Compatible matchings in geometric graphs

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Abstract. Two non-crossing geometric graphs on the same set of points are compatible if their union is also non-crossing. In this paper, we prove that every graph G that has an outerplanar embedding admits a non-crossing perfect matching compatible with G. Moreover, for non-crossing geometric trees and simple polygons, we study bounds on the minimum number of edges that a compatible non-crossing perfect matching must share with the tree or the polygon. We also give bounds on the maximal size of a compatible matching (not necessarily perfect) that is disjoint from the tree or the polygon.

Introduction

A geometric graph is a simple graph G, where the vertex set $V(G)$ is a finite set of points S in the plane and each edge in $E(G)$ is a closed straight-line segment connecting two points in S. A geometric graph is non-crossing if no two edges cross except at a common vertex.

Throughout the paper, all the graphs considered will be geometric and non-crossing. For this reason, we will use the term "graph" ("tree", "matching", ...) meaning that the graph (tree, matching, ...) is geometric and non-crossing. Moreover, we will assume that no three points are collinear.

Two graphs are said to be compatible if they have the same vertex set and their union is non-crossing. A graph that is compatible with a given graph G will be called G-compatible. In addition, if they have no edge in common, we call them disjoint.

Given a set S of n points in the plane and a graph G on S, in this paper we study the two following problems of compatibility. On one hand, to find a *perfect matching* M such that it is G-compatible and the number of common edges between M and G is minimum. On the other hand, to find a *matching* M such that it is G-compatible, disjoint from G, and the number of edges of M is maximum. Similar problems on compatible graphs have been studied in [**2, 3, 4**] and some related augmentation problems for geometric graphs appear in [**1, 5**].

Since these numbers depend on the set of points S and on the graph G, we have focused on bounding the values defined below. Given S –a set with an even number, n, of points–, $T(S)$ –a tree on S–, and M –a $T(S)$-compatible perfect matching–, let us define $m_{(T(S),M)}$ to be the number of edges of M not contained in $T(S)$. Let us also define $m_{Tree}(n) = \min_{|S|=n}\{\min_{T(S)}\{\max_M m_{(T(S),M)}\}\}$ for n even, i.e., the worst

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case of the maximal number of non-shared edges. In the case of non necessarily perfect matchings, let $d_{(T(S),M)}$ be the number of edges of a $T(S)$-compatible matching M that is disjoint from $T(S)$ and let $d_{Tree}(n) = \min_{|S|=n}\{\min_{T(S)}\{\max_M d_{(T(S),M)}\}\}$. Note that the definitions of $m_{Tree}(n)$ and $d_{Tree}(n)$ are identical, except that the maximum is taken over different families of matchings.

By defining in a similar way the values $m_{Polygon}(n)$ and $d_{Polygon}(n)$ for simple polygons, the main results that we have obtained are:

Theorem 1 *For n arbitrarily large,*

$$0 \leq m_{Tree}(n) \leq 13$$
$$n/10 \leq d_{Tree}(n) \leq n/4$$
$$n/20 \leq m_{Polygon}(n) \leq n/4$$
$$(n-3)/4 \leq d_{Polygon}(n) \leq n/3$$

1 Compatible perfect matchings

As a first result, we give the following theorem characterizing a set of graphs for which a compatible perfect matching always exists.

Theorem 2 *Given a set S of n (even) points and a graph G on S drawn as an outerplanar geometric graph on top of S, then there is always a G-compatible perfect matching.*

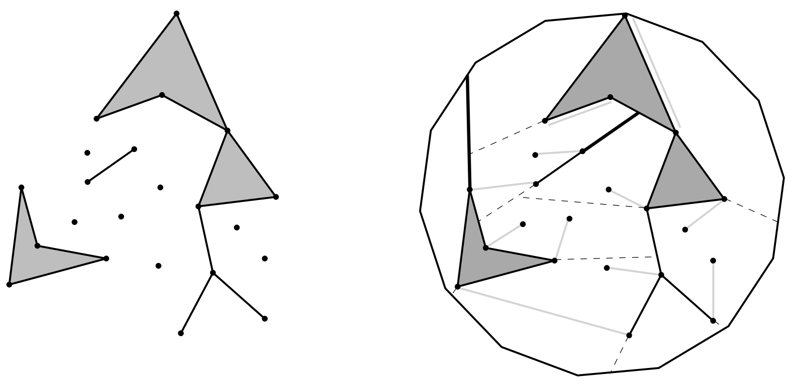


FIGURE 1. Obtaining a perfect matching for an outerplanar geometric graph.

Figure 1 shows an example of how to find this perfect matching. The method is similar to the one described in [**1**]. First, we add a big convex polygon passing through the top most point on S and containing all the points. Then, we join the non-trivial connected components of the graph by adding new edges (and consequently new vertices), until a (weakly) simple polygon is obtained. Given a component, the added edge (thick line in the figure) is part of the ray that emanates from the top most point of the component until it hits an edge. This ray bisects the reflex angle at the top most point. Note that the added vertices are convex in the polygon. Now, the (weakly) simple polygon is divided into convex regions by throwing rays (dashed lines in the figure) from the reflex vertices of the polygon. Using the dual graph associated to this subdivision, we can guarantee an assignment of an even number of vertices to each region. Then, the perfect matching (gray edges in the figure) is obtained by matching the vertices of each convex region.

If we drop the condition of outerplanarity (all the vertices in the unbounded face), then a G-compatible perfect matching does not always exist. Figure 2(a) shows an example of a graph G formed by a tree (which is outerplanar) plus an edge. The seven points $\{a, b, c, d, e, f, g\}$ can only be linked with one of the six points $\{1, 2, 3, 4, 5, 6\}$ and hence, there are no perfect matchings compatible with this graph G.

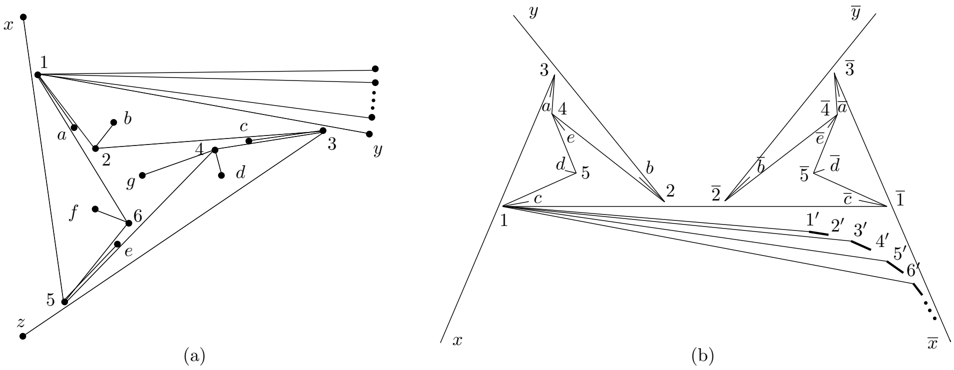
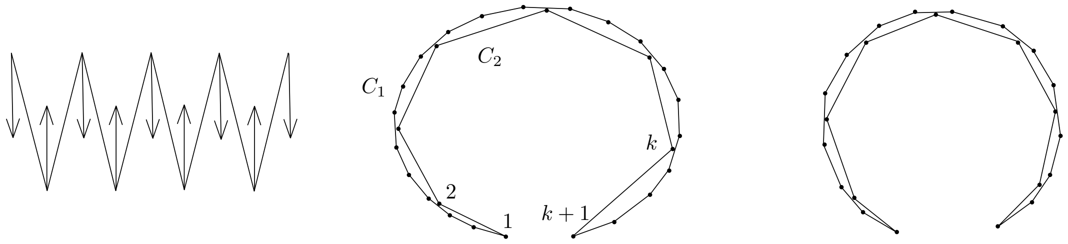


FIGURE 2. On the left, a graph G without any G-compatible perfect matching. On the right, a tree T for which any T-compatible perfect matching must share almost all its edges with the tree.

2 Compatible matchings for trees and simple polygons

In this section, we briefly explain how to obtain the bounds given in Theorem 1. The upper bounds are obtained by analyzing special cases of trees and simple polygons. Maybe, the most surprising bound is the upper bound for $m_{Tree}(n)$. This upper bound, which is a constant instead of a function of n, is based on the tree shown in Figure 2(b). For this tree, any compatible perfect matching must share almost all its edges with the tree (at least $n/2 - 13$ edges). The five points $\{1, 2, 3, 4, 5\}$ can only be matched with the points $\{a, b, c, d, e\}$ and the same for the points $\{\overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$ and $\{\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}\}$. Then, necessarily point $1'$ has to be matched to point $2'$, point $3'$ to point $4'$ and so on. Only the 24 points not in the convex chain $1', 2', 3'$... and the two last points of the chain, $m' - 1$ and m', can be matched with edges not in the tree.

Figure 3 shows the graphs used to obtain the upper bounds on $d_{Tree}(n)$ (Figure 3a), $m_{Polygon}(n)$ (Figure 3b) and $d_{Polygon}(n)$ (Figure 3c). In Figure 3a, to obtain a matching without sharing edges, we can link the points on the zig-zag path among them or link a point (or more) of an arrow with some point on the zig-zag path. Since the number of points on the zig-zag path is $n/4$, the size of a matching compatible and disjoint from the tree is at most $n/4$ edges. In Figure 3b, we have a simple polygon P formed by two convex chains, C_1 and C_2, with $3k - 1$ and $k + 1$ points, respectively. Observe that, to obtain a perfect matching M, we cannot join two non-consecutive points of C_1. Hence, we need to use internal diagonals of P joining a point of C_1 to a point of C_2 (except the first point), and then the number of this type of diagonals is at most k in M. Figure 3c is analyzed in a similar way.



(a) (b) (c)

FIGURE 3. Graphs used to obtain the upper bounds on $d_{Tree}(n)$, $m_{Polygon}(n)$ and $d_{Polygon}(n)$.

Regarding the lower bounds, we can obtain them by constructing specific (perfect) matchings for any tree or polygon. For instance, given a tree T, we construct three T-compatible matchings, disjoint from T, of sizes at least $\lfloor l/4 \rfloor$, $\lfloor (n-2l)/4 \rfloor$ and $\lfloor (n-l)/6 \rfloor$, respectively, where l is the number of leaves of T. So, if the number of leaves is for example 2, then we have a matching of size $\lfloor (n-4)/4 \rfloor$ (the maximum of the three previous sizes) and if the number of leaves is $n-1$, then we have a matching of size $\lfloor (n-1)/4 \rfloor$. With this method, the worst case appears when the number of leaves is $2n/5$. In this case, we can only guarantee a matching of size $n/10$.

The methods for obtaining the (perfect) matchings in the proof of Theorem 1 are based on several technical results on simple polygons. For example, let P be a simple polygon on S having r reflex vertices and c convex vertices, and let $S_0 \subseteq S$ be a subset of vertices containing all the reflex vertices and c_0 convex vertices. Then, we can show that there is a P-compatible matching among the vertices of S_0 of size at least $\lfloor (c_0 - 1)/2 \rfloor$ edges with all its edges being internal diagonals of P. We can also prove that if S_0 has h chains (a sequence $\{p_k, p_{k+1}, \ldots, p_{k+(l-1)}\}$ of consecutive vertices of P is a chain if all these vertices belong to S_0 and neither p_{k-1} nor p_{k+l} belong to S_0), then there is a P-compatible matching among the vertices of S_0 that is disjoint from P, its size is at least $\lfloor h/2 \rfloor$ edges and each edge of the matching is placed strictly inside P.

Finally, let us remark that, for n points in convex position, we can prove tight bounds for trees and paths. In the case of trees, $m_{Tree}(n) = \lceil (n-2)/6 \rceil$ and $d_{Tree}(n) = \lceil n/4 \rceil$. For paths, $m_{Path}(n) = \lceil n/4 \rceil$ and $d_{Path}(n) = \lceil (2n)/5 \rceil$, where $m_{Path}(n)$ and $d_{Path}(n)$ are defined in a similar way to $m_{Tree}(n)$ and $d_{Tree}(n)$.

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