On endomorphisms of torsion-free hyperbolic groups

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Abstract

Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. Let $g_1,\dots,g_n$ and $g_{1*},\dots,g_{n*}$ be elements of H such that g_{r*} is conjugate to g_r for each $r=1,\dots,n$. There is a uniform conjugator if and only if $W(g_{1*},\dots,g_{n*})$ is conjugate to $W(g_1,\dots,g_n)$ for every word W in n variables and length up to a computable constant depending only on $\delta, \sharp S$ and $\sum_{r=1}^{n}|g_r|$.

As a corollary we deduce, that there exists a computable constant $\mathcal{C}=\mathcal{C}(\delta,\sharp S)$ such that for any endomorphism φ of H if $\varphi(h)$ is conjugate to h for every element $h\in H$ of length up to $\mathcal{C}$, then φ is an inner automorphism.

Another corollary is the following: if H is a torsion-free conjugacy separable hyperbolic group, then the group $\mathrm{Out}(H)$ is residually finite.

1 Introduction

Let G be a group and B be a subset of G. An endomorphism φ of a group G is called *pointwise inner on B* if for every $g\in B$, the element $\varphi(g)$ is conjugate to g. We call φ *pointwise inner* if it is pointwise inner on G. The group of all pointwise inner automorphisms of G is denoted by $\mathrm{Aut}_{\mathrm{pi}}(G)$. Clearly, $\mathrm{Inn}(G)\trianglelefteq \mathrm{Aut}_{\mathrm{pi}}(G)\trianglelefteq \mathrm{Aut}(G)$.

There are groups with pointwise inner, but not inner automorphisms. For example, some finite groups [13], any free nilpotent group of class $c\geqslant 3$ (see [4]), some nilpotent Lie groups [5], and direct products of such groups with arbitrary groups. The fact that some nilpotent Lie groups have such phenomena was used to construct isospectral but not isometric Riemannian manifolds [5].

However, for free groups [7], for non-trivial free products [12], and for fundamental groups of closed surfaces of negative Euler characteristic [1] all pointwise inner automorphisms are indeed inner. (In the last paper this property was used to show that surface groups have a weak Magnus property.) The following corollary states that endomorphisms of torsion-free hyperbolic groups, which are pointwise inner on a ball of a uniformly bounded radius are inner automorphisms.

Corollary 1.1 *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. Then there exists a computable constant $\mathcal{C} = \mathcal{C}(\delta, \sharp S)$ such that for any endomorphism φ of H if $\varphi(h)$ is conjugate to h for every element h in the ball of radius $\mathcal{C}$, then φ is an inner automorphism.*

We deduce it from the following theorem.

Theorem 1.2 *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. Let $g_1, \ldots, g_n$ and $g_{1*}, \ldots, g_{n*}$ be elements of H such that g_{r*} is conjugate to g_r for each $r = 1, \ldots, n$. There is a uniform conjugator if and only if $W(g_{1*}, \ldots, g_{n*})$ is conjugate to $W(g_1, \ldots, g_n)$ for every word W in n variables and length up to a computable constant depending only on $\delta, \sharp S$ and $\sum_{r=1}^{n} |g_r|$.*

Note that Corollary 1.1 was formulated in [2]. Independently, D. Osin and A. Minasyan [11] proved a variant of Theorem 1.2 for relatively hyperbolic groups (but without the statement on a computable constant).

V. Metaftsis and M. Sykiotis [8, 9] proved that for any (relatively) hyperbolic group H the group $\mathrm{Inn}(H)$ has finite index in $\mathrm{Aut}_{\mathrm{pi}}(H)$. Their proof is not constructive: it uses ultrafilters and ideas of F. Paulin on limits of group actions.

Our Corollary 1.1 and Theorem 1.1 in [11] both imply that if H is a torsion-free hyperbolic group, then the groups $\mathrm{Inn}(H)$ and $\mathrm{Aut}_{\mathrm{pi}}(H)$ coinside. In [6], E.K. Grossman proved, that if G is a finitely generated conjugacy separable group, then the group $\mathrm{Aut}(G)/\mathrm{Aut}_{\mathrm{pi}}(G)$ is residually finite. From this one can immediately deduce the following corollary.

Corollary 1.3 *If H is a torsion-free conjugacy separable hyperbolic group, then the group $\mathrm{Out}(H)$ is residually finite.*

2 Notations and definitions. Quasi-geodesics in hyperbolic spaces

Let $\mathcal{X}$ be a geodesic metric space. If A, B are points or subsets of $\mathcal{X}$, the distance between them will be denoted by $d(A, B)$, or by $|A, B|$ or simply by $|AB|$. A geodesic segment between points A, B will be denoted by $[AB]$. By a geodesic n-gon $A_1A_2 \ldots A_n$, where $n \geqslant 3$, we mean the union of chosen geodesics $[A_1A_2], [A_2A_3], \ldots, [A_{n-1}A_n], [A_nA_1]$.

Let δ be a nonnegative real number. A geodesic triangle $A_1A_2A_3$ in $\mathcal{X}$ is called *δ-thin* if for any its vertex A_i and any two points $X \in [A_i, A_j]$, $Y \in [A_i, A_k]$ with

$$|A_iX| = |A_iY| \leqslant \frac{1}{2}(|A_iA_j| + |A_iA_k| - |A_jA_k|)$$

we have $|XY| \leqslant \delta$.

It is easy to prove that each side of a δ-thin triangle is contained in the δ-neighborhoods of the union of two other its sides.

The geodesic space $\mathcal{X}$ is called *δ-hyperbolic* if every geodesic triangle in $\mathcal{X}$ is δ-thin.

Let H be a group with a finite generating set S. The length of an element $g \in H$ with respect to S is denoted by $|g|$.

Let $\Gamma(H, S)$ be the geometric realization of the right Cayley graph of H with respect to S. We will consider $\Gamma(H, S)$ as a metric space with the metric, induced by the word metric on H. The ball of radius r about 1 in $\Gamma(H, S)$ is denoted by $\mathcal{B}(r)$. The cardinality of any subset $M \subseteq H$ is denoted by $\sharp M$. For brevity, the cardinality of the set $\mathcal{B}(r) \cap H$ is denoted by $\sharp\mathcal{B}(r)$.

The group H is called *δ-hyperbolic with respect to S* if the corresponding metric space $\Gamma(H, S)$ is δ-hyperbolic. A group is called *hyperbolic* if there exist a finite generating set S and a real number $\delta \geqslant 0$ such that this group is δ-hyperbolic with respect to S. It is known that if a group is hyperbolic with respect to some finite generating set, then it is hyperbolic with respect to any finite generating set.

Proposition 2.1 *If $A_1A_2 \dots A_n$ is a geodesic n-gon in a δ-hyperbolic geodesic space, then each of it sides is contained in the $((n-2)\delta)$-neighborhood of the union of all other its sides.*

Proposition 2.2 *(see Remark 1.21 in Chapter III.H of [3]) If $\mathcal{X}$ is a δ-hyperbolic geodesic space, then for any four points $A, B, C, D \in \mathcal{X}$ the following inequality holds*

$$|AC| + |BD| \leqslant \max\{|BC| + |AD|, |AB| + |CD|\} + 2\delta.$$

Definition 2.3 *Let $(\mathcal{X}, d)$ be a metric space and I be an interval of the real line (bounded or unbounded) or else the intersection of $\mathbb{Z}$ with such an interval. A map $p : I \to \mathcal{X}$ is called (λ, ϵ)-quasi-geodesic (where $\lambda \geqslant 1$ and $\epsilon \geqslant 0$) if*

$$\frac{1}{\lambda}|t - s| - \epsilon \leqslant d(p(t), p(s)) \leqslant \lambda|t - s| + \epsilon$$

for all $s, t \in I$.

Definition 2.4 *Let $(\mathcal{X}, d)$ be a metric space and fix $k > 0$. A path $p : [a, b] \to \mathcal{X}$ is said to be a k-local geodesic if $d(p(t), p(s)) = |t - s|$ for all $s, t \in [a, b]$ with $|t - s| \leqslant k$.*

Lemma 2.5 *(see Theorem 1.12 in Chapter III.Γ of [3]) Let H be a δ-hyperbolic group with respect to a finite generating set S. If $u, v \in H$ are conjugate, then the length of the shortest conjugator is bounded from above by a computable function of* $\max\{|u|, |v|\}$, *δ and $\sharp S$.*

The following proposition (without the statement on computability of λ, ϵ) is Corollary 3.10 of Chapter III.Γ in [3]. Since we need the computability and the statement is not obvious, we give a sketch of the proof.

Proposition 2.6 *Let H be a δ-hyperbolic group with respect to a finite generating set S, and let $g \in H$ has infinite order. Then the map $\mathbb{Z} \to H$ given by $n \mapsto g^n$ is a (λ, ϵ)-quasi-geodesic, where λ, ϵ are computable functions of $\delta, \sharp S, |g|$ only.*

Proof. First we note two easy facts.

1) Let k be a natural number. If the map $\mathbb{Z} \to H$ given by $n \mapsto g^{kn}$ is (λ_1, ϵ_1)-quasi-geodesic, then the map $\mathbb{Z} \to H$ given by $n \mapsto g^n$ is (λ, ϵ)-quasi-geodesic with $\lambda = k\lambda_1$ and $\epsilon = k|g| + \epsilon_1 + 1$. Thus, at any moment we can replace g by an appropriate g^k.

2) Let $g_0 = Short(g)$ be the shortest word among those, which are conjugate to g in H. Then there exists $h \in H$, such that $g = h^{-1}g_0h$ and $|h| \leqslant f(\delta, \sharp S, |g|)$, where f is a computable function by Lemma 2.5. If the map $\mathbb{Z} \to H$ given by $n \mapsto g_0^n$ is (λ_1, ϵ_1)-quasi-geodesic, then the map $\mathbb{Z} \to H$ given by $n \mapsto g^n$ is (λ, ϵ)-quasi-geodesic with $\lambda = \lambda_1$ and $\epsilon = \epsilon_1 + 2|h|$.

Since different powers of g are not conjugate, there is a natural $k \leqslant \sharp\mathcal{B}(8\delta + 1)$, such that $|Short(g^k)| > 8\delta$. Replacing g by $Short(g^k)$, we may assume that $|g| > 8\delta$ and g is the shortest word among those which are conjugate to g in H.

Then the bi-infinite path p_g, that begins at 1 and is labelled by powers of g is a k-local-geodesic with $k > 8\delta$. By [3, Theorem 1.13, Ch. III.H], p_g is $(2\delta, 3)$-quasi-geodesic. □

Corollary 2.7 *In notations of Proposition 2.6, for any natural n holds*

$$\frac{1}{\lambda}n - \epsilon \leqslant |g^n| \leqslant \lambda n + \epsilon.$$

The following proposition (without the statement on computability of R) is Theorem 1.7 in Chapter III.H of [3]. The computability of R can be extracted from the proof.

Proposition 2.8 *For all $\delta \geqslant 0, \lambda \geqslant 1, \epsilon \geqslant 0$ there exists a computable constant $R(\delta, \lambda, \epsilon)$ with the following property:*

If $\mathcal{X}$ is a δ-hyperbolic geodesic space, p is a (λ, ϵ)-quasi-geodesic in $\mathcal{X}$ and c is a geodesic segment joining the endpoints of p, then the Hausdorff distance between c and the image of p is less than R.

Corollary 2.9 *Let H be a δ-hyperbolic group with respect to a finite generating set S, and let $g \in H$ has infinite order.Then for any integers $i < j$ every geodesic segment $[g^i, g^j]$ lies in the μ-neighborhood of the set $\{g^i, g^{i+1}, \dots, g^j\}$ and this set lies in the μ-neighborhood of this segment, where $\mu = \mu(\delta, \sharp S, |g|)$ is a computable constant.*

Corollary 2.10 *Let H be a δ-hyperbolic group with respect to a finite generating set S, and let $g \in H$ has infinite order. For any natural numbers s, t holds:*

$$|g^{s+t}| \geqslant |g^s| + |g^t| - 2\mu,$$

where $\mu = \mu(\delta, \sharp S, |g|)$ is the constant from Corollary 2.9.

Proof. Set $A = 1$, $B = g^s$ and $C = g^{s+t}$. Choose geodesics $[AB]$, $[BC]$ and $[AC]$. By Corollary 2.9 the point B lies in the μ-neighborhood of $[AC]$, that is there exists $D \in [AC]$ such that $|BD| \leqslant \mu$. Then $|AC| \geqslant (|AB| - |BD|) + (|CB| - |BD|) \geqslant |AB| + |BC| - 2\mu$. □

The following proposition (without the statement on computability of n) is Proposition 3.20 of Ch. III.H [3]. The computability of n can be extracted from the proof.

Proposition 2.11 *Let H be a δ-hyperbolic group with respect to a finite generating set S. Then for every finite set of elements $h_1, \dots, h_r \in H$ there exists an integer $n > 0$ such that $h_1^n, \dots, h_r^n$ generate a free group of rank at most r. The integer n is a computable function of $\delta, \sharp S$ and $\sum_i^r |h_i|$.*

Notations. Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S.

1) For elements $u, v \in H$ and a real number $c > 0$ we will write $uv = u \underset{c}{\cdot} v$ if $\frac{1}{2}(|u|+|v|-|uv|) < c$. The last is equivalent to $|uv| > |u| + |v| - 2c$. If H is a free group, this means that the maximal terminal segment of u and the maximal initial segment of v which can be cancelled in the product uv are both of length smaller than c.

We will write $uvw = u \underset{c}{\cdot} v \underset{c}{\cdot} w$ if $uv = u \underset{c}{\cdot} v$ and $vw = v \underset{c}{\cdot} w$. By Lemma 9.1, if $|v| > 2c + \delta$, then

$$|u \underset{c}{\cdot} v \underset{c}{\cdot} w| > |u| + |v| + |w| - (4c + 2\delta).$$

2) Let $G = \{g_1, \dots, g_n\}$ be a finite subset of H. The symbol $u \approx_G v$ means that $|u^{-1}v|$ is bounded from above by a computable function, depending on δ, $\sharp S$ and $|g_1|, \dots, |g_n|$ only. The function will be clear from the context. Similarly, we write $|u| \approx_{|g|} |v|$ if $||u| - |v||$ is bounded from above by a computable function, depending on the same arguments.

3) It is known that for any nontrivial element $g \in H$ its centralizer $C_H(g)$ is infinite cyclic. Thus, the extraction of roots in H is unique. If the root of $g \in H$ of degree q exists, we denote it by $g^{\frac{1}{q}}$.

Agreement. For brevity, we will not write δ and $\sharp S$ in the arguments of computable functions.

3 The norm and the axis of an element

Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S.

Let g be a nontrivial element of H. The number $||g|| = \min_x \{d(x, gx)\}$, where x goes through all points of the Cayley graph $\Gamma(H, S)$ is called *the norm* of g. *The axis* $\mathcal{A}_g$ of g is the set of all points x in the Cayley graph $\Gamma(H, S)$ such that $d(x, gx) = ||g||$. It is easy to show that $||g||$ is a positive natural number, $\mathcal{A}_g \cap H$ is nonempty and A_g is g-invariant (it is even $C_H(g)$-invariant). Moreover, for any $x \in \mathcal{A}_g$ any geodesic segment $[x, gx]$ also lies in $\mathcal{A}_g$. Obviously, $||g|| \leqslant |g|$, $||hgh^{-1}|| = ||g||$ for any $h \in H$, and $\mathcal{A}_{hgh^{-1}} = h\mathcal{A}_g$.

Lemma 9.5 asserts that there exists a computable natural number $r = r(\delta, \sharp S, |g|)$ such that

$$\bigcup_{k=1}^{\infty} \mathcal{A}_{g^k} \subseteq \langle g \rangle \mathcal{B}(r).$$

From this, it is easy to deduce the following two corollaries.

Corollary 3.1 *For any nontrivial element $g \in H$ and any integer $k \neq 0$ there exists an element $x \in \mathcal{A}_{g^k} \cap H$ of length at most $r(|g|)$.*

Corollary 3.2 *For any nontrivial element $g \in H$ and any integer $k \neq 0$ the inequality $||g^k|| \geqslant |g^k| - 2r(|g|)$ holds.*

Proof. We take the element x from Corollary 3.1. Then $||g^k|| = d(x, g^k x) = |x^{-1} g^k x| \geqslant |g^k| - 2|x| \geqslant |g^k| - 2r(|g|)$. □

Corollary 3.3 *Let g be a nontrivial element of H. For any natural number C there exists a computable integer $k_0 = k_0(\delta, \sharp S, |g|, C)$, such that for any $k > k_0$ we have $||g^k|| > C$.*

Proof. The proof follows from Corollaries 3.2 and 2.7. □

Proposition 3.4 *There exist computable functions $f_1 : \mathbb{N} \to \mathbb{N}$ and $f_2 : \mathbb{N} \to \mathbb{N}$ such that for every nontrivial element $g \in H$ and for every natural numbers s, t we have*

$$||g^{s+t}|| - f_1(|g|) \leqslant ||g^s|| + ||g^t|| \leqslant ||g^{s+t}|| + f_2(|g|).$$

Proof. The first inequality follows from the inequalities

$$||g^{s+t}|| \leqslant |g^{s+t}| \leqslant |g^s| + |g^t| \leqslant ||g^s|| + ||g^t|| + 4r(|g|),$$

where we use Corollary 3.2. The second inequality follows from the inequalities

$$||g^s|| + ||g^t|| \leqslant |g^s| + |g^s| \leqslant |g^{s+t}| + 2\mu(|g|) \leqslant ||g^{s+t}|| + 2r(|g|) + 2\mu(|g|),$$

where we use Corollary 3.2 and Corollary 2.10. □

Proposition 3.5 *For any nontrivial element $g \in H$ and any point $P \in \Gamma(H, S)$ holds*

$$d(P, gP) \leqslant ||g|| + 2d(P, \mathcal{A}_g).$$

Recall, that to that moment we have introduced computable functions $\lambda, \epsilon, \mu, r, f_1, f_2$.

4 First technical lemma

Suppose that the product of conjugates of two powers of a given element equals the product of these powers. In this situation, the following technical lemma shows how the involved conjugators look like.

Lemma 4.1 *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. There exists a computable function $\hbar : \mathbb{N} \to \mathbb{R}_+$ with the following property: for any three elements $b, x, y \in H$ and any two positive integers s, t, which satisfy $\min\{||b^s||, ||b^t||, ||b^{s+t}||\} > 13\delta$ and*

$$(x \cdot b^s \cdot x^{-1})(y \cdot b^t \cdot y^{-1}) = b^{s+t}, \tag{1}$$

there exist integers n_1, n_2, n_3, n_4 and elements $v_x, v_y \in H$ of length at most $\hbar(|b|)$ such that

$$x = b^{n_1} v_x b^{n_2} \qquad \text{and} \qquad y = b^{n_3} v_y b^{n_4}.$$

Proof. Let b, x, y and s, t be as in the statement. Consider the axes $\mathcal{A}_{xb^sx^{-1}} = x\mathcal{A}_{b^s}$ and $\mathcal{A}_{yb^ty^{-1}} = y\mathcal{A}_{b^t}$. By Proposition 10.4 applied to the elements xb^sx^{-1} and yb^ty^{-1} (note that $||xb^sx^{-1}|| = ||b^s||$, $||yb^ty^{-1}|| = ||b^t||$ and $||(xb^sx^{-1})(yb^ty^{-1})|| = ||b^{s+t}||$ are all bigger than 13δ by hypothesis), the distance between $x\mathcal{A}_{b^s}$ and $y\mathcal{A}_{b^t}$ is at most

$$\max\{13\delta,\ \frac{1}{2}(||b^{s+t}|| - ||b^s|| - ||b^t||) + 16\delta\}.$$

And, by Proposition 3.4, this value does not exceed $\frac{1}{2}f_1(|b|) + 16\delta$, an upper bound which does not depend on s and t.

Now, take a point $\mathbf{Q} \in y\mathcal{A}_{b^t}$ such that $d(\mathbf{Q}, x\mathcal{A}_{b^s}) \leqslant \frac{1}{2}f_1(|b|) + 16\delta$, and set $\mathbf{P} = (yb^ty^{-1})^{-1}\mathbf{Q}$; in particular, $\mathbf{P} \in y\mathcal{A}_{b^t}$. Then we get

$$\begin{aligned}
d(\mathbf{P}, b^{s+t}\mathbf{P}) &= d(\mathbf{P}, (xb^sx^{-1})(yb^ty^{-1})\mathbf{P}) = d(\mathbf{P}, (xb^sx^{-1})\mathbf{Q}) \\
&\leqslant d(\mathbf{P}, \mathbf{Q}) + d(\mathbf{Q}, (xb^sx^{-1})\mathbf{Q}) \\
&\leqslant d(\mathbf{P}, \mathbf{Q}) + 2d(\mathbf{Q}, \mathcal{A}_{xb^sx^{-1}}) + ||b^s|| \\
&\leqslant ||b^t|| + ||b^s|| + f_1(|b|) + 32\delta \\
&\leqslant ||b^{s+t}|| + f_1(|b|) + f_2(|b|) + 32\delta,
\end{aligned}$$

where the last inequality uses Proposition 3.4. Next, apply Lemma 10.2 to conclude that $\mathbf{P} = b^{n_3}u$ for some $n_3 \in \mathbb{Z}$ and $u \in H$ with $|u| \leqslant \frac{1}{2}f_1(|b|) + \frac{1}{2}f_2(|b|) + r(|b|) + 19\delta$. And since $\mathbf{P} \in y\mathcal{A}_{b^t}$, we deduce from Lemma 9.5 that $y^{-1}\mathbf{P} = b^{-n_4}u'$, for some $n_4 \in \mathbb{Z}$ and $u' \in H$ with $|u'| \leqslant r(|b|)$. Hence,

$$y = b^{n_3}v_yb^{n_4},$$

where $v_y = uu'^{-1}$ has length bounded by

$$|v_y| = |uu'^{-1}| \leqslant |u| + |u'| \leqslant \frac{1}{2}f_1(|b|) + \frac{1}{2}f_2(|b|) + 2r(|b|) + 19\delta.$$

Finally, inverting and replacing b to b^{-1} in equation (1), we obtain again the same equation with x and y interchanged. So, the same argument shows that

$$x = b^{n_1}v_xb^{n_2},$$

for some $n_1, n_2 \in \mathbb{Z}$ and some $v_x \in H$ with the same upper bound for its length.

Hence, the function $\hbar(n) = \frac{1}{2}f_1(n) + \frac{1}{2}f_2(n) + 2r(n) + 19\delta$ satisfies the statement of the lemma. □

Corollary 4.2 *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set X. There exists a computable function $\hbar : \mathbb{N} \to \mathbb{R}_+$ with the following property.*

If $x_1b^{n_1}x_2b^{n_2}x_3b^{n_3} = 1$ is an equality in H, where $x_1x_2x_3 = 1$, $n_1 + n_2 + n_3 = 0$, all n_i are non-zero, and $||b^{n_i}|| > 13\delta$ for all i, then each of the x_i can be represented in the form $b^{m_1}ub^{m_2}vb^{m_3}$, where both u, v have length at most $\hbar(|b|)$.

Proof. Inverting the equality and cyclically permuting, we may assume that $n_1 > 0$ and $n_2 > 0$. Then we can apply Lemma 4.1. □

5 A special case of the main Theorem

Here we prove Theorem 8.2 in case, where $n = 2$ and g_1, g_2 generate a cyclic group. The proof contains ingredients, which will be used in general case.

Proposition 5.1 *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. Then for any $g \in H$ there is a constant $C = C(|g|)$ with the following property:*

Let a, b be some powers of g with $\min\{||a||, ||b||, ||ab^{-1}||\} > 13\delta$, and let b_* be a conjugate of b. If ab_*^s is conjugate to ab^s for every $s = -C, \ldots, C$, then $b_* = b$.

Proof. We will determine $C > 1$ dynamically in a finite number of steps. Let $a = g^n$, $b = g^m$, and $b_* = x^{-1}bx$ for some $x \in H$.

We may assume that $n, m > 0$. Indeed, if $n < 0$, we may first replace g by g^{-1} and n by $-n$, and m by $-m$, thus getting $n > 0$. If then $m < 0$, we may replace b by $b^{-1} = g^{-m}$ and b_* by b_*^{-1}, thus getting $m > 0$.

Suppose that ab_*^{-1} is conjugate to ab^{-1}, that is $g^n \cdot x^{-1}g^{-m}x = h^{-1}g^{n-m}h$ for some $h \in H$. We rewrite the last equation in two forms:

$$xh^{-1}g^{m-n}hx^{-1} \cdot xg^nx^{-1} = g^m, \tag{2}$$

$$h^{-1}g^{n-m}h \cdot x^{-1}g^mx = g^n. \tag{3}$$

We have $n \neq m$ (otherwise $a = b$, that contradicts to the assumption $||ab^{-1}|| > 13\delta$). If $m > n$, then from (2) and Lemma 4.1 we get that

$$x = g^pvg^q$$

for some $p, q \in \mathbb{Z}$ and $v \in H$ with $v \approx_g 1$. If $m < n$, then from (3) and Lemma 4.1 we get the same expression for x. Since x is defined (from the left) up to the centralizer of g, we may assume that $x = vg^q$. We have still $b_* = x^{-1}bx$. Replacing b_* by $g^qb_*g^{-q}$ (that does not change the hypothesis and the conclusion of the proposition), we may assume that $x = v$, and so $x \approx_g 1$.

Now we write ab_*^s as a conjugate of $ab^s = g^{n+sm}$. By Corollary 9.3, there exists $z_s \in H$ such that

$$g^n \cdot x^{-1}g^{sm}x = ab_*^s = z_s^{-1} \underset{c}{\cdot} g^{n+sm} \underset{c}{\cdot} z_s, \tag{4}$$

where the constant c depends on $|g|, \delta, \sharp S$ only. Comparing lengths and using Lemma 9.1 for sufficiently large s (to guarantee $|g^{n+sm}| > 2c + \delta$; for that apply Corollary 2.7), we have

$$|g^n| + |g^{sm}| + 2|x| > |g^{n+sm}| + 2|z_s| - (4c + 2\delta).$$

Recall that $n, m > 0$. For positive s we have also $|g^{n+sm}| \approx_{|g|} |g^n| + |g^{sm}|$ by Corollary 2.10. Recalling that $|x| \approx_{|g|} 0$, we get that $|z_s| \approx_{|g|} 0$. It follows, that if s goes over a sufficiently large set $\{\lfloor C/2 \rfloor, \ldots, C\}$, there must be repetitions: there exist $C/2 < s_1 < s_2 < C$ such that $z_{s_1} = z_{s_2}$ (denote it by z). We have

$$ab_*^{s_1} = z^{-1}g^{n+s_1m}z \tag{5}$$

and

$$ab_*^{s_2} = z^{-1}g^{n+s_2m}z.$$

From this we deduce

$$b_*^{s_1-s_2} = z^{-1}g^{m(s_1-s_2)}z.$$

This implies $b_* = z^{-1}g^m z$ and then (5) implies $a = z^{-1}g^n z$. Since $a = g^n$, the element z commutes with g, and so $b_* = b$. $\square$

6 Second technical lemma

The following lemma is a preliminary step in proving of Theorem 7.2. Equations (6) and (7) in its formulation have the following common form: the product of certain conjugates of two elements equals the product of these two elements.

Lemma 6.1 *Let H be a torsion-free δ-hyperbolic group with respect to a finite generationg set S, and let $b, b_*, w \in H$. There exists a computable natural number $M = M(|b|, |w|)$ such that the following holds:*

If b_* is conjugate to b (say $b_* = h^{-1}bh$), and wb_*^k is conjugate to wb^k for every $k = 1, \dots, M$, then there exists an element $d \in H$ and integers m, s, t, such that $s + t > 0$ and the following equations hold

$$(d \cdot b^s \cdot d^{-1})(dw \cdot b^t \cdot w^{-1}d^{-1}) = b^{s+t}, \tag{6}$$

$$(d^{-1}h \cdot w \cdot h^{-1}d)(d^{-1} \cdot b^m \cdot d) = wb^m. \tag{7}$$

Proof. We may assume that $b \neq 1$. By Corollary 9.3, there exists an element $h \in H$ such that for any integer k we have $b_*^k = h^{-1} \underset{c}{\cdot} b^k \underset{c}{\cdot} h$, where $c = 3\delta + \mu(|b|) + 1$. Since this expression remains valid while enlarging the constant, we shall consider it with $c = 3\delta + \mu(|b|) + |w| + 1$ in order to match with other calculations below. Thus,

$$wb_*^k = w(h^{-1} \underset{c}{\cdot} b^k \underset{c}{\cdot} h). \tag{8}$$

Suppose that wb_*^k is conjugate to wb^k for every $k = 1, \dots, M$ (the correct M will be chosen later). Then, by Lemma 9.2, for each of these k's there exist an element $e_k \in H$ and an integer l_k, such that $0 \leqslant l_k \leqslant k$ and

$$wb_*^k = e_k^{-1} \underset{c}{\cdot} (b^{k-l_k}wb^{l_k}) \underset{c}{\cdot} e_k. \tag{9}$$

Below we write $\approx$ instead of $\approx_{w,b}$. By Corollary 2.7 and by Lemma 9.6, there exists a natural number $K = K(|b|, |w|)$ such that $|b^k|$ and $|b^{k-l_k}wb^{l_k}|$ are bigger than $2c + \delta$ for all $k \geqslant K$. We consider $k \geqslant K$. Then from (8) and (9), with the help of Lemma 9.1, we deduce

$$|wb_*^k| \approx 2|h| + |b^k|$$

and

$$|wb_*^k| \approx 2|e_k| + |b^{k-l_k}wb^{l_k}| \approx 2|e_k| + |b^k|,$$

where the last approximation is due to Lemma 9.6. Therefore $|e_k| \approx |h|$.

Now we prove that $e_k \approx h$. For that we realize the right hand side of (8) in the Cayley graph $\Gamma(H, S)$ as the path starting at 1 and consisting of 4 consecutive geodesics with labels equal in

H to the elements w, h^{-1}, b^k, h. Analogously we realize the right hand side of (9) as the path starting at 1 and consisting of 3 consecutive geodesics with labels equal in H to the elements e_k^{-1}, $b^{k-l_k}wb^{l_k}$, e_k.

Both paths are (λ, ϵ)-quasigeodesics, connecting 1 and wb_*^k, where λ, ϵ depend on c only. By Proposition 2.8, these quasigeodesics are at bounded distance from each other. Since their last segments have labels equal to h and e_k in H, and since $|e_k| \approx |h|$, we deduce that $e_k \approx h$.

Thus e_k lies in the ball with center h and radius depending only on $|b|$ and $|w|$. Let M be the number of elements in this ball plus $(K+1)$.

Then there exist natural numbers $K \leqslant k_1 < k_2 \leqslant M$ such that $e_{k_1} = e_{k_2}$. Denote this element by e and, rewriting equation (9) for these two special values of k,

$$wb_*^{k_1} = e^{-1}(b^{k_1 - l_{k_1}} wb^{l_{k_1}})e \tag{10}$$

and

$$wb_*^{k_2} = e^{-1}(b^{k_2 - l_{k_2}} wb^{l_{k_2}})e,$$

we get

$$b_*^{k_2-k_1} = e^{-1}(b^{-l_{k_1}} w^{-1} b^{k_2-k_1+l_{k_1}-l_{k_2}} wb^{l_{k_2}})e.$$

Set $s = k_2 - k_1 + l_{k_1} - l_{k_2}$ and $t = l_{k_2} - l_{k_1}$ (so $s + t > 0$). Recalling that $b_*^{k_2-k_1} = h^{-1}b^{k_2-k_1}h$, we can rewrite the previous equation as

$$he^{-1}b^{-l_{k_1}}w^{-1}b^s wb^t b^{l_{k_1}} eh^{-1} = b^{s+t}.$$

Setting $d = he^{-1}b^{-l_{k_1}}w^{-1}$, we deduce $(db^sd^{-1})\cdot(dwb^tw^{-1}d^{-1}) = b^{s+t}$, which is equation (6). And using equation (10), the definition of d and $b_*^{k_1} = h^{-1}b^{k_1}h$, we obtain $(d^{-1}hwh^{-1}d)\cdot(d^{-1}b^{k_1}d) = wb^{k_1}$, which is equation (7) with $m = k_1$. □

Now, using (6) and (7) we obtain more information on relations between w, b and h.

Proposition 6.2 *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S and let b, w, d be elements of H that satisfy equation (6), where $s + t > 0$. Suppose additionally that $||b^k|| > 13\delta$ for all $k > 0$, and that $st \neq 0$. Then, there exist integers p, q, r and elements $u, v \in H$ of length at most $\hbar(|b|)$, such that*

$$w = b^p ub^r vb^q.$$

Proof. The proof follows from Corollary 4.2

Proposition 6.3 *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S and let b, w, d, h be elements of H that satisfy equations (6) and (7), where $s + t > 0$. Suppose additionally that $||b^k|| > 13\delta$ for all $k > 0$, and that $st = 0$. Then $h = b^p w^q$ for some rational numbers p, q.*

Proof. Let us distinguish the following two cases:

Case 1: $s = 0$. In this case, equation (6) says that dw commutes with b. So, $dw = b^p$ for some rational p. Plugging this into equation (7) we obtain $hwh^{-1} = b^{p+m}d^{-1}b^{-m} = b^{p+m}wb^{-p-m}$. Hence, $b^{-p-m}h$ commutes with w and the result follows.

Case 2: $t = 0$. In this case, equation (6) says that d commutes with b. So, $d = b^p$ for some rational p. Plugging this into equation (7) we obtain $b^{-p}hwh^{-1}b^p = w$. Hence, $b^{-p}h$ commutes with w and the result follows. □

7 Main theorem for two words

Now, we want to obtain some extra information by applying Lemma 6.1 to sufficiently many different elements w. To achieve this goal, given a pair of elements $a, b \in H$ we consider the finite set

$$\mathcal{W} = \{(a^i b)^{2j} \mid 1 \leqslant i \leqslant N,\ 1 \leqslant j \leqslant N^2\} \subseteq \langle a, b \rangle \leqslant H,$$

where

$$N = N(|b|) = 1 + (\sharp\mathcal{B}\,(\hbar(|b|)))^2$$

and the function $\hbar$ is defined in Lemma 4.1. Let us systematically apply Lemma 6.1 to every $w \in \mathcal{W}$.

Lemma 7.1 *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. Let $a, b \in H$ be elements, which generate a free subgroup of rank 2 and let $||b^k|| > 13\delta$ for all $k > 0$. Suppose that for every $w \in \mathcal{W}$, there exists a conjugate b_* of b such that the elements w, b, b_* satisfy the hypothesis of Lemma 6.1 (i.e. wb_*^k is conjugate to wb^k, for every integer $k = 1, \ldots, M(|b|, |w|)$). Then, for at least one such $w \in \mathcal{W}$, the conclusion of Lemma 6.1 holds with $st = 0$.*

Proof. Suppose the opposite and let us find a contradiction. We write $\mathcal{W} = \bigsqcup_{i=1}^N \mathcal{W}_i$, where $\mathcal{W}_i = \{(a^i b)^{2j} \mid 1 \leqslant j \leqslant N^2\}$. First we fix $i \in \{1, \ldots, N\}$. By Proposition 6.2, for every $w \in \mathcal{W}_i$, there exist integers p, q, r, and elements $u, v \in H$ of length at most $\hbar(|b|)$ such that

$$b^p w b^q = u b^r v. \tag{11}$$

Of course, these integers and elements depend on w. Since $\sharp\mathcal{W}_i > (\sharp\mathcal{B}(\hbar(|b|)))^2$ and the lengths of u and v are at most $\hbar(|b|)$, there exists a pair of elements $w_1, w_2 \in \mathcal{W}_i$ with the same u and v:

$$b^{p_1} w_1 b^{q_1} = u b^{r_1} v,$$

$$b^{p_2} w_2 b^{q_2} = u b^{r_2} v.$$

Combining these equations, we get

$$b^{p_2} w_2 b^{q_2 - q_1} w_1^{-1} b^{-p_1} = u b^{r_2 - r_1} u^{-1}. \tag{12}$$

Moreover, since $\sharp\mathcal{W}_i > (\sharp\mathcal{B}(\hbar(|b|)))^4$, there exists else one (disjoint) pair of elements $w_3, w_4 \in \mathcal{W}_i$ with the same u, v as above. And we similarly get

$$b^{p_4} w_4 b^{q_4 - q_3} w_3^{-1} b^{-p_3} = u b^{r_4 - r_3} u^{-1}. \tag{13}$$

Hence, the left sides of equations (12) and (13) commute. Write $w_1 = (a^i b)^\sigma$, $w_2 = (a^i b)^\tau$, $w_3 = (a^i b)^{\sigma'}$, $w_4 = (a^i b)^{\tau'}$. Simplifying notation, we can write the left sides of (12), (13) as

$$x = b^\alpha (a^i b)^\tau b^\beta (a^i b)^{-\sigma} b^\gamma$$

and

$$x' = b^{\alpha'}(a^ib)^{\tau'}b^{\beta'}(a^ib)^{-\sigma'}b^{\gamma'}$$

We have understood that x and x' commute. Since w_1, w_2, w_3, w_4 are different elements of $\mathcal{W}_i$, the exponents σ, τ, σ', τ' are positive and differ by at least 2. Note, that no specific information about the other integers $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ is known. The key point here is that this commutativity relation happens inside the free group $\langle a, b\rangle$.

Consider now the monomorphism $\langle a, b\rangle \to \langle a, b\rangle$ given by $a \mapsto a^ib$, $b \mapsto b$. Since x and x' both lie in its image, and commute, their preimages, namely $y = b^{\alpha}a^{\tau}b^{\beta}a^{-\sigma}b^{\gamma}$ and $y' = b^{\alpha'}a^{\tau'}b^{\beta'}a^{-\sigma'}b^{\gamma'}$, must also commute.

Suppose $\beta\beta' \neq 0$. Then, y is not a proper power in $\langle a, b\rangle$ (in fact, its cyclic reduction is either $a^{\tau}b^{\beta}a^{-\sigma}b^{\alpha+\gamma}$ with $\alpha+\gamma \neq 0$, or $a^{\tau-\sigma}b^{\beta}$, which are clearly not proper powers). Similarly, y' is not a proper power either. Then the commutativity of y and y' forces $y = y'^{\pm 1}$, which is obviously not the case. Hence, $\beta\beta' = 0$. Without loss of generality, we can assume $\beta = 0$.

Let us go back to equation (12) which, particularized to this special case, reads

$$b^{\alpha}(a^ib)^{\tau}b^0(a^ib)^{-\sigma}b^{\gamma} = ub^{\delta}u^{-1}, \tag{14}$$

that is

$$b^{\alpha}(a^ib)^{\rho}b^{\gamma} = ub^{\delta}u^{-1}, \tag{15}$$

where $\rho = \tau - \sigma$ and so $|\rho| \geqslant 2$. Recall, that the length of u is at most $\hbar(|b|)$.

Finally, it is time to move $i = 1, \ldots, N$. Since $N > \sharp\mathcal{B}(\hbar(|b|))$, there must exist two indices i_1 and i_2, with $1 \leqslant i_1 < i_2 \leqslant N$ and such that $u_{i_1} = u_{i_2}$ (call this element just u). Equation (15) in these two special cases says that

$$b^{\alpha}(a^{i_1}b)^{\rho}b^{\gamma} = ub^{\delta}u^{-1}$$

and

$$b^{\alpha'}(a^{i_2}b)^{\rho'}b^{\gamma'} = ub^{\delta'}u^{-1}.$$

Hence, $z = b^{\alpha}(a^{i_1}b)^{\rho}b^{\gamma}$ and $z' = b^{\alpha'}(a^{i_2}b)^{\rho'}b^{\gamma'}$ again commute, where $|\rho|, |\rho'| \geqslant 2$ and $1 \leqslant i_1 < i_2$. This implies that some positive power of z equals some positive power of z'. But it is straightforward to see that (after all possible reductions) the first a-syllable of any positive power of z is a^{i_1} (here we use $|\rho| \geqslant 2$); similarly the first a-syllable of any positive power of z' is a^{i_2}. Since $i_1 \neq i_2$, this is a contradiction which completes the proof. □

Now we prove the main Theorem 8.2 in case $n = 2$.

Theorem 7.2 *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S, and consider four elements $a, b, a_*, b_* \in H$ such that a_* is conjugate to a, and b_* is conjugate to b. There exists a computable constant C (only depending on $|a|$, $|b|$, δ and $\sharp S$), such that if $(a_*^ib_*^l)^jb_*^k$ is also conjugate to $(a^ib^l)^jb^k$ for every $i, j, k, l = -C, \ldots, C$ then there exists a uniform conjugator $g \in H$ with $a_* = g^{-1}ag$ and $b_* = g^{-1}bg$ (i.e. (a_*, b_*) is conjugate to (a, b)).*

Proof. The conclusion is obvious if a or b is trivial. So, let us assume $a \neq 1$ and $b \neq 1$. Note, that $\langle a\rangle = \langle b\rangle$ and even $a = b$ is allowed.

All along the proof, C will be an unspecified constant, and we shall prove the result imposing several times that C is big enough. At the end, collecting together all these requirements, we shall propose a valid value for C.

Since H is torsion-free, every nontrivial element has infinite cyclic centralizer. Let a_1, b_1 be generators of $C_H(a)$ and $C_H(b)$. We may assume that $a = a_1^p$ and $b = b_1^q$ for positive p and q. By Corollary 3.3, there exists a computable natural number r_0 such that for every $r \geqslant r_0$, $||a_1^r|| > 13\delta$ and $||b_1^r|| > 13\delta$. So, after replacing a, b, a_*, b_* by $a^{r_0}, b^{r_0}, a_*^{r_0}, b_*^{r_0}$, we can assume that $||a^r|| > 13\delta$ and $||b^r|| > 13\delta$ for every $r \neq 0$. Moreover, if a, b generate a cyclic group, then after the above replacement we have $a = b$ or $||ab^{-1}|| > 13\delta$.

For every word w on a and b, let us denote by w_* the corresponding word on a_* and b_*. Now, observe that we can uniformly conjugate a_* and b_* by any element of H (and abuse notation denoting the result a_* and b_* again), and both the hypothesis and conclusion of the theorem does not change. In particular, for any chosen word of the form $w = (a^i b^l)^j b^k$ (with $i, j, k, l \in \{-C, \ldots, C\}$), we can assume that $w_* = w$ (of course, with an underlying a_* and b_* now depending on w); when doing this, we say that we *center the notation on w*. Centering does not change a, b, it changs only a_*, b_*, therefore each time the constants depend only on a, b.

Let us distinguish two cases.

Case 1: $\langle a, b\rangle$ is a cyclic group. Centering the notation on a, we may assume that $a_* = a$. If $a = b$, then we use that ab_*^{-1} is conjugate to $ab^{-1} = 1$ and deduce immediately that $b_* = b$. Now we assume that $a \neq b$, and so $||ab^{-1}|| > 13\delta$. A part of our hypothesis says that $a_* b_*^s = ab_*^s$ is conjugate to ab^s for every $s = -C, \ldots, C$. Hence, for C as in Proposition 5.1, we have $b_* = b$. This concludes the proof in this case.

Case 2: $\langle a, b\rangle$ is not cyclic. By Proposition 2.11, there exists a suffciently big natural number p such that $\langle a^p, b^p\rangle$ is a free subgroup of H of rank 2. And note that proving the statement reduces to proving the same for the elements a^p, b^p, a_*^p, b_*^p. So, after replacing a, b, a_*, b_* by a^p, b^p, a_*^p, b_*^p, we can assume that $F_2 \simeq \langle a, b\rangle \leqslant H$.

With these gained assumptions, let us show that we can take

$$C = \max\{2N^2, \max_{w \in \mathcal{W}} M(|b|, |w|)\},$$

where the number N and the set $\mathcal{W}$ are defined at the beginning of section 7, and the function M is defined in Lemma 6.1. In fact, assume that $(a_*^i b_*^l)^j b_*^k$ is conjugate to $(a^i b^l)^j b^k$ for every $i, j, k, l = -C, \ldots, C$, and let us look for the required uniform conjugator.

Consider the set $\mathcal{W}$. Recall that for $w = (a^i b)^{2j}$ we have denoted $w_* = (a_*^i b_*)^{2j}$. In this language, (part of) our hypothesis says that $w_* b_*^k$ is conjugate to wb^k for every $w \in \mathcal{W}$, and every $k = 1, \ldots, M(|b|, |w|)$.

Fix $w \in \mathcal{W}$. Centering the notation on this w, we have that wb_*^k ($= w_* b_*^k$) is conjugate to wb^k for every $k = 1, \ldots, M(|b|, |w|)$. In particular, w satisfies the hypothesis of Lemma 6.1 (with the corresponding value of b_*). And this happens for every $w \in \mathcal{W}$. Thus, Lemma 7.1 ensures us that the conclusion of Lemma 6.1 holds with $st = 0$ for at least one $w_0 = (a^{i_0} b)^{2j_0} \in \mathcal{W}$, $1 \leqslant i_0 \leqslant N$, $1 \leqslant j_0 \leqslant N^2$ (note that Lemma 7.1 can be applied because we previously gained the assumptions $||b^m|| > 13\delta$ for every $m \neq 0$, and $F_2 \simeq \langle a, b\rangle \leqslant H$). For the rest of the proof, let us center the notation on this particular w_0.

Using Proposition 6.3 we conclude that every conjugator from b to b_* (say $b_* = h^{-1}bh$) is of the form $h = b^p w_0^q$ for some rational numbers p, q. Hence, $w_0^{-q} b w_0^q = b_*$ for some rational q. Then,

$$((w_0^{-q} a w_0^q)^{i_0} b_*)^{2j_0} = w_0^{-q} (a^{i_0} b)^{2j_0} w_0^q = w_0^{-q} w_0 w_0^q = w_0 = w_{0*} = (a_*^{i_0} b_*)^{2j_0}.$$

Extracting roots twice (here we use again the absence of torsion in H), we conclude that $w_0^{-q} a w_0^q = a_*$. Thus, w_0^q is a uniform right conjugator from (a, b) to (a_*, b_*) This concludes the proof. □

8 Main theorem for several words

It is known, that in a torsion-free hyperbolic group the centralizer of any nontrivial element is cyclic. An element $g \in H$ is called *root-free*, if it generates its centralizer, that is $\langle g \rangle = C_H(g)$.

Lemma 8.1 *(see Lemma 4.3 in [10]) Let H be a torsion-free hyperbolic group, and let a, b two elements, such that $b \notin C_H(a)$. Then there is a computable integer $k_0 = k_0(|a|, |b|) > 0$, such that for every $k > k_0$ the element ab^k is root-free.*

Theorem 8.2 *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S, and let $g_1, \dots, g_n$ and $g_{1*}, \dots, g_{n*}$ be elements of H such that g_{r*} is conjugate to g_r for each $r = 1, \dots, n$. There is a uniform conjugator if and only if $W(g_{1*}, \dots, g_{n*})$ is conjugate to $W(g_1, \dots, g_n)$ for every word W in n variables and length up to a computable constant depending only on $\delta, \sharp S$ and $\sum_{r=1}^{n} |g_r|$.*

Proof. Denote $G = \{g_1, \dots, g_n\}$. We may assume that all g_i are nontrivial. Because of Theorem 7.2 we may assume that $n > 2$. If the elements $g_1, \dots, g_n$ generate a cyclic group, we can apply Theorem 7.2 to every pair g_i, g_j and get the required conclusion.

Consider the case, where some two elements of G, say g_1, g_2, generate a noncyclic group. Using Proposition 2.11, we replace them by their big powers to ensure that g_1, g_2 generate a free group of rank 2. By Theorem 7.2, we may assume that $g_{1*} = g_1$ and $g_{2*} = g_2$. We will prove that $g_{r*} = g_r$ for every $g_r \in G$.

By Lemma 8.1, there exists k such that the element $g_1 g_2^k$ is root-free. Replacing g_1 by $g_1 g_2^k$, we may assume that g_1 is root-free.

Consider an element $g_r \in G \setminus \{g_1, g_2\}$. Applying Theorem 7.2 to the pair (g_1, g_r), we obtain $g_{r*} = x^{-1} g_r x$ for some $x \in C_H(g_1) = \langle g_1 \rangle$. Analogously, $g_{r*} = y^{-1} g_r y$ for some $y \in C_H(g_2) = \langle g_2 \rangle$. Then $x = g_1^l$ and $y = g_2^m$ for some integers l, m. If l or m is zero, we get $g_{r*} = g_r$.

Thus consider the case, where both l and m are nonzero. We have $xy^{-1} \in C_H(g_r)$, that is $g_r^k = g_1^l g_2^{-m}$ for some nonzero integers l, m and some rational k.

Again by Lemma 8.1, there exists $s > 2$, such that the elements $f_1 = g_1 g_2^s$ and $f_2 = g_2 (g_1 g_2^s)^s$ are root-free. Arguing with these elements as above with g_1, g_2, we deduce that either $g_{r*} = g_r$, or $g_r^t = f_1^p f_2^{-q}$ for some nonzero integers p, q and some rational t. Assuming the last, we deduce that the elements $f_1^p f_2^{-q}$ and $g_1^l g_2^{-m}$ commute in the free group $\langle g_1, g_2 \rangle$, that is impossible. Thus $g_{r*} = g_r$ for each $r = 1, \dots, n$. □

9 Some other technical lemmas

Lemma 9.1 *If $uvw = u \underset{c}{\cdot} v \underset{c}{\cdot} w$ and $|v| > 2c + \delta$ for some $c > 0$, then*

$$|u \underset{c}{\cdot} v \underset{c}{\cdot} w| > |u| + |v| + |w| - (4c + 2\delta).$$

Proof. Connect the points $A = 1$, $B = u$, $C = uv$, $D = uvw$ by geodesic segments and consider the geodesic rectangle $ABCD$. By assumption $|BC| > 2c + \delta$. From $u \underset{c}{\cdot} v$ and $v \underset{c}{\cdot} w$ we deduce

$$|AC| > |AB| + |BC| - 2c > |AB| + \delta$$

and

$$|BD| > |BC| + |CD| - 2c > |CD| + \delta.$$

respectively. From this and the rectangle inequality

$$|AC| + |BD| \leqslant \max\{|BC| + |AD|, |AB| + |CD|\} + 2\delta,$$

we deduce

$$(|AB| + |BC| - 2c) + (|BC| + |CD| - 2c) < |AC| + |BD| \leqslant |BC| + |AD| + 2\delta,$$

that implies

$$|AB| + |BC| + |CD| - (4c + 2\delta) < |AD|.$$

□

Lemma 9.2 *Let H be a δ-hyperbolic group with respect to a finite generating set S and let $w, b, g \in H$ with $b \neq 1$, and g being conjugated to wb^k for some natural k. Then there exist an element $x \in H$ and an integer $0 \leqslant l \leqslant k$, such that $g = x^{-1} \underset{c}{\cdot} b^{k-l}wb^l \underset{c}{\cdot} x$, where $c = 3\delta + \mu(|b|) + |w| + 1$.*

Proof. Let x be the shortest element for which there exists an l, $0 \leqslant l \leqslant k$, such that $g = x^{-1}b^{k-l}wb^l x$. We will prove that x and l satisfy the conclusion of the lemma. Suppose the opposite. Denote $A = 1$, $B = x^{-1}$, $C = x^{-1}b^{k-l}$, $D = x^{-1}b^{k-l}w$, $E = x^{-1}b^{k-l}wb^l$ and $F = x^{-1}b^{k-l}wb^l x$, and connect these points by geodesic segments.

Without loss of generality we may assume that $x^{-1}b^{k-l}wb^l$ cannot be written as $x^{-1} \underset{c}{\cdot} b^{k-l}wb^l$. Then, considering the geodesic triangle ABE, we conclude that $\frac{1}{2}(|AB| + |BE| - |AE|) \geqslant c$. By δ-hyperbolicity of H, there exist points $X_1 \in [AB]$ and $X_2 \in [BE]$ such that $|BX_1| = |BX_2| = c$ and $|X_1X_2| \leqslant \delta$. By Proposition 2.1, applied to the rectangle $BCDE$, there exists a point $X_3 \in [BC] \cup [CD] \cup [DE]$ such that $|X_2X_3| \leqslant 2\delta$.

Case 1. Suppose that $X_3 \in [BC]$. Since $C = Bb^{k-l}$, Corollary 2.9 implies that there exists an element $X_4 = Bb^s$, where $0 \leqslant s \leqslant k - l$, such that $|X_3X_4| \leqslant \mu(|b|)$.

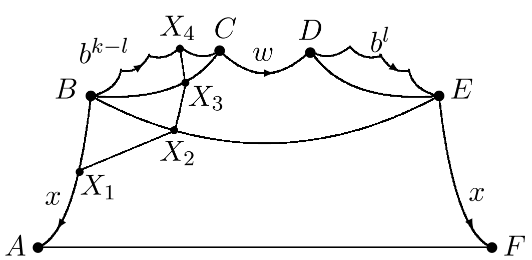

Figure 1

Case 2. Suppose that $X_3 \in [CD]$. Then for $X_4 = C = Bb^{k-l}$ we have $|X_3X_4| \leqslant |w|$.

Case 3. Suppose that $X_3 \in [DE]$. Since $E = Db^l$, Corollary 2.9 implies that there exist an element $X_4 = Db^s$, where $0 \leqslant s \leqslant l$, such that $|X_3X_4| \leqslant \mu(|b|)$.

In any case, $|X_1X_4| < c$. Since $|X_1B| = c$, we have $|AX_4| \leqslant |AX_1| + |X_1X_4| < |AX_1| + |X_1B| = |AB| = |x|$. Since $A = 1$, we have $|X_4| < |x|$. Now continue to the analyze the cases above.

In Cases 1 and 2 we have $X_4 = x^{-1}b^s$, where $0 \leqslant s \leqslant k-l$. Then $g = X_4b^{k-l-s}wb^{l+s}X_4^{-1}$. A contradiction with minimality of $|x|$.

In Case 3 we have $X_4 = x^{-1}b^{k-l}wb^s$, where $0 \leqslant s \leqslant l$. Then $g = X_4b^{k-s}wb^sX_4^{-1}$. Again a contradiction with minimality of $|x|$. □

Corollary 9.3 *Let H be a δ-hyperbolic group with respect to a finite generating set S and let $z, b \in H$. Then there exists an element $x \in H$, such that for any integer k holds $z^{-1}b^kz = x^{-1} \underset{c}{\cdot} b^k \underset{c}{\cdot} x$, where $c = 3\delta + \mu(|b|) + 1$.*

The following technical lemma asserts, that if $A_1A_2 \dots A_n$ is a broken line consisting of geodesic segments of large length and such that the union of each two consecutive segments is "almost geodesic", then this line itself is "almost geodesic".

Lemma 9.4 *Let $A_1, A_2, \dots, A_n$ be points in a δ-hyperbolic geodesic space, $n \geqslant 3$. Suppose that the following two conditions are satisfied:*

(i) $|A_{i-1}A_{i+1}| \geqslant |A_{i-1}A_i| + |A_iA_{i+1}| - 2\delta$ *for each* $2 \leqslant i \leqslant n-1$,

(ii) $|A_{j-1}A_j| > (2n-3)\delta$ *for each* $3 \leqslant j \leqslant n-1$.

Then

$$|A_1A_n| \geqslant \sum_{i=1}^{n-1} |A_iA_{i+1}| - (4n-10)\delta. \tag{16}$$

Proof. We will prove this lemma by induction on n. For $n = 3$ it is valid. So, assume it is valid for n and prove it for $n+1$. From formula (16) we deduce

$$|A_1A_n| \geqslant |A_1A_{n-1}| + |A_{n-1}A_n| - (4n-10)\delta$$

and from condition (i) we have

$$|A_{n-1}A_{n+1}| \geqslant |A_nA_{n+1}| + |A_{n-1}A_n| - 2\delta. \tag{17}$$

Summating and applying condition (ii) for $|A_{n-1}A_n|$, we get

$$|A_1A_n| + |A_{n-1}A_{n+1}| > |A_1A_{n-1}| + |A_nA_{n+1}| + 2\delta.$$

Therefore, from the rectangle inequality

$$|A_1A_n| + |A_{n-1}A_{n+1}| \leqslant \max\{|A_1A_{n-1}| + |A_nA_{n+1}|, \, |A_1A_{n+1}| + |A_{n-1}A_n|\} + 2\delta$$

we have that

$$|A_1A_n| + |A_{n-1}A_{n+1}| \leqslant |A_1A_{n+1}| + |A_{n-1}A_n| + 2\delta. \tag{18}$$

On the other hand, from the induction hypothesis (16) and inequality (17), we have

$$\begin{aligned} |A_1A_n| + |A_{n-1}A_{n+1}| &\geqslant \left(\textstyle\sum_{i=1}^{n-1} |A_iA_{i+1}| - (4n-10)\delta\right) + |A_nA_{n+1}| + |A_{n-1}A_n| - 2\delta \\ &= \textstyle\sum_{i=1}^{n} |A_iA_{i+1}| + |A_{n-1}A_n| - (4n-8)\delta. \end{aligned}$$

From this and inequality (18) we deduce that

$$|A_1A_{n+1}| \geqslant \sum_{i=1}^{n} |A_iA_{i+1}| - (4n-6)\delta,$$

and the proof is completed. □

Lemma 9.5 *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. For any nontrivial $g \in H$, there exists a computable natural number $r = r(|g|)$ such that*

$$\bigcup_{k=1}^{\infty} \mathcal{A}_{g^k} \subseteq \langle g \rangle \mathcal{B}(r).$$

Proof. First we prove that $\bigcup_{k=1}^{\infty} \mathcal{A}_{g^k}$ lies at bounded (in terms of $|g|$) distance from $C_H(g)$ and then that $C_H(g)$ lies at bounded distance from $\langle g \rangle$.

Take $x \in \mathcal{A}_{g^k} \cap H$. Then $|x^{-1}g^kx|$ is minimal among the lengths of all conjugates to g^k. In particular, $|x^{-1}g^kx| \leqslant |g^k|$. By Corollary 9.3, there exists $z \in H$, such that $x^{-1}g^kx = z^{-1} \underset{c}{\cdot} g^k \underset{c}{\cdot} z$ for some constant $c = c(|g|) > 0$. Thus, we have $|z^{-1} \underset{c}{\cdot} g^k \underset{c}{\cdot} z| \leqslant |g^k|$. Consider two cases.

Case 1. Suppose that $|g^k| > 2c + \delta$. By Lemma 9.1, $|z^{-1} \underset{c}{\cdot} g^k \underset{c}{\cdot} z| > 2|z| + |g^k| - (4c + 2\delta)$. Therefore $|z| < 2c + \delta$. Moreover, $x \in C_H(g)z$.

Case 2. Suppose that $|g^k| \leqslant 2c + \delta$. From $|x^{-1}g^kx| \leqslant |g^k|$ and Lemma 2.5, we conclude that there exists $y \in H$ such that $x^{-1}g^kx = y^{-1}g^ky$ and the length of y is bounded by a constant, depending on $2c + \delta$. Moreover, $x \in C_H(g)y$.

It remains to prove that $C_H(g)$ lies at bounded distance from $\langle g \rangle$. By Corollary 3.10 in Chapter III.Γ of [3], the group $\langle g \rangle$ has finite index in $C_H(g)$. As explained in the proof of that corollary, different positive powers of g are not conjugate to each other. Therefore, there exists a natural number $n \leqslant |\mathcal{B}(4\delta)| + 1$, such that g^n is not conjugate into the ball $\mathcal{B}(4\delta)$. In the proof, it is claimed that each element of $C_H(g)$ lies at distance at most $2|g^n| + 4\delta$ of $\langle g^n \rangle$ and hence of $\langle g \rangle$. This completes the proof. □

Lemma 9.6 *Let H be a torsion-free δ-hyperbolic group with respect to a finite generating set S. There exists a computable function $f : \mathbb{N}^2 \to \mathbb{N}$ such that for any two elements $u, v \in H$ and for any nonnegative integers l, k the following holds*

$$|u^kvu^l| > |u^{k+l}| - f(|u|, |v|).$$

Proof. We may assume that u is nontrivial. Let $\mu = \mu(|u|)$ be such a constant that for any two integers $s \leqslant t$ the quasi-geodesic $\{u^i \mid s \leqslant i \leqslant t\}$ is contained in the μ-neighborhood of any geodesic with endpoints u^s and u^t (see Corollary 2.9). Set $N = \sharp\mathcal{B}(2\mu + 2\delta + |v|)$ and $d = 2(\mu + 1)(N + 1)$.

We denote $A = 1$, $B = u^k$, $C = u^k v$, and $D = u^k v u^l$. Choose geodesics $[AB]$, $[AC]$ and $[CD]$. Let P be the point on $[CD]$ such that $|CP| = \frac{1}{2}(|CA| + |CD| - |AD|)$. We will consider two cases.

Case 1. Suppose that $|CP| < d$. Then

$$|u^k v u^l| = |AD| = |AC| + |CD| - 2|CP| \geqslant \big(|u^k| - |v|\big) + |u^l| - 2d \geqslant |u^{k+l}| - |v| - 2d.$$

Case 2. Suppose that $|CP| \geqslant d$. We will prove that in this case u and v commute and so $|u^k v u^l| \geqslant |u^{k+l}| - |v|$.

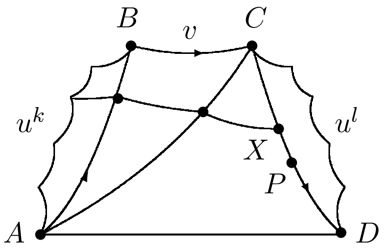

Figure 2

Let X be an arbitrary point on $[CP]$. Then X is at distance at most δ from the side $[AC]$ of the geodesic triangle ACD. But this side is in the $(\delta + |v|)$-neighborhood of the side $[AB]$ of the geodesic triangle ABC, and the last one is in the μ-neighborhood of the set $\{1, u, \dots, u^k\}$ by Corollary 2.9. Thus X is at distance at most $2\delta + \mu + |v|$ from a point $Y = u^p$, where $0 \leqslant p \leqslant k$.

By the same corollary, X is in the μ-neighborhood of $\{C, Cu, \dots, Cu^l\}$, that is X is at distance at most μ from a point $Z = Cu^q = u^k v u^q$, where $0 \leqslant q \leqslant l$. Thus $|u^{-p} \cdot u^k v u^q| = |Y, Z| \leqslant 2\delta + 2\mu + |v|$.

Now, let $X_1, \dots, X_{N+1}$ be points on $[CD]$, such that $|CX_i| = 2i(\mu + 1)$. Note that all X_i lie on $[CP]$. As above, X_i is at distance at most μ from a point $Z_i = Cu^{q_i}$. Note, that $q_i \neq q_j$ for $i \neq j$, otherwise $Z_i = Z_j$ and $|X_i X_j| \leqslant |X_i Z_i| + |Z_j X_j| \leqslant 2\mu$, a contradiction. Similarly, we have $|u^{-p_i} \cdot u^k v u^{q_i}| \leqslant 2\delta + 2\mu + |v|$ for some p_i. Thus all elements $u^{-p_i} \cdot u^k v u^{q_i}$ lie in the ball $\mathcal{B}(r)$, where $r = 2\delta + 2\mu + |v|$. Since the number of these elements is $N + 1$ and $N = \sharp\mathcal{B}(r)$, there are two coinciding elements of this form with different i, j:

$$u^{k-p_i} v u^{q_i} = u^{k-p_j} v u^{q_j}.$$

We get that $v u^{q_j - q_i} v^{-1} = u^{p_j - p_i}$. Since $q_i \neq q_j$, this implies that u, v commute. The proof is completed. □

10 Estimation of distance between axes

In this section, we assume that H is a δ-hyperbolic group with respect to a finite generating set S.

Lemma 10.1 *Let g be a nontrivial element of H. Let A be any point of $\Gamma(H,S)$ and B be a point on $\mathcal{A}_g$, nearest to A. Then for any geodesic segment $[BC] \subset \mathcal{A}_g$ holds*

$$|AC| \geqslant |AB| + |BC| - 2\delta.$$

Proof. Connect A, B, C by geodesic segments $[AB]$, $[BC]$ and $[AC]$. Let $X \in [BA]$ and $Y \in [BC]$ be such points, that $|BX| = |BY| = \frac{1}{2}(|BA| + |BC| - |AC|)$. Then $|XY| \leqslant \delta$. Since the point Y also lies on $\mathcal{A}_g$, we have that $|AB| \leqslant |AY|$. Therefore $|XB| \leqslant |XY| \leqslant \delta$. Hence

$$|AC| = |AB| + |BC| - 2|BX| \geqslant |AB| + |BC| - 2\delta.$$

□

Lemma 10.2 *Let $g \neq 1$ be an element of H and k be an integer number such that $||g^k|| > 5\delta$. Let A be an element of H and N be the real number such that $|A, g^k A| = ||g^k|| + N$. Then $A = g^t v$ for some $t \in \mathbb{Z}$ and $v \in H$ such that $|v| \leqslant \frac{1}{2}N + 3\delta + r(|g|)$.*

Proof. Let B be a point on $\mathcal{A}_{g^k}$ nearest to A. Denote $C = g^k B$, $D = g^k A$. By Lemma 10.1, we have

$$|AC| \geqslant |AB| + |BC| - 2\delta$$

and

$$|DB| \geqslant |CD| + |BC| - 2\delta.$$

Moreover, $|BC| = ||g^k|| > 5\delta$. Therefore, by Lemma 9.4, we get

$$\begin{aligned} |AD| &\geqslant |AB| + |BC| + |CD| - 6\delta \\ &= 2|AB| + |BC| - 6\delta \\ &= 2|AB| + ||g^k|| - 6\delta. \end{aligned}$$

Hence, from the hypothesis we have $|AB| \leqslant \frac{1}{2}N + 3\delta$. By Lemma 9.5 we are done. □

Lemma 10.3 *Let g be a nontrivial element of H with $||g|| > 5\delta$. Then the middle point of any geodesic segment $[X, gX]$, where X is a point of $\Gamma(H,S)$, lies in the 5δ-neighborhood of the axis $\mathcal{A}_g$.*

Proof. The point Y is the nearest to X on $\mathcal{A}_g$. Denote $X_1 = gX$, $Y_1 = gY$. By Lemma 10.1, we have

$$|XY_1| \geqslant |XY| + |YY_1| - 2\delta$$

and

$$|X_1Y| \geqslant |X_1Y_1| + |Y_1Y| - 2\delta.$$

Moreover, $|YY_1| = ||g|| > 5\delta$. Therefore, by Lemma 9.4 we get

$$2|XY| + |YY_1| \leqslant |XX_1| + 6\delta. \tag{19}$$

Let M be the middle point of the segment $[XX_1]$ and N be the middle point of the segment $[YY_1]$. Clearly $N \in \mathcal{A}_g$. We will estimate the distance $|NM|$. Consider the geodesic rectangle XMX_1N. By the rectangle inequality we have

$$
\begin{aligned}
|NM|+|XX_1| &\leqslant \max\{|XM|+|NX_1|, |MX_1|+|NX|\}+2\delta \\
&= \max\{\tfrac{1}{2}|XX_1|+|NX_1|, \tfrac{1}{2}|XX_1|+|NX|\}+2\delta.
\end{aligned}
$$

Note that $|NX| \leqslant |XY|+|YN| = |XY|+\frac{1}{2}|YY_1|$. Therefore from (19) we have $|NX| \leqslant \frac{1}{2}|XX_1|+3\delta$. Analogously $|NX_1| \leqslant \frac{1}{2}|XX_1|+3\delta$. From this we deduce that $|NM| \leqslant 5\delta$. $\square$

Proposition 10.4 *Let g and h be any elements of H such that $||g|| > 13\delta$, $||h|| > 13\delta$ and $||gh|| > 5\delta$. Then the distance between the axes $\mathcal{A}_g$ and $\mathcal{A}_h$ is at most*

$$
\max\{13\delta, \frac{1}{2}(||gh||-||g||-||h||)+16\delta\}.
$$

Proof. Let d be the distance between $\mathcal{A}_h$ and $\mathcal{A}_g$. If $d \leqslant 13\delta$, we are done. So, assume that $d > 13\delta$. Let X and Y be points of $\mathcal{A}_h$ and $\mathcal{A}_g$ such that $|XY| = d$. It is obvious, that

$$
|X, ghX| \leqslant |XY|+|Y, gY|+|gY, gX|+|gX, ghX| = d+||g||+d+||h||.
$$

By Lemmas 10.1 and 9.4, we have

$$
\begin{aligned}
|X, (gh)^2X| &\geqslant |XY|+|Y, gY|+|gY, gX|+|gX, ghX| \\
&+|ghX, ghY|+|ghY, ghgY|+|ghgY, ghgX|+|ghgX, ghghX|-22\delta \\
&= 2(d+||g||+d+||h||)-22\delta.
\end{aligned}
$$

Denote $A = X$, $B = ghX$ and $C = (gh)^2X$. Let M be the middle point of the geodesic segment $[AB]$. Then $M_1 = ghM$ is the middle point of the geodesic segment $(gh)[AB]$ connecting B and C, and we have

$$
\begin{aligned}
|M, ghM| &= |MM_1| \geqslant |AC|-|AM|-|CM_1| = |AC|-\tfrac{1}{2}|AB|-\tfrac{1}{2}|BC| \\
&= |AC|-|AB| \geqslant d+||g||+d+||h||-22\delta.
\end{aligned}
$$

By Lemma 10.3, M, as the middle point of the geodesic segment $[X, ghX]$, lies at distance at most 5δ from the axis $\mathcal{A}_{gh}$. Therefore $|M, ghM| \leqslant 10\delta+||gh||$. Hence $d \leqslant \frac{1}{2}(||gh||-||g||-||h||)+16\delta$. $\square$

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