda) = -2N\zeta^{(-)} + N'(x)$ . Hence, we have that

$$-2K_\lambda^{(-)} - 2L_\lambda^{(-)} + \text{div}(X_\lambda^{(-)}) = 0,$$

and similarly to Theorem 22(b) we have that

$$R_\lambda^{(-)} = \frac{1}{f_\lambda^{(-)2} F_\lambda^{(-)2}},$$

is an integrating factor for the vector field (25).

If  $\Psi_{(\lambda,1)}^{(-)}(x)$  a solution of the Schrödinger equation (23) then  $\zeta_{(\lambda,1)}^{(-)}$  is a solution of the Riccati equation (24). Then, according to Remark (25)(b) we have that  $\zeta_{(\lambda,2)}^{(-)}$  is another solution of the Riccati equation (24). In this case the vector field (25) admits the two invariant curves

$$\begin{aligned} f_{(\lambda,1)}^{(-)}(\zeta^{(-)}, x) &= -\zeta^{(-)} + \zeta_{(\lambda,1)}^{(-)}(x), \\ f_{(\lambda,2)}^{(-)}(\zeta^{(-)}, x) &= -\zeta^{(-)} + \zeta_{(\lambda,2)}^{(-)}(x) \end{aligned}$$

with generalized cofactors

$$\begin{aligned} K_{(\lambda,1)}^{(-)}(\zeta^{(-)}, x) &= -N(x) \left( \zeta^{(-)} + \zeta_{(\lambda,1)}^{(-)}(x) \right), \\ K_{(\lambda,2)}^{(-)}(\zeta^{(-)}, x) &= -N(x) \left( \zeta^{(-)} + \zeta_{(\lambda,2)}^{(-)}(x) \right). \end{aligned}$$

The two Generalized exponential factors

$$\begin{aligned} F_{(\lambda,1)}^{(-)}(\zeta^{(-)}, x) &= e^{\int \left( \frac{1}{2} \frac{N'(x)}{N(x)} + \zeta_{\lambda,1}^{(-)}(x) \right) dx}, \\ F_{(\lambda,2)}^{(-)}(\zeta^{(-)}, x) &= e^{\int \left( \frac{1}{2} \frac{N'(x)}{N(x)} + \zeta_{\lambda,2}^{(-)}(x) \right) dx} \end{aligned}$$

with generalized cofactors

$$\begin{aligned} L_{(\lambda,1)}^{(-)}(\zeta^{(-)}, x) &= N'(x)/2 + N(x)\zeta_{\lambda,1}^{(-)}(x), \\ L_{(\lambda,2)}^{(-)}(\zeta^{(-)}, x) &= N'(x)/2 + N(x)\zeta_{\lambda,2}^{(-)}(x). \end{aligned}$$

Note that it holds

$$-K_{(\lambda,1)}^{(-)} + K_{(\lambda,2)}^{(-)} - L_{(\lambda,1)}^{(-)} + L_{(\lambda,2)}^{(-)} = 0,$$

and similar to Theorem 22(a) we have that the vector field  $X_\lambda^{(-)}$  admits the first integral

$$I_{(\lambda)}^{(-)} = \frac{f_{(\lambda,2)}^{(-)} \cdot F_{(\lambda,2)}^{(-)}}{f_{(\lambda,1)}^{(-)} \cdot F_{(\lambda,1)}^{(-)}}.$$

Hence, the lemma is proved.  $\square$

**Proposition 29.** Consider  $\Psi_0^{(-)}(x)$  a particular solution of the Schrödinger equation  $H_- \Psi^{(-)} = \lambda \Psi^{(-)}$  for  $\lambda = 0$  and let  $\zeta_0^{(-)}(x) = \left( \ln \left( \Psi_0^{(-)}(x) \right) \right)'$ . Then after the Darboux transformation  $DT$  for all  $\lambda \neq 0$  equation (24) becomes

$$\zeta^{(+)\prime} = V_+ - \lambda - \zeta^{(+)^2}, \quad (27)$$

with  $V_+ = -\zeta_0^{(-)\prime} + \zeta_0^{(-)^2}$  and its associated vector field is

$$X_\lambda^{(+)} = \left( -\zeta_0^{(-)\prime} + \zeta_0^{(-)^2} - \lambda - \zeta^{(+)^2} \right) \frac{\partial}{\partial \zeta^{(+)}} + \frac{\partial}{\partial x}. \quad (28)$$

The vector field (28) for all  $\lambda \neq 0$  admits the following.

- Invariant curve

$$f_\lambda^{(+)}(\zeta^{(+)}, x) = -\zeta^{(+)} + \zeta_\lambda^{(-)} + \left( \ln \left( \zeta_\lambda^{(-)} - \zeta_0^{(-)} \right) \right)'$$

with generalized cofactor

$$K_\lambda^{(+)}(\zeta^{(+)}, x) = -\zeta^{(+)} - \zeta_\lambda^{(-)} - \left( \ln \left( \zeta_\lambda^{(-)} - \zeta_0^{(-)} \right) \right)'$$

- Generalized exponential factor

$$F_\lambda^{(+)}(\zeta^{(+)}, x) = (\zeta_\lambda^{(-)} - \zeta_0^{(-)}) e^{\int \zeta_\lambda^{(-)}}.$$

with generalized cofactor

$$L_\lambda^{(+)} = \zeta_\lambda^{(-)} + \left( \ln \left( \zeta_\lambda^{(-)} - \zeta_0^{(-)} \right) \right)'$$

- An integrating factor

$$R_\lambda^{(+)}(\zeta^{(+)}, x) = \frac{e^{-2 \int \zeta_\lambda^{(-)}}}{\left( \zeta_\lambda^{(-)} - \zeta_0^{(-)} \right)^2 \left( -\zeta^{(+)} + \zeta_\lambda^{(-)} + \left( \ln \left( \zeta_\lambda^{(-)} - \zeta_0^{(-)} \right) \right) \right)^2}, \quad \lambda \neq 0.$$

- A first integral of the form

$$I_\lambda^{(+)}(\zeta^{(+)}, x) = \frac{-\zeta^{(+)} + \zeta_{(\lambda,2)}^{(-)} + \left(\ln\left(\zeta_{(\lambda,2)}^{(-)} - \zeta_0^{(-)}\right)\right)'}{-\zeta^{(+)} + \zeta_{(\lambda,1)}^{(-)} + \left(\ln\left(\zeta_{(\lambda,1)}^{(-)} - \zeta_0^{(-)}\right)\right)'} \\ \left(\frac{\zeta_{(\lambda,2)}^{(-)} - \zeta_0^{(-)}}{\zeta_{(\lambda,1)}^{(-)} - \zeta_0^{(-)}}\right) e^{\int \left(\zeta_{(\lambda,2)}^{(-)} - \zeta_{(\lambda,1)}^{(-)}\right) dx}.$$

*Proof.* For  $\lambda = 0$  we consider  $\Psi_0^{(-)}$  a particular solution of the Schrödinger equation  $H_- \Psi^{(-)} = \lambda \Psi^{(-)}$ . We denote by  $\zeta_0^{(-)} = (\ln(\Psi_0^-))'$ . Then from Theorem 14 for  $\lambda \neq 0$  we have

$$\Psi_\lambda^{(+)} = \Psi_\lambda^{(-)'} - \zeta_0^{(-)} \Psi_\lambda^{(-)} = e^{\int \zeta_\lambda^{(-)}} (\zeta_\lambda^{(-)} - \zeta_0^{(-)}),$$

and we have use that  $\zeta_\lambda^{(-)} = \Psi_\lambda^{(-)'} / \Psi_\lambda^{(-)}$ . Note that  $\Psi_\lambda^{(+)}$  is a solution of the Schrödinger equation  $H_+ \Psi^{(+)} = \lambda \Psi^{(+)}$ . Let

$$\zeta_\lambda^{(+)} = \left(\ln(\Psi_\lambda^{(+)})\right)' = \left(\ln\left(e^{\int \zeta_\lambda^{(-)}} (\zeta_\lambda^{(-)} - \zeta_0^{(-)})\right)\right)'.$$

According to Theorem 14 we have

$$V_+ = \Psi_0^{(-)} \left(\frac{1}{\Psi_0^{(-)}}\right)'' = \Psi_0^{(-)} \left(-\frac{\zeta_0^{(-)}}{\Psi_0^{(-)}}\right)' = -\zeta_0^{(-)'} + \zeta_0^{(-)2},$$

and we additionally consider the change of variables  $\zeta^{(+)} = (\ln(\Psi^{(+)})\right)'$ . Then the Schrödinger equation  $H_+ \Psi^{(+)} = \lambda \Psi^{(+)}$  falls in equation (27).

Therefore, similar to Lemma 28, the vector field (28) for all  $\lambda \neq 0$  admits the invariant curve  $f_\lambda^{(+)}$  with generalized cofactor  $K_\lambda^{(+)}$  given by

$$f_\lambda^{(+)}(\zeta^{(+)}, x) = -\zeta^{(+)} + \zeta_\lambda^{(+)} = -\zeta^{(+)} + \zeta_\lambda^{(-)} + \left(\ln\left(\zeta_\lambda^{(-)} - \zeta_0^{(-)}\right)\right)', \\ K_\lambda^{(+)}(\zeta^{(+)}, x) = -\zeta^{(+)} - \zeta_\lambda^{(+)} = -\zeta^{(+)} - \zeta_\lambda^{(-)} - \left(\ln\left(\zeta_\lambda^{(-)} - \zeta_0^{(-)}\right)\right)'.$$

Additionally, for all  $\lambda \neq 0$  admits the generalized exponential factor

$$F_\lambda^{(+)}(\zeta^{(+)}, x) = e^{\int \zeta_\lambda^{(+)}} = e^{\int \left(\zeta_\lambda^{(-)} + \left(\ln\left(\zeta_\lambda^{(-)} - \zeta_0^{(-)}\right)\right)'\right)} = (\zeta_\lambda^{(-)} - \zeta_0^{(-)}) e^{\int \zeta_\lambda^{(-)}}.$$

with generalized cofactor

$$L_\lambda^{(+)} = \zeta_\lambda^{(+)} = \zeta_\lambda^{(-)} + \left(\ln\left(\zeta_\lambda^{(-)} - \zeta_0^{(-)}\right)\right)'.$$

Note that the vector field (28) has divergence  $\operatorname{div} X = -2\zeta^{(+)}$  and for all  $\lambda \neq 0$  it holds

$$-2K_\lambda^{(+)} - 2L_\lambda^{(+)} + \operatorname{div} X_\lambda^{(+)} = 0,$$

and similarly to Theorem 22(b) we have that

$$R_\lambda^{(+)} = \frac{1}{f_\lambda^{(+)} F_\lambda^{(+)^2}.$$

Note that if  $\Psi_{(\lambda,1)}^{(+)}$  is a solution of the Schrödinger equation  $H_+ \Psi^{(+)} = \lambda \Psi^{(+)}$  then  $\zeta_{(\lambda,1)}^{(+)}$  is a solution of the Riccati equation  $\zeta_{(\lambda,1)}^{(+)\prime} = V_+ - \lambda - \zeta^{(+)^2}$ . Following the same arguments as in the proof of Lemma (28) we obtain the expression of the first integral. Hence, the proof is completed.  $\square$

**Theorem 30.** *If Darboux transformation is strong isogaloisian for a potential  $V_- \in \mathbb{C}(x)$ , then  $f_\lambda^{(\pm)}(\zeta^{(\pm)}, x)$ ,  $K_\lambda^{(\pm)}(\zeta^{(\pm)}, x)$ ,  $L_\lambda^{(\pm)}(\zeta^{(\pm)}, x)$  and  $I_\lambda^{(\pm)}(\zeta^{(\pm)}, x)$  are of the same type in the variables  $\zeta^{(\pm)}$  and  $x$ . Furthermore, if  $V_-$  is a rational shape invariant potential, then  $f_\lambda^{(\pm)}(\zeta^{(\pm)}, x)$ ,  $K_\lambda^{(\pm)}(\zeta^{(\pm)}, x)$  and  $L_\lambda^{(\pm)}(\zeta^{(\pm)}, x)$  are rational functions in  $\zeta^{(\pm)}$  and  $x$ .*

*Proof.* The proof follows directly by Theorem 21 and Proposition 29.  $\square$

### 3. SEVERAL APPLICATIONS IN QUANTUM MECHANICS

In this section we present several examples that relate polynomial vector fields, using Lemma 28 and Proposition 29, with supersymmetric quantum mechanics. The potentials presented here are rational functions such as free particle, Square well potential, harmonic oscillator, three dimensional harmonic oscillator and the Coulomb potential.

**Example 31** (Free particle). *We consider the Schrödinger equation  $H_- \Psi^{(-)} = \lambda \Psi^{(-)}$  with potential  $V_- = 0$  and differential field<sup>1</sup>  $F = \mathbb{C}(x)$ . Thus, we have that  $\Lambda = \mathbb{C}$ .*

*If we choose  $\lambda_0 = 0$  and as particular solution of the Schrödinger equation the solution  $\Psi_0^{(-)} = x$  we have that  $\zeta_0^{(-)} = \Psi_0^{(-)\prime} / \Psi_0^{(-)} = 1/x$ . Since  $\zeta_0^{(-)} \in \mathbb{C}(x)$  from Proposition 16 we have that DT is strong isogaloisian. Additionally,*

$$\text{DT}(V_-) = V_+ = -\zeta_0^{(-)\prime} + \zeta_0^{(-)2} = \frac{2}{x^2},$$

*and so  $V_-$  is not shape invariant.*

*Then for  $\lambda \neq 0$  the general solution of  $H_- \Psi^{(-)} = \lambda \Psi^{(-)}$  is given by*

$$\Psi_\lambda^{(-)} = c_1 \Psi_{(\lambda,1)}^{(-)} + c_2 \Psi_{(\lambda,2)}^{(-)},$$

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<sup>1</sup>Although the smallest differential field containing the coefficients of the Schrödinger equation is  $\mathbb{C}$ , to avoid triviality, we choose  $\mathbb{C}(x)$  as the suitable differential field for such equation.

where we have consider  $\Psi_{(\lambda,1)}^{(-)} = e^{\sqrt{-\lambda}x}$  and  $\Psi_{(\lambda,2)}^{(-)} = e^{-\sqrt{-\lambda}x}$ . Hence we have that

$$\begin{aligned}\zeta_{\lambda}^{(-)} &= \left( \ln \left( \Psi_{\lambda}^{(-)} \right) \right)' = \sqrt{-\lambda} \frac{c_1 e^{\sqrt{-\lambda}x} - c_2 e^{-\sqrt{-\lambda}x}}{c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}}, \\ \zeta_0^{(-)} &= \left( \ln \left( \Psi_0^{(-)} \right) \right)' = \frac{1}{x}.\end{aligned}$$

Additionally, we have that

$$\zeta_{(\lambda,1)}^{(-)} = \left( \ln \left( \Psi_{(\lambda,1)}^{(-)} \right) \right)' = \sqrt{-\lambda} \quad \text{and} \quad \zeta_{(\lambda,2)}^{(-)} = \left( \ln \left( \Psi_{(\lambda,2)}^{(-)} \right) \right)' = -\sqrt{-\lambda}.$$

Here in the expression of the vector field (25) we have that  $T(x) = 0$  and  $N(x) = 1$ . Hence the polynomial vector field (25) associated to the Schrödinger equation  $H_- \Psi^{(-)} = \lambda \Psi^{(-)}$  with potential  $V_- = 0$ , is

$$X_{\lambda}^{-} = (-\lambda - \zeta^{(-)^2}) \frac{\partial}{\partial \zeta^{(-)}} + \frac{\partial}{\partial x},$$

and is quadratic. Then according to Lemma 28 for all  $\lambda \in \Lambda \setminus \{0\}$ , the vector field  $X_{\lambda}^{(-)}$  admits the following.

- Invariant curve

$$f_{\lambda}^{(-)}(\zeta^{(-)}, x) = -\zeta^{(-)} + \sqrt{-\lambda} \frac{c_1 e^{\sqrt{-\lambda}x} - c_2 e^{-\sqrt{-\lambda}x}}{c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}} \in \mathbb{C}(\zeta^{(-)}, x, e^{\sqrt{-\lambda}x}),$$

with generalized cofactor

$$K_{\lambda}^{(-)}(\zeta^{(-)}, x) = -\zeta^{(-)} - \sqrt{-\lambda} \frac{c_1 e^{\sqrt{-\lambda}x} - c_2 e^{-\sqrt{-\lambda}x}}{c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}} \in \mathbb{C}(\zeta^{(-)}, x, e^{\sqrt{-\lambda}x}).$$

- Generalized exponential factor

$$F_{\lambda}^{(-)}(\zeta^{(-)}, x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x},$$

with generalized cofactor

$$L_{\lambda}^{(-)}(\zeta^{(-)}, x) = \sqrt{-\lambda} \frac{c_1 e^{\sqrt{-\lambda}x} - c_2 e^{-\sqrt{-\lambda}x}}{c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}} \in \mathbb{C}(\zeta^{(-)}, x, e^{\sqrt{-\lambda}x}).$$

- Generalized Darboux integrating factor

$$R_{\lambda}^{(-)}(\zeta^{(-)}, x) = \frac{1}{\left( c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x} \right)^2 \left( -\zeta^{(-)} + \sqrt{-\lambda} \frac{c_1 e^{\sqrt{-\lambda}x} - c_2 e^{-\sqrt{-\lambda}x}}{c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}} \right)^2},$$

and note that  $R_{\lambda}^{(-)}(\zeta^{(-)}, x) \in \mathbb{C}(\zeta^{(-)}, x, e^{\sqrt{-\lambda}x})$ .

- Generalized Darboux first integral

$$I_{\lambda}^{(-)}(\zeta^{(-)}, x) = \frac{-\zeta^{(-)} - \sqrt{-\lambda}}{-\zeta^{(-)} + \sqrt{-\lambda}} e^{-2\sqrt{-\lambda}x} \in \mathbb{C}(\zeta^{(-)}, x, e^{\sqrt{-\lambda}x}), \quad \lambda \neq 0.$$

Applying the Darboux transformation DT for  $\lambda \neq 0$  (and according to Proposition 29) we have

$$\begin{aligned} \text{DT}(\Psi_\lambda^{(-)}) &= \Psi_\lambda^{(+)} = \Psi_\lambda^{(-)'} - \zeta_0^{(-)} \Psi_\lambda^{(-)} \\ &= \frac{c_1(\sqrt{-\lambda}x - 1)e^{\sqrt{-\lambda}x}}{x} - \frac{c_2(\sqrt{-\lambda}x + 1)e^{-\sqrt{-\lambda}x}}{x}, \\ \zeta_\lambda^{(+)} &= \left( \ln \left( \Psi_\lambda^{(+)} \right) \right)' = \\ &= \frac{\lambda(c_2e^{-\sqrt{-\lambda}x} + c_1e^{\sqrt{-\lambda}x})x^2 + \sqrt{-\lambda}(c_1e^{\sqrt{-\lambda}x} - c_2e^{-\sqrt{-\lambda}x})x - c_1e^{\sqrt{-\lambda}x} - c_2e^{-\sqrt{-\lambda}x}}{x(\sqrt{-\lambda}(c_1e^{\sqrt{-\lambda}x} - c_2e^{-\sqrt{-\lambda}x})x - c_1e^{\sqrt{-\lambda}x} - c_2e^{-\sqrt{-\lambda}x})}. \end{aligned}$$

Additionally we have

$$\begin{aligned} \zeta_{(\lambda,1)}^{(+)} &= \sqrt{-\lambda} + \left( \ln \left( \sqrt{-\lambda} - \frac{1}{x} \right) \right)' \\ &= \sqrt{-\lambda} + \frac{1}{\sqrt{-\lambda}x^2 - x}, \\ \zeta_{(\lambda,2)}^{(+)} &= -\sqrt{-\lambda} + \left( \ln \left( -\sqrt{-\lambda} - \frac{1}{x} \right) \right)' \\ &= -\sqrt{-\lambda} - \frac{1}{\sqrt{-\lambda}x^2 + x}. \end{aligned}$$

We can see that for all  $\lambda \in \Lambda$  the Picard-Vessiot extensions are given by  $L_0 = \tilde{L}_0 = \mathbb{C}(x)$ ,  $L_\lambda = \tilde{L}_\lambda = \mathbb{C}(x, e^{\sqrt{-\lambda}x})$  for  $\lambda \in \mathbb{C}^*$ . In this way, we have that  $\text{DGal}(L_0/F) = \text{DGal}(\tilde{L}_0/F) = e$ ; for  $\lambda \neq 0$ , we have  $\text{DGal}(L_\lambda/F) = \text{DGal}(\tilde{L}_\lambda/F) = \mathbb{G}_m$ , see [1, 2].

Now according to Proposition 29 we can compute  $\text{DT}(X_\lambda^-) = X_\lambda^+$  and we obtain the rational vector field

$$X_\lambda^+ = \left( \frac{2}{x^2} - \lambda - \zeta^{(+)^2} \right) \frac{\partial}{\partial \zeta^{(+)}} + \frac{\partial}{\partial x}, \quad \forall \lambda \neq 0,$$

and equivalently we can consider the polynomial vector field of degree four

$$\overline{X}_\lambda^+ = \left( 2 - \lambda x^2 - x^2 \zeta^{(+)^2} \right) \frac{\partial}{\partial \zeta^{(+)}} + x^2 \frac{\partial}{\partial x}, \quad \forall \lambda \neq 0,$$

and admits the following.

- Invariant curve

$$\begin{aligned} \overline{f}_\lambda^{(+)}(\zeta^{(+)}, x) &= f_\lambda^{(+)}(\zeta^{(+)}, x) = -\zeta^{(+)} + \zeta_\lambda^{(-)} + \frac{\zeta_\lambda^{(-)'} + \frac{1}{x^2}}{\zeta_\lambda^{(-)} - \frac{1}{x}} \\ &= -\zeta^{(+)} - \frac{c_1(\sqrt{-\lambda}x - 1 + \lambda x^2)e^{\sqrt{-\lambda}x} + c_2(-1 + \lambda x^2 - \sqrt{-\lambda}x)e^{-\sqrt{-\lambda}x}}{x(c_1(\sqrt{-\lambda}x - 1)e^{\sqrt{-\lambda}x} + c_2(-\sqrt{-\lambda}x - 1)e^{-\sqrt{-\lambda}x})}, \end{aligned}$$

and note that  $f_\lambda^{(+)}(\zeta^{(+)}, x) \in \mathbb{C}(\zeta^{(+)}, x, e^{\sqrt{-\lambda}x})$

with generalized cofactor

$$\begin{aligned} \overline{K}_\lambda^{(+)}(\zeta^{(+)}, x) &= x^2 K_\lambda^{(+)}(\zeta^{(+)}, x) = x^2 \left( -\zeta^{(+)} - \zeta_\lambda^{(-)} - \frac{\zeta_\lambda^{(-)'} + \frac{1}{x^2}}{\zeta_\lambda^{(-)} - \frac{1}{x}} \right) \\ &= x^2 \left( -\zeta^{(+)} + \frac{c_1(\sqrt{-\lambda}x - 1 + \lambda x^2)e^{\sqrt{-\lambda}x} + c_2(-1 + \lambda x^2 - \sqrt{-\lambda}x)e^{-\sqrt{-\lambda}x}}{x(c_1(\sqrt{-\lambda}x - 1)e^{\sqrt{-\lambda}x} + c_2(-\sqrt{-\lambda}x - 1)e^{-\sqrt{-\lambda}x})} \right), \end{aligned}$$

and note that  $K_\lambda^{(+)} \in \mathbb{C}(\zeta^{(+)}, x, e^{\sqrt{-\lambda}x})$ .

- Generalized exponential factor

$$\begin{aligned} \overline{F}_\lambda^{(+)}(\zeta^{(+)}, x) &= F_\lambda^{(+)}(\zeta^{(+)}, x) = \left( \zeta_\lambda^{(-)} - \zeta_0^{(-)} \right) e^{\int \zeta_\lambda^{(-)}} \\ &= \frac{c_1(\sqrt{-\lambda}x - 1)e^{\sqrt{-\lambda}x} + c_2(-1 + \lambda x^2 - \sqrt{-\lambda}x)e^{-\sqrt{-\lambda}x}}{x} \end{aligned}$$

with generalized cofactor

$$\begin{aligned} \overline{L}_\lambda^{(+)} &= x^2 L_\lambda^{(+)} = x^2 \zeta_\lambda^{(+)} = x^2 \left( \zeta_\lambda^{(-)} + \frac{\zeta_\lambda^{(-)'} + \frac{1}{x^2}}{\zeta_\lambda^{(-)} - \frac{1}{x}} \right) \\ &= -x^3 \frac{c_1(\sqrt{-\lambda}x - 1 + \lambda x^2)e^{\sqrt{-\lambda}x} + c_2(-1 + \lambda x^2 - \sqrt{-\lambda}x)e^{-\sqrt{-\lambda}x}}{(c_1\sqrt{-\lambda}x - 1)e^{\sqrt{-\lambda}x} + c_2(-\sqrt{-\lambda}x - 1)e^{-\sqrt{-\lambda}x}}. \end{aligned}$$

and note that  $L_\lambda^{(+)} \in \mathbb{C}(\zeta^{(+)}, x, e^{\sqrt{-\lambda}x})$ .

- Generalized Darboux integrating factor

$$\begin{aligned} \overline{R}_\lambda^{(+)}(\zeta^{(+)}, x) &= \frac{1}{x^2} R_\lambda^{(+)}(\zeta^{(+)}, x) \\ &= \frac{(c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x})^{-2}}{x^2 \left( \zeta_\lambda^{(-)} - \frac{1}{x} \right)^2 \left( -\zeta^{(+)} + \zeta_\lambda^{(-)} + \frac{\zeta_\lambda^{(-)'} + \frac{1}{x^2}}{\zeta_\lambda^{(-)} - \frac{1}{x}} \right)^2} \in \mathbb{C}(\zeta^{(+)}, x, e^{\sqrt{-\lambda}x}). \end{aligned}$$

- Generalized Darboux first integral

$$\begin{aligned} \overline{I}_\lambda^{(+)}(\zeta^{(+)}, x) &= I_\lambda^{(+)}(\zeta^{(+)}, x) \\ &= \frac{-\zeta^{(+)} - \frac{1}{\sqrt{-\lambda}x^2 + x} - \sqrt{-\lambda}}{-\zeta^{(+)} + \frac{1}{\sqrt{-\lambda}x^2 - x} + \sqrt{-\lambda}} \left( \frac{\sqrt{-\lambda}x + 1}{\sqrt{-\lambda}x - 1} \right) e^{-2\sqrt{-\lambda}x} \in \mathbb{C}(\zeta^{(+)}, x, e^{\sqrt{-\lambda}x}). \end{aligned}$$

We could consider another case where  $\zeta_{(\lambda,1)}^{(-)} = -\sqrt{\lambda} \tan \sqrt{\lambda}x$ , being  $\Psi_{(\lambda,1)}^{(-)} = \cos \sqrt{\lambda}x$  another particular solution and  $\Psi_0^{(-)} = x$ . Thus, in this case we obtain the Schrödinger polynomial vector field

$$X_{(\lambda,1)}^{(-)} = (-\lambda - \zeta^{(-)2}) \frac{\partial}{\partial \zeta^{(-)}} + \frac{\partial}{\partial x},$$

of degree two. According to Lemma 28 for all  $\lambda \in \Lambda \setminus \{0\}$ , the vector field  $X_{(\lambda,1)}^{(-)}$  admits the following.

- *Invariant curve*

$$f_{(\lambda,1)}^{(-)}(\zeta^{(-)}, x) = -\zeta^{(-)} - \sqrt{\lambda} \tan(\sqrt{\lambda}x) \in \mathbb{C}(\zeta^{(-)}, x, \tan(\sqrt{\lambda}x)),$$

with generalized cofactor

$$K_{(\lambda,1)}^{(-)}(\zeta^{(-)}, x) = -\zeta^{(-)} + \sqrt{\lambda} \tan(\sqrt{\lambda}x) \in \mathbb{C}(\zeta^{(-)}, x, \tan(\sqrt{\lambda}x)).$$

- *Generalized exponential factor*

$$F_{(\lambda,1)}^{(-)}(\zeta^{(-)}, x) = e^{-\sqrt{\lambda} \int \tan(\sqrt{\lambda}x) dx} = \frac{1}{\sqrt{1 + \tan^2(\sqrt{\lambda}x)}},$$

with generalized cofactor

$$L_{(\lambda,1)}^{(-)}(\zeta^{(-)}, x) = -\sqrt{\lambda} \tan(\sqrt{\lambda}x) \in \mathbb{C}(\zeta^{(-)}, x, \tan(\sqrt{\lambda}x)).$$

- *Generalized Darboux integrating factor*

$$R_{(\lambda,1)}^{(-)}(\zeta^{(-)}, x) = \frac{1 + \tan^2(\sqrt{\lambda}x)}{\left(-\zeta^{(-)} - \sqrt{\lambda} \tan(\sqrt{\lambda}x)\right)^2} \in \mathbb{C}\left(\zeta^{(-)}, x, \tan(\sqrt{\lambda}x)\right) \quad \text{for } \lambda \neq 0.$$

- *Generalized Darboux first integral*

$$I_{(\lambda,1)}^{(-)}(\zeta^{(-)}, x) = \frac{\zeta^{(-)} \tan(\sqrt{\lambda}x) - \sqrt{\lambda}}{(\zeta^{(-)} + \sqrt{\lambda} \tan(\sqrt{\lambda}x))\sqrt{\lambda}} \in \mathbb{C}\left(\zeta^{(-)}, x, \tan(\sqrt{\lambda}x)\right).$$

After the Darboux transformation the vector field  $X_{(\lambda,1)}^-$  according to Proposition 29 we obtain  $X_{(\lambda,1)}^+ = DT(X_{(\lambda,1)}^-)$  and is given by the expression

$$X_{(\lambda,1)}^+ = \left(\frac{2}{x^2} - \lambda - \zeta^{(+)^2}\right) \frac{\partial}{\partial \zeta^{(+)}} + \frac{\partial}{\partial x},$$

and we can work with the polynomial vector field of degree four

$$\bar{X}_{(\lambda,1)}^+ = \left(2 - \lambda x^2 - x^2 \zeta^{(+)^2}\right) \frac{\partial}{\partial \zeta^{(+)}} + x^2 \frac{\partial}{\partial x},$$

that admits

- *The invariant curve*

$$\bar{f}_{(\lambda,1)}^+(\zeta^{(+)}, x) = -\zeta^{(+)} + \frac{-\sqrt{\lambda} x \tan(\sqrt{\lambda}x) + \lambda x^2 - 1}{x \left(\sqrt{\lambda} x \tan(\sqrt{\lambda}x) + 1\right)},$$

with generalized cofactor

$$\bar{K}_{(\lambda,1)}^+(\zeta^{(+)}, x) = x^2 \left( -\zeta^{(+)} - \frac{-\sqrt{\lambda} x \tan(\sqrt{\lambda}x) + \lambda x^2 - 1}{x \left(\sqrt{\lambda} x \tan(\sqrt{\lambda}x) + 1\right)} \right).$$

- *The generalized exponential factor*

$$\bar{F}_{(\lambda,1)}^+(\zeta^{(+)}, x) = \frac{-\sqrt{\lambda} \tan(\sqrt{\lambda}x) - \frac{1}{x}}{\sqrt{1 + \tan^2(\sqrt{\lambda}x)}},$$

with generalized cofactor

$$\bar{L}_{(\lambda,1)}^+(\zeta^{(+)}, x) = x \frac{-\sqrt{\lambda} x \tan(\sqrt{\lambda}x) + \lambda x^2 - 1}{\sqrt{\lambda} x \tan(\sqrt{\lambda}x) + 1}.$$

The integrating factor

$$\bar{R}_{(\lambda,1)}^+(\zeta^{(+)}, x) = \frac{1 + \tan^2(\sqrt{\lambda}x)}{x^2 \left( -\zeta^{(+)} + \frac{-\sqrt{\lambda} x \tan(\sqrt{\lambda}x) + \lambda x^2 - 1}{x(\sqrt{\lambda} x \tan(\sqrt{\lambda}x) + 1)} \right)^2 \left( -\sqrt{\lambda} \tan(\sqrt{\lambda}x) - \frac{1}{x} \right)^2}.$$

We could also considered the partial solution  $\Psi_{(\lambda,2)}^{(-)} = \sin(\sqrt{\lambda}x)$  and we could work in a similarly way as before.

**Remark 32.** As in the same philosophy of the original Darboux transformation, we can iterate it to obtain families of new potentials. Starting with  $V = 0$ , the following potentials can be obtained using Darboux iteration  $\text{DT}_n$  (see [4, 5, 20]).

$$I) V_n = \frac{n(n-1)b^2}{(bx+c)^2}, \quad II) V_n = \frac{m^2n(n-1)(b^2-a^2)}{(a \cosh(mx) + b \sinh(mx))^2},$$

$$III) V_n = \frac{-4abm^2n(n-1)}{(ae^{mx} + be^{-mx})^2}, \quad IV) V_n = \frac{m^2n(n-1)(b^2+a^2)}{(a \cos(mx) + b \sin(mx))^2}.$$

In particular for the rational potential given in I), see [1, 2], we have  $F = F_n = \mathbb{C}(x)$  and for  $\lambda_n = \lambda = 0$ , we have

$$\Psi_0^{(n)} = \frac{c_1}{(bx+c)^n} + c_2(bx+c)^{n+1}, \text{ so that } \text{DGal}(L_0/F) = \text{DGal}(L_0^{(n)}/F) = e,$$

whilst for  $\lambda \neq 0$  and  $\lambda_n = 0$ , the general solution  $\Psi_\lambda^{(n)}$  is given by

$$\Psi_\lambda^{(n)}(x) = c_1 A_n(x, \lambda) C_n(\sin(\sqrt{\lambda}x) + c_2 B_n(x, \lambda) D_n(\cos(\sqrt{\lambda}x))),$$

where  $A_n, B_n, C_n, D_n \in \mathbb{C}(x)$ , so that

$$\text{DGal}(L_\lambda/F) = \text{DGal}(L_\lambda^{(n)}/F) = \mathbb{G}_m.$$

**Example 33** (Square Well potential). We now consider the potential  $V_- = \ell(\ell+1)/r^2 \in \mathbb{C}(r)$  with  $\ell \geq 1$  natural number, which is a particular case of square well potential. For  $\lambda = 0$  we consider  $\Psi_0^{(-)} = r^{\ell+1}$  as particular solution of the Schrödinger equation  $H_- \Psi^{(-)} = \lambda \Psi^{(-)}$  and we set  $\zeta_0^{(-)} =$

$\Psi_0^{(-)'} / \Psi_0^{(-)} = (\ell + 1)/r$ . Since  $\zeta_0^{(-)} \in \mathbb{C}(x)$  from Proposition 16 we have that  $DT$  is strong isogalossian. Additionally,

$$DT(V_-) = V_+ = -\zeta_0^{(-)'} + \zeta_0^{(-)2} = \frac{(\ell + 1)(\ell + 2)}{r^2},$$

which means that  $V_{\pm}$  differs only in parameters and for instance  $V_-$  is a shape invariant potential. Due to this potential is obtained by iteration of Darboux transformation over the free particle potential, the algebraic spectrum is the same, i.e.,  $\Lambda = \mathbb{C}$ . The Schrödinger equation  $H_- \Psi^{(-)} = \lambda \Psi^{(-)}$  through the changes  $\Psi \mapsto \sqrt{x} \Psi$ ,  $\ell \mapsto \ell - \frac{1}{2}$  and  $r \mapsto \frac{r}{\sqrt{\lambda}}$  is transformable to a Bessel equation with parameter  $\mu = \ell + \frac{1}{2}$ , which obviously is a half integer. Thus, the general solution of the Schrödinger equation with square well potential is given by

$$\Psi_{\lambda}^{(-)} = c_1 \Psi_{(\lambda,1)}^{(-)} + c_2 \Psi_{(\lambda,2)}^{(-)},$$

being  $\Psi_{(\lambda,1)}^{(-)}$  and  $\Psi_{(\lambda,2)}^{(-)}$  Bessel functions. Thus, we can get

$$\zeta_{(\lambda,1)}^{(-)} = \frac{d}{dx} \ln(\Psi_{(\lambda,1)}^{(-)}), \quad \zeta_{(\lambda,2)}^{(-)} = \frac{d}{dx} \ln(\Psi_{(\lambda,2)}^{(-)}).$$

Here in the expression of the vector field (25) we have that  $T(r) = \ell(\ell + 1)$  and  $N(r) = r^2$ . Hence, the vector field (25) is written

$$\bar{X}_{\lambda}^{(-)} = \left( \ell(\ell + 1) - \lambda r^2 - r^2 \zeta^{(-)2} \right) \frac{\partial}{\partial \zeta^{(-)}} + r^2 \frac{\partial}{\partial r}.$$

Due to the square well potential satisfies the shape invariance condition, by Theorem 21 and Theorem 30, the differential fields, Picard-Vessiot extensions are preserved as well the form of the vector fields and its elements given in Lemma (28). To illustrate this fact, we consider the particular case  $\ell = 1$ , which lead us to  $\Psi_0 = r^2$  and

$$\Psi_{\lambda}^{(-)} = c_1 \Psi_{(\lambda,1)}^{(-)} + c_2 \Psi_{(\lambda,2)}^{(-)},$$

being

$$\Psi_{(\lambda,1)}^{(-)} = \frac{\sqrt{-\lambda r} + 1}{r} e^{-\sqrt{-\lambda r}}, \quad \Psi_{(\lambda,2)}^{(-)} = \frac{\sqrt{-\lambda r} - 1}{r} e^{\sqrt{-\lambda r}},$$

obtaining

$$\zeta_{(\lambda,1)}^{(-)} = -\frac{-\lambda r^2 + \sqrt{-\lambda r} + 1}{\sqrt{-\lambda r^2} + r}, \quad \zeta_{(\lambda,2)}^{(-)} = \frac{-\lambda r^2 - \sqrt{-\lambda r} + 1}{\sqrt{-\lambda r^2} - r}.$$

Here in the expression of the vector field (25) we have that  $T(r) = 2$  and  $N(r) = r^2$ . Hence, the vector field (25) is written

$$\bar{X}_{\lambda}^{-} = \left( 2 - \lambda r^2 - r^2 \zeta^{(-)2} \right) \frac{\partial}{\partial \zeta^{(-)}} + r^2 \frac{\partial}{\partial r}.$$

Thus, the vector field  $X_{\lambda}^{-}$  admits

- *Invariant curve*

$$f_\lambda^{(-)}(\zeta^{(-)}, r) = -\zeta^{(-)} + \frac{c_1 \zeta_{(\lambda,1)}^{(-)} \Psi_{(\lambda,1)}^{(-)} + c_2 \zeta_{(\lambda,2)}^{(-)} \Psi_{(\lambda,2)}^{(-)}}{c_1 \Psi_{(\lambda,1)}^{(-)} + c_2 \Psi_{(\lambda,2)}^{(-)}} \in \mathbb{C}(r, e^{\sqrt{-\lambda}r}, \zeta^{(-)}),$$

with generalized cofactor

$$K_\lambda^{(-)}(\zeta^{(-)}, r) = -r^2 \left( \zeta^{(-)} + \frac{c_1 \zeta_{(\lambda,1)}^{(-)} \Psi_{(\lambda,1)}^{(-)} + c_2 \zeta_{(\lambda,2)}^{(-)} \Psi_{(\lambda,2)}^{(-)}}{c_1 \Psi_{(\lambda,1)}^{(-)} + c_2 \Psi_{(\lambda,2)}^{(-)}} \right) \in \mathbb{C}(r, e^{\sqrt{-\lambda}r}, \zeta^{(-)}),$$

- *Generalized exponential factor*

$$F_\lambda^{(-)}(\zeta^{(-)}, r) = r e^{\int \zeta_\lambda^{(-)}} = r \Psi_\lambda^{(-)} \in \mathbb{C}(r, e^{\sqrt{-\lambda}r}),$$

with generalized cofactor

$$L_\lambda^{(-)}(\zeta^{(-)}, r) = r + r^2 \zeta_\lambda^{(-)} \in \mathbb{C}(r, e^{\sqrt{-\lambda}r})$$

- *Generalized Darboux integrating factor*

$$R_\lambda^{(-)}(\zeta^{(-)}, r) = \frac{e^{-2 \int \zeta_\lambda^{(-)}}}{r^2 \left( -\zeta^{(-)} + \zeta_\lambda^{(-)} \right)^2} \in \mathbb{C}(r, e^{\sqrt{-\lambda}r}, \zeta^{(-)}).$$

- *Generalized Darboux first integral*

$$I_\lambda^{(-)}(\zeta^{(-)}, r) = \frac{-\zeta^{(-)} + \zeta_{(\lambda,2)}^{(-)} \frac{\sqrt{-\lambda}r + 1}{\sqrt{-\lambda}r - 1}}{-\zeta^{(-)} + \zeta_{(\lambda,1)}^{(-)} \frac{\sqrt{-\lambda}r + 1}{\sqrt{-\lambda}r - 1}} e^{-2\sqrt{-\lambda}r} \in \mathbb{C}(r, \zeta^{(-)}).$$

Now according to Proposition 29 we can compute  $\text{DT}(X_\lambda^-) = X_\lambda^+$  and we obtain the rational vector field

$$X_\lambda^+ = \left( \frac{6}{r^2} - \lambda - \zeta^{(+)^2} \right) \frac{\partial}{\partial \zeta^{(+)}} + \frac{\partial}{\partial r},$$

and equivalently we can work with the polynomial vector field  $\bar{X}_\lambda^+$

$$\bar{X}_\lambda^+ = \left( 6 - \lambda r^2 - \zeta^{(-)^2} r^2 \right) \frac{\partial}{\partial \zeta^{(+)}} + r^2 \frac{\partial}{\partial r},$$

and admits the following

- *Invariant curve*

$$\bar{f}_\lambda^{(+)}(\zeta^{(+)}, r) = f_\lambda^{(+)}(\zeta^{(+)}, r) = -\zeta^{(+)} + \zeta_\lambda^{(-)} + \frac{\zeta_\lambda^{(-)'} + \frac{2}{r^2}}{\zeta_\lambda^{(-)} - \frac{2}{r}} \in \mathbb{C}(r, e^{\sqrt{-\lambda}r}, \zeta^{(+)})$$

with generalized cofactor

$$\bar{K}_\lambda^{(+)}(\zeta^{(+)}, r) = r^2 K_\lambda^{(+)}(\zeta^{(+)}, r) = r^2 \left( -\zeta^{(+)} - \zeta_\lambda^{(-)} - \frac{\zeta_\lambda^{(-)'} + \frac{2}{r^2}}{\zeta_\lambda^{(-)} - \frac{2}{r}} \right) \in \mathbb{C}(r, e^{\sqrt{-\lambda}r}, \zeta^{(+)}).$$

- *Generalized exponential factor*

$$\overline{F}_\lambda^{(+)}(\zeta^{(+)}, r) = F_\lambda^{(+)}(\zeta^{(+)}, r) = \left(\zeta_\lambda^{(-)} - \frac{2}{r}\right) e^{\int \zeta_\lambda^{(-)}} \in \mathbb{C}(r, e^{\sqrt{-\lambda}r})$$

with generalized cofactor

$$\overline{L}_\lambda^{(+)} = r^2 L_\lambda^{(+)} = r^2 \zeta_\lambda^{(+)} = r^2 \left( \frac{\zeta_\lambda^{(-)} + \zeta_\lambda^{(-)'} + \frac{2}{r^2}}{\zeta_\lambda^{(-)} - \frac{2}{r}} \right) \in \mathbb{C}(r, e^{\sqrt{-\lambda}r}).$$

- *Generalized Darboux integrating factor*

$$\begin{aligned} \overline{R}_\lambda^{(+)}(\zeta^{(+)}, r) &= \frac{1}{r^2} R_\lambda^{(+)}(\zeta^{(+)}, r) \\ &= \frac{1}{r^2} \frac{e^{-2 \int \zeta_\lambda^{(-)}}}{\left(\zeta_\lambda^{(-)} - \frac{2}{r}\right)^2 \left(-\zeta^{(+)} + \zeta_\lambda^{(-)} + \left(\ln\left(\zeta_\lambda^{(-)} - \frac{2}{r}\right)\right)'\right)^2} \in \mathbb{C}(r, e^{\sqrt{-\lambda}r}, \zeta^{(+)}). \end{aligned}$$

- *Generalized Darboux first integral*

$$\begin{aligned} \overline{I}_\lambda^{(+)}(\zeta^{(+)}, r) &= I_\lambda^{(+)}(\zeta^{(+)}, r) \\ &= \frac{-\zeta^{(+)} + \zeta_{(\lambda,2)}^{(-)} + \left(\ln\left(\zeta_{(\lambda,2)}^{(-)} - \frac{2}{r}\right)\right)'}{-\zeta^{(+)} + \zeta_{(\lambda,1)}^{(-)} + \left(\ln\left(\zeta_{(\lambda,1)}^{(-)} - \frac{2}{r}\right)\right)'} \in \mathbb{C}(r, \zeta^{(+)}). \end{aligned}$$

The following examples correspond to other rational shape invariant potentials. For physic interest we consider only solutions satisfying some analytic conditions, in particular we consider square integrable wave functions<sup>2</sup>, see [31, 36].

**Example 34** (Harmonic Oscillator). *We consider the Schrödinger equation  $H_- \Psi^{(-)} = \lambda \Psi^{(-)}$  with potential  $V_- = x^2 - 1$  (note that in system (25) we have  $N(x) = 1$  and  $T(x) = x^2 - 1$ ). For  $\lambda_0 = 0$  a particular solution is  $\Psi_0^{(-)} = e^{-\frac{x^2}{2}}$  and so  $\zeta_0^{(-)} = \left(\ln\left(\Psi_0^{(-)}\right)\right)' = -W = -x \in \mathbb{C}(x)$ . From Proposition 16 we have that  $DT$  is strong isogalossian. Additionally*

$$V_+ = DT(V_-) = -\zeta_0^{(-)'} + \zeta_0^{(-)2} = x^2 + 1,$$

and so  $V_-$  is shape invariant.

For  $\lambda \neq 0$  we have

$$\Psi_{(\lambda,1)}^{(-)} = H_{\frac{\lambda}{2}} e^{-\frac{x^2}{2}}, \quad \lambda \in 2\mathbb{N}, \quad \zeta_{(\lambda,1)}^{(-)} = \frac{H'_{\lambda/2}}{H_{\lambda/2}} - x \in \mathbb{C}(x),$$

where  $H_{\lambda/2}$  are the Hermite polynomials, see [31], defined as

$$H_{\lambda/2} = (-1)^{\lambda/2} e^{x^2} \frac{d^{\lambda/2}}{dx^{\lambda/2}} e^{-x^2}, \quad \lambda \in 2\mathbb{N},$$

<sup>2</sup>The functional space of the square integrable functions is denoted  $L^2$

and satisfying the recurrence property

$$H_{n+1} = 2xH_n - H_{n-1}, \quad \lambda = 2n, \quad n \in \mathbb{N}.$$

Here we have (see also Remark 25(b))

$$\zeta_{(\lambda,1)}^{(-)} = \frac{H'_{\lambda/2}}{H_{\lambda/2}} - x, \quad \zeta_{(\lambda,2)}^{(-)} = \frac{e^{x^2}}{H_{\lambda/2}^2 \int \frac{e^{x^2}}{H_{\lambda/2}^2 dx}} + \frac{H'_{\lambda/2}}{H_{\lambda/2}} - x.$$

Note that the quadratic vector field (25) can be written as

$$X_{\lambda}^{-} = \left( x^2 - 1 - \lambda - \zeta^{(-)^2} \right) \frac{\partial}{\partial \zeta^{(-)}} + \frac{\partial}{\partial x},$$

and admits the invariant curve

$$f_{\lambda}^{(-)}(\zeta^{(-)}, x) = -\zeta^{(-)} + \frac{H'_{\lambda/2}}{H_{\lambda/2}} - x \in \mathbb{C}(\zeta^{(-)}, x),$$

with generalized cofactor

$$K_{\lambda}^{(-)}(\zeta^{(-)}, x) = -\zeta^{(-)} - \frac{H'_{\lambda/2}}{H_{\lambda/2}} + x \in \mathbb{C}(\zeta^{(-)}, x).$$

The generalized exponential factor

$$F_{\lambda}^{(-)}(\zeta^{(-)}, x) = e^{-\frac{x^2}{2} + \int (\ln H_{\lambda/2})'} = H_{\lambda/2} e^{-\frac{x^2}{2}} \in L^2,$$

with generalized cofactor

$$L_{\lambda}^{(-)}(\zeta^{(-)}, x) = \frac{H'_{\lambda/2}}{H_{\lambda/2}} - x \in \mathbb{C}(x).$$

Moreover,  $X_{\lambda}^{-}$  admits the generalized integrating factor

$$R_{\lambda}^{(-)}(\zeta^{(-)}, x) = \frac{e^{x^2}}{H_{\lambda/2}^2 \left( -\zeta^{(-)} + \frac{H'_{\lambda/2}}{H_{\lambda/2}} - x \right)^2}, \in \mathbb{C}(x, e^{x^2/2})$$

and the first integral

$$I_{\lambda}^{(-)}(\zeta^{(-)}, x) = \frac{e^{x^2} + \left( H_{\lambda/2} H'_{\lambda/2} - H_{\lambda/2}^2 (\zeta^{(-)} + x) \right) \int \frac{e^{x^2}}{H_{\lambda/2}^2} dx}{H_{\lambda/2} H'_{\lambda/2} - H_{\lambda/2}^2 (\zeta^{(-)} + x)} \in F.$$

After the Darboux transformation  $DT$  the vector field  $X_{\lambda}^{-}$  becomes

$$X_{\lambda}^{+} = (x^2 + 1 - \lambda - \zeta^{(+)^2}) \frac{\partial}{\partial \zeta^{(+)}} + \frac{\partial}{\partial x},$$

and is polynomial of degree two. Additionally, we have

$$\zeta_{(\lambda,1)}^{(+)} = \frac{H''_{\lambda/2}}{H_{\lambda/2}} - x, \quad \zeta_{(\lambda,2)}^{(+)} = \frac{xe^{x^2} + (H_{\lambda/2}H''_{\lambda/2} - xH_{\lambda/2}H'_{\lambda/2}) \int \frac{e^{x^2}}{H_{\lambda/2}^2} dx}{e^{x^2} + H_{\lambda/2}H'_{\lambda/2} \int \frac{e^{x^2}}{H_{\lambda/2}^2} dx}.$$

The vector field  $X_\lambda^+$  admits the invariant curve

$$f_\lambda^{(+)}(\zeta^{(+)}, x) = -\zeta^{(+)} + \frac{H''_{\lambda/2}}{H'_{\lambda/2}} - x \in \mathbb{C}(\zeta^{(+)}, x),$$

with generalized cofactor

$$K_\lambda^{(+)}(\zeta^{(+)}, x) = -\zeta^{(+)} - \frac{H''_{\lambda/2}}{H'_{\lambda/2}} + x \in \mathbb{C}(\zeta^{(+)}, x).$$

The generalized exponential factor

$$F_\lambda^{(+)}(\zeta^{(+)}, x) = e^{-\frac{x^2}{2} + \int (\ln H'_{\lambda/2})'} \in L^2,$$

with generalized cofactor

$$L_\lambda^{(+)}(\zeta^{(+)}, x) = -\frac{H''_{\lambda/2}}{H'_{\lambda/2}} + x \in \mathbb{C}(x).$$

Additionally,  $X_\lambda^+$  admits the generalized Darboux integrating factor

$$R_\lambda^{(+)}(\zeta^{(+)}, x) = \frac{e^{x^2}}{H_{\lambda/2}'^2 \left( -\zeta^{(+)} + \frac{H''_{\lambda/2}}{H'_{\lambda/2}} - x \right)^2} \in \mathbb{C}(x, e^{x^2/2}),$$

and the first integral  $I^{(+)}(\zeta^{(+)}, x) = I^{(+)}$

$$I^{(+)} = \frac{\left( H'_{\lambda/2} e^{x^2} (-\zeta^{(+)} + x) + \left( H'_{\lambda/2} H_{\lambda/2} H''_{\lambda/2} - H_{\lambda/2}'^2 H_{\lambda/2} (\zeta^{(+)} + x) \right) \int \frac{e^{x^2}}{H_{\lambda/2}^2} dx \right) e^{\int \Phi}}{\left( e^{x^2} + H_{\lambda/2} H'_{\lambda/2} \int \frac{e^{x^2}}{H_{\lambda/2}^2} dx \right) \left( H''_{\lambda/2} - (\zeta^{(+)} + x) H'_{\lambda/2} \right)} \in F,$$

with

$$\Phi = \frac{e^{x^2} (2xH'_{\lambda/2} - H''_{\lambda/2})}{H_{\lambda/2}' \left( e^{x^2} + H_{\lambda/2} H'_{\lambda/2} \int \frac{e^{x^2}}{H_{\lambda/2}^2} dx \right)}.$$

We recall that the differential field  $F$  in  $I^{(\pm)}$  is the same due to is an extension of  $\mathbb{C}(x, \zeta^{(\pm)}(x))$  adding the solutions of the Schrödinger equation.

**Example 35** (3D Harmonic Oscillator). We take  $V_- = r^2 + \frac{\ell(\ell+1)}{r^2} - (2\ell+3)$ . Additionally, we have

$$\begin{aligned}\Psi_0^{(-)} &= r^{\ell+1} e^{-\frac{r^2}{2}}, & \zeta_0^{(-)} &= \frac{\ell+1}{r} - r \in \mathbb{C}(r) \\ \Psi_\lambda^{(-)} &= r^{\ell+1} P_{\lambda/4} e^{-\frac{r^2}{2}}, & \zeta_{(\lambda,1)}^{(-)} &= \frac{\ell+1}{r} + \frac{P'_{\lambda/4}}{P_{\lambda/4}} - r \\ \zeta_{(\lambda,2)}^{(-)} &= \frac{e^{r^2}}{r^{2\ell+2} P_{\lambda/4}^2} \int \frac{e^{r^2}}{r^{2\ell+2} P_{\lambda/4}^2} dx + \frac{P'_{\lambda/4}}{P_{\lambda/4}} - r + \frac{\ell+1}{r},\end{aligned}$$

where  $P_{\lambda/4}$  are polynomials which are related to the Generalized Laguerre Polynomials, see [31]. According to Proposition 16  $DT$  is strong isogaloisian. Since,

$$DT(V_-) = V_+ = r^2 + \frac{(\ell+1)(\ell+2)}{r^2} - (2\ell+1),$$

we have that the potential  $V_-$  is also shape invariant.

Note that  $T(r) = r^4 + \ell(\ell+1) - (2\ell+3)r^2$  and  $N(r) = r^2$ . Hence, system (25) can be written as

$$X_\lambda^- = \left( r^4 + \ell(\ell+1) - (2\ell+3)r^2 - \lambda r^2 - r^2 \zeta^{(-)2} \right) \frac{\partial}{\partial \zeta^{(-)}} + r^2 \frac{\partial}{\partial r}.$$

According to Lemma 28 admits the invariant curve

$$f_\lambda^{(-)}(\zeta^{(-)}, r) = -\zeta^{(-)} + \frac{\ell+1}{r} + \frac{P'_{\lambda/4}}{P_{\lambda/4}} - r \in \mathbb{C}(\zeta^{(-)}, r),$$

with generalized cofactor

$$K_\lambda^{(-)}(\zeta^{(-)}, r) = -r^2 \zeta^{(-)} - (\ell+1)r - \frac{P'_{\lambda/4}}{P_{\lambda/4}} r + r^2 \in \mathbb{C}(\zeta^{(-)}, r).$$

The generalized exponential factor

$$F_\lambda^{(-)}(\zeta^{(-)}, r) = r^{\ell+2} P_{\lambda/4} e^{-\frac{r^2}{2}} \in L^2,$$

with generalized cofactor

$$L_\lambda^{(-)}(\zeta^{(-)}, r) = r + (\ell+1)r + \frac{P'_{\lambda/4}}{P_{\lambda/4}} r^2 - r^3 \in \mathbb{C}(r).$$

Moreover,  $X_\lambda^-$  admits the generalized Darboux integrating factor

$$R_\lambda^{(-)}(\zeta^{(-)}, r) = \frac{e^{r^2}}{P_{\lambda/4}^2 r^{2(\ell+2)} \left( -\zeta^{(-)} + \frac{\ell+1}{r} + \frac{P'_{\lambda/4}}{P_{\lambda/4}} - r \right)^2} \in \mathbb{C}(r, e^{r^2/2}),$$

and the first integral

$$I_\lambda^{(-)}(\zeta^{(-)}, r) = \frac{e^{r^2}}{r^{2\ell+2} P_{\lambda/4} P'_{\lambda/4} - r^{2\ell+1} P_{\lambda/4}^2 (r^2 + \zeta^{(-)} r - \ell - 1)} + \int \frac{e^{r^2}}{r^{2\ell+1} P_{\lambda/4}^2} dr \in F.$$

According to Proposition 29 using the Darboux transformation we obtain the rational vector field

$$X_\lambda^+ = \left( r^2 + \frac{(\ell+1)(\ell+2)}{r^2} - (2\ell+1) - \lambda - \zeta^{(+2)} \right) \frac{\partial}{\partial \zeta^{(+)}} + \frac{\partial}{\partial r},$$

or equivalently we can consider the polynomial vector field

$$\bar{X}_\lambda^+ = \left( r^4 + (\ell+1)(\ell+2) - (2\ell+1)r^2 - \lambda r^2 - r^2 \zeta^{(+2)} \right) \frac{\partial}{\partial \zeta^{(+)}} + r^2 \frac{\partial}{\partial r}.$$

Moreover, we have

$$\begin{aligned} \zeta_{(\lambda,1)}^{(+)} &= \frac{P''_{\lambda/4}}{P'_{\lambda/4}} - r + \frac{\lambda+1}{r}, \\ \zeta_{(\lambda,2)}^{(+)} &= r^{2\ell+2} P_{\lambda/4} \left( r P''_{\lambda/4} - (r^2 - \ell - 1) P'_{\lambda/4} \right) \int \frac{e^{r^2}}{r^{2\ell+2} P_{\lambda/4}^2} + e^{r^2} (r^2 - \ell - 1). \end{aligned}$$

The vector field  $\bar{X}_\lambda^+$  admits the invariant curve

$$\bar{f}_\lambda^{(+)}(\zeta^{(+)}, r) = -\zeta^{(+)} + \frac{\ell+1}{r} + \frac{P''_{\lambda/4}}{P'_{\lambda/4}} - r \in \mathbb{C}(\zeta^{(+)}, r),$$

with generalized cofactor

$$\bar{K}_\lambda^{(+)}(\zeta^{(+)}, r) = -r^2 \zeta^{(+)} - r(\ell+1) - \frac{P''_{\lambda/4}}{P'_{\lambda/4}} r^2 + r^3 \in \mathbb{C}(\zeta^{(+)}, r).$$

The generalized exponential factor

$$\bar{F}_\lambda^{(+)}(\zeta^{(+)}, r) = r^{\ell+2} P'_{\lambda/4} e^{-\frac{r^2}{2}} \in L^2,$$

with generalized cofactor

$$\bar{L}_\lambda^{(+)}(\zeta^{(+)}, r) = r + (\ell+1)r - \frac{P''_{\lambda/4}}{P'_{\lambda/4}} r^2 + r^3 \in \mathbb{C}(r).$$

Additionally, the vector field  $\bar{X}_\lambda^+$  admits the generalized Darboux integrating factor

$$R_\lambda^{(+)}(\zeta^{(+)}, r) = \frac{e^{r^2}}{P_{\lambda/4}^2 r^{2(\ell+2)} \left( -\zeta^{(+)} + \frac{\ell+1}{r} + \frac{P''_{\lambda/4}}{P'_{\lambda/4}} - r \right)^2} \in \mathbb{C}(r, e^{r^2/2}),$$

and the first integral

$$I_\lambda^{(+)}(\zeta^{(+)}, r) = \frac{P'_{\lambda/4} P_{\lambda/4} r^{2\ell+2} \left( r P''_{\lambda/4} - (r^2 + \zeta^{(+)} r - \ell - 1) P'_{\lambda/4} \right) \int \frac{e^{r^2}}{r^{2\ell+2} P_{\lambda/4}^2} dr + A}{\left( r P''_{\lambda/4} - (r^2 + \zeta^{(+)} r - \ell - 1) P'_{\lambda/4} \right) \left( e^{r^2} + r^{2\ell+2} P_{\lambda/2} P'_{\lambda/2} \int \frac{e^{r^2}}{r^{2\ell+2} P_{\lambda/4}^2} dr \right)}$$

with

$$A = P'_{\lambda/4} e^{r^2} (r^2 - \zeta^{(+)} r - \ell - 1) e^{\int \Phi},$$

$$\Phi = \frac{e^{r^2} \left( (2r^2 - 2\ell - 2) P'_{\lambda/4} - r P''_{\lambda/4} \right)}{r P'_{\lambda/4} \left( e^{r^2} + r^{2\ell+2} P_{\lambda/4} P'_{\lambda/4} \int \frac{e^{r^2}}{r^{2\ell+2} P_{\lambda/4}^2} dr \right)}.$$

Note that  $I_{\lambda}^{(\pm)}(\zeta^{(\pm)}, r) \in F$ , being  $F$  an extension of  $\mathbb{C}(r, \zeta^{(\pm)})$  adding the solutions of the Schrödinger equation  $H^{\pm}\Psi = \lambda\Psi$ .

**Example 36** (Coulomb). In this case we have  $V_- = \frac{\ell(\ell+1)}{r^2} - \frac{2(\ell+1)}{r} + 1$  and we can take

$$\Psi_0^{(-)} = r^{\ell+1} e^{-r}, \quad \zeta_0^{(-)} = \frac{\ell+1}{r} - 1 \in \mathbb{C}(r).$$

Note that due to Proposition 16 we have that  $DT$  is strong isogaloisian. Since

$$DT(V_-) = V_+ = \frac{(\ell+1)(\ell+2)}{r^2} - \frac{2(\ell+1)}{r} + 1,$$

we have that the potential  $V_-$  is shape invariant.

$$\text{It is hold that } \Lambda = \left\{ 1 - \left( \frac{\ell+1}{\ell+1+n} \right)^2 : n \in \mathbb{Z}_+ \right\} \cup \left\{ 1 - \left( \frac{\ell+1}{\ell-n} \right)^2 : n \in \mathbb{Z}_+ \right\},$$

see [1, 2]. In particular, for  $\lambda = 1 - \left( \frac{\ell+1}{\ell+1+n} \right)^2$ ,  $\lambda \neq 0$ , we have

$$\Psi_{\lambda}^{(-)} = r^{\ell+1} P_n e^{\frac{-(\ell+1)r}{\ell+1+n}},$$

$$\zeta_{(\lambda,1)}^{(-)} = \frac{\ell+1}{r} + \frac{P'_n}{P_n} - \frac{\ell+1}{\ell+1+n} \in \mathbb{C}(r),$$

$$\zeta_{(\lambda,2)}^{(+)} = \frac{\ell+1}{r} + \frac{P'_n}{P_n} - \frac{\ell+1}{\ell+n+1} - 1 + \frac{e^{\frac{4r-2nr}{\ell+n+1}}}{r^{2\ell+2} P_n^2 \int \frac{e^{\frac{4r-2nr}{\ell+n+1}}}{r^{\ell+2} P_n^2}},$$

where  $P_n$  are polynomials, which are related to the Generalized Laguerre Polynomials, see [31].

According to Lemma 28 for  $T(r) = \ell(\ell+1) - 2(\ell+1)r + r^2$  and  $N(r) = r^2$  the vector field (25) becomes

$$X_{\lambda}^{(-)} = \left( \ell(\ell+1) - 2(\ell+1)r + r^2 - \lambda r^2 - r^2 \zeta^{(-)2} \right) \frac{\partial}{\partial \zeta^{(-)}} + r^2 \frac{\partial}{\partial r},$$

and admits.

- the invariant curve

$$f_{\lambda}^{(-)}(\zeta^{(-)}, r) = -\zeta^{(-)} + \frac{\ell+1}{r} + \frac{P'_n}{P_n} - \frac{\ell+1}{\ell+1+n} \in \mathbb{C}(\zeta^{(-)}, r),$$

with generalized cofactor

$$K_\lambda^{(-)}(\zeta^{(-)}, r) = -r^2\zeta^{(-)} - (\ell + 1)r - \frac{P'_n}{P_n}r^2 + \frac{\ell + 1}{\ell + 1 + n}r^2 \in \mathbb{C}(\zeta^{(-)}, r).$$

- The generalized exponential factor

$$F_\lambda^{(-)}(\zeta^{(-)}, r) = r^{\ell+2}P_n e^{\frac{-(\ell+1)r}{\ell+1+n}} \in L^2,$$

with generalized cofactor

$$L_\lambda^{(-)}(\zeta^{(-)}, r) = r + (\ell + 1)r + \frac{P'_n}{P_n}r^2 - \frac{\ell + 1}{\ell + 1 + n}r^2 \in \mathbb{C}(r),$$

- We note that the vector field  $X_\lambda^-$  admits the generalized Darboux integrating factor

$$R_\lambda^{(-)}(\zeta^{(-)}, r) = \frac{e^{\frac{2(\ell+1)r}{\ell+1+n}}}{P_n^2 r^{2(\ell+2)} \left( -\zeta^{(-)} + \frac{\ell+1}{r} + \frac{P'_n}{P_n} - \frac{\ell+1}{\ell+1+n} \right)^2} \in \mathbb{C}(\zeta^{(-)}, r, e^{\frac{(\ell+1)r}{\ell+1+n}}),$$

- the first integral

$$I_\lambda^{(-)}(\zeta^{(-)}, r) = \left( \frac{(\ell + n + 1)r^{-2\ell-1} e^{\frac{4r-2nr}{\ell+n+1}}}{P_n((\ell + n + 1)rP'_n + (\ell + 1)(\ell + n + 1) - AI_1)} + 1 \right) e^{\int \phi} \in F,$$

with

$$A = \ell r(\zeta^{(-)} + 2) + nr(\zeta^{(-)} + 1) + r(\zeta^{(-)} + 2)P_n$$

$$I_1 = \int \frac{e^{\frac{4r-2nr}{\ell+n+1}}}{r^{2\ell+2}P_n^2} dr,$$

$$\Phi = \frac{e^{\frac{4r-2nr}{\ell+n+1}}}{r^{2\ell+2}P_n^2 \int \frac{e^{\frac{4r-2nr}{\ell+n+1}}}{r^{2\ell+2}P_n^2}}.$$

Note that

$$\zeta_{(\lambda,1)}^{(+)} = \frac{\ell + 1}{r} - \frac{3\ell + n + 3}{\ell + n + 1} + \frac{(\ell + n + 1)P''_n}{((\ell + n + 1)P'_n - (\ell + 1)P_n)} - \frac{(\ell + 1)^2 P_n}{(\ell + n + 1)((\ell + n + 1)P'_n - (\ell + 1)P_n)}.$$

For suitability let assume  $\mu = \ell + n + 1$ , thus we also have

$$\zeta_{(\lambda,2)}^{(+)} = \frac{e^{\frac{2nr}{\mu}} r^{2\ell+2} P_n \chi A_1 P''_n + (\ell + 1)(r(2\ell + n + 2) - (\ell + 1)\mu)P_n + e^{4r}\mu^2(r - \ell - 1)}{r\mu(e^{\frac{2nr}{\mu}} r^{2\ell+2} P_n \chi (\mu P'_n - (\ell + 1)P_n) + \mu e^{4r})},$$

with

$$A_1 = r\mu^2 P_n'' + \mu((\ell+1)\mu - r(3\ell+n+3)),$$

$$\chi = \int \frac{e^{\frac{4r-2nr}{\mu}}}{r^{2\ell+2} P_n^2} dr.$$

After the Darboux transformation  $DT$  the vector field  $X_\lambda^-$  becomes

$$X_\lambda^{(+)} = \left( \frac{(\ell+1)(\ell+2)}{r^2} - \frac{2(\ell+1)}{r} + 1 - \lambda - \zeta^{(+)^2} \right) \frac{\partial}{\partial \zeta^{(+)}} + \frac{\partial}{\partial r}.$$

The vector field  $X_\lambda^+$  admits the invariant curve

$$f_\lambda^{(+)}(\zeta^{(+)}, r) = -\zeta^{(+)} + \frac{\ell+1}{r} + \frac{P_n''}{P_n'} - \frac{\ell+1}{\ell+1+n} \in \mathbb{C}(\zeta^{(+)}, r),$$

with generalized cofactor

$$K_\lambda^{(+)}(\zeta^{(+)}, r) = -\zeta^{(+)} - \frac{\ell+1}{r} - \frac{P_n''}{P_n'} + \frac{\ell+1}{\ell+1+n} \in \mathbb{C}(\zeta^{(+)}, r).$$

The generalized exponential factor

$$F_\lambda^{(+)}(\zeta^{(+)}, r) = r^{\ell+1} P_n' e^{\frac{-(\ell+1)r}{\ell+1+n}} \in L^2,$$

with generalized cofactor

$$L_\lambda^{(+)}(\zeta^{(+)}, r) = \frac{\ell+1}{r} + \frac{P_n''}{P_n'} - \frac{\ell+1}{\ell+1+n} \in \mathbb{C}(r).$$

Hence, the vector field  $X_\lambda^+$  admits the generalized Darboux integrating factor

$$R_\lambda^{(+)}(\zeta^{(+)}, r) = \frac{e^{\frac{2(\ell+1)r}{\ell+1+n}}}{P_n^2 r^{2(\ell+1)} \left( -\zeta^{(+)} + \frac{\ell+1}{r} + \frac{P_n''}{P_n'} - \frac{\ell+1}{\ell+1+n} \right)^2} \in \mathbb{C}(\zeta^{(+)}, r, e^{\frac{(\ell+1)r}{\ell+1+n}}).$$

and the first integral is given by relation

$$I(\zeta^{(+)}, r) = \frac{-\zeta^{(+)} + \zeta_{(\lambda,2)}^{(+)}}{-\zeta^{(+)} + \zeta_{(\lambda,1)}^{(+)}} e^{\int \left( \zeta_{(\lambda,2)}^{(+)} - \zeta_{(\lambda,1)}^{(+)} \right) dr} \in F.$$

Note that  $I_\lambda^{(\pm)}(\zeta^{(\pm)}, r) \in F$ , being  $F$  an extension of  $\mathbb{C}(r, \zeta^{(\pm)})$  adding the solutions of the Schrödinger equation  $H^\pm \Psi = \lambda \Psi$ .

#### 4. FINAL REMARKS

This work is a first approach to apply Darboux transformation into polynomial vector fields that its foliation is of Riccati type. The strong isogallosian structure of the Darboux transformation guarantees that the transformed vector field is also of Riccati type. Additionally, we show that the shape invariant potential preserve the rational structure of the invariant objects such that invariant curves, generalized exponential factors, generalized darbox integrating factors. We remark that in this work we analyze the

case of algebraically solvable potentials, in particular potentials satisfying the shape invariance condition. It arises a natural question: What happens with the quasi-algebraically solvable potentials? A more general study could be also done for non rational potentials.

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