Exponentially and non-exponentially small splitting of separatrices for the pendulum with a fast meromorphic perturbation

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Abstract

In this paper we study the splitting of separatrices phenomenon which arises when one considers a Hamiltonian System of one degree of freedom with a fast periodic or quasiperiodic and meromorphic in the state variables perturbation. The obtained results are different from the previous ones in the literature, which mainly assume algebraic or trigonometric polynomial dependence on the state variables. As a model, we consider the pendulum equation with several meromorphic perturbations and we show the sensitivity of the size of the splitting on the width of the analyticity strip of the perturbation with respect to the state variables. We show that the size of the splitting is exponentially small if the strip of analyticity is wide enough. Furthermore, we see that the splitting grows as the width of the analyticity strip shrinks, even becoming non-exponentially small for very narrow strips. Our results prevent from using polynomial truncations of the meromorphic perturbation to compute the size of the splitting of separatrices.

1   Introduction

Exponentially small splitting of separatrices appears in analytic dynamical systems with different time scales. A paradigmatic example are analytic Hamiltonian systems of one degree of freedom with a fast non-autonomous periodic or quasiperiodic perturbation. Namely, systems of the form

$$
\begin{aligned}
H\left(x, y, \frac{t}{\varepsilon}\right) &= H_0(x,y) + \mu\varepsilon^\eta H_1\left(x, y, \frac{t}{\varepsilon}\right) \\
&= \frac{y^2}{2} + V(x) + \mu\varepsilon^\eta H_1\left(x, y, \frac{t}{\varepsilon}\right),
\end{aligned}
\tag{1}
$$

where $\varepsilon > 0$ is a small parameter, $\mu \in \mathbb{R}$, $\eta \geq 0$, and H_0 has a hyperbolic critical point whose invariant manifolds coincide along a separatrix.

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This phenomenon was first pointed out by Poincaré [Poi99] but it was not until the last decades when this problem started to be studied rigorously (see for instance [HMS88, SMH91, DS92, Fon93, CG94, Gel94, Fon95, Sau95, DGJS97, DS97, Gel97b, Tre97, GGM99, Gel00, Sau01, DGS04, BF04, BF05, Bal06, Oli06, GOS10, BFGS11]). Nevertheless, the results which obtain asymptotic formulas for the splitting only deal with Hamiltonian systems whose perturbation is an algebraic or trigonometric polynomial with respect to the state variables x and y. In all these cases the splitting of separatrices is exponentially small with respect to the parameter ε. Moreover, the imaginary part of the complex singularity of the time-parameterization of the unperturbed separatrix closest to the real axis plays a significant role.

All the previous works dealing with the periodic case, show that under certain non-degeneracy conditions, the distance between the invariant manifolds is of order

$$d \sim \mu\varepsilon^q e^{-\frac{a}{\varepsilon}}, \tag{2}$$

where a is the imaginary part of the complex singularity of the time-parameterization of the unperturbed separatrix closest to the real axis and $q \in \mathbb{R}$. Moreover, for $\eta > \eta^*$, where η^* depends on the properties of both H_0 and H_1, the splitting is well predicted by the Poincaré-Arnol'd-Melnikov method (see [Mel63], and [GH83] for a more modern exposition of this method). This case is usually called *regular case*. In the *singular case* $\eta = \eta^*$ the splitting is exponentially small as (2) but the first order does not coincide with the Melnikov prediction (see [BFGS11] and references therein). In the quasiperiodic case, under certain hypotheses, one can also show that the Melnikov method predicts correctly the splitting and the size of both the Melnikov function and the splitting depends strongly on a [DGJS97, Sau01, DGS04]. However, this case is much less understood and there are very few results.

Nevertheless, many of the models known, for instance in celestial mechanics, are not algebraic or trigonometric polynomials in the state variables but involve functions with a finite strip of analyticity (see, for instance, [LS80, Xia92, MP94, FGKR11]). As far as the authors know, the only result dealing with the exponentially small splitting of separatrices in the periodic case for non-entire perturbations is [Gel97a]. However, the author considers models with a strip of analyticity very big with respect to ε so that he can deal with them as if they were polynomial. In the quasiperiodic case, as far as the authors know, there are not previous results.

The goal of this paper is to study how the splitting of separatrices behavior depends on the width of the analyticity strip when one considers a meromorphic perturbation. Essentially, we see that the size of the splitting depends strongly on this width and that, in general, the singularity of the separatrix does not play any role in this size. We consider also the case when the strip tends to infinity as $\varepsilon \to 0$ and we see how the size of the splitting tends to the size known for the entire cases. In the other limiting case, namely when the strip of analyticity shrinks to the real line as $\varepsilon \to 0$, we see that even if the perturbation is still analytic, the splitting becomes algebraic in ε both in the periodic and the quasiperiodic case.

We focus our study in particular examples, which allow us to analyze in great detail the behavior of the splitting. Nevertheless, we expect the same to happen for fairly general systems.

We work with time periodic and quasiperiodic perturbations of the classical pendulum. More concretely, we consider the following model,

$$\ddot{x} = \sin x + \mu\varepsilon^\eta \frac{\sin x}{(1 + \alpha \sin x)^2} f\left(\frac{t}{\varepsilon}\right), \tag{3}$$

where $f(\tau)$ is an analytic function which depends either periodically or quasiperiodically on τ.

The associated system

$$\begin{cases} \dot{x} = y \\ \dot{y} = \sin x + \mu\varepsilon^\eta \dfrac{\sin x}{(1 + \alpha \sin x)^2} f\left(\dfrac{t}{\varepsilon}\right) \end{cases} \tag{4}$$

is Hamiltonian with Hamiltonian function

$$H\left(x, y, \frac{t}{\varepsilon}\right) = \frac{y^2}{2} + \cos x - 1 + \mu\varepsilon^\eta \psi(x) f\left(\frac{t}{\varepsilon}\right), \tag{5}$$

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where $\psi(x)$ is defined by $\psi'(x) = -\sin x/(1 + \alpha \sin x)^2$ and $\psi(0) = 0$. Here $\alpha \in [0,1)$ is a parameter which changes the width of the analyticity strip of ψ, which is given by

$$|\mathrm{Im}\, x| \le \ln\left(\frac{1 + \sqrt{1 - \alpha^2}}{\alpha}\right). \tag{6}$$

When $\alpha = 0$, the system is entire in x and y and has been previously studied for particular choices of f in [Tre97, DGJS97, OSS03, Oli06], whereas when $\alpha = 1$, ψ is not defined in $x = 3\pi/2$. In this paper, we consider any $\alpha \in (0,1)$ either independent of or dependent on ε.

In the periodic case, as experts know, the only important property to obtain the asymptotic formula for the splitting is that one has to require that the first harmonics of f are different from zero. Thus, we choose

$$f(\tau) = \sin \tau.$$

Dealing with any other function with non-zero first Fourier coefficients is analogous. In this setting, one can rephrase system (4) as a Hamiltonian system of two degrees of freedom considering $\tau = t/\varepsilon$ as a new angle and I its conjugate action, which gives the Hamiltonian

$$\begin{aligned}
K(x,y,\tau,I) &= \frac{I}{\varepsilon} + H(x,y,\tau) \\
&= \frac{I}{\varepsilon} + \frac{y^2}{2} + \cos x - 1 + \mu \varepsilon^\eta \psi(x) \sin \tau.
\end{aligned} \tag{7}$$

In the quasiperiodic case, we consider the same model (3) with $f(\tau) = F(\tau, \gamma\tau)$, where

$$\gamma = \frac{\sqrt{5} + 1}{2} \tag{8}$$

is the golden mean number and $F : \mathbb{T}^2 \to \mathbb{R}$. Note that if one takes $\alpha = 0$, one recovers the model considered in [DGJS97]. On the function F we assume the same hypotheses that are assumed in that article. Namely, if one considers its Fourier expansion in the angles $\theta = (\theta_1, \theta_2)$,

$$F(\theta_1, \theta_2) = \sum_{k \in \mathbb{Z}^2} F^{[k]} e^{ik \cdot \theta}, \tag{9}$$

we assume that there exist constants $r_1, r_2 > 0$ such that

$$\sup_{k = (k_1, k_2) \in \mathbb{Z}^2} \left| F^{[k]} e^{r_1 |k_1| + r_2 |k_2|} \right| < \infty. \tag{10}$$

Furthermore, we assume that there exist a and k_0 such that

$$F^{[k]} > a e^{-r_1 |k_1| - r_2 |k_2|} \tag{11}$$

for all $|k_1|/|k_2|$ which are continuous fraction convergents of γ and $|k_2| > k_0$. An example of function satisfying these hypotheses is

$$F(\theta_1, \theta_2) = \frac{\cos \theta_1 \cos \theta_2}{(\cosh r_1 - \cos \theta_1)(\cosh r_2 - \cos \theta_2)}.$$

Introducing the angle coordinates (θ_1, θ_2) and their conjugate actions (I_1, I_2), system (4) can be seen as a 3 degrees of freedom Hamiltonian System with Hamiltonian

$$\begin{aligned}
K(x,y,\theta,I) &= \frac{\omega \cdot I}{\varepsilon} + H(x,y,\theta) \\
&= \frac{\omega \cdot I}{\varepsilon} + \frac{y^2}{2} + \cos x - 1 + \mu \varepsilon^\eta \psi(x) F(\theta_1, \theta_2),
\end{aligned} \tag{12}$$

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where $\omega = (1, \gamma)$ is the frequency vector.

We have chosen these particular models for several reasons. First, the hyperbolic critical point $(0,0)$ of the unperturbed pendulum persists when the perturbation is added. This fact is not crucial but simplifies the computations. Second, with the chosen function ψ, the size of the Melnikov function depends on the strip of analyticity of the perturbations, as is expected to happen for general systems. In Remark 2.5 in Section 2, we consider the non-generic model

$$\ddot{x} = \sin x + \mu \varepsilon^\eta \frac{\sin x}{(1 - \alpha \cos x)^2} \sin \frac{t}{\varepsilon},$$

which has the same strip of analyticity (6). However, due to certain cancellations, the size of Melnikov function does not depend on this strip.

In the quasiperiodic case, we have chosen a very specific function F. On one hand, we have chosen the frequency vector $\omega = (1, \gamma)$, where γ is the golden mean (8). The size of the splitting strongly depends on the diophantine properties of the chosen frequency and, in fact, its rigorous study has been only done, as far as the authors know, for quadratic frequencies (see [Sau01, LMS03, DG03]). On the other hand, the chosen function F has finite strip of analyticity in the angles (θ_1, θ_2). This is the only kind of systems for which it is known that the Melnikov function predicts correctly the size of the splitting (see [Sim94, SV01]). In fact, in the quasiperiodic case, the width of the strip of analyticity of F also plays a crucial role in the size of the splitting.

Finally, as we have already explained, this particular choice of the perturbation makes everything easily computable. This allows us to obtain explicit formulas for the first order of the splitting of separatrices, using the Melnikov function, and see how it depends on the width of the analyticity strip of ψ. Then, we can compare our results for α small with the existing previous ones for $\alpha = 0$.

As we have already said, when $\alpha = 1$ system (4) is not defined at $x = 3\pi/2$. Therefore, it has no sense to study the splitting problem for α too close to 1. Indeed, the perturbation is small in the real line provided

$$\frac{\varepsilon^\eta}{(1 - \alpha)^2} \ll 1.$$

Nevertheless, as usually happen in the exponentially small splitting problems (see [GOS10]), we will see that the splitting problem has sense under the slightly weaker hypothesis

$$\frac{\varepsilon^\eta}{(1 - \alpha)^{3/2}} \ll 1.$$

When $\mu = 0$, the system is the classical pendulum. It has a hyperbolic critical point at $(0,0)$ whose invariant manifolds coincide along two separatrices (see Figure 1). We focus our attention on the positive one, which can be parameterized as

$$x_0(u) = 4 \arctan{(e^u)}, \quad y_0(u) = \dot{x}_0(u) = \frac{2}{\cosh u}, \tag{13}$$

whose singularities are at $u = i\pi/2 + ik\pi$, $k \in \mathbb{Z}$.

The goal of this paper is to measure how this separatrix splits when one takes $\mu > 0$, paying special attention on how this splitting depends on the relative size between α and ε. The previous results in the periodic case [Tre97, OSS03, Oli06] consider $\alpha = 0$. They obtain an asymptotic formula for the distance between the invariant manifolds, which is of the form

$$d \sim \mu \varepsilon^{\eta - 2} e^{-\frac{\pi}{2\varepsilon}}.$$

In this paper, we see that in the periodic case

- If $\alpha \leq \varepsilon^2$, that is when the strip of analyticity is of order $2 \ln(1/\varepsilon)$ or bigger, the distance between the invariant manifolds coincides with the results obtained in the case $\alpha = 0$, when the perturbation is a trigonometric polynomial.
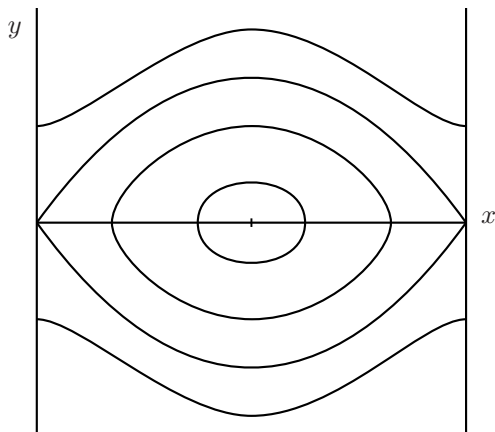
Figure 1: Phase portrait of the pendulum. In it, one can see the two separatrices. We focus our study on the upper one.

- If $\varepsilon^2 \ll \alpha$ and $1 - \alpha \gg \varepsilon^2$, that is when the strip of analyticity is between $2\ln(1/\varepsilon)$ and ε, the distance is exponentially small but the first order is given by a completely different formula

$$d \sim \mu\varepsilon^q e^{-\frac{c}{\varepsilon}},$$

  where $q \in \mathbb{R}$ and $0 < c < \pi/2$ are constants which depend on α. Therefore, the splitting is bigger than it was in the polynomial case. Moreover, we see that it monotonously increases with α.

- Finally, if $0 < 1 - \alpha \geq \varepsilon^2$, that is when the strip of analyticity is narrower than ε, the distance between the invariant manifolds is non-exponentially small,

$$d \sim \mu\varepsilon^q,$$

  where $q \in \mathbb{R}$ is a constant which depend on α.

In particular, the second statement shows that even if α is a small parameter, our study prevents of using the classical approach in perturbation theory. It consists in expanding the perturbation term in powers of α and studying the splitting of the first order, which has a polynomial perturbation. We will see that even for $\alpha \sim \varepsilon$, this approach leads to a wrong result for the splitting. Therefore, we show that the study of the splitting of separatrices for meromorphic perturbations cannot be reduced to the study of simplified polynomial models. In the quasiperiodic case, an analogous phenomenon happens.

The structure of the paper goes as follows. First in Section 2 we give the main results considering the periodic case. In Proposition 2.1 we show the behavior of the Melnikov function with respect to α. In Corollary 2.6 we analyze the range of the parameter α for which the Melnikov function is not exponentially small. Then, in Theorems 2.8 and 2.10, we show for which range of parameters ε, α and η, the Melnikov function predicts correctly the splitting.

In Section 3 we consider the quasiperiodic case. First in Proposition 3.1 we study the size of the Melnikov function and in Theorem 3.5 we prove its validity. As in the periodic case, we also show that if $1 - \alpha \geq \varepsilon^2$ both the Melnikov function and the splitting of separatrices are not exponentially small. These results are given in Corollary 3.3 and Theorem 3.7 respectively.

Section 4 is devoted to prove Theorems 2.8 and 2.10, and Section 5 is devoted to prove Theorems 3.5 and 3.7.

Finally, in Appendix A we give some heuristic ideas about how to deal with the so-called singular periodic case, namely, when the parameter η reaches a certain limiting value and therefore the Melnikov function does not predict correctly the size of the splitting.

1.1 Heuristics on the relation between regularity and Arnol'd diffusion

Even if in this paper we study the so called isochronous case, where the frequency of the perturbation is fixed, the same kind of study would apply to an anisochronous case (see [Sau01]). In the anisochronous case one can encounter both situations, the case of rationally dependent frequencies, which leads to a periodic in time perturbation, and the case of rationally independent frequencies, which leads to a quasiperiodic perturbation.

Anisochronous systems have attracted a lot of attention because they are good models to study the phenomenon called Arnol'd diffusion. The name comes from the fact that was V. Arnol'd who produced in 1964 [Arn64] the first example showing a possible mechanism that leads to global instabilities in nearly integrable Hamiltonian Systems.

Arnol'd proved the presence of instabilities in the following particular model

$$H(I_1, I_2, \varphi_1, \varphi_2, t) = \frac{I_1^2}{2} + \frac{I_2^2}{2} + \varepsilon(\cos\varphi_1 - 1) + \mu\varepsilon(\cos\varphi_1 - 1)(\sin\varphi_2 + \sin t),$$

showing the existence of orbits whose action I_2 changes drastically.

Nevertheless, it is expected that instabilities exist in fairly general nearly integrable Hamiltonian Systems. Chierchia and Gallavotti in [CG94] proposed the study of the following generalization of the Arnol'd model

$$H(I_1, I_2, \varphi_1, \varphi_2, t) = \frac{I_1^2}{2} + \frac{I_2^2}{2} + \varepsilon(\cos\varphi_1 - 1) + \mu\varepsilon h(\varphi_1, \varphi_2, t; \varepsilon). \tag{14}$$

In this setting, they coined the terminology *a priori stable* versus *a priori unstable*. A priori stable refers to consider $\mu = \varepsilon^\eta$ with $\eta \geq 0$ and a priori unstable refers to $\varepsilon = 1$ and μ small. In the a priori stable case the unperturbed system, $\varepsilon = 0$ is completely integrable in the sense that it is written in global action-angle variables. In the a priori unstable case the unperturbed system, $\mu = 0$, even if it is integrable in the sense that it has conserved quantities, presents some hyperbolicity, namely has partially hyperbolic tori with homoclinic trajectories.

There have been some recent works proving the existence of instabilities for a priori unstable systems using the ideas proposed by V. Arnol'd (see [DdlLS06, DH09] and see [CY04, Tre04, Ber08] for proofs using other methods). One of the main steps in the proof is to see that the stable and unstable manifolds of the partially hyperbolic tori, which coincide when $\mu = 0$, split producing chains of heteroclinic orbits. To detect the splitting of these invariant manifolds becomes an essential step in these geometric methods. The main reason that makes very difficult to detect this splitting when μ and ε are small is that one expects this splitting to be exponentially small in ε and, thus, very difficult to study. This was the origin of the distinction between a priori stable and unstable systems, Arnol'd overcame this difficulty taking the parameter μ exponentially small in ε. It is an open problem to see whether Arnol'd mechanism works in the a priori stable setting, namely taking $\mu = \varepsilon^\eta$ with $\eta \geq 0$.

Nonetheless, the regularity of the system plays a crucial role in the classification between a priori stable and unstable systems. The first observation, that is commonly accepted, is that in the $\mathcal{C}^r$ case, even in the a priori stable setting, the splitting is not exponentially small. Nevertheless, as far as the authors know, the only proof of this fact is the paper [DJSG99] where it is seen that for quasiperiodic $\mathcal{C}^r$ perturbations the splitting is polynomially small in ε. Therefore the distinction between a priori stable and unstable systems regarding the splitting problem only has sense for analytic perturbations. Moreover, the results stated in Theorems 2.10 and 3.7 show that for analytic perturbations with narrow strip of analyticity, the size of the splitting is not exponentially small. In particular, we see that the splitting is polynomial in ε in both the periodic and the quasiperiodic case. Therefore, one would expect that for anisochronous systems (14) with a narrow strip of analyticity the splitting at the resonances and in the nonresonant zones are of the same order and polynomial with respect to ε.

On the other hand, it is a commonly accepted fact that the bigger the size of the splitting the faster the diffusion. In fact, Nekhroshev type lower bounds of the diffusion time (or upper bounds of the stability time) increase with the regularity of the system [Nek77, MS02, Bou10] and this agrees with the fact that the splitting decreases with the regularity. For analytic Hamiltonians, these bounds are bigger for bigger strips of analyticity [Pös93, DG96, ?]. The results in this paper show that this is also consistent with the fact that the splitting decreases when the strip of analyticity increases.

6

2 The Melnikov function and its validity in the periodic case

As we want to deal with a time periodic perturbation of the pendulum equation, the dynamics is better understood in the three-dimensional extended phase space $(x, y, \tau) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}$. In this space, $\Lambda = \{(0, 0, \tau); \tau \in \mathbb{T}\}$ is a hyperbolic periodic orbit and, for $\mu = 0$, its 2-dimensional stable and unstable invariant manifolds coincide along the homoclinic manifold

$$\mathcal{W}^u(\Lambda) = \mathcal{W}^s(\Lambda) = \{(x, y, \tau) : H_0(x, y) = 0\} = \{(x, y, \tau) = (x_0(u), y_0(u), \tau); (u, \tau) \in \mathbb{R} \times \mathbb{T}\},$$

where $(x_0(u), y_0(u))$ is the parameterization of the separatrix given in (13).

Our goal is to study how these manifolds split when $\mu \neq 0$. To this end we need to introduce some notion of distance between them. As the manifolds are graphs for $\mu = 0$, the same happens for $\mu \varepsilon^\eta > 0$ small enough in suitable domains. More concretely, in Section 4 we will see that one can parameterize the perturbed stable and unstable manifolds as

$$\begin{cases} x = x_0(u) \\ y = y^{u,s}(u, \tau) \\ \tau = \tau. \end{cases}$$

Here, $(u, \tau) \in (-\infty, U) \times \mathbb{T}$, for certain $U > 0$, for the unstable manifold and $(u, \tau) \in (-U, \infty) \times \mathbb{T}$ for the stable one. Taking into account that the invariant manifolds are Lagrangian, in Section 4 we use the Hamilton-Jacobi equation to see that the functions $y^{u,s}$ can be given as

$$y^{u,s}(u, \tau) = \frac{1}{y_0(u)} \partial_u T^{u,s}(u, \tau),$$

for certain generating functions $T^{u,s}$.

Therefore, a natural way to measure the difference between the manifolds is to compute

$$D(u, \tau) = \partial_u T^s(u, \tau) - \partial_u T^u(u, \tau).$$

If one considers a perturbative approach taking μ as small parameter, one can easily see that the first order in μ of the function $D(u, \tau)$ is given by the Melnikov function

$$\mathcal{M}(u, \tau) = \int_{-\infty}^{+\infty} \{H_0, H_1\} \left(x_0(u+s), y_0(u+s), \tau + \frac{s}{\varepsilon} \right) ds. \tag{15}$$

In other words, one has that

$$D(u, \tau) = \mu \varepsilon^\eta \mathcal{M}(u, \tau) + \mathcal{O}\left(\mu^2 \varepsilon^{2\eta}\right). \tag{16}$$

To see that the manifolds split, we can choose a transversal section to the unperturbed separatrix to measure their distance. The simplest one is $x = \pi$, which corresponds to compute $D(0, \tau)$ (see (13)). Then, the zeros of $D(0, \tau)$ correspond to homoclinic orbits of the perturbed system and the distance between the invariant manifolds is given by

$$d(\tau) = y^s(0, \tau) - y^u(0, \tau) = \frac{1}{2} D(0, \tau). \tag{17}$$

Nevertheless, when α is close to 1, namely $\alpha = 1 - C\varepsilon^r$ with $r > 0$ and $C > 0$, the perturbative term in (3) has a non uniform bound for $x \in [0, 2\pi]$. Indeed, its maximum (in absolute value), which is $\mu \varepsilon^\eta/(1-\alpha)^2 = \mu \varepsilon^{\eta-2r}/C^2$, is reached at $x = 3\pi/2$. Therefore, it is natural to expect that the invariant manifolds of the perturbed system remain $\mu \varepsilon^\eta$-close to the unperturbed separatrix only before they reach a neighborhood of the section $x = 3\pi/2$. For this reason, in this case we will measure the distance in this section, where one expects that the perturbed manifolds are closer. This section, by (13), corresponds to $u = \ln(1 + \sqrt{2})$. Therefore, we define the $\widetilde{d}(\tau)$, the distance between the perturbed invariant manifolds at the section $x = 3\pi/2$, which is given by

$$\widetilde{d}(\tau) = y^s \left(\ln\left(1 + \sqrt{2}\right), \tau \right) - y^u \left(\ln\left(1 + \sqrt{2}\right), \tau \right) = 2\sqrt{2} D \left(\ln\left(1 + \sqrt{2}\right), \tau \right). \tag{18}$$

For α close to 1, one would expect that the distance $d(\tau)$ is bigger than $\widetilde{d}(\tau)$ since the manifolds deviate from the homoclinic after they cross the section $x = 3\pi/2$. Nevertheless, when α is not close to 1, both quantities are equivalent.

2.1 The Melnikov function

In this section we compute the Melnikov function associated to Hamiltonian (5) with $f(\tau) = \sin \tau$ and we study its dependence on ε and α. Let us recall that the Melnikov function (15) is given by

$$\mathcal{M}(u, \tau; \varepsilon, \alpha) = 4 \int_{-\infty}^{+\infty} \frac{\sinh(u+s)\cosh(u+s)}{\left(\cosh^2(u+s) - 2\alpha\sinh(u+s)\right)^2} \sin\left(\tau + \frac{s}{\varepsilon}\right) ds. \tag{19}$$

As it is well known, the evaluation of this integral can be done using Residuum Theory. To this end, one has to look for the poles of the function

$$\beta(u) = \frac{\sinh(u)\cosh(u)}{\left(\cosh^2(u) - 2\alpha\sinh(u)\right)^2} \tag{20}$$

closest to the real axis. For $\alpha = 0$, $\beta(u)$ has order 3 poles at $\rho_{\pm}^0 = \pm i\pi/2$. Nevertheless, for $\alpha > 0$ these poles bifurcate into a combination of zeros and poles. Since the size of the Melnikov function (19) depends on the location of these poles, one has to study their dependence on α. We will see that the relative size between the parameter α and the period $2\pi\varepsilon$ of the perturbation will lead to a significantly different size of (19).

Considering the denominator of $\beta(u)$, namely $(\cosh^2(u) - 2\alpha\sinh(u))^2$, one can easily see that the poles of (20) are the solutions of

$$\sinh u = \alpha \pm i\sqrt{1-\alpha^2},$$

This equation has four families of solutions given by $\rho_- + 2\pi ki$, $\rho_+ + 2\pi ki$, which are solutions of

$$\sinh u = \alpha + i\sqrt{1-\alpha^2}.$$

and their conjugate families $\overline{\rho}_- + 2\pi ki$ and $\overline{\rho}_+ + 2\pi ki$ for $k \in \mathbb{Z}$. The singularities $\rho_-, \rho_+, \overline{\rho}_-$ and $\overline{\rho}_+$ are the closest to the real axis (see Figure 2) and they satisfy

$$0 < \operatorname{Im}\rho_- < \frac{\pi}{2} < \operatorname{Im}\rho_+ < \pi,$$

for $\alpha \in (0, 1)$.

In particular, if one considers α as a small parameter, they have the expansions

$$\rho_{\pm}(\alpha) = i\frac{\pi}{2} \pm (-1+i)\sqrt{\alpha} + \mathcal{O}(\alpha) \quad \text{as } \alpha \to 0. \tag{21}$$

In the other limiting regime $\alpha \to 1$, their expansions are

$$\begin{aligned}
\rho_-(\alpha) &= \ln(1 + \sqrt{2}) + i(1-\alpha)^{1/2} + \mathcal{O}(1-\alpha) \\
\rho_+(\alpha) &= -\ln(1 + \sqrt{2}) + \pi i - i(1-\alpha)^{1/2} + \mathcal{O}(1-\alpha)
\end{aligned} \qquad \text{as } \alpha \to 1. \tag{22}$$

Then, one can see that the function β for $-\pi < |\operatorname{Im} u| < \pi$ and $-1 < \operatorname{Re} u < 1$ behaves as

$$\beta(u) \sim \frac{(u - i\pi/2)(u + i\pi/2)}{(u-\rho_-)^2(u-\rho_+)^2(u-\overline{\rho}_-)^2(u-\overline{\rho}_+)^2}. \tag{23}$$

Using this fact, one can easily compute the size of the Melnikov function in (19) using Residuum Theory.

Proposition 2.1. *There exists $\varepsilon_0 > 0$ such that $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in (0, 1)$,*
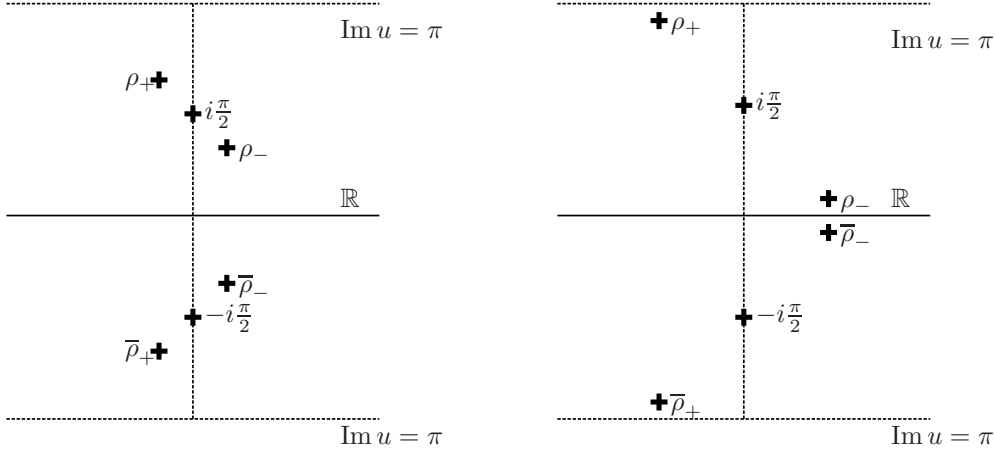
Figure 2: Location of the singularities $u = \rho_-, \rho_+, \overline{\rho}_-, \overline{\rho}_-$. In the left picture we show them for α small, in such a way that they are close to the singularities of the separatrix of the pendulum $u = \pm i\pi/2$. In the right picture we show them for α close to 1. Then, ρ_- and $\overline{\rho}_-$ approach the real axis while ρ_+ and $\overline{\rho}_+$ approach the lines $\operatorname{Im} u = \pi$ and $\operatorname{Im} u = -\pi$ respectively.

- If α satisfies $0 < \alpha \leq C\varepsilon^\nu$ where $\nu > 2$ and $C > 0$ are constants independent of ε, the Melnikov function (19) satisfies the asymptotic formula

$$\mathcal{M}(u, \tau; \varepsilon, \alpha) = \frac{4\pi}{\varepsilon^2} e^{-\frac{\pi}{2\varepsilon}} \left(\cos(\tau - u/\varepsilon) + \mathcal{O}\left(\frac{\alpha}{\varepsilon^2}, e^{-\frac{\pi}{2\varepsilon}} \right) \right). \tag{24}$$

- If $\alpha = \alpha_* \varepsilon^2 + \mathcal{O}\left(\varepsilon^3 \right)$ for some constant $\alpha_* > 0$, the Melnikov function (19) satisfies the asymptotic formula

$$\mathcal{M}(u, \tau; \varepsilon, \alpha) = \frac{\lambda}{\varepsilon^2} e^{-\frac{\pi}{2\varepsilon}} \sin\left(\tau - \phi_* - u/\varepsilon \right) + \mathcal{O}\left(\frac{1}{\varepsilon} e^{-\frac{\pi}{2\varepsilon}} \right) \tag{25}$$

for certain constants $\lambda = \lambda(\alpha_*)$ and $\phi_* = \phi_*(\varepsilon, \alpha_*)$.

- If α satisfies $C\varepsilon^\nu < \alpha < 1$ where $\nu \in (0, 2)$ and $C > 0$ are constants independent of ε, the Melnikov function (19) satisfies the asymptotic formula

$$\mathcal{M}(u, \tau; \varepsilon, \alpha) = \left| \frac{\delta_2(\alpha)}{\varepsilon} + \delta_1(\alpha) \right| e^{-\frac{\operatorname{Im} \rho_-}{\varepsilon}} \sin\left(\tau - \phi - u/\varepsilon \right) + \mathcal{O}\left(e^{-\frac{\pi}{2\varepsilon}} \right). \tag{26}$$

where $\delta_i(\alpha)$ are given by

$$\delta_1(\alpha) = \frac{2\pi}{(1 - \alpha^2)^{3/2}} \left(\sinh \rho_- - i \left(1 - \alpha^2 \right)^{1/2} \right) \tag{27}$$

$$\delta_2(\alpha) = \frac{2\pi \sinh \rho_-}{(1 - \alpha^2) \cosh \rho_-} \tag{28}$$

with $\sinh \rho_- = \alpha + i\sqrt{1 - \alpha^2}$, and $\phi = \phi(\varepsilon, \alpha)$.

When $\alpha \ll \varepsilon^2$, the first statement of Proposition 2.1 ensures that the Melnikov function is non-degenerate. In the case $\alpha \sim \varepsilon^2$, one would have to analyze the behavior of the constant $\lambda(\alpha_*)$, which appears in formula (25), to check that it does not vanish. Even if this constant is computable in our example, we will not study it, since it is not the purpose of this paper to study this particular case. We just want to remark that for general α_*, the first asymptotic order of the Melnikov function has changed

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from (24) to (25). In fact, as can be seen in formula (26), $\alpha \sim \varepsilon^2$ is the transition value in which the size of the Melnikov function changes drastically. Indeed, formula (26) and the expansion of ρ_- given in (22) show that the size of the Melnikov function becomes bigger when α approaches 1. This particular behavior will be analyzed in Section 2.1.1.

Next corollary, whose proof is straightforward, analyzes the behavior of formula (26) for α in the range $\varepsilon^2 \ll \alpha \leq \alpha_0 < 1$ for any fixed $\alpha_0 \in (0,1)$. Let us observe that ρ_- has the asymptotic expansion (21) when $\alpha \to 0$. Next corollary shows that the Melnikov function is exponentially small for this range of the parameter α.

Corollary 2.2. *Let us fix any $\alpha_0 \in (0,1)$. Then, the Melnikov function in (26) has the following asymptotic formulas for $\varepsilon^2 \ll \alpha \leq \alpha_0 < 1$.*

- *If α satisfies $\alpha = C\varepsilon^\nu$ for certain $\nu \in (0,2)$ and $C > 0$,*

$$\mathcal{M}(u,\tau;\varepsilon, C\varepsilon^\nu) = \frac{\left|\delta_2^0\right|}{C\varepsilon^{1+\nu/2}} e^{-\frac{\operatorname{Im}\rho_-}{\varepsilon}} \left(\sin\left(\tau - \phi - u/\varepsilon\right) + \mathcal{O}\left(\varepsilon^{\nu/2}\right)\right),$$

  where

$$\delta_2^0 = -\pi(1+i).$$

  Therefore, in this case the Melnikov function is non-degenerate since $\delta_2^0 \neq 0$.

- *If $\alpha \in (0,\alpha_0]$ is independent of ε,*

$$\mathcal{M}(u,\tau;\varepsilon,\alpha) = \frac{\left|\delta_2(\alpha)\right|}{\varepsilon} e^{-\frac{\operatorname{Im}\rho_-}{\varepsilon}} \left(\sin\left(\tau - \phi - u/\varepsilon\right) + \mathcal{O}\left(\varepsilon\right)\right).$$

  It can be checked that $\delta_2(\alpha) \neq 0$ for any $\alpha \in (0,1)$, and therefore, in this case the Melnikov function is also non-degenerate.

Remark 2.3. *If one takes $\alpha \sim \varepsilon^\nu$ and $\nu \to 2$ in formula (26) does not obtain formula (25). The reason is that formula (26) only takes into account the residuum of ρ_- since the residuum of ρ_+ is exponentially small with respect to the one of ρ_-. Nevertheless, this is not the case when $\alpha \sim \varepsilon^2$ (see (21)). Then, the residuums of both singularities ρ_- and ρ_+ make a contribution to the Melnikov function of the same exponentially small order $\mathcal{O}(\varepsilon^{-2}e^{-\pi/2\varepsilon})$.*

Remark 2.4. *In the first statement of Proposition 2.1, one can see that, when the strip of analyticity (6) is wide enough, namely taking $\alpha \sim \varepsilon^\nu$ with $\nu > 2$, the Melnikov function at first order behaves as in the entire case $\alpha = 0$ [Gel97a]. In other words, the Melnikov function does not realize the finiteness of the strip of analyticity and the exponentially small coefficient is given by the imaginary part of the singularities of the separatrix.*

On the other hand, when the strip of analyticity (6) is independent of ε or not extremely big with respect to ε, namely taking $\alpha \sim \varepsilon^\nu$ with $\nu \in [0,2)$, the exponentially small coefficient does not coincide with the imaginary part of the singularity of the unperturbed separatrix. Instead it is given by the imaginary part of this new singularity ρ_-, which appears when one evaluates the perturbation along the unperturbed separatrix. Note that even if one takes, for instance, $\alpha = \varepsilon$, the strip of analyticity is of order $\ln(1/\varepsilon)$ and one has that

$$\mathcal{M}(u,\tau;\varepsilon,\varepsilon) \sim \varepsilon^{-\frac{3}{2}} e^{-\frac{\pi - 2\sqrt{\varepsilon}}{2\varepsilon}}.$$

That is, even for perturbations with a wide strip of analyticity with respect to ε, it appears a correcting term in the exponential.

The case $\alpha \sim \varepsilon^2$ is the boundary between these two different behaviors. In this case, the exponential coefficient is given by the imaginary part of the singularity of the unperturbed separatrix but the constant in front of the exponential is not the one given in the first statement of Proposition 2.1 but a different one, which is given in the second statement for the value $\alpha = \alpha_* \varepsilon^2 + \mathcal{O}(\varepsilon^3)$.

Proof of Proposition 2.1. The classical way to compute the Melnikov function is to change the path of integration to $\operatorname{Im} s = \pm \pi$ and apply Residuum Theory, using that the function β has order two poles at ρ_-, ρ_+, $\overline{\rho}_-$ and $\overline{\rho}_+$ in the strip $-\pi < \operatorname{Im} s < \pi$. Nevertheless, to prove the first statement, instead of computing the residuums of both $\rho_\pm$ and $\overline{\rho}_\pm$ and look for the cancellations, we just expand the Melnikov integral in power series of α, since it is uniformly convergent as a real integral. Thus, we obtain,

$$\mathcal{M}(u, \tau; \varepsilon, \alpha) = 4 \sum_{n=0}^{\infty} (n+1) 2^n \alpha^n \int_{-\infty}^{+\infty} \frac{\sinh^{n+1}(u+s)}{\cosh^{2n+3}(u+s)} \sin\left(\tau + \frac{s}{\varepsilon}\right) ds.$$

Then, the term for $n = 0$ gives the first asymptotic order and can be computed using again Residuum Theory. To bound the other terms goes as follows. We Fourier-expand the terms in the series of the Melnikov integral in τ and we change the path of integration to $\operatorname{Im} s = \pi/2 - \varepsilon$ or $\operatorname{Im} s = -(\pi/2 - \varepsilon)$ depending on the harmonic. This gives us the exponentially small term $e^{-\frac{\pi}{2\varepsilon}}$. To bound the other terms in the integral, we use that

$$\left| \frac{\sinh^{n+1}(u+s)}{\cosh^{2n+1}(u+s)} \right| \le \left(\frac{K}{\varepsilon} \right)^{2n+1},$$

for certain constant $K > 0$ independent of ε. Moreover using that

$$\frac{1}{\left| \cosh^2(u+s) \right|}$$

decays exponentially as $\operatorname{Re} s \to \pm\infty$ and that it has poles of order two $u + s = \pm i\pi/2$, one can see that

$$\int_{-\infty}^{\infty} \frac{1}{\left| \cosh^2(u + s \pm (i\pi/2 - \varepsilon)) \right|} ds \le \frac{K}{\varepsilon}.$$

Then, one has that

$$\left| \int_{-\infty}^{+\infty} \frac{\sinh^{n+1}(u+s)}{\cosh^{2n+3}(u+s)} \sin\left(\tau + \frac{s}{\varepsilon}\right) ds \right| \le \left(\frac{K}{\varepsilon} \right)^{2n+2} e^{-\frac{\pi}{2\varepsilon}}.$$

Therefore, the remainder can be easily bounded provided $0 < \alpha \ll \varepsilon^2$.

When $\alpha \sim \varepsilon^2$ we cannot ensure that the preceding series is convergent and therefore we use directly Residuum Theory in the strip $0 < \operatorname{Im} s < \pi$ or $-\pi < \operatorname{Im} s < 0$ depending on the harmonic. For instance, for the positive harmonic, we take the strip $0 < \operatorname{Im} s < \pi$, which contains the singularities ρ_- and ρ_+, that are at a distance of order $\mathcal{O}(\varepsilon)$ from $i\pi/2$ (see (21)). Then, it is enough to compute the residuum of $\beta(s) e^{is/\varepsilon}$ at these singularities, which using (23) satisfies

$$\operatorname{Res}\left(\beta(s) e^{is/\varepsilon}, s = \rho_\pm \right) = \frac{A_\pm}{\varepsilon^2} e^{-\frac{\pi}{2\varepsilon}} + \mathcal{O}\left(\frac{1}{\varepsilon} e^{-\frac{\pi}{2\varepsilon}} \right).$$

for certain computable constants $A_\pm \in \mathbb{C}$ independent of ε.

In the case $\alpha \sim \varepsilon^\nu$ with $0 \le \nu < 2$, we have that $\pi/2 - \operatorname{Im}\rho_-, \operatorname{Im}\rho_+ - \pi/2 \sim \varepsilon^{\nu/2} \gg \varepsilon$. Therefore,

$$e^{-\frac{\operatorname{Im}\rho_+}{\varepsilon}} \ll e^{-\frac{\pi}{2\varepsilon}} \ll e^{-\frac{\operatorname{Im}\rho_-}{\varepsilon}}.$$

Using these facts, we can compute the Melnikov integral changing the path up to $\operatorname{Im} s = \pi/2$ and we just need to consider the residuum at the singularity ρ_-, which can be explicitly computed. $\square$

Remark 2.5. *All the singularities of the function $\beta(u)$ in (20) have different imaginary part for any $\alpha \in (0,1)$. This is one of the reasons of the choice of the perturbation in (3), since this is not always the case. Let us consider, for instance, the pendulum with a different perturbation as*

$$\ddot{x} = \sin x + \mu \varepsilon^\eta \frac{\sin x}{(1 - \alpha \cos x)^2} \sin \frac{t}{\varepsilon}, \tag{29}$$

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which has Hamiltonian function

$$\widetilde{H}\left(x, y, \frac{t}{\varepsilon}\right) = \frac{y^2}{2} + \cos x - 1 + \mu\varepsilon^\eta \frac{1}{\alpha(1 - \alpha\cos x)} \sin\frac{t}{\varepsilon}.$$

Then, the Melnikov function is given by

$$\widetilde{\mathcal{M}}\left(u, \frac{t}{\varepsilon}\right) = 4 \int_{-\infty}^{+\infty} \frac{\cosh(u+s)\sinh(u+s)}{\left((1-\alpha)\cosh^2(u+s) + 2\alpha\right)^2} \sin\left(\frac{t+s}{\varepsilon}\right)$$

To compute its size, one has to study the singularities of

$$\widetilde{\beta}(u) = \frac{\cosh u \sinh u}{\left((1-\alpha)\cosh^2 u + 2\alpha\right)^2}.$$

The closest ones to the real axis with positive imaginary part are

$$\widetilde{\rho}_\pm = i\frac{\pi}{2} \pm \operatorname{arcsinh}\sqrt{\frac{2\alpha}{1-\alpha}}$$

and therefore both have the same imaginary part for any $\alpha \in (0,1)$. In these cases, the Melnikov function, which is given by the residuums of both singularities, has the same size for any α satisfying $0 \le \alpha \le \alpha_0 < 1$ for any α_0 independent of ε, but is divergent for $\alpha = 1$.

2.1.1 Narrow strip of analyticity: a drastic change on the size of the Melnikov function

In Corollary 2.2 we have seen that the Melnikov function is exponentially small with respect to ε provided $0 < \alpha \le \alpha_0 < 1$ for any fixed $\alpha_0 \in (0,1)$. We devote this section to study this function when α is close to 1, which will lead to a Melnikov function which is either exponentially small with a different exponential dependence on ε or even to a non-exponentially small Melnikov function.

We consider $\alpha = 1 - C\varepsilon^r$ with $r > 0$ and $C > 0$. In this setting, one can see that the analyticity strip (6) of the Hamiltonian system (4) is very small, of order $\mathcal{O}(\varepsilon^{r/2})$. As a consequence, the singularity $\rho_-(\alpha)$ is very close to the real line. Indeed, from (22), one has

$$\operatorname{Im}\rho_-(\alpha) = C^{1/2}\varepsilon^{r/2} + \mathcal{O}\left(\varepsilon^r\right).$$

Corollary 2.6. *If $\alpha = 1 - C\varepsilon^r$ with $r > 0$ and $C > 0$, the Melnikov function in (26) has the following asymptotic formulas*

- *If $0 < r < 2$*

$$\mathcal{M}\left(u, \tau; \varepsilon, 1 - C\varepsilon^r\right) = \frac{\pi}{\sqrt{2}C\varepsilon^{1+r}} e^{-\frac{\operatorname{Im}\rho_-}{\varepsilon}} \left(\sin\left(\tau - \phi - u/\varepsilon\right) + \mathcal{O}\left(\varepsilon^{r/2}, \varepsilon^{1-r/2}\right)\right).$$

- *If $r = 2$,*

$$\mathcal{M}\left(u, \tau; \varepsilon, 1 - C\varepsilon^2\right) = \frac{\pi e^{-\sqrt{C}}}{\sqrt{2}C^{3/2}\varepsilon^3} \left(\sin\left(\tau - \phi - u/\varepsilon\right) + \mathcal{O}(\varepsilon)\right),$$

- *If $r > 2$,*

$$\mathcal{M}\left(u, \tau; \varepsilon, 1 - C\varepsilon^r\right) = \frac{\pi}{\sqrt{2}C^{3/2}\varepsilon^{3r/2}} \left(\sin\left(\tau - \phi - u/\varepsilon\right) + \mathcal{O}\left(\varepsilon^{r/2-1}\right)\right),$$

where $\phi = \phi(\varepsilon, \alpha)$ is the constant given in Proposition 2.1.

Therefore, in all these cases the Melnikov function is non-degenerate. Moreover, in the last two cases, it is non-exponentially small.

Remark 2.7. *To illustrate which size has the Melnikov function for the range of α considered in the first statement of Corollary 2.6, we can take $\alpha = 1 - \varepsilon$. Then, using the expansion (22), one can see that*

$$\mathcal{M}\left(u, \tau; \varepsilon, 1 - \varepsilon\right) \sim \varepsilon^{-2} e^{-\frac{1}{\sqrt{\varepsilon}}}.$$

Namely, the Melnikov function is still exponentially small but it has a different exponential dependence on ε.

2.2 Validity of the Melnikov function

Once we have computed the Melnikov function in Proposition 2.1 provided $0 < \alpha \leq \alpha_0 < 1$ for a fixed α_0, we can compute the prediction it gives for the distance between the manifolds at the section $x = \pi$, which we call $d_0(\tau)$. Recall that by (16) it is the first order in μ of the function $d(\tau)$ given in (17).

If α satisfies $0 < \alpha \leq C\varepsilon^\nu$ where $\nu > 2$ and $C > 0$ are constants independent of ε, $d_0(\tau)$ is given by

$$d_0(\tau) = 2\pi\mu\varepsilon^{\eta-2}e^{-\frac{\pi}{2\varepsilon}}\left(\cos\tau + \mathcal{O}\left(\frac{\alpha}{\varepsilon^2}, e^{-\frac{\pi}{2\varepsilon}}\right)\right).$$

If $\alpha = \alpha_*\varepsilon^2 + \mathcal{O}(\varepsilon^3)$ for any constant $\alpha_* > 0$, it is given by,

$$d_0(\tau) = \frac{\lambda(\alpha_*)}{2}\mu\varepsilon^{\eta-2}e^{-\frac{\pi}{2\varepsilon}}\sin(\tau - \phi_*) + \mathcal{O}\left(\mu\varepsilon^{\eta-1}e^{-\frac{\pi}{2\varepsilon}}\right).$$

Finally, if α satisfies $C\varepsilon^\nu < \alpha \leq \alpha_0 < 1$ where $\nu \in (0,2)$, $C > 0$ and $\alpha_0 \in (0,1)$ are constants independent of ε, it is given by

$$d_0(\tau) = \frac{|\delta_2(\alpha)|}{2}\mu\varepsilon^{\eta-1}e^{-\frac{\operatorname{Im}\rho_-}{\varepsilon}}\sin(\tau - \phi) + \mathcal{O}\left(\mu\varepsilon^\eta e^{-\frac{\operatorname{Im}\rho_-}{\varepsilon}}\right).$$

As we have already explained, direct application of Melnikov theory only ensures that the distance between the manifolds is given by

$$d(\tau) = d_0(\tau) + \mathcal{O}\left(\mu^2\varepsilon^{2\eta}\right).$$

Therefore, the Melnikov function is the first order of the splitting provided μ is exponentially small with respect to ε. However, it is well known that often, even if Melnikov theory cannot be applied, the Melnikov function is the true first order of the splitting (see [HMS88, DS92, Gel94, DS97, Gel97a, BF04, BF05, Gel00, GOS10, BFGS11]). Next theorem shows that, under certain conditions, the Melnikov function gives the true first order of the distance between the manifolds. We want to point out that the only available proof of the correct prediction of the Melnikov function for meromorphic perturbations is [Gel97a], in which the author considers systems with analyticity strip wide enough with respect to ε. All the other references deal with polynomial perturbations. Thus, this theorem is, as far as the authors know, the first one which shows the dependence of the size of the splitting on the width of the analyticity strip.

Theorem 2.8. *Let us consider any $\mu_0 > 0$ and $\alpha_0 \in (0,1)$. Then, there exists $\varepsilon_0 > 0$ such that for $|\mu| < \mu_0$, $\varepsilon \in (0,\varepsilon_0)$ and $\alpha \in (0,\alpha_0]$ such that $\varepsilon^{\eta-1}(\varepsilon + \sqrt{\alpha})$ is small enough,*

- *If α satisfies $0 < \alpha \leq C\varepsilon^\nu$ where $\nu > 2$ and $C > 0$ are constants independent of ε, the invariant manifolds split and their distance at the section $x = \pi$ is given by*

$$d(\tau) = 2\pi\mu\varepsilon^{\eta-2}e^{-\frac{\pi}{2\varepsilon}}\left(\cos\tau + \mathcal{O}\left(\frac{\alpha}{\varepsilon^2}, \mu\varepsilon^\eta\right)\right).$$

- *If $\alpha = \alpha_*\varepsilon^2 + \mathcal{O}(\varepsilon^3)$ for some constant $\alpha_* > 0$ and the constant $\lambda(\alpha_*)$ introduced in Proposition 2.1 satisfies $\lambda(\alpha_*) \neq 0$, the invariant manifolds split and their distance at the section $x = \pi$ is given by*

$$d(\tau) = \frac{|\lambda(\alpha_*)|}{2}\mu\varepsilon^{\eta-2}e^{-\frac{\pi}{2\varepsilon}}\left(\sin(\tau - \phi_*) + \mathcal{O}\left(\mu\varepsilon^\eta\right)\right).$$

- *If α satisfies and $C\varepsilon^\nu < \alpha < \alpha_0$ where $\nu \in (0,2]$ and $C > 0$ are constants independent of ε, the invariant manifolds split and their distance at the section $x = \pi$ is given by*

$$d(\tau) = \frac{|\delta_2(\alpha)|}{2}\mu\varepsilon^{\eta-1}e^{-\frac{\operatorname{Im}\rho_-}{\varepsilon}}\left(\sin(\tau - \phi) + \mathcal{O}\left(\mu\varepsilon^{\eta-1}\sqrt{\alpha}, \varepsilon\right)\right).$$

The proof of this theorem is deferred to Section 4.

Remark 2.9. *If one considers $\alpha \sim \varepsilon^\nu$ with $\nu > 2$ the condition $\varepsilon^{\eta-1}(\varepsilon + \sqrt{\alpha})$ small enough is equivalent to requiring $\eta > 0$. This condition is the same that has to be required if one considers the polynomial case $\alpha = 0$ (see [Tre97, BFGS11]). In other words, in the case $\nu > 2$ the validity of the Melnikov function is the same as in the polynomial case.*

On the other hand, if one considers $\alpha \sim \varepsilon^\nu$ with $\nu < 2$, the condition $\varepsilon^{\eta-1}(\varepsilon + \sqrt{\alpha})$ small enough is

$$\eta - 1 + \frac{\nu}{2} > 0.$$

In particular, if one considers α as a parameter independent of ε, one has to require $\eta > 1$.

The limit cases in the previous settings, $\eta = 0$ and $\eta - 1 + \nu/2 = 0$ respectively, are usually called singular cases (see [BFGS11]). In Appendix A we make some remarks about how these cases could be studied.

2.2.1 Narrow strip of analyticity: validity of the Melnikov function

In Section 2.1.1 we have seen how the Melnikov function changes drastically its size when $\alpha = 1 - C\varepsilon^r$ with $r > 0$, even becoming non-exponentially small if $r \geq 2$. In this section we prove that it gives the correct first order for the splitting of separatrices. Note that, in the range of parameters for which the Melnikov function is not exponentially small, one can just apply classical perturbation techniques to prove that this function gives the first order of the splitting.

First, we give the prediction of the distance given by the Melnikov function when $\alpha = 1 - C\varepsilon^r$ with $r > 0$ and $C > 0$, which can be deduced from Corollary 2.6. Recall that, as we have explained in Section 2, in this case we study the distance between the manifolds at the section $x = 3\pi/2$. We have called $\widetilde{d}(\tau)$ to this distance (see (18)) and we call $\widetilde{d}_0(\tau)$ to the Melnikov prediction of this distance.

If $r \in (0, 2)$, calling $u^* = \ln(1 + \sqrt{2})$,the Melnikov prediction is given by

$$\widetilde{d}_0(\tau) = \mu\varepsilon^{\eta-r-1}\frac{2\pi}{C}e^{-\frac{\operatorname{Im}\rho_-}{\varepsilon}}\left(\sin\left(\tau - \phi - u_*/\varepsilon\right) + \mathcal{O}\left(\varepsilon^{r/2}, \varepsilon^{1-r/2}\right)\right), \tag{30}$$

whereas if $r = 2$ is given by

$$\widetilde{d}_0(\tau) = \mu\varepsilon^{\eta-3}\frac{2\pi e^{-\sqrt{C}}}{C^{3/2}}\left(\sin(\tau - \phi - u^*/\varepsilon) + \mathcal{O}(\varepsilon)\right). \tag{31}$$

Finally, if $r > 2$ is given by

$$\widetilde{d}_0(\tau) = \mu\varepsilon^{\eta-3r/2}\frac{2\pi}{C^{3/2}}\left(\sin(\tau - \phi - u^*/\varepsilon) + \mathcal{O}\left(\varepsilon^{r/2-1}\right)\right). \tag{32}$$

In these cases, direct application of Melnikov theory only ensures that the distance between the manifolds is given by

$$d(\tau) = d_0(\tau) + \mathcal{O}\left(\mu^2\varepsilon^{2(\eta-2r)}\right),$$

since the perturbation has size $\mathcal{O}(\mu\varepsilon^{\eta-2r})$. Next theorem widens the range of the validity of this prediction under certain hypotheses.

Theorem 2.10. *Let us consider any $\mu_0 > 0$ and let us assume $\eta > \max\{r+1, 3r/2\}$. Then, there exists $\varepsilon_0 > 0$ such that for $|\mu| < \mu_0$, $\varepsilon \in (0, \varepsilon_0)$ and $\alpha = 1 - C\varepsilon^r$ with $r > 0$ and $C > 0$, the invariant manifolds split and their distance on the section $x = 3\pi/2$ is given by,*

- *If $0 < r < 2$,*

$$\widetilde{d}(\tau) = \mu\varepsilon^{\eta-r-1}\frac{2\pi}{C}e^{-\frac{\operatorname{Im}\rho_-}{\varepsilon}}\left(\sin(\tau - \phi - u^*/\varepsilon) + \mathcal{O}\left(\mu\varepsilon^{\eta-r-1}, \varepsilon^{1-r/2}, \varepsilon^{r/2}\right)\right).$$

- *If $r = 2$,*

$$\widetilde{d}(\tau) = \mu\varepsilon^{\eta-3}\frac{2\pi e^{-\sqrt{C}}}{C^{3/2}}\left(\sin(\tau - \phi - u^*/\varepsilon) + \mathcal{O}\left(\varepsilon, \mu\varepsilon^{\eta-3}\right)\right).$$

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- If $r > 2$,

$$\widetilde{d}(\tau) = \mu \varepsilon^{\eta - 3r/2} \frac{2\pi}{C^{3/2}} \left(\sin(\tau - \phi - u^*/\varepsilon) + \mathcal{O}\left(\varepsilon^{r/2-1}, \mu \varepsilon^{\eta - 3r/2}\right) \right),$$

where $u^* = \ln(1 + \sqrt{2})$ and $\phi = \phi(\varepsilon, \alpha)$ is the constant given in Corollary 2.6.

3 The Melnikov function and its validity in the quasiperiodic case

Analogously to the periodic case, we work in the extended phase space that now is $(x, y, \theta_1, \theta_2) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}^2$. Note that we do not work in the whole symplectic space, namely we omit the actions (I_1, I_2), since they do not have any dynamical interest. In this setting $\mathcal{T} = \{(0, 0, \theta_1, \theta_2); (\theta_1, \theta_2) \in \mathbb{T}^2\}$ is a normally hyperbolic torus and, for $\mu = 0$, its stable and unstable invariant manifolds coincide along the homoclinic manifold

$$\mathcal{W}^u(\mathcal{T}) = \mathcal{W}^s(\mathcal{T}) = \{(x, y, \theta_1, \theta_2); H_0(x, y) = 0\}$$
$$= \left\{(x, y, \theta_1, \theta_2) = (x_0(u), y_0(u), \theta_1, \theta_2); (u, \theta_1, \theta_2) \in \mathbb{R} \times \mathbb{T}^2\right\},$$

where $(x_0(u), y_0(u))$ is the time parameterization of the homoclinic orbit, which is given in (13).

Then, we look for the perturbed manifolds as

$$\begin{cases} x = x_0(u) \\ y = y^{u,s}(u, \theta_1, \theta_2). \end{cases}$$

Here $(u, \theta_1, \theta_2) \in (-\infty, U) \times \mathbb{T}^2$, for certain $U > 0$, for the unstable manifold and $(u, \theta_1, \theta_2) \in (-U, +\infty) \times \mathbb{T}^2$ for the stable one. Again, the Lagrangian character of the manifolds implies that the functions $y^{u,s}$ are given by

$$y^{u,s}(u, \theta_1, \theta_2) = \frac{1}{y_0(u)} \partial_u T^{u,s}(u, \theta_1, \theta_2).$$

Therefore, as in the periodic case, a natural way to measure the difference between the manifolds is to compute

$$D(u, \theta_1, \theta_2) = \partial_u T^s(u, \theta_1, \theta_2) - \partial_u T^u(u, \theta_1, \theta_2),$$

whose first order in μ is given by the Melnikov function

$$\mathcal{M}(u, \theta_1, \theta_2) = \int_{-\infty}^{+\infty} \{H_0, H_1\}\left(x_0(u+s), y_0(u+s), \theta_1 + \frac{s}{\varepsilon}, \theta_2 + \gamma\frac{s}{\varepsilon}\right) ds.$$

Namely, one has that

$$D(u, \theta_1, \theta_2) = \mu \varepsilon^{\eta} M(u, \theta_1, \theta_2) + \mathcal{O}\left(\mu^2 \varepsilon^{2\eta}\right).$$

As in the periodic case, for α small or fixed independently of ε, we measure the distance at the section $x = \pi$, which is given by

$$d(\theta_1, \theta_2) = y^s(0, \theta_1, \theta_2) - y^u(0, \theta_1, \theta_2) = \frac{1}{2}D(0, \theta_1, \theta_2), \tag{33}$$

whereas in the case $\alpha = 1 - C\varepsilon^r$ with $r > 0$ and $C > 0$, we measure it at the section $x = 3\pi/2$, which is given by

$$\widetilde{d}(\theta_1, \theta_2) = y^s\left(\ln\left(1 + \sqrt{2}\right), \theta_1, \theta_2\right) - y^u\left(\ln\left(1 + \sqrt{2}\right), \theta_1, \theta_2\right) = 2\sqrt{2}D\left(\ln\left(1 + \sqrt{2}\right), \theta_1, \theta_2\right). \tag{34}$$

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3.1 The Melnikov function

If one applies the Poincaré-Melnikov method to Hamiltonian (12), obtains

$$\mathcal{M}\left(u, \theta_1, \theta_2; \varepsilon, \alpha\right) = 4 \int_{-\infty}^{+\infty} \beta(u+s) F\left(\theta_1 + \frac{s}{\varepsilon}, \theta_2 + \frac{\gamma s}{\varepsilon}\right) ds, \tag{35}$$

where β is the function defined in (20) and F is the quasiperiodic perturbation (9) satisfying (10) and (11).

In the quasiperiodic case, it is a well known fact that the size of the Melnikov function is not given by its first harmonic (see [Sim94, DGJS97, DJSG99]). Instead, the leading harmonic depends on ε. For this reason, we need to compute carefully the size of all harmonics. We compute them using Residuum Theory and the properties of the function β given in Section 2. We follow the same approach of [DGJS97]. For this reason, let us first introduce certain functions and constants defined in that paper.

We define the $2 \ln \gamma$-periodic function $c(\delta)$ defined by

$$c(\delta) = C_0 \cosh\left(\frac{\delta - \delta_0}{2}\right) \quad \text{for } \delta \in [\delta_0 - \ln \gamma, \delta_0 + \ln \gamma], \tag{36}$$

where

$$C_0 = 2\sqrt{\frac{(\gamma r_1 + r_2)}{\gamma + \gamma^{-1}}}, \quad \delta_0 = \ln \varepsilon^*, \quad \varepsilon^* = \frac{\left(\gamma + \gamma^{-1}\right)}{\gamma^2(r_1 \gamma + r_2)}$$

and continued by $2 \ln \gamma$-periodicity onto the whole real axis. Note that the constant C_0 is slightly different from the one considered in [DGJS97] since in that paper, it includes a $\pi/2$ coefficient coming from the imaginary part of the singularity of the unperturbed separatarix. Since in this paper the singularity changes with respect to α, we have defined a new constant which is independent of it. Then, the formulas of the splitting, given in Proposition 3.1 and Theorems 3.5 and 3.7, will contain the dependence on the imaginary part of the singularity explicitly. Following the lines in [DGJS97], one can see that the function $c(\delta)$ oscillates and has lower and upper bounds independent of ε.

One can use the function $c(\delta)$ to give the size of the Melnikov function as done in [DGJS97]. In the next proposition we see how this size changes depending on the relation between ε and α. In all the results in the quasiperiodic case we include the case $\alpha = 0$, since the proof we present is also valid in this case and slightly improves the results in the literature [DGJS97, Sau01].

Proposition 3.1. *Let us assume (10) and (11) and fix $u_0 > 0$. Then, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in [0, 1)$,*

- *If α satisfies $0 \leq \alpha \leq C\varepsilon^\nu$ where $\nu > 1$ and $C > 0$ are constants independent of ε, the Melnikov function (35) satisfies that*

$$\frac{C_1}{\varepsilon} e^{-c(\ln(2\varepsilon/\pi))\sqrt{\frac{\pi}{2\varepsilon}}} \leq \sup_{(u,\theta_1,\theta_2)\in(-u_0,u_0)\times\mathbb{T}^2} |\mathcal{M}\left(u, \theta_1, \theta_2; \varepsilon, \alpha\right)| \leq \frac{C_2}{\varepsilon} e^{-c(\ln(2\varepsilon/\pi))\sqrt{\frac{\pi}{2\varepsilon}}} \tag{37}$$

for certain constants $u_0, C_1, C_2 > 0$.

- *In the intermediate case $\alpha = \alpha_* \varepsilon + \mathcal{O}(\varepsilon^2)$, one can only give upper bounds of type*

$$\sup_{(u,\theta_1,\theta_2)\in(-u_0,u_0)\times\mathbb{T}^2} |\mathcal{M}\left(u, \theta_1, \theta_2; \varepsilon, \alpha\right)| \leq \frac{C_2}{\varepsilon} e^{-c(\ln(2\varepsilon/\pi))\sqrt{\frac{\pi}{2\varepsilon}}}$$

for certain constant $u_0, C_2 > 0$.

- *If α satisfies $C\varepsilon^\nu < \alpha < 1 - C\varepsilon^r$ where $\nu \in (0,1]$, $r \in [0,2)$ and $C > 0$ are constants independent of ε, the Melnikov function satisfies*

$$\frac{C_1}{\sqrt{\varepsilon\alpha}(1-\alpha)^{5/4}} e^{-c(\ln(\varepsilon/\operatorname{Im}\rho_-))\sqrt{\frac{\operatorname{Im}\rho_-}{\varepsilon}}} \leq \sup_{(u,\theta_1,\theta_2)\in(-u_0,u_0)\times\mathbb{T}^2} |\mathcal{M}(u,\theta_1,\theta_2;\varepsilon,\alpha)|$$
$$\leq \frac{C_2}{\sqrt{\varepsilon\alpha}(1-\alpha)^{5/4}} e^{-c(\ln(\varepsilon/\operatorname{Im}\rho_-))\sqrt{\frac{\operatorname{Im}\rho_-}{\varepsilon}}} \quad (38)$$

for certain constants $C_1, C_2 > 0$.

Proof. As in the periodic case, to prove the first statement, instead of computing the residuums of both $\rho_\pm$ and look for cancellations, we just expand the Melnikov integral, which is uniformly convergent as a real integral, in power series of α as

$$\mathcal{M}(u,\theta_1,\theta_2;\varepsilon,\alpha) = \sum_{n=0}^\infty \alpha^n \mathcal{M}_n(u,\theta_1,\theta_2;\varepsilon), \quad (39)$$

where

$$\mathcal{M}_n(u,\theta_1,\theta_2;\varepsilon) = 4(n+1)2^n \int_{-\infty}^{+\infty} \frac{\sinh^{n+1}(u+s)}{\cosh^{2n+3}(u+s)} F\left(\theta_1 + \frac{s}{\varepsilon}, \theta_2 + \frac{\gamma s}{\varepsilon}\right) ds.$$

The function $\mathcal{M}_0$ was computed in [DGJS97], and in that paper it was shown that it can be bounded as formula (37). The rest of the functions $\mathcal{M}_n$ can be bounded as follows. We Fourier-expand $\mathcal{M}_n$ in (θ_1,θ_2) and we change the path of integration to $\operatorname{Im} s = \pi/2 - \sqrt{\varepsilon}$ or $\operatorname{Im} s = -(\pi/2 - \sqrt{\varepsilon})$ depending on the sign of $k \cdot \omega = k_1 + \gamma k_2$. Then, we can bound each harmonic as

$$\left|\mathcal{M}_n^{[k]}(u;\varepsilon)\right| \leq \left(\frac{K}{\varepsilon}\right)^{n+1} \left|F^{[k]}\right| e^{-\frac{|k\cdot\omega|}{\varepsilon}\left(\frac{\pi}{2} - \sqrt{\varepsilon}\right)}.$$

Therefore, proceeding as in [DGJS97], one can see that $\mathcal{M}_n(u,\theta_1,\theta_2;\varepsilon)$ can be bounded as

$$|\mathcal{M}_n(u,\theta_1,\theta_2;\varepsilon)| \leq \left(\frac{K}{\varepsilon}\right)^{n+1} e^{-c(\ln(2\varepsilon/\pi))\sqrt{\frac{\pi}{2\varepsilon}}},$$

where the function c is given in (36).

Therefore, if $\varepsilon \ll \alpha$, the series (39) is decreasing, and therefore, the leading term is given by the first order.

For the second and third statements, we just apply the Residuum Theory to the Fourier harmonics of the Melnikov function (35). In can be easily seen that they have size

$$\left|\mathcal{M}^{[k]}(u)\right| = \left|F^{[k]}\right| \left|\frac{k\cdot\omega}{\varepsilon}\delta_2(\alpha) + \delta_1(\alpha)\right| e^{-\frac{|k\cdot\omega|\operatorname{Im}\rho_-}{\varepsilon}}, \quad (40)$$

where $\delta_1(\alpha)$ and $\delta_2(\alpha)$ are the functions defined in (27) and (28) respectively. For the the range of α we are considering, the leading term is δ_2, which satisfies

$$\delta_2(\alpha) \sim \frac{1}{\sqrt{\alpha}(1-\alpha)}.$$

Finally one has to proceed as in [DGJS97]. Even if the method in that paper does not apply directly since now ρ_- depends on ε, it is enough to consider as a new parameter $q = \varepsilon/\operatorname{Im}\rho_-$, which is still small since we are assuming $C\varepsilon^\nu < \alpha < 1 - C\varepsilon^r$ where $\nu \in (0,1]$, $r \in [0,2)$ and $C > 0$ and therefore $\operatorname{Im}\rho_- \gg \varepsilon$. With this new parameter q, it is straightforward to obtain the size of the Melnikov function with the techniques in [DGJS97]. $\square$

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Remark 3.2. *Note that, as in the periodic case, the size of the Melnikov function depends strongly on α. When the strip of analyticity (6) is wide enough, that is $\alpha \sim \varepsilon^\nu$ with $\nu > 1$, the Melnikov function behaves as in the entire case $\alpha = 0$ [DGJS97]. In other words, the Melnikov function does not notice the finiteness of the strip of analyticity and the exponentially small coefficient is given by the imaginary part of the singularities of the separatrix. Nevertheless, note that the condition is different. In the periodic case was needed $\alpha \ll \varepsilon^2$ instead of $\alpha \ll \varepsilon$.*

When the strip of analyticity is independent of ε or not extremely big with respect to ε, namely taking $\alpha \sim \varepsilon^\nu$ with $\nu \in [0,1)$, the exponentially small coefficient multiplying the periodic function $c(\delta)$ does not coincide with the imaginary part of the singularity of the unperturbed separatrix. Instead, it is given by the imaginary part of this new singularity ρ_-, which appears when one evaluates the perturbation along the unperturbed separatrix. Note that even if one takes, for instance, $\alpha = \sqrt{\varepsilon}$, which gives a strip of analyticity of order $\frac{1}{2}\ln\frac{1}{\varepsilon}$, one has that, using (21),

$$\mathcal{M}\left(u,\theta_1,\theta_2;\varepsilon,\sqrt{\varepsilon}\right) \sim \varepsilon^{-\frac{3}{4}} e^{-c(\ln(\varepsilon/\operatorname{Im}\rho_-))\sqrt{\frac{\pi}{2\varepsilon}}\left(1-\frac{\varepsilon^{1/4}}{\pi}\right)}.$$

That is, even for perturbations with a wide strip of analyticity with respect to ε, there appears a correcting term in the exponential.

Notice that the case $\alpha \sim \varepsilon$ is the boundary between these two different behaviors.

3.1.1 Narrow strip of analyticity: a drastic change on the size of the Melnikov function

In this section, we study how the Melnikov increases when the analyticity strip shrinks, namely when $\alpha = 1 - C\varepsilon^r$ with $r \geq 0$ and $C > 0$. Next corollary gives upper and lower bounds for it in this case.

Corollary 3.3. *Let us assume (10) and (11) and fix $u_0 > 0$. Then, if one takes $\alpha = 1 - C\varepsilon^r$ with $C > 0$ and $r > 0$, the Melnikov function in (38) has the following upper and lower bounds*

- *If $0 < r < 2$,*

$$\frac{C_1}{\sqrt{\alpha}\varepsilon^{1/2+5r/4}} e^{-c(\ln(\varepsilon/\operatorname{Im}\rho_-))\sqrt{\frac{\operatorname{Im}\rho_-}{\varepsilon}}} \leq \sup_{(u,\theta_1,\theta_2)\in(-u_0,u_0)\times\mathbb{T}^2} |\mathcal{M}\left(u,\theta_1,\theta_2;\varepsilon,\alpha\right)|$$

$$\leq \frac{C_2}{\sqrt{\alpha}\varepsilon^{1/2+5r/4}} e^{-c(\ln(\varepsilon/\operatorname{Im}\rho_-))\sqrt{\frac{\operatorname{Im}\rho_-}{\varepsilon}}} \quad (41)$$

- *If $r \geq 2$,*

$$\frac{C_1}{\varepsilon^{3r/2}} \leq \sup_{(u,\theta_1,\theta_2)\in(-u_0,u_0)\times\mathbb{T}^2} |\mathcal{M}\left(u,\theta_1,\theta_2;\varepsilon,1-C\varepsilon^r\right)| \leq \frac{C_2}{\varepsilon^{3r/2}}$$

for certain constants $C_1, C_2 > 0$.

The first statement of this corollary is just a rewriting of the third statement of Proposition 3.1. The second one is a direct consequence of formula (40), if one takes into account the definition of δ_1 and δ_2 in (27) and (28) and the asymptotics for ρ_- in (22).

Remark 3.4. *When the strip of analyticity shrinks, which ocurrs when α approaches 1, the Melnikov function increases exponentially. For instance, if one takes $\alpha = 1 - \varepsilon$, using (22),*

$$\mathcal{M}\left(u,\theta_1,\theta_2;\varepsilon,1-\varepsilon\right) \sim \varepsilon^{-\frac{7}{4}} e^{-\frac{c(\ln(\varepsilon/\operatorname{Im}\rho_-))}{\varepsilon^{1/4}}}.$$

The exponentially small Melnikov function keeps increasing until the limiting case $\alpha \sim 1 - \varepsilon^2$ when it becomes non-exponentially small.

3.2 Validity of the Melnikov function

Once we have computed the Melnikov function in Proposition 3.1, we compute the prediction it gives for the distance between the manifolds at the section $x = \pi$ for $0 < \alpha \le \alpha_0 < 1$ for any fixed α_0

If α satisfies $0 \le \alpha \le C\varepsilon^\nu$ where $\nu > 1$ and $C > 0$ are constants independent of ε, we can ensure that d_0 satisfies

$$C_1|\mu|\varepsilon^{\eta-1}e^{-c(\ln(2\varepsilon/\pi))\sqrt{\frac{\pi}{2\varepsilon}}} \le \max_{(\theta_1,\theta_2)\in\mathbb{T}^2}|d_0(\theta_1,\theta_2)| \le C_2|\mu|\varepsilon^{\eta-1}e^{-c(\ln(2\varepsilon/\pi))\sqrt{\frac{\pi}{2\varepsilon}}}. \tag{42}$$

If $\alpha = \alpha_*\varepsilon + \mathcal{O}(\varepsilon^2)$ for any constant $\alpha_* > 0$, d_0 satisfies

$$\max_{(\theta_1,\theta_2)\in\mathbb{T}^2}|d_0(\theta_1,\theta_2)| \le C_2|\mu|\varepsilon^{\eta-1}e^{-c(\ln(2\varepsilon/\pi))\sqrt{\frac{\pi}{2\varepsilon}}}. \tag{43}$$

Finally, if $C\varepsilon^\nu < \alpha \le \alpha_0 < 1$ where $\nu \in (0,2)$ and $C > 0$ are constants independent of ε, d_0 satisfies

$$\frac{C_1|\mu|\varepsilon^{\eta-\frac{1}{2}}}{\sqrt{\alpha}}e^{-c(\ln(\varepsilon/\operatorname{Im}\rho_-))\sqrt{\frac{\operatorname{Im}\rho_-}{\varepsilon}}} \le \max_{(\theta_1,\theta_2)\in\mathbb{T}^2}|d_0(\theta_1,\theta_2)|$$

$$\le \frac{C_2|\mu|\varepsilon^{\eta-\frac{1}{2}}}{\sqrt{\alpha}}e^{-c(\ln(\varepsilon/\operatorname{Im}\rho_-))\sqrt{\frac{\operatorname{Im}\rho_-}{\varepsilon}}}. \tag{44}$$

In all three formulas $C_1, C_2 > 0$ are constants independent of α and ε.

As we have already explained, direct application of Melnikov theory only ensures that the distance between the invariant manifolds on the section $x = \pi$ is given by

$$d(\theta_1,\theta_2) = d_0(\theta_1,\theta_2) + \mathcal{O}\left(\mu^2\varepsilon^{2\eta}\right).$$

Therefore, the Melnikov function (35), is, in principle, the first order provided μ is exponentially small with respect to ε. Next theorem shows for which range of parameters α, ε and η, the Melnikov function gives the true first order. In the quasiperiodic case, it is not known whether this range is the optimal one for which the Melnikov function predicts the splitting correctly.

Theorem 3.5. *Let us consider any $\mu_0 > 0$ and $\alpha_0 \in (0,1)$. Then, there exists $\varepsilon_0 > 0$ such that for $|\mu| < \mu_0$, $\varepsilon \in (0,\varepsilon_0)$ and $\alpha \in [0,\alpha_0]$ such that $\varepsilon^{\eta-1}(\sqrt{\varepsilon} + \sqrt{\alpha})$ is small enough,*

- *If α satisfies $0 \le \alpha \le C\varepsilon^\nu$ where $\nu \ge 1$ and $C > 0$ are constants independent of ε, the distance between the invariant manifolds on the section $x = \pi$ is given by*

$$d(\theta_1,\theta_2) = d_0(\theta_1,\theta_2) + \mathcal{O}\left(|\mu|^2\varepsilon^{2\eta-2}e^{-c(\ln(\varepsilon/\operatorname{Im}\rho_-))\sqrt{\frac{\operatorname{Im}\rho_-}{\varepsilon}}}\right).$$

- *If α satisfies $C\varepsilon^\nu \le \alpha \le \alpha_0 < 1$ where $\nu \in [0,1)$ and $C > 0$ are constants independent of ε, the distance between the invariant manifolds on the section $x = \pi$ is given by*

$$d(\theta_1,\theta_2) = d_0(\theta_1,\theta_2) + \mathcal{O}\left(|\mu|^2\varepsilon^{2\eta-2}e^{-c(\ln(\varepsilon/\operatorname{Im}\rho_-))\sqrt{\frac{\operatorname{Im}\rho_-}{\varepsilon}}}\right).$$

The proof of this theorem is deferred to Section 5.

Remark 3.6. *Note that even if this theorem holds true provided $\varepsilon^{\eta-1}(\sqrt{\varepsilon} + \sqrt{\alpha})$ is small enough, the Melnikov prediction is the true first order only if one imposes a more restrictive condition.*

Comparing the size of the Melnikov prediction (42) and the size of the remainder, in the case $\alpha \ll \varepsilon$, one has to impose the condition $\eta > 1$. Let us observe that when $\alpha = 0$, this theorem, which gives that Melnikov provides the true first order of the distance if $\eta > 1$, is an improvement with respect to the sharpest previous result [DGJS97], which needed the condition $\eta > 3$ (see also [Sau01]).

In the case $\alpha \gg \varepsilon$ one has to impose $\varepsilon^{\eta-\frac{3}{2}}\sqrt{\alpha} \ll 1$.

Finally, note that the intermediate case $\alpha = \alpha_* \varepsilon + \mathcal{O}(\varepsilon^2)$ is included in the first statement of the theorem. Nevertheless, if one would check that d_0 is nondegenerate, Melinkov theory would give also in this case the correct prediction.

3.2.1 Narrow strip of analyticity: validity of the Melnikov function

In Section 3.1.1 we have studied the size of the Melnikov function when $\alpha = 1 - C\varepsilon^r$ with $C, r > 0$. We devote this section to show for which range of parameters the Melnikov function gives the correct first order of the splitting.

First, we give the prediction of the distance given by the Melnikov function if $\alpha = 1 - C\varepsilon^r$ with $r > 0$ at the section $x = 3\pi/2$. If $0 < r < 2$,

$$C_1 |\mu| \varepsilon^{\eta - \frac{1}{2} - \frac{5r}{4}} e^{-c(\ln(\varepsilon/\operatorname{Im}\rho_-))\sqrt{\frac{\operatorname{Im}\rho_-}{\varepsilon}}} \le \max_{(\theta_1,\theta_2)\in\mathbb{T}^2} \left|\widetilde{d}_0(\theta_1,\theta_2)\right|$$

$$\le C_2 |\mu| \varepsilon^{\eta - \frac{1}{2} - \frac{5r}{4}} e^{-c(\ln(\varepsilon/\operatorname{Im}\rho_-))\sqrt{\frac{\operatorname{Im}\rho_-}{\varepsilon}}} \quad (45)$$

whereas if $r > 2$

$$C_1 |\mu| \varepsilon^{\eta - 3r/2} \le \max_{(\theta_1,\theta_2)\in\mathbb{T}^2} \left|\widetilde{d}_0(\theta_1,\theta_2)\right| \le C_2 |\mu| \varepsilon^{\eta-3r/2}. \quad (46)$$

Next theorem proves the validity of this prediction under certain hypotheses.

Theorem 3.7. *Let us consider any $\mu_0 > 0$ and $\alpha = 1 - C\varepsilon^r$ with $r > 0$, $C > 0$, and let us assume $\eta > \max\{r + 1, 3r/2\}$. Then, there exists $\varepsilon_0 > 0$ such that for $|\mu| < \mu_0$ and $\varepsilon \in (0, \varepsilon_0)$, the invariant manifolds split and the distance between them on the section $x = 3\pi/2$ is given by,*

- *If $0 < r < 2$,*

$$\widetilde{d}(\theta_1,\theta_2) = \widetilde{d}_0(\theta_1,\theta_2) + \mathcal{O}\left(|\mu|^2 \varepsilon^{2\eta-2r-2} e^{-c(\ln(\varepsilon/\operatorname{Im}\rho_-))\sqrt{\frac{\operatorname{Im}\rho_-}{\varepsilon}}}\right).$$

- *If $r > 2$,*

$$\widetilde{d}(\theta_1,\theta_2) = \widetilde{d}_0(\theta_1,\theta_2) + \mathcal{O}\left(\mu\varepsilon^{2\eta-3r}\right).$$

4 The periodic case: Proof of Theorem 2.8 and 2.10

In Section 4.1 we prove Theorem 2.8, that is when the parameter α is bounded away from 1. Nevertheless, we prove the result for a wider range of the parameter α. Therefore, our proof deals also with the first statement of Theorem 2.10. For this reason, during Section 4.1 we assume the condition

$$0 \le \alpha \ll 1 - \varepsilon^2. \quad (47)$$

Note, that this range of parameters correspond to an exponentially small Melnikov function. The second statement of Theorem 2.10 is proved in Section 4.2

4.1 Proof of Theorem 2.8

To study the splitting of separatrices we follow the approach proposed in [LMS03, Sau01, GOS10], which was inspired by Poincaré. Namely we use the fact that the invariant manifolds are Lagrangian graphs. This allows us to look for parameterizations of the invariant manifolds as graphs of the gradient of certain generating functions.

First, let us point out that we will consider τ as a complex variable to take advantage of the analyticity of the Hamiltonian with respect to this variable. To this end, we consider a fixed $\sigma > 0$ and we take

$$\tau \in \mathbb{T}_\sigma = \{\tau \in \mathbb{C}/\mathbb{Z} : |\operatorname{Im}\tau| < \sigma\}.$$

Note that since the Hamiltonian is entire in τ, we can take any σ. From now on, all the constants appearing in this section will depend on σ. As we will see during the proof, in the periodic case, the analyticity strip with respect to τ does not play any role in the size of the splitting.

Let us consider consider the symplectic change of variables

$$\begin{cases} x = x_0(u) = 4\arctan(e^u) \\ y = \dfrac{w}{y_0(u)} = \dfrac{\cosh u}{2}w, \end{cases} \tag{48}$$

which was introduced in [Bal06] (see also [LMS03, Sau01]) and reparameterize time as $\tau = t/\varepsilon$. With these new variables, the Hamiltonian function (5) reads

$$\varepsilon\overline{H}(u,w,\tau) = \varepsilon H\left(x_0(u), \frac{w}{y_0(u)}, \tau\right) \tag{49}$$

and the unperturbed separatrix can be parameterized as a graph as

$$w = y_0^2(u) = \frac{4}{\cosh^2 u}.$$

Then, one can look for the perturbed invariant manifolds as graphs of the gradient of generating functions $T^{u,s}(u,\tau)$. Namely, we look for functions $T^{u,s}(u,\tau)$ such that the invariant manifolds are given by $w = \partial_u T^{u,s}(u,\tau)$. Moreover, these functions are solutions of the Hamilton-Jacobi equation

$$\varepsilon^{-1}\partial_\tau T + \overline{H}(u, \partial_u T, \tau) = 0. \tag{50}$$

This equation reads

$$\varepsilon^{-1}\partial_\tau T + \frac{\cosh^2 u}{8}(\partial_u T)^2 - \frac{4}{\cosh^2 u} + \mu\varepsilon^\eta\Psi(u)\sin\tau = 0 \tag{51}$$

with

$$\Psi(u) = \psi(x_0(u)), \tag{52}$$

where ψ is the function considered in (5) and $x_0(u)$ is the first component of the time-parameterization of the separatrix (see (48)).

Moreover, we impose the asymptotic conditions

$$\lim_{\operatorname{Re}u\to-\infty} y_0^{-1}(u)\cdot\partial_u T^u(u,\tau) = 0 \quad \text{(for the unstable manifold)} \tag{53}$$

$$\lim_{\operatorname{Re}u\to+\infty} y_0^{-1}(u)\cdot\partial_u T^s(u,\tau) = 0 \quad \text{(for the stable manifold).} \tag{54}$$

One can easily see that for $\mu = 0$, the solution of equation (51) is just

$$T_0(u) = 4\frac{e^u}{\cosh u}, \tag{55}$$

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which is the generating function of the unperturbed separatrix and has singularities at $u = \pm i\pi/2$. Nevertheless, as we have explained in Section 2, the term $\Psi(u)$ has singularities at $u = \rho_\pm, \overline{\rho}_\pm$. In particular, $u = \rho_-, \overline{\rho}_-$ are closer to the real axis than $u = \pm i\pi/2$, $u = \rho_+$ and $u = \overline{\rho}_+$. Therefore, to study the exponentially small splitting of separatrices for this kind of systems, one has to look for parameterizations of the invariant manifolds, namely solutions of equation (51), in complex domains up to a distance of order $\mathcal{O}(\varepsilon)$ of the singularities $u = \rho_-, \overline{\rho}_-$. To this end, we define the domains

$$
\begin{aligned}
D_\kappa^u &= \{u \in \mathbb{C}; |\mathrm{Im}\, u| < -\tan\beta_1\,(\mathrm{Re}\, u - \mathrm{Re}\, \rho_-) + \mathrm{Im}\, \rho_- - \kappa\varepsilon\} \\
D_\kappa^s &= \{u \in \mathbb{C}; |\mathrm{Im}\, u| < \tan\beta_1\,(\mathrm{Re}\, u - \mathrm{Re}\, \rho_-) + \mathrm{Im}\, \rho_- - \kappa\varepsilon\},
\end{aligned}
\tag{56}
$$

where $\beta_1 > 0$ is an angle independent of ε and $\kappa > 0$ is such that $\mathrm{Im}\, \rho_- - \kappa\varepsilon > 0$ (see Figure 3).
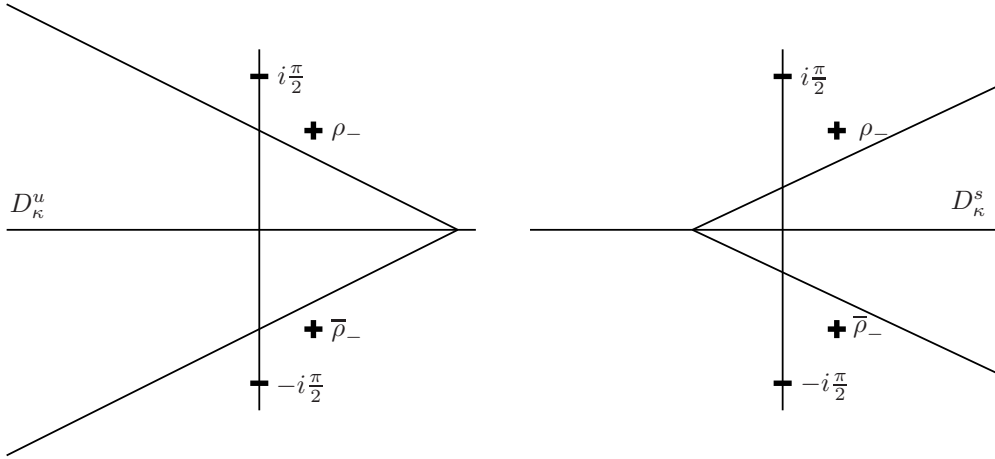

Figure 3: The domains D_κ^u and D_κ^s defined in (56).

Note that this is a significant difference in comparison with the papers dealing with the exponentially small splitting of separatrices problem for algebraic or trigonometric polynomial perturbations (see [DS97, Gel97b, BF04, BFGS11]). In these cases, one needs to look for the parameterizations up to a distance of order $\mathcal{O}(\varepsilon)$ of the singularities of the unperturbed separatrix. Nevertheless, in this paper, since the perturbation changes the location of the singularities of the perturbed invariant manifolds so drastically, one has to study the invariant manifolds close to these new singularities.

Let us recall that when α is close to 1, namely, when the analyticity strip of the Hamiltonian is very narrow, the singularity ρ_- is given by

$$
\rho_- = \ln(1 + \sqrt{2}) + i(1 - \alpha)^{1/2} + \mathcal{O}(1 - \alpha).
$$

Therefore, $\mathrm{Im}\, \rho_- \ll \varepsilon$ if $1 - \alpha \ll \varepsilon^2$. In this case, as the Melnikov function is not exponentially small (see Corollary 2.6), we can use a classical perturbative approach to prove its validity. We leave this easier yet surprising case to Section 4.2. Thus, from now on, we assume the condition (47) and we proceed to prove Theorem 2.8. This condition ensures that the intersection $D_\kappa^u \cap D^s\kappa \cap \mathbb{R}$ contains a fundamental domain, since it is of size $\mathcal{O}(\sqrt{1 - \alpha})$. Note that if α is bounded away from 1, one can choose the angle β_1 so that the domain $D_\kappa^u \cap D^s\kappa \cap \mathbb{R}$ contains the point $u = 0$ (which corresponds to the section $x = \pi$). If $\alpha = 1 - C\varepsilon^r$ with $r \in (0, 2)$, the domain $D_\kappa^u \cap D^s\kappa \cap \mathbb{R}$ contains the point $u = \ln(1 + \sqrt{2})$ (which corresponds to the section $x = 3\pi/2$). One could change slightly the domain so that it would contain $u = 0$ also in this latter case. Nevertheless, in this case one would have to take $\beta_1 \sim \varepsilon^{r/2}$. This would lead to worse estimates which would require a stronger condition on η for the Melnikov function to be the true first order of the splitting.

The next theorem gives the existence of the invariant manifolds in the domains D_κ^* with $* = u, s$ defined in (56). We state the results for the unstable invariant manifold. The stable one has analogous properties.

Theorem 4.1. *Let us fix $\kappa_1 > 0$. Then, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, $\alpha \in (0, 1)$, $\mu \in B(\mu_0)$, if $\frac{\varepsilon + \sqrt{\alpha}}{1 - \alpha} \varepsilon^{\eta - 1}$ is small enough and (47) is satisfied, the Hamilton-Jacobi equation (50) has a unique (modulo an additive constant) real-analytic solution in $D_{\kappa_1}^u \times \mathbb{T}_\sigma$ satisfying the asymptotic condition (53).*

Moreover, there exists a real constant $b_1 > 0$ independent of ε, μ and α, such that for $(u, \tau) \in D_{\kappa_1}^u \times \mathbb{T}_\sigma$,

$$|\partial_u T^u(u, \tau) - \partial_u T_0(u)| \leq \frac{b_1 |\mu| \varepsilon^{\eta - 1}}{(\varepsilon + \sqrt{\alpha})(1 - \alpha)}$$

$$\left| \partial_u^2 T^u(u, \tau) - \partial_u^2 T_0(u) \right| \leq \frac{b_1 |\mu| \varepsilon^{\eta - 2}}{(\varepsilon + \sqrt{\alpha})(1 - \alpha)}.$$

Furthermore, if we define the half Melnikov function

$$\mathcal{M}^u(u, \tau) = -4 \int_{-\infty}^{0} \frac{\sinh(u + s) \cosh(u + s)}{\left(\cosh^2(u + s) - 2\alpha \sinh(u + s) \right)^2} \sin\left(\tau + \frac{s}{\varepsilon} \right) ds, \tag{57}$$

the generating function T^u satisfies that, for $(u, \tau) \in D_{\kappa_1}^u \times \mathbb{T}_\sigma$,

$$|\partial_u T^u(u, \tau) - \partial_u T_0(u) - \mu \varepsilon^\eta \mathcal{M}^u(u, \tau)| \leq b_1 |\mu|^2 \frac{\varepsilon^{2\eta - 2}}{(1 - \alpha)^2}. \tag{58}$$

The proof of this theorem is deferred to Section 4.1.1. The parameterization of the stable manifold has analogous properties. In particular, we can define

$$\mathcal{M}^s(u, \tau) = 4 \int_{0}^{+\infty} \frac{\sinh(u + s) \cosh(u + s)}{\left(\cosh^2(u + s) - 2\alpha \sinh(u + s) \right)^2} \sin\left(\tau + \frac{s}{\varepsilon} \right) ds, \tag{59}$$

and then, for $(u, \tau) \in D_{\kappa_1}^s \times \mathbb{T}_\sigma$,

$$|\partial_u T^s(u, \tau) - \partial_u T_0(u) - \mu \varepsilon^\eta \mathcal{M}^s(u, \tau)| \leq |\mu|^2 \frac{\varepsilon^{2\eta - 2}}{(1 - \alpha)^2}. \tag{60}$$

Note that the Melnikov function defined in (19) is simply

$$\mathcal{M}(u, \tau; \varepsilon, \alpha) = \mathcal{M}^s(u, \tau) - \mathcal{M}^u(u, \tau).$$

From now on, we omit the dependence on ε and α of the Melnikov function $\mathcal{M}$, which we denote by $\mathcal{M}(u, \tau)$.

Once we know the existence of parameterizations of the invariant manifolds, the next step is to study their difference. To this end, we define

$$\Delta(u, \tau) = T^s(u, \tau) - T^u(u, \tau). \tag{61}$$

This function is defined in $R_{\kappa_1} \times \mathbb{T}_\sigma$, where R_κ, for any $\kappa > 0$, is the romboidal domain

$$R_\kappa = D_\kappa^u \cap D_\kappa^s. \tag{62}$$

Subtracting equation (51) for both T^s and T^u, one can easily see that $\Delta \in \operatorname{Ker} \widetilde{\mathcal{L}}_\varepsilon$ for

$$\widetilde{\mathcal{L}}_\varepsilon = \varepsilon^{-1} \partial_\tau + \left(\frac{\cosh^2 u}{8} \left(\partial_u T^s(u, \tau) + \partial_u T^u(u, \tau) \right) \right) \partial_u. \tag{63}$$

Since Theorem 4.1 ensures that the perturbed invariant manifolds are well approximated by the unperturbed separatrix in the domains $D_{\kappa_1}^u$ and $D_{\kappa_1}^s$, we know that the operator $\widetilde{\mathcal{L}}_\varepsilon$ is close to the constant coefficients operator

$$\mathcal{L}_\varepsilon = \varepsilon^{-1} \partial_\tau + \partial_u. \tag{64}$$

As it is well known, any function which is defined for $(u, \tau) \in \{u \in \mathbb{C} : \operatorname{Re} u = a, \operatorname{Im} u \in [-r_0, r_0]\} \times \mathbb{T}_\sigma$ and belongs to the kernel of $\mathcal{L}_\varepsilon$, it is defined in the strip $\{|\operatorname{Im} u| < r_0\} \times \mathbb{T}_\sigma$ and has exponentially small bounds for real values of the variables. This fact is summarized in the next lemma, whose proof follows the same lines as the one of the slightly different Lemma 3.10 of [GOS10].

Lemma 4.2. *Let us consider a function $\zeta(u,\tau)$ analytic in $(u,\tau) \in \{u \in \mathbb{C} : \operatorname{Re} u = a, \operatorname{Im} u \in (-r_0, r_0)\} \times \mathbb{T}_\sigma$ which is solution of $\mathcal{L}_\varepsilon \zeta = 0$. Then, ζ can be extended analytically to $\{|\operatorname{Im} u| < r_0\} \times \mathbb{T}_\sigma$ and its mean value*

$$\langle \zeta \rangle = \frac{1}{2\pi} \int_0^{2\pi} \zeta(u,\tau) d\tau$$

does not depend on u. Moreover, for $r \in (0, r_0)$ and $\sigma' \in (0, \sigma)$, we define

$$M_r = \max_{(u,\tau) \in [-ir, ir] \times \overline{\mathbb{T}}_{\sigma'}} |\partial_u \zeta(u,\tau)|. \tag{65}$$

Then, provided ε is small enough, for $(u,\tau) \in \mathbb{R} \times \mathbb{T}_{\sigma'}$ the following inequality holds

$$|\partial_u \zeta(u,\tau)| \leq 4 M_r e^{-\frac{r}{\varepsilon}}.$$

To apply this lemma to study the difference between the invariant manifolds, we follow [Sau01] (see also [GOS10]). Namely, we look for a change of variables which conjugates $\widetilde{\mathcal{L}}_\varepsilon$ in (63) with $\mathcal{L}_\varepsilon$ in (64).

Theorem 4.3. *Let us consider the constant κ_1 defined in Theorem 4.1 and let us fix any $\kappa_3 > \kappa_2 > \kappa_1$. Then, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in (0,1)$ satisfying (47) and that $\varepsilon^{\eta-1} \frac{\varepsilon + \sqrt{\alpha}}{1-\alpha}$ is small enough, there exists a real-analytic function $\mathcal{C}$ defined in $R_{\kappa_2} \times \mathbb{T}_\sigma$ such that the change*

$$(u,\tau) = (v + \mathcal{C}(v,\tau), \tau) \tag{66}$$

conjugates the operators $\widetilde{\mathcal{L}}_\varepsilon$ and $\mathcal{L}_\varepsilon$ defined in (63) and (64) respectively. Moreover, for $(v,\tau) \in R_{\kappa_2} \times \mathbb{T}_\sigma$, $v + \mathcal{C}(v,\tau) \in R_{\kappa_1}$ and there exists a constant $b_2 > 0$ such that

$$|\mathcal{C}(v,\tau)| \leq b_2 |\mu| \varepsilon^\eta \frac{\varepsilon + \sqrt{\alpha}}{1-\alpha}$$

$$|\partial_v \mathcal{C}(v,\tau)| \leq b_2 |\mu| \varepsilon^{\eta-1} \frac{\varepsilon + \sqrt{\alpha}}{1-\alpha}.$$

Furthermore, $(u,\tau) = (v + \mathcal{C}(v,\tau), \tau)$ is invertible and its inverse is of the form $(v,\tau) = (u + \mathcal{V}(u,\tau), \tau)$ where $\mathcal{V}$ is a function defined for $(u,\tau) \in R_{\kappa_3} \times \mathbb{T}_\sigma$, which satisfies

$$|\mathcal{V}(u,\tau)| \leq b_2 |\mu| \varepsilon^\eta \frac{\varepsilon + \sqrt{\alpha}}{1-\alpha}$$

and that $u + \mathcal{V}(u,\tau) \in R_{\kappa_2}$ for $(u,\tau) \in R_{\kappa_3} \times \mathbb{T}_\sigma$.

The proof of this theorem is deferred to Section 4.1.2.

Once we have obtained the change of variables $\mathcal{C}$, we are able to prove the validity of the Melnikov prediction. We prove it by bounding $\partial_u \Delta(u,\tau) - \mathcal{M}(u,\tau)$, where Δ and $\mathcal{M}$ are the functions defined in (61) and (19) respectively.

As a first step, we consider

$$\partial_v \left(\Delta(v + \mathcal{C}(v,\tau), \tau) \right) - \mathcal{M}(v,\tau)$$

where $\mathcal{C}$ is the function obtained in Theorem 4.3.

Theorem 4.4. *There exists $\varepsilon_0 > 0$ and $b_3 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in (0,1)$ satisfying (47) and that $\varepsilon^{\eta-1} \frac{\varepsilon + \sqrt{\alpha}}{1-\alpha}$ is small enough, the following bound is satisfied*

$$|\partial_v \left(\Delta(v + \mathcal{C}(v,\tau), \tau) \right) - \mathcal{M}(v,\tau)| \leq b_3 |\mu|^2 \frac{\varepsilon^{2\eta-2}}{(1-\alpha)^2} e^{-\frac{\operatorname{Im} \rho_-}{\varepsilon}}$$

for $v \in R_{\kappa_2} \cap \mathbb{R}$ and $\tau \in \mathbb{T}$,

Proof. First, we define the Melnikov potential L (see [DG00]), namely a function such that $\partial_v L(v, \tau) = \mathcal{M}(v, \tau)$, and we study $\Delta(v + \mathcal{C}(v, \tau), \tau) - L(v, \tau)$.

As $\Delta \in \mathrm{Ker}\widetilde{\mathcal{L}}_\varepsilon$, where $\widetilde{\mathcal{L}}_\varepsilon$ is the operator in (63), by Theorem 4.3, the function $\Phi(v, \tau) = \Delta(v + \mathcal{C}(v, \tau), \tau) - L(v, \tau) \in \mathrm{Ker}\mathcal{L}_\varepsilon$, where $\mathcal{L}_\varepsilon$ is the operator defined in (64). Therefore we can apply Lemma 4.2. To this end, we have to bound $\partial_v \Phi(v, \tau)$ in the domain $R_{\kappa_2} \times \mathbb{T}_\sigma$. We split Φ as $\Phi(v, \tau) = \Phi_1(v, \tau) + \Phi_2(v, \tau)$ where

$$\Phi_1(v, \tau) = \Delta\left(v + \mathcal{C}(v, \tau), \tau\right) - \Delta(v, \tau)$$
$$\Phi_2(v, \tau) = \Delta(v, \tau) - L(v, \tau).$$

where Δ is the functions defined in (61) and L is the Melnikov potential.

To bound $\partial_v \Phi_1$, one has to take into account that $\Delta = (T^s - T_0) - (T^u - T_0)$ and therefore, it is enough to consider the bounds obtained in Theorems 4.1 and 4.3 to obtain

$$|\partial_v \Phi_1(v, \tau)| \leq K|\mu|\varepsilon^{2\eta - 2} \frac{1}{(1 - \alpha)^2}$$

for $(v, \tau) \in R_{\kappa_2} \times \mathbb{T}_\sigma$. For the second term, it is enough to use bounds (58) and (60) to obtain

$$|\partial_v \Phi_2(v, \tau)| \leq K|\mu|\varepsilon^{2\eta - 2} \frac{1}{(1 - \alpha)^2}$$

for $(v, \tau) \in R_{\kappa_2} \times \mathbb{T}_\sigma$.

Therefore, we have that

$$|\partial_v \Phi(v, \tau)| \leq K|\mu|\varepsilon^{2\eta - 2} \frac{1}{(1 - \alpha)^2}$$

for $(v, \tau) \in R_{\kappa_2} \times \mathbb{T}_\sigma$ and then, it is enough to apply Lemma 4.2 to finish the proof of Theorem 4.4. $\square$

From this result and the exponential smallness of $\mathcal{M}$, given in Proposition 2.1, and considering the inverse change $(v, \tau) = (u + \mathcal{V}(u, \tau), \tau)$ obtained in Theorem 4.3, it is straightforward to obtain an exponentially small bound for $\partial_u \Delta(u, \tau) - \mathcal{M}(u, \tau)$. It is stated in the next corollary, whose proof is straightforward.

Corollary 4.5. *There exists $\varepsilon_0 > 0$ and $b_4 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in (0, 1)$ satisfying (47) and that $\varepsilon^{\eta - 1}\frac{\varepsilon + \sqrt{\alpha}}{1 - \alpha}$ is small enough, the following bound is satisfied*

$$|\partial_u \Delta(u, \tau) - \mathcal{M}(u, \tau)| \leq b_4|\mu|^2 \frac{\varepsilon^{2\eta - 2}}{(1 - \alpha)^2} e^{-\frac{\mathrm{Im}\,\rho_-}{\varepsilon}}$$

for $u \in R_{\kappa_3} \cap \mathbb{R}$ and $\tau \in \mathbb{T}$.

From this corollary and using that, by Proposition 2.1 and Corollary 2.2, the Melnikov function has non-degenerate zeros, one can see that if α is bounded away from 1 and satisfies that $\varepsilon^{\eta - 1}(\varepsilon + \sqrt{\alpha})$ is small enough, the manifolds intersect transversally and their distance satisfy the desired asymptotic formula. This finishes the proof of Theorem 2.8 (see [BFGS11]).

4.1.1 The invariant manifolds: Proof of Theorem 4.1

Since the proof for both invariant manifolds is analogous, we only deal with the unstable case. We look for a solution of equation (51) satisfying the asymptotic condition (53). We look for it as a perturbation of the unperturbed separatrix T_0 in (55) and therefore we work with

$$Q(u, \tau) = T(u, \tau) - T_0(u). \tag{67}$$

Replacing T in equation (51), it is straightforward to see that the equation for Q reads

$$\mathcal{L}_\varepsilon Q = \mathcal{F}(\partial_u Q, u, \tau) \tag{68}$$

where $\mathcal{L}_\varepsilon$ is the operator defined in (64) and

$$\mathcal{F}(h, u, \tau) = -\frac{\cosh^2 u}{8} h^2 - \mu \varepsilon^\eta \Psi(u) \sin \tau. \tag{69}$$

where $\Psi(u)$ is the function defined in (52).

We devote the rest of the section to obtain a solution of equation (68) which is defined in $D_\kappa^u \times \mathbb{T}_\sigma$ and satisfies the asymptotic condition (53).

We start defining a norm for functions defined in the domain D_κ^u with $\kappa > 0$. Since we want to capture their behavior both as $\operatorname{Re} u \to -\infty$, namely its exponential decay, and for u close to $u = ia$, we consider weighted norms with different weights. For this reason we consider $U > 0$ to divide D_κ^u by the vertical line $\operatorname{Re} u = -U$. Then, given $\kappa > 0$ and an analytic function $h : D_\kappa^u \to \mathbb{C}$, we consider

$$
\begin{aligned}
\|h\| = & \sup_{u \in D_\kappa^u \cap \{\operatorname{Re} u < -U\}} \left| e^{-2u} h(u) \right| \\
& + \sup_{u \in D_\kappa^u \cap \{\operatorname{Re} u > -U\}} \left| (u - \rho_-)^2 (u - \rho_+)(u - \overline{\rho}_-)^2 (u - \overline{\rho}_+) h(u) \right|.
\end{aligned}
\tag{70}
$$

For analytic functions $h : D_\kappa^u \times \mathbb{T}_\sigma \to \mathbb{C}$, we consider the corresponding Fourier norm

$$\|h\|_\sigma = \sum_{k \in \mathbb{Z}} \left\| h^{[k]} \right\| e^{|k|\sigma}.$$

and the following Banach space

$$\mathcal{E}_{\kappa,\sigma} = \left\{ h : D_\kappa^u \times \mathbb{T}_\sigma \to \mathbb{C}; \text{real-analytic}, \|h\|_\sigma < \infty \right\}. \tag{71}$$

To obtain the solutions of equation (68), we need to solve an equation of the form $\mathcal{L}_\varepsilon h = g$, where $\mathcal{L}_\varepsilon$ is the differential operator defined in (64). Note that $\mathcal{L}_\varepsilon$ is invertible in $\mathcal{E}_{\kappa,\sigma}$. It turns out that its inverse is $\mathcal{G}_\varepsilon$ defined by

$$\mathcal{G}_\varepsilon(h)(u, \tau) = \int_{-\infty}^0 h(u + s, \tau + \varepsilon^{-1} s) \, ds. \tag{72}$$

Lemma 4.6. *The operator $\mathcal{G}_\varepsilon$ in (72) satisfies the following properties.*

1. *$\mathcal{G}_\varepsilon$ is linear from $\mathcal{E}_{\kappa,\sigma}$ to itself, commutes with ∂_u and satisfies $\mathcal{L}_\varepsilon \circ \mathcal{G}_\varepsilon = \operatorname{Id}$.*

2. *If $h \in \mathcal{E}_{\kappa,\sigma}$, then*
$$\|\mathcal{G}_\varepsilon(h)\|_\sigma \le K \|h\|_\sigma.$$
   Furthermore, if $\langle h \rangle = 0$, then
$$\|\mathcal{G}_\varepsilon(h)\|_\sigma \le K \varepsilon \|h\|_\sigma.$$

3. *If $h \in \mathcal{E}_{\kappa,\sigma}$, then $\partial_u \mathcal{G}_\varepsilon(h) \in \mathcal{E}_{\kappa,\sigma}$ and*
$$\|\partial_u \mathcal{G}_\varepsilon(h)\|_\sigma \le K \|h\|_\sigma.$$

Proof. It follows the same lines as the proof of Lemma 5.5 in [GOS10]. $\square$

Once we have obtained an inverse of the operator $\mathcal{L}_\varepsilon$ defined in (64) we can obtain solutions of equation (68) using a fixed point argument. Then, Theorem 4.1 is a straightforward consequence of the following proposition.

Proposition 4.7. *Let us fix $\kappa_1 > 0$. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in (0, 1)$ satisfying (47) and that $\frac{\varepsilon + \sqrt{\alpha}}{1 - \alpha} \varepsilon^{\eta - 1}$ is small enough, there exists a function Q defined in $D_{\kappa_1}^u \times \mathbb{T}_\sigma$ such that $\partial_u Q \in \mathcal{E}_{\kappa_1, \sigma}$ is a fixed point of the operator*

$$\overline{\mathcal{F}}^u(h) = \partial_u \mathcal{G}_\varepsilon \mathcal{F}(h), \tag{73}$$

where $\mathcal{G}_\varepsilon$ and $\mathcal{F}$ are the operators defined in (72) and (69) respectively. Furthermore, there exists a constant $b_1 > 0$ such that,

$$\|\partial_u Q\|_\sigma \le b_1 |\mu| \varepsilon^{\eta+1},$$
$$\|\partial_u^2 Q\|_\sigma \le b_1 |\mu| \varepsilon^{\eta}.$$

Moreover, if we consider the half Melnikov function defined in (57), we have that

$$\|\partial_u Q - \mu \varepsilon^\eta \mathcal{M}^u\|_\sigma \le K|\mu|^2 \varepsilon^{2\eta} \frac{\varepsilon + \sqrt{\alpha}}{1-\alpha}. \tag{74}$$

Proof. Let us consider $\kappa_0 < \kappa_1$. It is straightforward to see that $\overline{\mathcal{F}}^u$ is well defined from $\mathcal{E}_{\kappa_0,\sigma}$ to itself. We are going to prove that there exists a constant $b_1 > 0$ such that $\overline{\mathcal{F}}^u$ sends $\overline{B}(b_1|\mu|\varepsilon^{\eta+1}) \subset \mathcal{E}_{\kappa_0,\sigma}$ to itself and is contractive there.

Let us first consider $\overline{\mathcal{F}}^u(0)$. From the definition of $\overline{\mathcal{F}}^u$ in (73), the definition of $\mathcal{F}$ in (69) and using Lemma 4.6, we have that

$$\overline{\mathcal{F}}^u(0)(u,\tau) = \partial_u \mathcal{G}_\varepsilon \mathcal{F}(0)(u,\tau) = -\mu \varepsilon^\eta \mathcal{G}_\varepsilon \left(\Psi'(u) \sin\tau \right). \tag{75}$$

To bound it, first let us point out that $\Psi'(u) = \beta(u)$ where $\beta(u)$ is the function defined in (20). We bound each term for $\operatorname{Im} u \ge 0$, the other case is analogous. Using (23) and taking into account that

$$\left| \frac{u - i\pi/2}{u - \rho_+} \right| < 1, \tag{76}$$

one can easily see that $\|\Psi'(u)\| < K$. Then, taking into account that $\sin\tau$ has zero average and applying Lemma 4.6, there exists a constant $b_1 > 0$ such that

$$\left\| \overline{\mathcal{F}}^u(0) \right\|_\sigma \le \frac{b_1}{2} |\mu| \varepsilon^{\eta+1}.$$

To bound the Lipschitz constant, let us consider $h_1, h_2 \in \overline{B}(b_1|\mu|\varepsilon^{\eta+1}) \in \mathcal{E}_{\kappa_0,\sigma}$. To bound $\|\overline{\mathcal{F}}^u(h_2) - \overline{\mathcal{F}}^u(h_1)\|_\sigma$, we need first the following bounds for $u \in D_{\kappa_0}^u$,

$$\left| \frac{u - i\pi/2}{u - i\rho_-} \right| \le 1 + \left| \frac{i\pi/2 - \rho_-}{u - i\rho_-} \right| \le 1 + K \frac{\sqrt{\alpha}}{\varepsilon} \tag{77}$$

and

$$\left| \frac{1}{(u - i\rho_-)(u - i\overline{\rho}_-)^2} \right| \le \frac{K}{\varepsilon(1-\alpha)}, \tag{78}$$

which are a direct consequence of (21) and (22).

Then, it is easy to see that

$$\left\| \overline{\mathcal{F}}^u(h_2) - \overline{\mathcal{F}}^u(h_1) \right\|_\sigma \le \frac{K}{\varepsilon(1-\alpha)} \left(1 + \frac{\sqrt{\alpha}}{\varepsilon} \right) \|h_2 + h_1\|_\sigma \|h_2 - h_1\|_\sigma$$
$$\le K|\mu| \frac{\varepsilon + \sqrt{\alpha}}{1-\alpha} \varepsilon^{\eta-1} \|h_2 - h_1\|_\sigma.$$

Then, using that by hypothesis $\frac{\varepsilon+\sqrt{\alpha}}{1-\alpha} \varepsilon^{\eta-1}$ is small enough,

$$\operatorname{Lip} \overline{\mathcal{F}}^u = K|\mu| \frac{\varepsilon + \sqrt{\alpha}}{1-\alpha} \varepsilon^{\eta-1} < 1/2$$

and therefore $\overline{\mathcal{F}}^u$ is contractive from the ball $\overline{B}(b_1|\mu|\varepsilon^{\eta+1}) \subset \mathcal{E}_{\kappa_0,\sigma}$ into itself, and it has a unique fixed point h^*. Moreover, since it has exponential decay as $\operatorname{Re} u \to -\infty$, we can take

$$Q(u,\tau) = \int_{-\infty}^u h^*(v,\tau) dv.$$

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To obtain the bound for $\partial_u^2 Q$, it is enough to apply Cauchy estimates to the nested domains $D_{\kappa_1}^u \subset D_{\kappa_0}^u$ (see, for instance, [GOS10]), and rename b_1 if necessary.

Finally, to prove (74), it is enough to point out that $\overline{\mathcal{F}}^u(0) = \mu \varepsilon^\eta \mathcal{M}^u$ and consider the obtained bound for the Lipschitz constant. □

4.1.2 Straightening the operator $\widetilde{\mathcal{L}}_\varepsilon$: proof of Theorem 4.3

We devote this section to prove Theorem 4.3. It is a well known fact, see for instance Lemma 6.3 of [GOS10], that looking for a change (66) which conjugates $\widetilde{\mathcal{L}}_\varepsilon$ and $\mathcal{L}_\varepsilon$ defined in (63) and (64) is equivalent to looking for a function $\mathcal{C}$ solution of the equation

$$\mathcal{L}_\varepsilon \mathcal{C}(v,\tau) = \left. \frac{\cosh^2 u}{8} \left(\partial_u T^s(u,\tau) + \partial_u T^u(u,\tau) \right) \right|_{u=v+\mathcal{C}(v,\tau)} - 1.$$

Taking into account the definition of T_0 and Q in (55) and (67), this equation can be written as

$$\mathcal{L}_\varepsilon \mathcal{C} = \mathcal{J}(\mathcal{C}) \tag{79}$$

where

$$\mathcal{J}(h)(v,\tau) = \left. \frac{\cosh^2 u}{8} \left(\partial_u Q^s(u,\tau) + \partial_u Q^u(u,\tau) \right) \right|_{u=v+h(v,\tau)}. \tag{80}$$

To look for a solution of this equation, we start by defining some norms and Banach spaces. Given $n \in \mathbb{N}$ and a function $h : R_\kappa \to \mathbb{C}$, we define

$$\|h\|_n = \sup_{v \in R_\kappa} \left| (v - \rho_-)^n (v - \overline{\rho}_-)^n h(u) \right|. \tag{81}$$

Moreover for analytic functions $h : R_\kappa \times \mathbb{T}_\sigma \to \mathbb{C}$, we define the corresponding Fourier norm

$$\|h\|_{n,\sigma} = \sum_{k \in \mathbb{Z}} \left\| h^{[k]} \right\|_n e^{|k|\sigma}$$

and the Banach space

$$\mathcal{X}_{n,\sigma} = \{ h : R_\kappa \times \mathbb{T}_\sigma \to \mathbb{C}; \text{ real-analytic}, \|h\|_{n,\sigma} < \infty \}.$$

To obtain a solution of equation (79) in the domain R_κ, we need to solve equations of the form $\mathcal{L}_\varepsilon h = g$, where $\mathcal{L}_\varepsilon$ is the operator defined in (64). To find a right-inverse of this operator in $\mathcal{X}_{n,\sigma}$ let us consider u_1 the upper vertex of R_κ and u_0 the left endpoint of R_κ. Then, we define the operator $\widetilde{\mathcal{G}}_\varepsilon$ as

$$\widetilde{\mathcal{G}}_\varepsilon(h)(v,\tau) = \sum_{k \in \mathbb{Z}} \widetilde{\mathcal{G}}_\varepsilon(h)^{[k]}(v) e^{ik\tau}, \tag{82}$$

where its Fourier coefficients are given by

$$\widetilde{\mathcal{G}}_\varepsilon(h)^{[k]}(v) = \int_{-v_1}^v e^{i\frac{k}{\varepsilon}(w-v)} h^{[k]}(w)\, dw \qquad \text{if } k < 0$$

$$\widetilde{\mathcal{G}}_\varepsilon(h)^{[0]}(v) = \int_{v_0}^v h^{[0]}(w)\, dw$$

$$\widetilde{\mathcal{G}}_\varepsilon(h)^{[k]}(v) = -\int_v^{v_1} e^{i\frac{k}{\varepsilon}(w-v)} h^{[k]}(w)\, dw \qquad \text{if } k > 0.$$

The following lemma, which is proved in [GOS10] (see Lemma 8.3 of this paper), gives some properties of this operator.

Lemma 4.8. *The operator $\widetilde{\mathcal{G}}_\varepsilon$ in (82) satisfies the following properties.*

1. If $h \in \mathcal{X}_{n,\sigma}$, then $\widetilde{\mathcal{G}}_\varepsilon(h) \in \mathcal{X}_{n,\sigma}$ and

$$\left\|\widetilde{\mathcal{G}}_\varepsilon(h)\right\|_{n,\sigma} \le K\|h\|_{n,\sigma}.$$

Moreover, if $\langle h \rangle = 0$,

$$\left\|\widetilde{\mathcal{G}}_\varepsilon(h)\right\|_{n,\sigma} \le K\varepsilon\|h\|_{n,\sigma}.$$

2. If $h \in \mathcal{X}_{n,\sigma}$ with $n > 1$, then $\widetilde{\mathcal{G}}_\varepsilon(h) \in \mathcal{X}_{n-1,\sigma}$ and

$$\left\|\widetilde{\mathcal{G}}_\varepsilon(h)\right\|_{n-1,\sigma} \le \frac{K}{\sqrt{1-\alpha}}\|h\|_{n,\sigma}.$$

In next proposition we obtain a solution of equation (79) using a fixed point argument.

Proposition 4.9. *Let us consider the constant $\kappa_1 > 0$ defined in Theorem 4.1 and let us consider any $\kappa_2 > \kappa_1$. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in (0,1)$ satisfying (47) and that $\frac{\varepsilon + \sqrt{\alpha}}{1-\alpha}\varepsilon^{\eta-1}$ is small enough, there exists a function $\mathcal{C} \in \mathcal{X}_{1,\sigma}$ defined in $R_{\kappa_2} \times \mathbb{T}_\sigma$ such that is a fixed point of the operator*

$$\overline{\mathcal{J}}(h) = \widetilde{\mathcal{G}}_\varepsilon \mathcal{J}(h) \tag{83}$$

where $\widetilde{\mathcal{G}}_\varepsilon$ and $\mathcal{J}$ are the operators defined in (82) and (80) respectively. Furthermore, $v + \mathcal{C}(v,\tau) \in R_{\kappa_1}$ for $(v,\tau) \in R_{\kappa_2} \times \mathbb{T}_\sigma$ and there exists a constant $b_2 > 0$ such that

$$\|\mathcal{C}\|_{1,\sigma} \le b_2|\mu|\varepsilon^{\eta+1}\frac{\sqrt{\alpha}+\varepsilon}{(1-\alpha)^{1/2}}$$

$$\|\partial_v\mathcal{C}\|_{1,\sigma} \le b_2|\mu|\varepsilon^{\eta}\frac{\sqrt{\alpha}+\varepsilon}{(1-\alpha)^{1/2}}.$$

Proof. It is straightforward to see that $\overline{\mathcal{J}}$ is well defined from $\mathcal{X}_{1,\sigma}$ to itself. We are going to prove that there exists a constant $b_2 > 0$ such that $\overline{\mathcal{J}}$ sends $\overline{B}(b_2|\mu|\varepsilon^{\eta+1}\frac{\sqrt{\alpha}+\varepsilon}{(1-\alpha)^{1/2}}) \subset \mathcal{X}_{1,\sigma}$ to itself and is contractive there.

Let us first consider $\overline{\mathcal{J}}(0)$. From the definition of $\overline{\mathcal{J}}$ in (83), the definition of $\mathcal{J}$ in (80), we have that

$$\overline{\mathcal{J}}(0)(v,\tau) = \widetilde{\mathcal{G}}_\varepsilon\mathcal{J}(0)(v,\tau) = \widetilde{\mathcal{G}}_\varepsilon\left(\frac{\cosh^2 v}{8}\left(\partial_v Q^s(v,\tau) + \partial_v Q^u(v,\tau)\right)\right).$$

Using that $\partial_v Q^u$ is a fixed point of the operator $\overline{\mathcal{F}}^u$ in (73) and that $\partial_v Q^s$ is a fixed point of an analogous operator $\overline{\mathcal{F}}^s$, we can split $\overline{\mathcal{J}}(0)(v,\tau) = \mathcal{B}_1(v,\tau) + \mathcal{B}_2(v,\tau)$ with

$$\mathcal{B}_1(v,\tau) = \widetilde{\mathcal{G}}_\varepsilon\left(\frac{\cosh^2 v}{8}\left(\overline{\mathcal{F}}^u(0)(v,\tau) + \overline{\mathcal{F}}^s(0)(v,\tau)\right)\right)$$

$$\mathcal{B}_2(v,\tau) = \widetilde{\mathcal{G}}_\varepsilon\left(\frac{\cosh^2 v}{8}\left(\overline{\mathcal{F}}^u(\partial_v Q^u)(v,\tau) - \overline{\mathcal{F}}^u(0)(v,\tau) + \overline{\mathcal{F}}^s(\partial_v Q^s)(v,\tau) - \overline{\mathcal{F}}^s(0)(v,\tau)\right)\right)$$

To bound $\mathcal{B}_1$, it is enough to recall that, in the proof of Proposition 4.7, we have seen that $\|\overline{\mathcal{F}}(0)\|_\sigma \le K|\mu|\varepsilon^{\eta+1}$. Then, taking into account (76) and (77), one can see that

$$\left\|\frac{\cosh^2 v}{8}\left(\overline{\mathcal{F}}^u(0)(v,\tau) + \overline{\mathcal{F}}^s(0)(v,\tau)\right)\right\|_{1,\sigma} \le K|\mu|\varepsilon^{\eta}\frac{\sqrt{\alpha}+\varepsilon}{(1-\alpha)^{1/2}}.$$

Moreover, taking into account that $\left\langle\overline{\mathcal{F}}^{u,s}(0)\right\rangle = 0$ and applying Lemma 4.8, we have that

$$\|\mathcal{B}_1\|_{1,\sigma} \le K|\mu|\varepsilon^{\eta+1}\frac{\sqrt{\alpha}+\varepsilon}{(1-\alpha)^{1/2}}.$$

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For the second term, let us first point out that

$$\overline{\mathcal{F}}^*(\partial_v Q^*)(v,\tau) - \overline{\mathcal{F}}^*(0)(v,\tau) = -\partial_v \mathcal{G}_\varepsilon \left(\frac{\cosh^2 v}{8} (\partial_v Q^*)^2 \right), \quad * = u,s.$$

Using Proposition 4.7 and (76), one has that

$$\left| \frac{\cosh^2 v}{8} (\partial_v Q^*(v,\tau))^2 \right| \leq \frac{K|\mu|^2 \varepsilon^{2\eta+2}}{(v-\rho_-)^4 (v-\overline{\rho}_-)^4}.$$

Therefore, analogously to Lemma 4.6, one can easily see that

$$\left| \partial_v \mathcal{G}_\varepsilon \left(\frac{\cosh^2 v}{8} (\partial_v Q^*(v,\tau))^2 \right) \right| \leq \frac{K|\mu|^2 \varepsilon^{2\eta+2}}{(v-\rho_-)^4 (v-\overline{\rho}_-)^4}.$$

Using inequalities (77) and (78),

$$\left\| \frac{\cosh^2 v}{8} \left(\overline{\mathcal{F}}^*(\partial_v Q^*)(v,\tau) - \overline{\mathcal{F}}^*(0)(v,\tau) \right) \right\|_{2,\sigma} \leq K|\mu|^2 \eta^{2\eta+2} \frac{(\sqrt{\alpha}+\varepsilon)^2}{1-\alpha}.$$

Then, using Lemma 4.8, one has that

$$\|\mathcal{B}_2\|_{1,\sigma} \leq K|\mu|^2 \varepsilon^{2\eta+2} \frac{(\sqrt{\alpha}+\varepsilon)^2}{(1-\alpha)^{3/2}}.$$

Therefore, since $\varepsilon^{\eta-1} \frac{\sqrt{\alpha}+\varepsilon}{1-\alpha} \ll 1$, there exists a constant $b_2 > 0$ such that

$$\left\| \overline{\mathcal{J}}(0) \right\|_{1,\sigma} \leq \frac{b_2}{2} |\mu| \varepsilon^{\eta+1} \frac{\sqrt{\alpha}+\varepsilon}{(1-\alpha)^{1/2}}.$$

To bound the Lipschitz constant, it is enough to apply the mean value theorem, use the bounds of $\partial_u Q^{u,s}$ and $\partial_u^2 Q^{u,s}$ given in Proposition 4.7 and Lemma 4.8 to see that

$$\text{Lip} \leq K|\mu| \varepsilon^{\eta-1} \frac{\sqrt{\alpha}+\varepsilon}{1-\alpha}.$$

Then, using that $\varepsilon^{\eta-1} \frac{\sqrt{\alpha}+\varepsilon}{1-\alpha} \ll 1$, the operator $\overline{\mathcal{J}}$ is contractive from $\overline{B}(b_2|\mu|\varepsilon^{\eta+1} \frac{\sqrt{\alpha}+\varepsilon}{(1-\alpha)^{1/2}}) \subset \mathcal{X}_{1,\sigma}$ to itself and it has a unique fixed point $\mathcal{C}$. Finally, to obtain a bound for $\partial_v \mathcal{C}$ it is enough to apply Cauchy estimates reducing slightly the domain and renaming b_2 if necessary. $\square$

Proof of Theorem 4.3. Once we have proved Proposition 4.9, it only remains to obtain the inverse change given by the function $\mathcal{V}$, which is straightforward using a fixed point argument. $\square$

4.2 Proof of Theorem 2.10

The first statement of Theorem 2.10 is a direct consequence of Corollary 4.5 taking $\alpha = 1 - C\varepsilon^r$ with $r \in (0,2)$. Note that the condition $\varepsilon^{\eta-1} \frac{\varepsilon+\sqrt{\alpha}}{1-\alpha}$ becomes $\eta > r + 1$. The proof of the second and third statements, which correspond to $r \geq 2$, are considerably simpler, since we do not need to prove any exponential smallness. The first observation is that $\text{Im}\,\rho_- \sim \varepsilon^{r/2} \leq \varepsilon$. Even though in this case it is not necessary, we keep the analyticity properties of the parameterizations of the invariant manifolds and we work in the domains $D_\kappa^u \times \mathbb{T}_\sigma$ and $D_\kappa^s \times \mathbb{T}_\sigma$ (see (56)).

Theorem 4.10. *Let us fix $\kappa_1 > 0$. Then, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0,\varepsilon_0)$, $\alpha = 1 - C\varepsilon^r$ with $C > 0$ and $r \geq 2$, $\mu \in B(\mu_0)$, if $\eta - 3r/2 > 0$, the Hamilton-Jacobi equation (50) has a unique (modulo an additive constant) real-analytic solution in $\mathcal{D}_{\kappa_1}^u \times \mathbb{T}_\sigma$ satisfying the asymptotic condition (53).*

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Moreover, there exists a real constant $b_4 > 0$ independent of ε and μ, such that for $(u, \tau) \in \mathcal{D}_{\kappa_1}^u \times \mathbb{T}_\sigma$,

$$|\partial_u T^u(u, \tau) - \partial_u T_0(u)| \leq b_4 |\mu| \varepsilon^{\eta - 3r/2}.$$

Furthermore, for $(u, \tau) \in \mathcal{D}_{\kappa_1}^u \times \mathbb{T}_\sigma$, the generating function T^u satisfies that

$$|\partial_u T^u(u, \tau) - \partial_u T_0(u) - \mu \varepsilon^\eta \mathcal{M}^u(u, \tau)| \leq b_4 |\mu|^2 \varepsilon^{2\eta - 3r} \tag{84}$$

where $\mathcal{M}^u$ is the function defined in (57).

Proof. The proof follows the same lines of Theorem 4.1. We use the modified norm

$$\|h\| = \sup_{u \in \mathcal{D}^u \cap \{\operatorname{Re} u < -U\}} \left| e^{-2u} h(u) \right| + \sup_{u \in \mathcal{D}^u \cap \{\operatorname{Re} u > -U\}} |h(u)|. \tag{85}$$

Then, using (75), one can bound $\overline{\mathcal{F}}(0)$ as

$$\left\| \overline{\mathcal{F}}(0) \right\| = K |\mu| \varepsilon^\eta \left(K + \int_{-U}^u \frac{1}{|v - \rho_-|^2 |v - \overline{\rho}_-|^2} \right) \leq \frac{b_4}{2} |\mu| \varepsilon^{\eta - 3r/2}.$$

Proceeding as before, it is straightforward to see that $\overline{\mathcal{F}}$ is contractive from the ball $\overline{B}(\mu \varepsilon^{\eta - 3r/2})$ to itself with Lipschitz constant satisfying

$$\operatorname{Lip} \leq |\mu| \varepsilon^{\eta - 3r/2},$$

which gives the desired result. $\qquad \square$

The function T^s satisfies the same properties in the symmetric domain $\mathcal{D}_{\kappa_1}^s$. Note that $D_{\kappa_1}^u \cap D_{\kappa_1}^s \cap \mathbb{R}$ is an interval of size $\mathcal{O}(\varepsilon^r/2)$ centered at $u = \ln(1 + \sqrt{2})$. Therefore, $D_{\kappa_1}^u \cap D_{\kappa_1}^s \cap \mathbb{R}$ does not contain a fundamental domain. Nevertheless, it suffices to deal with this domain to compute the distance between the manifolds in the section $x = 3\pi/2$.

From Theorem 4.10 the formula of the distance follows. Thus, to finish the proof of Theorem 2.10, it is enough to use Corollary 2.6, for $r \geq 2$, to check that the Melnikov function $\mathcal{M}(\ln(1 + \sqrt{2}), \tau)$ has simple zeros so that the manifolds intersect transversally.

5 The quasiperiodic case: Proof of Theorems 3.5 and 3.7

As we did in the periodic case, the proof in next section includes at the same time the results in Theorem 3.5 and in the first statement of Theorem 3.7. Then, Section 5.2 contains the proof of the second and third statement of Theorem 3.7.

5.1 Proof of Theorem 3.5

We follow the same approach as in the periodic case. Therefore, we only point out the main differences with respect to it. We perform the symplectic change of variables (48) to Hamiltonian (12). In the new variables, it reads

$$\varepsilon \overline{K}(u, w, \theta_1, \theta_2, I_1, I_2) = \varepsilon K\left(x_0(u), \frac{w}{y_0(u)}, \theta_1, \theta_2, I_1, I_2\right). \tag{86}$$

As in the periodic case, we look for the perturbed invariant manifolds as graphs of the gradient of generating functions $T^{u,s}(u, \theta_1, \theta_2)$, which are solutions of the Hamilton-Jacobi equation

$$\overline{K}(u, \partial_u T, \theta_1, \theta_2, \partial_{\theta_1} T, \partial_{\theta_2} T) = 0. \tag{87}$$

This equation reads

$$\varepsilon^{-1} \partial_{\theta_1} T + \varepsilon^{-1} \gamma \partial_{\theta_2} T + \frac{\cosh^2 u}{8} (\partial_u T)^2 - \frac{4}{\cosh^2 u} + \mu \varepsilon^\eta \Psi(u) F(\theta_1, \theta_2) = 0, \tag{88}$$

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where $\Psi(u)$ is the function defined in (52).

We impose the same asymptotic conditions (53) and (54) as in the periodic case. For $\mu = 0$, the solution of equation (88) is (55). We study the existence of solutions of equation (88) in complex domains which have points close to the singularities $u = \rho_-, \overline{\rho}_-$. Nevertheless, in this case, we only need to stay at a distance of order $\mathcal{O}(\sqrt{\varepsilon})$ of the singularity instead of $\mathcal{O}(\varepsilon)$ as happened in the periodic case (see [DGJS97]). To this end, we define the modified domains

$$
\begin{aligned}
D_\kappa^u &= \left\{ u \in \mathbb{C}; |\operatorname{Im} u| < -\tan \beta_1 \left(\operatorname{Re} u - \operatorname{Re} \rho_-\right) + \operatorname{Im} \rho_- - \kappa\sqrt{\varepsilon} \right\} \\
D_\kappa^s &= \left\{ u \in \mathbb{C}; |\operatorname{Im} u| < \tan \beta_1 \left(\operatorname{Re} u - \operatorname{Re} \rho_-\right) + \operatorname{Im} \rho_- - \kappa\sqrt{\varepsilon} \right\}.
\end{aligned}
\tag{89}
$$

Note that the only difference with respect to the domains (56) is the change of ε by $\sqrt{\varepsilon}$.

In the quasiperiodic case, the complex domain of the angular variables plays a crucial role. As it was done in [DGJS97, Sau01], we will prove the existence of these generating functions in very concrete domains in (θ_1, θ_2). To this end let us define the complexified torus for any $\sigma = (\sigma_1, \sigma_2) \in \mathbb{R}^2$ with $\sigma_1, \sigma_2 > 0$,

$$
\mathbb{T}_\sigma^2 = \left\{ (\theta_1, \theta_2) \in (\mathbb{C}/\mathbb{Z})^2 : |\operatorname{Im} \theta_i| \leq \sigma_i \right\},
$$

with

$$
\sigma_i = r_i - d_i \sqrt{\varepsilon},
\tag{90}
$$

where r_i are the constants defined in (10) and $d_i > 0$ are any constants independent of ε.

The next theorem gives the existence of the invariant manifolds in the domains $D_\kappa^* \times \mathbb{T}_\sigma^2$ with $* = u, s$ (see (89)). We state the results for the unstable invariant manifold. The stable one has analogous properties.

Theorem 5.1. *Let us fix any $\kappa_1, d_1, d_2 > 0$. Then, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, $\alpha \in (0, 1)$, $\mu \in B(\mu_0)$, satisfying (47) and that $\varepsilon^{\eta-1} \frac{\sqrt{\varepsilon}+\sqrt{\alpha}}{1-\alpha}$ is small enough the Hamilton-Jacobi equation (88) has a unique (modulo an additive constant) real-analytic solution in $D_{\kappa_1}^u \times \mathbb{T}_\sigma^2$, with σ defined in (90), satisfying the asymptotic condition (53).*

Moreover, there exists a real constant $b_5 > 0$ independent of ε and μ, such that for $(u, \theta_1, \theta_2) \in D_{\kappa_1}^u \times \mathbb{T}_\sigma^2$,

$$
\left|\partial_u T^u(u, \theta_1, \theta_2) - \partial_u T_0(u)\right| \leq \frac{b_5 |\mu| \varepsilon^{\eta-1}}{(\sqrt{\varepsilon} + \sqrt{\alpha})(1-\alpha)}
$$

$$
\left|\partial_u^2 T^u(u, \theta_1, \theta_2) - \partial_u^2 T_0(u)\right| \leq \frac{b_5 |\mu| \varepsilon^{\eta-\frac{3}{2}}}{(\sqrt{\varepsilon} + \sqrt{\alpha})(1-\alpha)}.
$$

Furthermore, if we define the half Melnikov function

$$
\mathcal{M}^u(u, \theta_1, \theta_2) = -4 \int_{-\infty}^0 \frac{\sinh(u+s)\cosh(u+s)}{\left(\cosh^2(u+s) - 2\alpha\sinh(u+s)\right)^2} F\left(\theta_1 + \frac{s}{\varepsilon}, \theta_2 + \frac{\gamma s}{\varepsilon}\right) ds,
\tag{91}
$$

the generating function T^u satisfies that, for $(u, \theta_1, \theta_2) \in D_{\kappa_1}^u \times \mathbb{T}_\sigma^2$,

$$
\left|\partial_u T^u(u, \theta_1, \theta_2) - \partial_u T_0(u) - \mu\varepsilon^\eta \mathcal{M}^u(u, \theta_1, \theta_2)\right| \leq \frac{b_5 |\mu|^2 \varepsilon^{2\eta-2}}{(1-\alpha)^2}.
\tag{92}
$$

The proof of this theorem is deferred to Section 5.1.1. The parameterization of the stable manifold has analogous properties. In particular, we can define

$$
\mathcal{M}^s(u, \theta_1, \theta_2) = 4 \int_0^{+\infty} \frac{\sinh(u+s)\cosh(u+s)}{\left(\cosh^2(u+s) - 2\alpha\sinh(u+s)\right)^2} F\left(\theta_1 + \frac{s}{\varepsilon}, \theta_2 + \frac{\gamma s}{\varepsilon}\right) ds,
\tag{93}
$$

and then, for $(u, \theta_1, \theta_2) \in D_{\kappa_1}^s \times \mathbb{T}_\sigma^2$,

$$
\left|\partial_u T^s(u, \theta_1, \theta_2) - \partial_u T_0(u) - \mu\varepsilon^\eta \mathcal{M}^s(u, \theta_1, \theta_2)\right| \leq \frac{b_5 |\mu|^2 \varepsilon^{2\eta-2}}{(1-\alpha)^2}.
\tag{94}
$$

We consider the function
$$\Delta(u, \theta_1, \theta_2) = T^s(u, \theta_1, \theta_2) - T^u(u, \theta_1, \theta_2). \tag{95}$$

This function is defined in $R_{\kappa_1} \times \mathbb{T}^2_\sigma$, where R_κ is the romboidal domain defined by
$$R_\kappa = D^u_\kappa \cap D^s_\kappa, \tag{96}$$

where D^*_κ are the domains defined in (89).

Subtracting equation (88) for both T^s and T^u, one can easily see that $\Delta \in \mathrm{Ker}\widetilde{\mathcal{L}}_\varepsilon$ for

$$\widetilde{\mathcal{L}}_\varepsilon = \varepsilon^{-1}\partial_{\theta_1} + \varepsilon^{-1}\gamma\partial_{\theta_2} + \left(\frac{\cosh^2 u}{8}\left(\partial_u T^s(u, \theta_1, \theta_2) + \partial_u T^u(u, \theta_1, \theta_2)\right)\right)\partial_u. \tag{97}$$

Since Theorem 5.1 ensures that the perturbed invariant manifolds are well approximated by the unperturbed separatrix in the domains $D^u_{\kappa_1} \times \mathbb{T}^2_\sigma$ and $D^s_{\kappa_1} \times \mathbb{T}^2_\sigma$, we know that the operator $\widetilde{\mathcal{L}}_\varepsilon$ is close to the constant coefficients operator
$$\mathcal{L}_\varepsilon = \varepsilon^{-1}\partial_{\theta_1} + \varepsilon^{-1}\gamma\partial_{\theta_2} + \partial_u \tag{98}$$

in the domain $R_{\kappa_1} \times \mathbb{T}^2_\sigma$.

Any function which is defined in $\{u \in \mathbb{C}; \mathrm{Re}\, u = a, \mathrm{Im}\, u \in [-r_0, r_0]\} \times \mathbb{T}^2_\sigma$, for any $a \in \mathbb{R}$, and belongs to the kernel of $\mathcal{L}_\varepsilon$ is defined in all the strip $\{|\mathrm{Im}\, u| < r_0\} \times \mathbb{T}^2_\sigma$ and has exponentially small bounds for real values of the variables. This fact is summarized in the next lemma, whose proof follows the same lines as the one of Lemma 4.1 of [Sau01].

Lemma 5.2. *Let us consider a function $\zeta(u, \theta_1, \theta_2)$ analytic in $(u, \theta_1, \theta_2) \in \{u \in \mathbb{C} : \mathrm{Re}\, u = \mathrm{Re}\, \rho_-, |\mathrm{Im}\, u| < \mathrm{Im}\, \rho_- - \kappa\sqrt{\varepsilon}\} \times \mathbb{T}^2_\sigma$, where $\sigma = (\sigma_1, \sigma_2)$ with $\sigma_i = r_i - d_i\sqrt{\varepsilon}$, which is solution of $\mathcal{L}_\varepsilon\zeta = 0$. Then, ζ can be extended analytically to $\{|\mathrm{Im}\, u| < \mathrm{Im}\, \rho_- - \kappa\sqrt{\varepsilon}\} \times \mathbb{T}^2_\sigma$ and its mean value*

$$\langle \zeta \rangle = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\zeta(u, \theta_1, \theta_2)d\theta_1 d\theta_2$$

does not depend on u. Moreover, for $\kappa' > \kappa$ and $d'_i > d_i$, we define

$$M = \max_{(u, \theta_1, \theta_2) \in [-\mathrm{Im}\, \rho_- + \kappa'\sqrt{\varepsilon}, \mathrm{Im}\, \rho_- - \kappa'\sqrt{\varepsilon}] \times \overline{\mathbb{T}}^2_{\sigma'}} |\partial_u\zeta(u, \theta_1, \theta_2)| \tag{99}$$

where $\sigma' = (\sigma'_1, \sigma'_2)$ with $\sigma'_i = r_i - d'_i\sqrt{\varepsilon}$. Then, provided ε is small enough, for $(u, \theta_1, \theta_2) \in \mathbb{R} \times \mathbb{T}^2$,

$$|\partial_u\zeta(u, \theta_1, \theta_2)| \leq 4Me^{-c(\ln(\varepsilon/\mathrm{Im}\, \rho_-))\sqrt{\frac{\mathrm{Im}\, \rho_-}{\varepsilon}}}.$$

where c is the periodic function defined in (36).

To apply this lemma to study the difference between the invariant manifolds, following [Sau01] (see also [GOS10]) we look for a change of variables which conjugates $\widetilde{\mathcal{L}}_\varepsilon$ in (97) with $\mathcal{L}_\varepsilon$ in (98).

Theorem 5.3. *Let us consider the constant $\kappa_1 > 0$ defined in Theorem 4.1 and let us fix any $\kappa_3 > \kappa_2 > \kappa_1$. Then, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, $\alpha \in (0, 1)$ and $\mu \in B(\mu_0)$ satisfying (47) and that $\varepsilon^{\eta-1}\frac{\sqrt{\varepsilon}+\sqrt{\alpha}}{1-\alpha}$ is small enough, there exists a real-analytic function $\mathcal{C}$ defined in $R_{\kappa_2} \times \mathbb{T}^2_\sigma$ such that the change*

$$(u, \theta_1, \theta_2) = (v + \mathcal{C}(v, \theta_1, \theta_2), \theta_1, \theta_2) \tag{100}$$

conjugates the operators $\widetilde{\mathcal{L}}_\varepsilon$ and $\mathcal{L}_\varepsilon$ defined in (97) and (98). Moreover, for $(v, \theta_1, \theta_2) \in R_{\kappa_2} \times \mathbb{T}_\sigma$, $v + \mathcal{C}(v, \theta_1, \theta_2) \in R_{\kappa_1}$ and there exists a constant $b_6 > 0$ such that

$$|\mathcal{C}(v, \theta_1, \theta_2)| \leq b_6|\mu|\varepsilon^{\eta-\frac{1}{2}}\frac{\sqrt{\alpha}+\sqrt{\varepsilon}}{1-\alpha}|\ln\varepsilon|$$

$$|\partial_v\mathcal{C}(v, \theta_1, \theta_2)| \leq b_6|\mu|\varepsilon^{\eta-1}\frac{\sqrt{\alpha}+\sqrt{\varepsilon}}{1-\alpha}|\ln\varepsilon|.$$

Furthermore, $(u, \theta_1, \theta_2) = (v + \mathcal{C}(v, \theta_1, \theta_2), \theta_1, \theta_2)$ is invertible and its inverse is of the form $(v, \theta_1, \theta_2) = (u + \mathcal{V}(u, \theta_1, \theta_2), \theta_1, \theta_2)$ where $\mathcal{V}$ is a function defined for $(u, \theta_1, \theta_2) \in R_{\kappa_3} \times \mathbb{T}^2_\sigma$ and satisfies

$$|\mathcal{V}(u, \theta_1, \theta_2)| \leq b_6 |\mu| \varepsilon^{\eta - \frac{1}{2}} \frac{\sqrt{\alpha} + \sqrt{\varepsilon}}{1 - \alpha} |\ln \varepsilon|$$

and that $u + \mathcal{V}(u, \theta_1, \theta_2) \in R_{\kappa_2}$ for $(u, \theta_1, \theta_2) \in R_{\kappa_3} \times \mathbb{T}^2_\sigma$.

The proof of this theorem is deferred to Section 5.1.2.

Next step is to prove the validity of the Melnikov function. To this end, we bound $\partial_u \Delta(u, \theta_1, \theta_2) - \mathcal{M}(u, \theta_1, \theta_2)$, where Δ and $\mathcal{M}$ are the functions defined in (95) and (35) respectively. As in the periodic case, as a first step, we bound $\partial_v (\Delta(v + \mathcal{C}(v, \theta_1, \theta_2), \theta_1, \theta_2)) - \mathcal{M}(v, \theta_1, \theta_2)$ where $\mathcal{C}$ is the function obtained in Theorem 5.3.

Theorem 5.4. *There exists ε_0 and $b_7 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $\alpha \in (0, 1)$ and $\mu \in B(\mu_0)$, satisfying (47) and that $\varepsilon^{\eta-1} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1 - \alpha}$ is small enough, the following bound is satisfied*

$$|\partial_v (\Delta(v + \mathcal{C}(v, \theta_1, \theta_2), \theta_1, \theta_2)) - \mathcal{M}(v, \theta_1, \theta_2)| \leq \frac{b_7 |\mu|^2 \varepsilon^{2\eta-2} |\ln \varepsilon|}{(1 - \alpha)^2} e^{-c(\ln(\varepsilon / \operatorname{Im} \rho_-))} \sqrt{\frac{\operatorname{Im} \rho_-}{\varepsilon}}$$

for $v \in R_{\kappa_3} \cap \mathbb{R}$ and $(\theta_1, \theta_2) \in \mathbb{T}^2$,

Proof. First we define the Melnikov potential L, namely a function such that $\partial_u L = \mathcal{M}$ (see [DG00]). As $\Delta \in \operatorname{Ker} \widetilde{\mathcal{L}}_\varepsilon$, where $\widetilde{\mathcal{L}}_\varepsilon$ is the operator in (97), by Theorem 5.3, the function $\Phi(v, \theta_1, \theta_2) = \Delta(v + \mathcal{C}(v, \theta_1, \theta_2), \theta_1, \theta_2) - L(v, \tau) \in \operatorname{Ker} \mathcal{L}_\varepsilon$, where $\mathcal{L}_\varepsilon$ is the operator defined in (98). Therefore we can apply Lemma 5.2.

To this end, we have to bound $\partial_v \Phi(v, \theta_1, \theta_2)$ in the domain $R_{\kappa_2} \times \mathbb{T}^2_\sigma$. We split Φ as $\Phi(v, \theta_1, \theta_2) = \Phi_1(v, \theta_1, \theta_2) + \Phi_2(v, \theta_1, \theta_2)$ where

$$\Phi_1(v, \theta_1, \theta_2) = \Delta(v + \mathcal{C}(v, \theta_1, \theta_2), \theta_1, \theta_2) - \Delta(v, \theta_1, \theta_2)$$
$$\Phi_2(v, \theta_1, \theta_2) = \Delta(v, \theta_1, \theta_2) - L(v, \theta_1, \theta_2),$$

where Δ is the function defined in (95) and L is the Melnikov potential.

To bound $\partial_v \Phi_1$, one has to take into account that $\Delta = (T^s - T_0) - (T^u - T_0)$ and therefore, it is enough to consider the bounds obtained in Theorems 5.1 and 5.3 to obtain

$$|\partial_v \Phi_1(v, \theta_1, \theta_2)| \leq \frac{K |\mu|^2 \varepsilon^{2\eta-2}}{(1 - \alpha)^2} |\ln \varepsilon|$$

for $(v, \theta_1, \theta_2) \in R_{\kappa_2} \times \mathbb{T}^2_\sigma$. For the second term, it is enough to use bounds (94) and (92) to obtain

$$|\partial_v \Phi_2(v, \theta_1, \theta_2)| \leq \frac{K |\mu|^2 \varepsilon^{2\eta-2}}{(1 - \alpha)^2} |\ln \varepsilon|$$

for $(v, \theta_1, \theta_2) \in R_{\kappa_2} \times \mathbb{T}^2_\sigma$.

Therefore, we have that

$$|\partial_v \Phi(v, \theta_1, \theta_2)| \leq \frac{K |\mu|^2 \varepsilon^{2\eta-2}}{(1 - \alpha)^2} |\ln \varepsilon|$$

for $(v, \theta_1, \theta_2) \in R_{\kappa_2} \times \mathbb{T}^2_\sigma$ and then, it is enough to apply Lemma 5.2 to finish the proof of Theorem 5.4. $\qquad\square$

From this result and considering the inverse change $(v, \theta_1, \theta_2) = (u + \mathcal{V}(u, \theta_1, \theta_2), \theta_1, \theta_2)$ obtained in Theorem 5.3, it is straightforward to obtain exponentially small bounds for $\partial_u \Delta(u, \theta_1, \theta_2) - \mathcal{M}(u, \theta_1, \theta_2)$ and its derivative. They are stated in the next corollary, whose proof is straightforward.

Corollary 5.5. *Let us consider any $\kappa_4 > \kappa_3$. Then, there exists ε_0 and $b_7 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in (0, 1)$, satisfying (47) and that $\varepsilon^{\eta-1} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1 - \alpha}$ is small enough, the following bound is satisfied*

$$|\partial_u \Delta(u, \theta_1, \theta_2) - \mathcal{M}(u, \theta_1, \theta_2)| \leq \frac{b_7 |\mu|^2 \varepsilon^{2\eta-2} |\ln \varepsilon|}{(1 - \alpha)^2} e^{-c(\ln(\varepsilon/\operatorname{Im} \rho_-))} \sqrt{\frac{\operatorname{Im} \rho_-}{\varepsilon}}$$

for $u \in R_{\kappa_4} \cap \mathbb{R}$ and $(\theta_1, \theta_2) \in \mathbb{T}^2$.

This corollary finishes the proof of Theorem 3.5. Note that when α is bounded away from 1, we just need the simpler condition $\varepsilon^{\eta-1}(\sqrt{\varepsilon} + \sqrt{\alpha})$ small enough.

5.1.1 The invariant manifolds: proof of Theorem 5.1

We look for a solution of equation (88) satisfying the asymptotic condition (53) as a perturbation of T_0 in (55). As in the periodic case, we define

$$Q(u, \theta_1, \theta_2) = T(u, \theta_1, \theta_2) - T_0(u), \tag{101}$$

which is solution of

$$\mathcal{L}_\varepsilon Q = \mathcal{F}(\partial_u Q, u, \theta_1, \theta_2), \tag{102}$$

where $\mathcal{L}_\varepsilon$ is the operator defined in (98) and

$$\mathcal{F}(h, u, \theta_1, \theta_2) = -\frac{\cosh^2 u}{8} h^2 - \mu \varepsilon^\eta \Psi(u) F(\theta_1, \theta_2), \tag{103}$$

where $\Psi(u)$ is the function defined in (52).

We devote the rest of the section to obtain a solution of equation (102) which is defined in $D_\kappa^u \times \mathbb{T}_\sigma^2$ and satisfies the asymptotic condition (53). Recall that D_κ^u has been defined in (89) and σ in (90).

We use analogous norms as the ones in the periodic case. For analytic functions $h : D_\kappa^u \times \mathbb{T}_\sigma^2 \to \mathbb{C}$, we define the Fourier norm

$$\|h\|_\sigma = \sum_{k \in \mathbb{Z}^2} \left\| h^{[k]} \right\| e^{|k_1|\sigma_1 + |k_2|\sigma_2},$$

where $\|\cdot\|$ is the norm defined in (70). We consider the Banach space

$$\mathcal{E}_{\kappa,\sigma} = \left\{ h : D_\kappa^u \times \mathbb{T}_\sigma^2 \to \mathbb{C}; \text{real-analytic}, \|h\|_\sigma < \infty \right\}. \tag{104}$$

First we solve the equation $\mathcal{L}_\varepsilon h = g$, where $\mathcal{L}_\varepsilon$ is the differential operator defined in (98). This operator is invertible in $\mathcal{E}_{\kappa,\sigma}$ and its inverse can be defined as

$$\mathcal{G}_\varepsilon(h)(u, \theta) = \int_{-\infty}^0 h(u + s, \theta_1 + \varepsilon^{-1} s, \theta_2 + \gamma \varepsilon^{-1} s) \, dt. \tag{105}$$

Lemma 5.6. *The operator $\mathcal{G}_\varepsilon$ in (105) satisfies the following properties.*

1. *$\mathcal{G}_\varepsilon$ is linear from $\mathcal{E}_{\kappa,\sigma}$ to itself, commutes with ∂_u and satisfies $\mathcal{L}_\varepsilon \circ \mathcal{G}_\varepsilon = \mathrm{Id}$.*

2. *If $h \in \mathcal{E}_{\kappa,\sigma}$, then*
$$\|\mathcal{G}_\varepsilon(h)\|_\sigma \leq K \|h\|_\sigma.$$

   Furthermore, one can bound each Fourier coefficient $\mathcal{G}_\varepsilon^{[k]}(h)$ with $k \neq 0$ as
$$\left\| \mathcal{G}^{[k]}(h) \right\| \leq \frac{K\varepsilon}{|k \cdot \omega|} \left\| h^{[k]} \right\|.$$

3. *If $h \in \mathcal{E}_{\kappa,\sigma}$, then $\partial_u \mathcal{G}_\varepsilon(h) \in \mathcal{E}_{\kappa,\sigma}$ and*
$$\|\partial_u \mathcal{G}_\varepsilon(h)\|_\sigma \leq K \|h\|_\sigma.$$

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Proof. The proof is analogous to the proof of Lemma 5.5 of [GOS10] $\qquad\qquad\square$

We can obtain solutions of equation (102) using a fixed point argument. Theorem 5.1 is a straightforward consequence of the following proposition.

Proposition 5.7. *Let us fix $\kappa_1 > 0$. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in (0, 1)$ satisfying condition (47) and that $\varepsilon^{\eta-1} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1-\alpha}$ is small enough, there exists a function Q defined in $D_{\kappa_1}^u \times \mathbb{T}_\sigma^2$ such that $\partial_u Q \in \mathcal{E}_{\kappa_1, \sigma}$ is a fixed point of the operator*

$$\overline{\mathcal{F}}^u(h) = \partial_u \mathcal{G}_\varepsilon \mathcal{F}(h), \tag{106}$$

where $\mathcal{G}_\varepsilon$ and $\mathcal{F}$ are the operators defined in (105) and (103) respectively. Furthermore, there exists a constant $b_5 > 0$ such that,

$$\|\partial_u Q\|_\sigma \le b_5 |\mu| \varepsilon^\eta,$$
$$\left\|\partial_u^2 Q\right\|_\sigma \le b_5 |\mu| \varepsilon^{\eta - \frac{1}{2}}.$$

Moreover, if we consider the half Melnikov function defined in (91),

$$\|\partial_u Q - \mu \varepsilon^\eta \mathcal{M}^u\|_\sigma \le b_5 |\mu|^2 \varepsilon^{2\eta-1} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1-\alpha}. \tag{107}$$

Proof. Let us consider $\kappa_0 < \kappa_1$. It is straightforward to see that $\overline{\mathcal{F}}^u$ is well defined from $\mathcal{E}_{\kappa_0, \sigma}$ to itself. We are going to prove that there exists a constant $b_5 > 0$ such that $\overline{\mathcal{F}}^u$ sends $\overline{B}(b_5 |\mu| \varepsilon^\eta) \subset \mathcal{E}_{\kappa_0, \sigma}$ to itself and is contractive there.

Let us first consider $\overline{\mathcal{F}}^u(0)$. From the definition of $\overline{\mathcal{F}}^u$ in (106), the definition of $\mathcal{F}$ in (103) and using Lemma 5.6, we have that

$$\overline{\mathcal{F}}^u(0)(u, \theta_1, \theta_2) = \partial_u \mathcal{G}_\varepsilon \mathcal{F}(0)(u, \theta_1, \theta_2) = -\mu \varepsilon^\eta \mathcal{G}_\varepsilon \left(\Psi'(u) F(\theta_1, \theta_2)\right).$$

To bound it, we bound first each Fourier coefficient and we take advantage of the fact that $\langle F \rangle = 0$. By Lemma 5.6, formula (10) and recalling that $\Psi'(u) = \beta(u)$, where β is the function defined in (20), satisfies $\|\Psi'\| \le K$, we have that

$$\left\|\overline{\mathcal{F}}^u(0)^{[k]}\right\| \le \frac{K |\mu| \varepsilon^{\eta+1}}{|k \cdot \omega|} e^{-r_1 |k_1| - r_2 |k_2|}$$

and therefore we have that

$$\left\|\overline{\mathcal{F}}^u(0)\right\|_\sigma \le K |\mu| \varepsilon^{\eta+1} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{|k \cdot \omega|} e^{-d_1 |k_1| \sqrt{\varepsilon} - d_2 |k_2| \sqrt{\varepsilon}}.$$

To bound this sum, we split it in two depending $|k \cdot \omega| > 1/2$ or $|k \cdot \omega| < 1/2$. For the first one,

$$\sum_{\substack{k \in \mathbb{Z}^2 \setminus \{0\} \\ |k \cdot \omega| > 1/2}} \frac{1}{|k \cdot \omega|} e^{-d_1 |k_1| \sqrt{\varepsilon} - d_2 |k_2| \sqrt{\varepsilon}} \le 2 \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} e^{-d_1 |k_1| \sqrt{\varepsilon} - d_2 |k_2| \sqrt{\varepsilon}} \le \frac{K}{\varepsilon}.$$

For the second one, we take into account that, for a fixed k_2, there exists only one k_1 satisfying $|k \cdot \omega| < 1/2$. Moreover, it satisfies

$$-|k_1| \le -\gamma |k_2| + |k \cdot \omega| \le -\gamma |k_2| + \frac{1}{2}.$$

Therefore, with this inequality and taking into account that for any $k \in \mathbb{Z}^2 \setminus \{0\}$,

$$|k \cdot \omega| > \frac{C}{|k|}$$

for certain $C > 0$, we can see that

$$\sum_{\substack{k \in \mathbb{Z}^2 \setminus \{0\} \\ |k \cdot \omega| < 1/2}} \frac{1}{|k \cdot \omega|} e^{-d_1 |k_1| \sqrt{\varepsilon} - d_2 |k_2| \sqrt{\varepsilon}} \le K \sum_{k_2 \in \mathbb{Z} \setminus \{0\}} |k_2| e^{-(d_1 \gamma + d_2)|k_2|\sqrt{\varepsilon} + \frac{d_1}{2}\sqrt{\varepsilon}} \le \frac{K}{\varepsilon}.$$

Then, one can easily see that there exists a constant $b_5 > 0$ such that

$$\left\| \overline{\mathcal{F}}^u(0) \right\|_\sigma \le \frac{b_5}{2} |\mu| \varepsilon^\eta.$$

To bound the Lipschitz constant, let us consider $h_1, h_2 \in \overline{B}(b_5 |\mu| \varepsilon^\eta) \in \mathcal{E}_{\kappa_0, \sigma}$. Then, one can proceed as in the periodic case recalling that now

$$\left| \frac{u - i\pi/2}{u - \rho_-} \right| \le 1 + \left| \frac{i\pi/2 - \rho_-}{u - \rho_-} \right| \le 1 + K \frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}$$

and

$$\left| \frac{1}{u - \overline{\rho}_-} \right| \le \frac{K}{\sqrt{1 - \alpha}}.$$

Therefore,

$$\begin{aligned}
\left\| \overline{\mathcal{F}}^u(h_2) - \overline{\mathcal{F}}^u(h_1) \right\|_\sigma &\le \frac{K}{\sqrt{\varepsilon}(1 - \alpha)} \left(1 + \frac{\sqrt{\alpha}}{\sqrt{\varepsilon}} \right) \|h_2 + h_1\|_\sigma \|h_2 - h_1\|_\sigma \\
&\le K |\mu| \varepsilon^{\eta-1} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1 - \alpha} \|h_2 - h_1\|_\sigma.
\end{aligned}$$

Then, using that by hypothesis $\varepsilon^{\eta-1} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1 - \alpha}$ is small enough,

$$\mathrm{Lip} \, \overline{\mathcal{F}}^u = K |\mu| \varepsilon^{\eta-1} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1 - \alpha} < 1/2$$

and therefore $\overline{\mathcal{F}}^u$ is contractive from the ball $\overline{B}(b_5 |\mu| \varepsilon^\eta) \subset \mathcal{E}_{\kappa_0, \sigma}$ into itself, and it has a unique fixed point h^*. Moreover, since it has exponential decay as $\mathrm{Re} \, u \to -\infty$, we can take

$$Q(u, \tau) = \int_{-\infty}^{u} h^*(v, \tau) dv.$$

To obtain the bound for $\partial_u^2 Q$, it is enough to apply Cauchy estimates to the nested domains $D_{\kappa_1}^u \subset D_{\kappa_0}^u$ (see, for instance, [GOS10]), and rename b_5 if necessary.

Finally, to prove (74), it is enough to point out that $\overline{\mathcal{F}}^u(0) = \mu \varepsilon^\eta \mathcal{M}^u$ and consider the obtained bound for the Lipschitz constant. $\qquad \square$

5.1.2 Straightening the operator $\widetilde{\mathcal{L}}_\varepsilon$: proof of Theorem 5.3

As we have explained in the periodic case, we just need to look for a function $\mathcal{C}$ solution of the equation

$$\mathcal{L}_\varepsilon \mathcal{C}(v, \theta_1, \theta_2) = \frac{\cosh^2 u}{8} \left(\partial_u T^s(u, \theta_1, \theta_2) + \partial_u T^u(u, \theta_1, \theta_2) \right) \Big|_{u = v + \mathcal{C}(v, \theta_1, \theta_2)} - 1.$$

Taking into account the definition of T_0 and Q in (55) and (101), this equation can be written as

$$\mathcal{L}_\varepsilon \mathcal{C} = \mathcal{J}(\mathcal{C}) \tag{108}$$

where

$$\mathcal{J}(h)(v, \theta_1, \theta_2) = \frac{\cosh^2 u}{8} \left(\partial_u Q^s(u, \theta_1, \theta_2) + \partial_u Q^u(u, \theta_1, \theta_2) \right) \Big|_{u = v + h(v, \theta_1, \theta_2)}. \tag{109}$$

To look for a solution of this equation, we start by defining some norms and Banach spaces. Given $n \in \mathbb{N}$ and an analytic function $h : R_\kappa \times \mathbb{T}^2_\sigma \to \mathbb{C}$, we define the Fourier norm

$$\|h\|_{n,\sigma} = \sum_{k \in \mathbb{Z}^2} \left\| h^{[k]} \right\|_n e^{|k_1|\sigma_1 + |k_2|\sigma_2},$$

where $\| \cdot \|_n$ is the norm defined in (81). Thus, we introduce the following Banach spaces

$$\mathcal{X}_{n,\sigma} = \{ h : R_\kappa \times \mathbb{T}^2_\sigma \to \mathbb{C}; \text{ real-analytic}, \|h\|_{n,\sigma} < \infty \}.$$

To find a right-inverse of the operator $\mathcal{L}_\varepsilon$ in (98) in $R_\kappa \times \mathbb{T}^2_\sigma$ we consider u_1 the upper vertex of R_κ and u_0 the left endpoint of R_κ. Then, we define the operator $\widetilde{\mathcal{G}}_\varepsilon$ as

$$\widetilde{\mathcal{G}}_\varepsilon(h)(v,\theta) = \sum_{k \in \mathbb{Z}^2} \widetilde{\mathcal{G}}_\varepsilon(h)^{[k]}(v) e^{ik\cdot\theta}, \tag{110}$$

where its Fourier coefficients are given by

$$\widetilde{\mathcal{G}}_\varepsilon(h)^{[k]}(v) = \int_{-v_1}^{v} e^{i\frac{k\cdot\omega}{\varepsilon}(s-v)} h^{[k]}(s)\, ds \qquad\qquad \text{if } k \cdot \omega < 0$$

$$\widetilde{\mathcal{G}}_\varepsilon(h)^{[0]}(v) = \int_{v_0}^{v} h^{[0]}(s)\, ds$$

$$\widetilde{\mathcal{G}}_\varepsilon(h)^{[k]}(v) = -\int_{v}^{v_1} e^{i\frac{k\cdot\omega}{\varepsilon}\omega(s-v)} h^{[k]}(s)\, ds \qquad\qquad \text{if } k \cdot \omega > 0.$$

The following lemma, which can be proved analogously to Lemma 8.3 of [GOS10], gives some properties of this operator.

Lemma 5.8. *The operator $\widetilde{\mathcal{G}}_\varepsilon$ in (110) satisfies if $h \in \mathcal{X}_{1,\sigma}$, then $\widetilde{\mathcal{G}}_\varepsilon(h) \in \mathcal{X}_{0,\sigma}$ and*

$$\left\| \widetilde{\mathcal{G}}_\varepsilon(h) \right\|_{0,\sigma} \leq K \frac{|\ln \varepsilon|}{\sqrt{1-\alpha}} \|h\|_{1,\sigma}.$$

Next, Theorem 5.3 is a straightforward consequence of the following proposition.

Proposition 5.9. *Let us consider the constant $\kappa_1 > 0$ defined in Theorem 5.1 and let us consider any $\kappa_2 > \kappa_1$. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in (0,1)$ satisfying condition (47) and that $\varepsilon^{\eta-1}\frac{\sqrt{\varepsilon}+\sqrt{\alpha}}{1-\alpha}$ is small enough, there exists a function $\mathcal{C} \in \mathcal{X}_{1,\sigma}$ defined in $R_{\kappa_2} \times \mathbb{T}^2_\sigma$ such that is a fixed point of the operator*

$$\overline{\mathcal{J}}(h) = \widetilde{\mathcal{G}}_\varepsilon \mathcal{J}(h), \tag{111}$$

where $\widetilde{\mathcal{G}}_\varepsilon$ and $\mathcal{J}$ are the operators defined in (110) and (109) respectively. Furthermore, $v + \mathcal{C}(v, \theta_1, \theta_2) \in R_{\kappa_1}$ for $(v, \theta_1, \theta_2) \in R_{\kappa_2} \times \mathbb{T}^2_\sigma$ and there exists a constant $b_6 > 0$ such that

$$\|\mathcal{C}\|_{0,\sigma} \leq b_6 |\mu| \varepsilon^{\eta-\frac{1}{2}} \frac{\sqrt{\varepsilon}+\sqrt{\alpha}}{1-\alpha} |\ln \varepsilon|$$

$$\|\partial_v \mathcal{C}\|_{0,\sigma} \leq b_6 |\mu| \varepsilon^{\eta-1} \frac{\sqrt{\varepsilon}+\sqrt{\alpha}}{1-\alpha} |\ln \varepsilon|.$$

Proof. It is straightforward to see that $\overline{\mathcal{J}}$ is well defined from $\mathcal{X}_{1,\sigma}$ to itself. We are going to prove that there exists a constant $b_6 > 0$ such that $\overline{\mathcal{J}}$ sends $\overline{B}(b_6 |\mu| \varepsilon^{\eta-\frac{1}{2}} \frac{\sqrt{\varepsilon}+\sqrt{\alpha}}{1-\alpha} |\ln \varepsilon|) \subset \mathcal{X}_{1,\sigma}$ to itself and is contractive there.

Let us first consider $\overline{\mathcal{J}}(0)$. From the definition of $\overline{\mathcal{J}}$ in (111), the definition of $\mathcal{J}$ in (109), we have that

$$\overline{\mathcal{J}}(0)(v, \theta_1, \theta_2) = \widetilde{\mathcal{G}}_\varepsilon \mathcal{J}(0)(v, \theta_1, \theta_2) = \widetilde{\mathcal{G}}_\varepsilon \left(\frac{\cosh^2 v}{8} \left(\partial_v Q^s(v, \theta_1, \theta_2) + \partial_v Q^u(v, \theta_1, \theta_2) \right) \right).$$

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Then, it is enough to apply Lemma 5.8 and Proposition 5.7 to see that there exists a constant $b_6 > 0$ such that

$$\left\| \overline{\mathcal{J}}(0) \right\|_{0,\sigma} \leq \frac{b_6}{2} |\mu| \varepsilon^{\eta - \frac{1}{2}} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1 - \alpha} |\ln \varepsilon|.$$

To bound the Lipschitz constant, it is enough to apply the mean value theorem, use the bound of $\partial_u^2 Q$ of Proposition 5.7 and Lemma 5.8 to see that

$$\text{Lip} \leq K |\mu| \varepsilon^{\eta - 1} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1 - \alpha} |\ln \varepsilon|.$$

Then, using that $\varepsilon^{\eta - 1} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1 - \alpha} |\ln \varepsilon| \ll 1$, the operator $\overline{\mathcal{J}}$ is contractive from $\overline{B}(b_6 |\mu| \varepsilon^{\eta - \frac{1}{2}} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1 - \alpha} |\ln \varepsilon|) \subset \mathcal{X}_{1,\sigma}$ to itself and it has a unique fixed point $\mathcal{C}$. Finally, to obtain a bound for $\partial_v \mathcal{C}$ it is enough to apply Cauchy estimates reducing slightly the domain and renaming b_6 if necessary. $\qquad\square$

Proof of Theorem 5.3. Once we have proved Proposition 5.9, it only remains to obtain the inverse change given by the function $\mathcal{V}$, which is straightforward using a fixed point argument. $\qquad\square$

5.2 Proof of Theorem 3.7

The first statement of Theorem 3.7 is a direct consequence of Corollary 5.5 taking $\alpha = 1 - C \varepsilon^r$ with $r \in (0, 2)$. Note that the condition $\varepsilon^{\eta - 1} \frac{\sqrt{\varepsilon} + \sqrt{\alpha}}{1 - \alpha}$ becomes, as in the periodic cas, $\eta > r + 1$. The proof of the second and third statements, which correspond to $r \geq 2$, are considerably simpler, since we do not need to prove any exponential smallness. It follows the same lines as the proof of Theorem 2.10.

Theorem 5.10. *Let us fix $\kappa_1 > 0$. Then, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, $\alpha = 1 - C \varepsilon^r$ with $C > 0$ and $r \geq 2$, $\mu \in B(\mu_0)$, if $\eta - 3r/2 > 0$, the Hamilton-Jacobi equation (87) has a unique (modulo an additive constant) real-analytic solution in $\mathcal{D}_{\kappa_1}^u \times \mathbb{T}_\sigma^2$ satisfying the asymptotic condition (53).*

Moreover, there exists a real constant $b_8 > 0$ independent of ε and μ, such that for $(u, \theta_1, \theta_2) \in \mathcal{D}_{\kappa_1}^u \times \mathbb{T}_\sigma^2$,

$$|\partial_u T^u(u, \theta_1, \theta_2) - \partial_u T_0(u)| \leq b_8 |\mu| \varepsilon^{\eta - 3r/2}.$$

Furthermore, for $(u, \theta_1, \theta_2) \in \mathcal{D}_{\kappa_1}^u \times \mathbb{T}_\sigma^2$, the generating function T^u satisfies that

$$|\partial_u T^u(u, \theta_1, \theta_2) - \partial_u T_0(u) - \mu \varepsilon^\eta \mathcal{M}^u(u, \theta_1, \theta_2)| \leq b_8 |\mu|^2 \varepsilon^{2\eta - 3r}, \tag{112}$$

where $\mathcal{M}^u$ is the function defined in (91).

Proof. It follows the same lines of the proof of Theorem 4.10. We use the modified norm (85) and we bound $\overline{\mathcal{F}}(0)$ as

$$\left\| \overline{\mathcal{F}}(0) \right\| = K |\mu| \varepsilon^\eta \left(K + \int_{-U}^u \frac{1}{|v - \rho_-|^2 |v - \overline{\rho}_-|^2} \right) \leq \frac{b_4}{2} |\mu| \varepsilon^{\eta - 3r/2}.$$

Finally, it is straightforward to see that $\overline{\mathcal{F}}$ is contractive from the ball $\overline{B}(\mu \varepsilon^{\eta - 3r/2})$ to itself with Lipschitz constant equal to

$$\text{Lip} \leq |\mu| \varepsilon^{\eta - 3r/2},$$

which gives the desired result. $\qquad\square$

The function T^s satisfies the same properties in the symmetric domain $\mathcal{D}_{\kappa_1}^s$. From this theorem, the formula of the distance $\widetilde{d}(\theta_1, \theta_2)$ follows.

Acknowledgements

A   Some remarks on the singular periodic case

We devote this section to give some remarks and conjectures about the singular case, namely when the Melnikov function does not predict correctly the splitting of separatrices. We restrict this discussion to the case $0 < \alpha \le \alpha_0 < 1$ for any fixed α_0, where Theorem 2.8 holds.

The main point in the proof of the exponentially small splitting of separatrices is to give a good approximation of the invariant manifolds not only in the real line but in a complex strip wich reaches a neighborhood of order $\mathcal{O}(\varepsilon)$ of the singularities. In this paper, this is the result given in Theorem 4.1. To prove this theorem, one has to impose the condition $\varepsilon^{\eta-1}(\varepsilon + \sqrt{\alpha})$ small enough. Looking at the first order of the perturbed invariant manifolds given by the half Melnikov functions (57) and (59) one can see that this condition is necessary. In fact, if $\varepsilon^{\eta-1}(\varepsilon + \sqrt{\alpha})$ is of order 1, the half Melnikov functions have the same size as the separatrix when $u - \rho_- \sim \varepsilon$. A more careful analysis shows that $\partial_u T_0(u)$ and the remainder $\partial_u T(u, \tau) - \partial_u T_0(u)$ also become of the same size when $u - \rho_- \sim \varepsilon$. This implies that, when $\varepsilon^{\eta-1}(\varepsilon + \sqrt{\alpha})$ is not small, at a distance $\mathcal{O}(\varepsilon)$ of the singularities the unperturbed separatrix is not a good approximation of the perturbed invariant manifolds and then in these cases Melnikov fails to predict correctly the splitting of separatrices. The correct approach when $\varepsilon^{\eta-1}(\varepsilon + \sqrt{\alpha}) \sim 1$ is to look for the first order of the invariant manifolds at a distance $\mathcal{O}(\varepsilon)$ of the singularities in a different way. As was first pointed out in [Laz84] in the study of the Standard Map, these new first orders are solutions of a different equation usually called *inner equation*, which is independent of ε (see [Gel97b, Gel00, GS01, BS08, MSS10, GG11]). For classical Hamiltonian Systems, the inner equation is a new Hamilton-Jacobi equation, which has been studied in several models in [OSS03, Bal06, GOS10, BFGS11]. Nevertheless, in all these works the inner equation was considered for points at a distance $\mathcal{O}(\varepsilon)$ of the singularity of the unperturbed separatrix since it was this singularity which was giving the exponentially small coefficient in the splitting. Nevertheless, for system (4) one has to proceed more carefully since this coefficient depends on α (see Proposition 2.1 and Theorem 2.8). Therefore, we will obtain different inner equations depending on the relation between α and ε. Namely, in some cases $(0 < \alpha \le \varepsilon^2)$ we will have to study the inner equation close to $u = i\pi/2$ and in others $(\varepsilon^2 \ll \alpha \le \alpha_0 < 1)$ close to $u = \rho_-$.

We start with the case $0 < \alpha \ll \varepsilon^2$, which corresponds to a wide analyticity strip. As we have explained Proposition 2.1, the exponential coefficient in the Melinkov function is given by the imaginary part of the singularity of the unperturbed separatrix. Namely, the analyticity strip is so wide that the Melnikov function behaves as in the entire case $\alpha = 0$. Then, it can be easily seen that the singular change that one has to perform is given by $u = i\pi/2 + \varepsilon z$ and $\varphi^{u,s}(z, \tau) = \varepsilon T^{u,s}(i\pi/2 + \varepsilon z, \tau)$ (see [OSS03]). Now the function φ is the solution of a new Hamilton-Jacobi equation. If we let $\varepsilon \to 0$ in this equation we obtain the inner equation

$$\partial_\tau \varphi_0 - \frac{z^2}{8} \left(\partial_z \varphi_0\right)^2 + \frac{2}{z^2}(1 - \mu \sin \tau) = 0, \tag{113}$$

which does not depend either on ε or α.

Certain solutions $\varphi_0^{u,s}$ of this equation are the candidates to be the first order of the functions $\varphi^{u,s}$, namely the first order of the parameterizations of the invariant manifolds close to the singularity. Then, the study of their difference would give the first order of the difference between the invariant manifolds. This equation was already studied in [OSS03] using Resurgence Theory and in [Bal06] using classical functional analysis techniques.

The case $\alpha \sim \varepsilon^2$ is the transition case. In Proposition 2.1 we have seen that the exponential coefficient is given by $\pi/2$ but the residuums of both $u = \rho_\pm$ make a contribution to the Melnikov function. Let us assume that

$$\alpha(\varepsilon) = \alpha_* \varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right).$$

The natural change to inner variable is given by $u = i\pi/2 + \varepsilon z$ and the rescaling in the generating function by $\varphi^{u,s}(z, \tau) = \varepsilon T^{u,s}(i\pi/2 + \varepsilon z, \tau)$, obtaining , taking into account (21), a new inner equation

$$\partial_\tau \varphi_0 - \frac{z^2}{8} \left(\partial_z \varphi_0\right)^2 + \frac{2}{z^2} + \mu \frac{i}{2\left(z^2 + 2i\alpha_*\right)} \sin \tau = 0.$$

Finally we deal with the case $\alpha \gg \varepsilon^2$. We first consider the case α independent of ε, and from it we will deduce the case $\alpha \sim \varepsilon^\nu$ with $\nu \in (0,1)$. If we take a fixed α, we have seen that one has to study the parameterizations of the invariant manifolds close to $u = \rho_-$ and the singularity $u = \rho_+$ does not play any role in the size of the splitting. Note that now the limiting case is $\eta = 1$, that is, we deal with equation

$$\varepsilon^{-1}\partial_\tau T + \frac{\cosh^2 u}{8}\left(\partial_u T\right)^2 - \frac{4}{\cosh^2 u} + \mu\varepsilon\Psi(u)\sin\tau = 0.$$

As a first step, we can expand the parameterization of the invariant manifold $T^{u,s}(u,\tau)$ as a power series of ε. It can be easily seen that

$$T^{u,s}(u,\tau) \sim \sum_{k\geq 0} \varepsilon^k T_k(u,\tau)$$

where T_0 corresponds to the separatrix (55). One can see that the terms of the series satisfy that for $k \geq 0$,

$$\partial_u T_k(u,\tau) \sim \frac{1}{(u-\rho_-)^k}$$

and therefore they all become of the same size at a distance ε of the singularity. Nevertheless, in this case one has to be more careful, since if one considers the asymptotic size of the power series terms of the generating function T instead of $\partial_u T$, we have that

$$T_0(u) \sim 1 \quad \text{and} \quad T_k(u,\tau) \sim \frac{1}{(u-\rho_-)^{k-1}} \quad \text{for } k \geq 1.$$

Namely, at a distance of order $\mathcal{O}(\varepsilon)$ of $u = \rho_-$ all the terms with $k \geq 1$ become of the same order but T_0 is still bigger. Therefore, it is more convenient to deal with the function $Q = T - T_0$. Now, one can consider the change to inner variable $u = \rho_- + \varepsilon z$ and the rescaling in the generating function by $\phi(z,\tau) = \varepsilon Q(\rho_- + \varepsilon z, \tau)$, obtaining the inner equation

$$\partial_\tau\phi + \partial_z\phi + \frac{\cosh^2 \rho_-}{8}\left(\partial_z\phi\right)^2 - \mu\frac{\delta_2(\alpha)}{z}\sin\tau = 0,$$

where $\delta_2(\alpha)$ is the funtion defined in (28).

Proceeding analogously, one can deduce the inner equation for the case $\alpha = \alpha_*\varepsilon^\nu$ with $\nu \in (0,2)$. Recall that for this range of α the limiting case was $\eta = 1 - \nu/2$ (see Remark 2.9). Namely, we deal with the equation following equation in $Q(u,\tau) = T(u,\tau) - T_0(u)$,

$$\varepsilon^{-1}\partial_\tau Q + \partial_u Q + \frac{\cosh^2 u}{8}\left(\partial_u T\right)^2 + \mu\varepsilon^{1-\frac{\nu}{2}}\Psi(u)\sin\tau = 0.$$

As in the previous case, we study the inner equation close to $u = \rho_-$ and therefore the change to inner variable is still $u = \rho_- + \varepsilon z$. Nevertheless, now the rescaling in the generating function is given by $\phi(z,\tau) = \varepsilon^{1-\nu/2}Q(\rho_- + \varepsilon z, \tau)$. Then, proceeding as before and taking into account the definition of $\delta_2(\alpha)$ in (28), one can obtain the following inner equation

$$\partial_\tau\phi + \partial_z\phi - \frac{i\alpha_0}{4}\left(\partial_z\phi\right)^2 + \frac{\mu}{(1+i)\sqrt{\alpha_0}}\frac{1}{z}\sin\tau = 0.$$

One could expect that studying all these inner equations, one could obtain the true first asymptotic order of the difference between the perturbed invariant manifolds.

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