

# Synthesis of Spatial RPRP Closed Linkages for a Given Screw System

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*The dimensional synthesis of spatial chains for a prescribed set of positions can be applied to the design of parallel robots by joining the solutions of each serial chain at the end effector. This design method does not provide with the knowledge about the trajectory between task positions and, in some cases, may yield a system with negative mobility. These problems can be avoided for some overconstrained but movable linkages if the finite screw system associated to the motion of the linkage is known. The finite screw system defining the motion of the robot is generated by a set of screws, which can be related to the set of finite tasks positions traditionally used in the synthesis theory. The interest of this paper lies in presenting a method to define the whole workspace of the linkage as the input task for the exact dimensional synthesis problem. This method is applied to the spatial RPRP closed linkage, for which one solution exists.*

## Nomenclature

- S Line or dual vector, defined using Plücker coordinates.
- s Vector
- $\hat{S}$  Quaternion or dual quaternion
- $\hat{w}$  Dual number
- [ $M$ ] Matrix

## 1 Introduction

Synthesis of parallel robots has focused mainly on type or structural synthesis, using group theory, screw theory, or geometric methods, see for instance [1], [2], [3]. Dimensional synthesis examples exist, mainly for optimizing performance indices [4], [5], [6] or for reachable workspace sizing [7], [8], [9]; see also [10] for a comprehensive approach.

The dimensional synthesis of spatial serial chains for a prescribed set of positions can be used for the design of parallel robots by synthesizing all supporting legs for the same set of positions. There are a few examples of finite-position dimensional synthesis of parallel robots in the lit-

erature, most of them doing partial synthesis. Wolbrecht et al. [11] perform synthesis of 3-RRS, 4-RRS and 5-RRS symmetric parallel manipulators; Kim and Tsai [12] and Rao [13] solve the partial kinematic synthesis of a 3-RPS parallel manipulator. This method has been successfully applied mainly to special parallel systems with imposed symmetry. In general, the method does not allow the control of the final trajectory of the parallel system; in the most extreme cases, it may yield a system with negative mobility, that can be assembled at each task positions but cannot be driven from task position to task position.

The kinematic mapping is used for the synthesis of planar and spherical linkages in order to state design equations and to provide a tool for visualizing the workspace and trajectories of the linkage. See Ravani and Roth [14] and more recent applications by Hayes [15], Schröcker [16] and Wu and Ge [17]. For spatial motion, Study's kinematic mapping is used to obtain simplified equations for analysis and synthesis, see Husty et al. [18] and [19]. However, the kinematic image for the spatial motion is a six-dimensional quadric and that makes the visualization of workspaces and trajectories difficult for the designer.

In this paper, the workspace of the linkage is visualized as a set of finite screws corresponding to the set of finite displacements of the end-effector. One interesting question is whether the finite-screw surfaces generated by a set of task positions can give any information for the synthesis of parallel robots and, in particular, of some overconstrained closed linkages. Using Parkin's definition for pitch [20], the screws corresponding to finite displacements of some linkages can form screw systems. Huang [21] showed that the single RR chain forms a finite screw system of third order; however, the set of finite displacements of the coupler of the Bennett linkage form a cylindroid, which is a general 2-system of screws [22]. Baker [23] has also studied the motion of the Bennett linkage. Perez and McCarthy [24] used two arbitrary displacements to generate the cylindroid of finite screws associated to the Bennett linkage in order to perform dimen-

sional synthesis.

In this paper, the focus is on the simplest of the over-constrained linkages, the closed spatial RPRP linkage. This linkage is one of the Delassus linkages, later studied by Waldron [25] and Baker [26]. Recently, Huang [27] has shown that the set of screws corresponding to displacements of this linkage forms a 2-screw system. We use this result in order to synthesize RPRP linkages with positive mobility and for a given shape of the screw system of relative displacements. In order to do so, we state the design equations using the Clifford algebra of dual quaternions [28]. The dual quaternion expression can be easily related to the screw system and it is also used to assign the magnitude to the screws in order to obtain the correspondence between the screw system and the trajectory of the end-effector. The design yields a single RPRP linkage.

## 2 Clifford Algebra Equations for the Synthesis

The approach used in this paper for stating design equations is based on the method of Lee and Mavroidis [29]. They equate the forward kinematics of a serial chain to a set of goal displacements and consider the Denavit-Hartenberg parameters as variables. A more efficient formulation for our purposes consists of stating the forward kinematics of relative displacements using the even Clifford subalgebra of the projective space  $C^+(P^3)$ , also known as dual quaternions [28].

The Plücker coordinates  $S = (s, \mathbf{c} \times \mathbf{s})$  of a line can be identified with the Clifford algebra element  $S = \mathbf{s} + \epsilon \mathbf{c} \times \mathbf{s}$ , with  $\epsilon^2 = 0$ . Similarly, a screw  $J$  becomes the element  $J = (1 + \mu \epsilon)S$ , where  $\mu$  is the pitch. Using the Clifford product we can compute the exponential of the screw  $\frac{\theta}{2}J$ ,

$$e^{\frac{\theta}{2}J} = \left( \cos \frac{\theta}{2} - \frac{d}{2} \sin \frac{\theta}{2} \epsilon \right) + \left( \sin \frac{\theta}{2} + \frac{d}{2} \cos \frac{\theta}{2} \epsilon \right) S = \cos \frac{\hat{\theta}}{2} + \sin \frac{\hat{\theta}}{2} S. \quad (1)$$

The exponential of a screw defines a unit dual quaternion, which can be identified with a relative displacement from an initial position to a final position in terms of a rotation around and a slide along an axis.

### 2.1 Forward Kinematics

For a serial chain with  $n$  joints, in which each joint can rotate an angle  $\theta_i$  and slide a distance  $d_i$ , around and along the axis  $S_i$ ,  $i = 1, \dots, n$ , the forward kinematics of relative displacements (with respect to a reference position) can be expressed as the composition of Clifford algebra elements corresponding to the motion of each joint. Let  $\vec{\theta}_0$  and  $\vec{d}_0$  be the joint parameters of this chain when in the reference configuration, so we have  $\vec{\Delta\hat{\theta}} = (\vec{\theta} - \vec{\theta}_0 + (\vec{d} - \vec{d}_0)\epsilon)$ . Then, the movement from this reference configuration is defined by

$$\hat{Q}(\vec{\Delta\hat{\theta}}) = e^{\frac{\Delta\hat{\theta}_1}{2} S_1} e^{\frac{\Delta\hat{\theta}_2}{2} S_2} \dots e^{\frac{\Delta\hat{\theta}_n}{2} S_n} = \left( \cos \frac{\Delta\hat{\theta}_1}{2} + \sin \frac{\Delta\hat{\theta}_1}{2} S_1 \right) \left( \cos \frac{\Delta\hat{\theta}_2}{2} + \sin \frac{\Delta\hat{\theta}_2}{2} S_2 \right) \dots \left( \cos \frac{\Delta\hat{\theta}_n}{2} + \sin \frac{\Delta\hat{\theta}_n}{2} S_n \right). \quad (2)$$

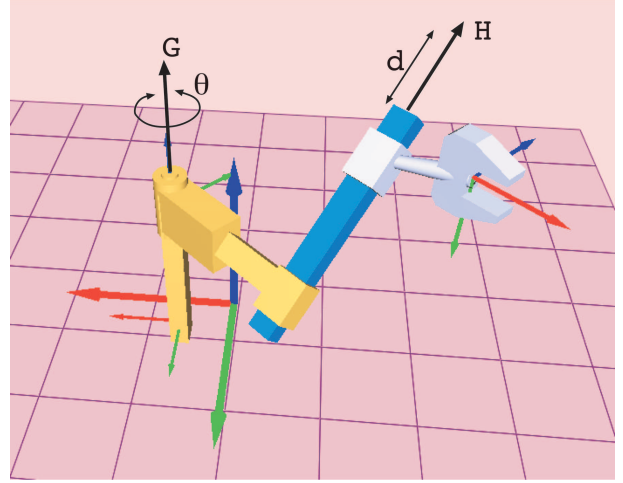


Fig. 1. The RP serial chain

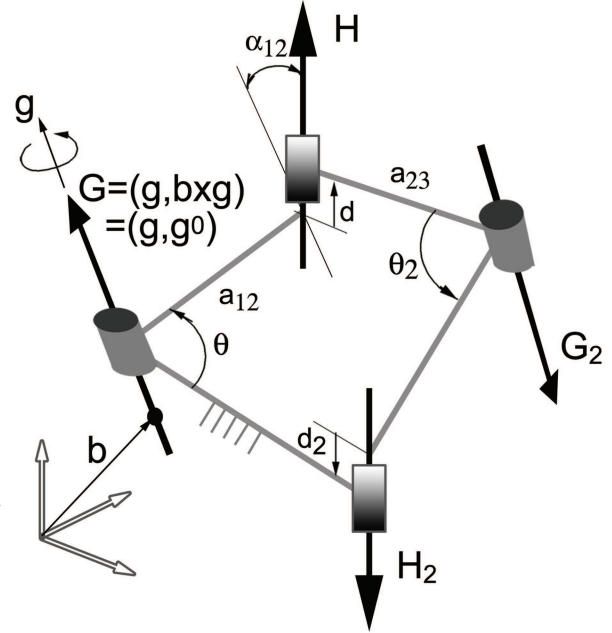


Fig. 2. The RPRP closed linkage

The RPRP linkage has a mobility  $M = -2$  using the Kutzbach-Gruebler formula; however, for certain dimensions of the links, it moves with one degree of freedom. Waldron [25] shows that the RPRP linkage will have positive mobility only if both revolute joints are parallel, and the directions of the prismatic joints are symmetric with respect to the plane containing the revolute joints.

The RPRP linkage can be seen as a serial RP chain and a serial PR chain joined at their end-effectors. The RP serial chain consists of a revolute joint followed by a prismatic joint. Figure 1 shows the RP serial chain and Figure 2 shows a sketch of the RPRP linkage with its axes. In the PR serial chain, the order of the joints in the chain is switched.

For both the RP and PR serial chains, let  $G = \mathbf{g} + \epsilon \mathbf{g}^0$  be the revolute joint axis, with rotation  $\theta$ , and  $H = \mathbf{h} + \epsilon \mathbf{h}^0$  the prismatic joint axis, with slide  $d$ . Notice that, for synthesis

purposes, the location of the slider, given by  $\mathbf{h}^0$ , is irrelevant. The Clifford algebra forward kinematics equations for the RP chain are

$$\begin{aligned}\hat{Q}_{RP}(\Delta\theta, \Delta d) &= (\cos \frac{\Delta\theta}{2} + \sin \frac{\Delta\theta}{2} \mathbf{G})(1 + \epsilon \frac{\Delta d}{2} \mathbf{H}) \\ &= (\cos \frac{\Delta\theta}{2} + \sin \frac{\Delta\theta}{2} \mathbf{g}) + \epsilon (\frac{\Delta d}{2} \cos \frac{\Delta\theta}{2} \mathbf{h} + \\ &\quad \sin \frac{\Delta\theta}{2} \mathbf{g}^0 + \frac{\Delta d}{2} \sin \frac{\Delta\theta}{2} (\mathbf{g} \times \mathbf{h} - \mathbf{g} \cdot \mathbf{h})).\end{aligned}\quad (3)$$

For the PR chain, the only difference is a negative sign in the cross product. In Eq. (3), the angle and slide are measured from a reference configuration.

## 2.2 Design Equations and Counting

Given a set of  $m$  task positions expressed as relative displacements,  $\hat{P}_{1j} = \cos \frac{\Delta\hat{\theta}_{1j}}{2} + \sin \frac{\Delta\hat{\theta}_{1j}}{2} \mathbf{P}_{1j}$ ,  $j = 2, \dots, m$ , we equate them to the forward kinematics equations in Eq. (2),

$$\hat{P}_{1j} = e^{\frac{\Delta\hat{\theta}_{1j}}{2} S_1} e^{\frac{\Delta\hat{\theta}_{2j}}{2} S_2} \dots e^{\frac{\Delta\hat{\theta}_{nj}}{2} S_n}, \quad j = 2, \dots, m \quad (4)$$

in order to create the design equations.

The result is  $8(m-1)$  design equations. The design variables that determine the dimensions of the chain are the  $n$  joint axes  $S_i$ ,  $i = 1, \dots, n$ , in the reference configuration. In addition, the equations contain the  $n(m-1)$  pairs of joint parameters  $\Delta\hat{\theta}_{ij} = \Delta\theta_{ij} + \Delta d_{ij}\epsilon$ , which are also unknown.

For the RP (and similarly for the PR) serial chain, the design equations are

$$\hat{Q}_{RP}(\Delta\theta^j, \Delta d^j) = \hat{P}_{1j}, \quad j = 2, \dots, m. \quad (5)$$

The counting of independent equations and unknowns defines the maximum number of arbitrary positions  $m$  that can be reached, based only on the type and number of joints of the serial chain, see [30] for details. Consider a serial chain with  $r$  revolute and  $p$  prismatic joints. The maximum number of task positions is given by

$$m = \frac{3r + p + 6}{6 - (r + p)}. \quad (6)$$

For serial chains with less than three revolute joints, the structure of semi-direct product of the composition of displacements needs to be considered, and the maximum number of rotations  $m_R$  needs to be calculated too. Assuming that the orientations are given and that both the directions of the revolute joints and the angles to reach the task orientations are known, we can count, in a similar fashion, the number of translations  $m_T$  that the chain can be defined for.

$$m_R = \frac{3+r}{3-r}, \quad m_T = \frac{2r+p+3}{3-p}. \quad (7)$$

In order to determine the maximum number of task positions for the RP and PR chains, we apply Eq. (6) to obtain  $m = 2.5$  task positions. Additional information is obtained using Eq. (7) to compute  $m_R = 2$  task rotations, and  $m_T = 3$  task translations. Hence, we can define one arbitrary relative displacement and a second relative displacement whose orientation is not general.

## 3 Screw System for the RPRP Linkage

In the context of this paper, a finite-screw surface is a ruled surface in which the lines with their associated pitch correspond to relative displacements. A screw surface will be a screw system if it is closed under addition and scalar multiplication, that is, if every screw of the set can be written as a linear combination of screws belonging to it.

### 3.1 The finite Screw System Generated by the Motion of the RPRP Linkage

The linear combination of two arbitrary screws representing relative displacements form a 2-system known as the cylindroid, which turns out to be the manifold for the relative displacements of the closed 4R linkage. Huang [27], by intersecting the 3-systems associated with the finite displacements of the RP and PR dyads, shows that the screw surface of the closed RPRP linkage forms a 2-system of a special type, the fourth special type according to Hunt [31], also known as 2-IB [32]. The screws of this system are parallel, coplanar screws whose pitches vary linearly with their distance.

The screw system corresponding to the RPRP linkage can be obtained in a quite straightforward way by using the composition of relative displacements at each joint, as expressed in Eq.(3). Let us denote the resulting displacements

$$\begin{aligned}\hat{Q}_{RP}(\Delta\theta, \Delta d) &= \cos \frac{\Delta\psi}{2} + \sin \frac{\Delta\psi}{2} S_{RP} = \\ &= (\cos \frac{\Delta\psi}{2} - \epsilon \frac{\Delta t}{2} \sin \frac{\Delta\psi}{2}) + \sin \frac{\Delta\psi}{2} (1 + \epsilon \frac{\Delta t}{2} \frac{1}{\tan \frac{\Delta\psi}{2}}) S_{RP},\end{aligned}\quad (8)$$

where  $\Delta\psi$  and  $\Delta t$  are the angle about and slide along the screw axis  $S_{RP}$  of the resulting relative displacement. Expand the product in Eq.(3) and separate the dual scalar and the dual vector part, so that

$$\begin{aligned}\cos \frac{\Delta\psi}{2} - \epsilon \frac{\Delta t}{2} \sin \frac{\Delta\psi}{2} &= \cos \frac{\Delta\theta}{2} - \epsilon \frac{\Delta d}{2} \sin \frac{\Delta\theta}{2} \mathbf{G} \cdot \mathbf{H} \\ \sin \frac{\Delta\psi}{2} (1 + \epsilon \frac{\Delta t}{2} \frac{1}{\tan \frac{\Delta\psi}{2}}) S_{RP} &= \\ \sin \frac{\Delta\theta}{2} \mathbf{G} + \epsilon \frac{\Delta d}{2} \cos \frac{\Delta\theta}{2} \mathbf{H} + \epsilon \frac{\Delta d}{2} \sin \frac{\Delta\theta}{2} \mathbf{G} \times \mathbf{H}.\end{aligned}\quad (9)$$

For studying the screw system, we are only interested in the dual-vector part of Eq.(9), which defines the lines with their associated pitch.

The resulting screw is parameterized by joint variables  $\Delta\theta$  of the revolute joint and  $\Delta d$  corresponding to the prismatic joint. We denote the expression in Eq.(9) as the *finite kinematic generator* of the screw system. In the case of finite displacements, the screw systems correspond to subspaces of a projective space, and so we take the unit screws as representatives by dividing the dual vector of Eq.(9) by  $\sin \frac{\Delta\Psi}{2}$ . From the real part of the first equation in (9) we can see that  $\sin \frac{\Delta\Psi}{2} = \pm \sin \frac{\Delta\theta}{2}$ , so that

$$\pm(1 + \varepsilon \frac{\frac{\Delta r}{2}}{\tan \frac{\Delta\Psi}{2}})S_{RP} = G + \varepsilon \frac{\frac{\Delta d}{2}}{\tan \frac{\Delta\theta}{2}}H + \varepsilon \frac{\Delta d}{2}G \times H. \quad (10)$$

If we give values to the joint variables  $\Delta\theta$  and  $\Delta d$ , we will generate a screw surface of dimension 3 of a special type, in which all the screws have parallel directions.

It is easy to show that the set of screws generated by the motion of the RP serial chain is indeed a screw system, that is, closed under addition and multiplication by scalar. Generate two unit screws  $S_1$  and  $S_2$  using Eq.(10) and perform the linear combination with real constants  $k_1$  and  $k_2$ . We normalize the resulting screw by dividing by  $k_1 + k_2$  to obtain

$$(1 + \varepsilon \frac{\frac{\Delta r_3}{2}}{\tan \frac{\Delta\Psi_3}{2}})S_3 = \frac{1}{k_1 + k_2}(k_1S_1 + k_2S_2) \\ = \mathbf{g} + \varepsilon(\mathbf{g}^0 + \varepsilon \frac{\frac{\Delta d_c}{2}}{\tan \frac{\Delta\theta_c}{2}}\mathbf{h} + \varepsilon \frac{\Delta d_c}{2}\mathbf{g} \times \mathbf{h}), \quad (11)$$

where

$$\Delta d_c = \frac{k_1}{k_1 + k_2}\Delta d_1 + \frac{k_2}{k_1 + k_2}\Delta d_2, \\ \tan \frac{\Delta\theta_c}{2} = \frac{k_1 \frac{\Delta d_1}{2} + k_2 \frac{\Delta d_2}{2}}{k_1 \frac{\frac{\Delta d_1}{2}}{\tan \frac{\Delta\theta_1}{2}} + k_2 \frac{\frac{\Delta d_2}{2}}{\tan \frac{\Delta\theta_2}{2}}}, \quad (12)$$

hence the linear combination belongs to the screw system generated by the relative motion of the RP chain. Next is to calculate the dimension of this screw system. Notice that the direction of all unit screws generated is equal to  $\mathbf{g}$ , then we just need to look at the dimension of the dual part. Generate three screws with values  $\Delta\theta_i$  and  $\Delta d_i$ ,  $i = 1, 2, 3$ . For the three vectors corresponding to the dual part to be linearly independent, the determinant of the column vector matrix must be different from zero. We obtain the value for the determinant

$$\det([s_i^0]) = \mathbf{g} \cdot \mathbf{h} \quad \mathbf{g}^0 \cdot \mathbf{h} \left( \frac{\frac{\Delta d_1}{2}}{\tan \frac{\Delta\theta_1}{2}} \left( \frac{\Delta d_3}{2} - \frac{\Delta d_2}{2} \right) + \right. \\ \left. \frac{\frac{\Delta d_2}{2}}{\tan \frac{\Delta\theta_2}{2}} \left( \frac{\Delta d_1}{2} - \frac{\Delta d_3}{2} \right) + \frac{\frac{\Delta d_3}{2}}{\tan \frac{\Delta\theta_3}{2}} \left( \frac{\Delta d_2}{2} - \frac{\Delta d_1}{2} \right) \right), \quad (13)$$

which is different from zero when the angle  $\theta$  and the slide  $d$  are independent, except for special parallel or perpendicular arrangements of the joints. Hence, a maximal set of independent screws has cardinality 3.

In order to generate the screw system of the RPRP linkage, we impose the motion constraint relations between the joint variables. The conditions are derived in [27] from the closure equations of the linkage, to obtain

$$d = d_2, \\ \theta = \pm(\theta_2 - \pi), \\ \frac{\tan \frac{\theta}{2}}{d} = \frac{\sin \alpha_{12}}{a_{12} \pm a_{23}} = k, \quad (14)$$

where the twist angle  $\alpha_{12}$  and link lengths  $a_{12}, a_{23}$  are shown in Figure 2; the angles  $\theta, \theta_2$  and slides  $d, d_2$ , also shown in Figure 2, have to be measured according to the convention [27] from the previous common normal line. The plus/minus sign corresponds to folded and unfolded linkages. For our purposes, only the third condition is needed. It is important to notice that this condition applies to absolute values of the joint variables, measured from the previous common normal line as stated before.

In order to use this relation, substitute  $\Delta d = d - d_0$ ,  $\Delta\theta = \theta - \theta_0$  to apply Eq.(14) so that the resulting expression depends only on  $d$ , for instance. For all possible values of the slide  $d$ , this generates a 2-IB system of screws [31, 32]. This can be checked numerically by generating 6 random screws and computing the rank of the matrix that has the screws as columns. In this case, the rank is 2. It can also be shown that when the third condition in Eq.(14) holds, the determinant in Eq.(13) is equal to zero.

The unit screws of the system can be related to spatial displacements if we add the value of the magnitude of the screw. For finite displacement screws, the value of the magnitude related to each screw is unique; this is due to the fact that finite screw systems are projective subspaces [33]. The information about the magnitude corresponding to each screw can be extracted from the scalar part of the dual quaternion product in Eq.(9). This equation relates the rotation associated to the resulting screw,  $\Delta\Psi$ , to the joint variables of the kinematic chain. For the RPRP chain, this yields  $\sin \frac{\Delta\Psi}{2} = \pm \sin \frac{\Delta\theta}{2}$  as noted before.

Figure 3 illustrates the nonlinear relation between the screws generated by an RPRP closed chain and the corresponding set of absolute positions of the end effector (for those positions we assume that the reference configuration is the identity). We use as values for the linkage those of an example from [27].

### 3.2 The Finite Screw System Generated as a Linear Combination of Two Screws

It has been shown that the relative displacements of the RPRP chain generate a 2-IB finite screw system. By definition, this same screw system can be generated as the linear combination of two screws with same direction and arbitrary

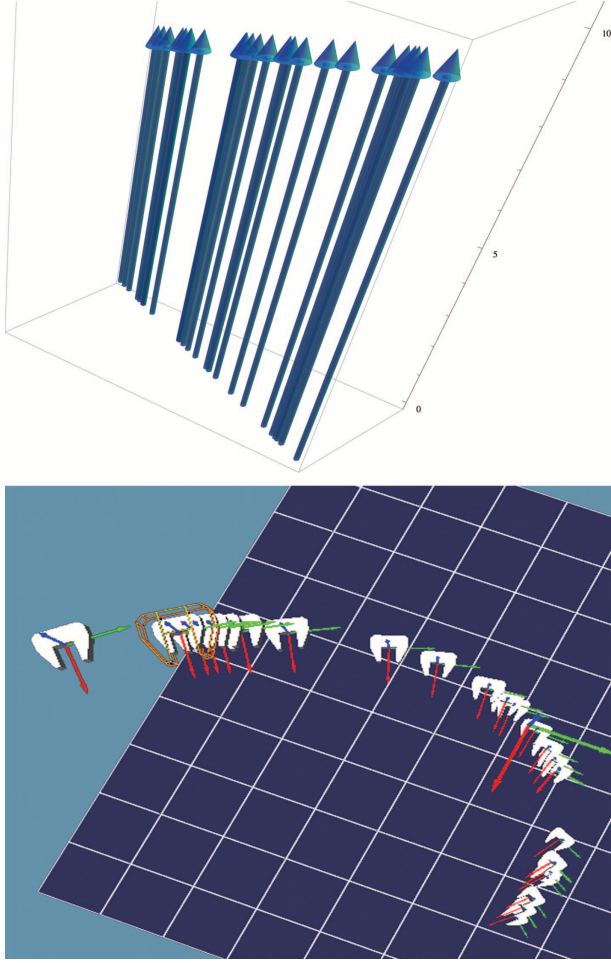


Fig. 3. Screw system generated by the RPRP linkage, above; corresponding absolute displacements, below

location and (possibly) finite pitches. From a synthesis point of view, the key is that this coincides with the results of the counting in section 2.2. The task positions defined for the synthesis of the RP (or PR) chain are two relative displacements with same direction and, in general, finite pitches.

The screw system is characterized by the pitch distribution as a linear function of the distance between screws along a common normal. In this derivation we can parameterize the results as a function of the angle  $\theta$  or the slide  $d$ .

Consider the screw  $S_{RP} = (1 + \varepsilon p)S_{RP} = \mathbf{s}_{RP} + \varepsilon \mathbf{s}_{RP}^0$ . The pitch is obtained by computing  $p = \frac{\mathbf{s}_{RP} \cdot \mathbf{s}_{RP}^0}{\mathbf{s}_{RP} \cdot \mathbf{s}_{RP}}$  in Eq.(10),

$$p = \frac{1 + k^2 dd_0}{2k} \mathbf{g} \cdot \mathbf{h} = \frac{1}{2k} (1 + \tan \frac{\theta_0}{2} \tan \frac{\theta}{2}) \mathbf{g} \cdot \mathbf{h}. \quad (15)$$

For two screws  $S_A$  and  $S_B$ , the difference in the pitches is given by

$$p_B - p_A = \frac{kd_0 \mathbf{g} \cdot \mathbf{h}}{2} (d_B - d_A) = \frac{\tan \frac{\theta_0}{2} \mathbf{g} \cdot \mathbf{h}}{2k} (\tan \frac{\theta_B}{2} - \tan \frac{\theta_A}{2}) \quad (16)$$

The distance between two screws along the common normal is calculated by finding the perpendicular point on

the axis,  $\mathbf{c}_{RP} = \frac{\mathbf{s}_{RP} \times \mathbf{s}_{RP}^0}{\mathbf{s}_{RP} \cdot \mathbf{s}_{RP}}$  and computing the norm of the difference for two of them,

$$\begin{aligned} \mathbf{c}_B - \mathbf{c}_A &= \left( \frac{kd_0}{2} \mathbf{g} \times \mathbf{h} + \frac{1}{2} (\mathbf{g} \mathbf{g} \cdot \mathbf{h} - \mathbf{h} \mathbf{h}) \right) (d_B - d_A) \\ &= \frac{1}{2k} (\tan \frac{\theta_0}{2} \mathbf{g} \times \mathbf{h} + \mathbf{g} \mathbf{g} \cdot \mathbf{h} - \mathbf{h} \mathbf{h}) (\tan \frac{\theta_B}{2} - \tan \frac{\theta_A}{2}), \end{aligned} \quad (17)$$

and

$$\begin{aligned} \|\mathbf{c}_B - \mathbf{c}_A\| &= \frac{1}{2} |d_B - d_A| \sqrt{\sin^2 \alpha_{12} (1 + d_0^2 k^2)} \\ &= \frac{1}{2} \left| \tan \frac{\theta_B}{2} - \tan \frac{\theta_A}{2} \right| \sqrt{\frac{\sin^2 \alpha_{12}}{k^2 \cos^2 \frac{\theta_0}{2}}} \end{aligned} \quad (18)$$

The slope of the (linear) pitch distribution is computed as

$$K = \frac{p_B - p_A}{\|\mathbf{c}_B - \mathbf{c}_A\|}, \quad (19)$$

and for the RPRP chain we can simplify the expression to

$$K = \pm \frac{\cos \alpha_{12}}{\sqrt{d_0^2 \sin^2 \alpha_{12} + (a_{12} + a_{23})^2}} d_0 = \pm \frac{\cos \alpha_{12}}{\sin \alpha_{12}} \sin \frac{\theta_0}{2}. \quad (20)$$

Notice that the slope is constant and depends on the initial configuration. The sign is negative if  $d_B < d_A$  or if  $\tan \frac{\theta_B}{2} < \tan \frac{\theta_A}{2}$ .

The screw system is related to the displacement of the RPRP linkage by computing the distribution of the magnitude with respect to the known pitch, distance and magnitude of the screws  $S_A$  and  $S_B$  used to define the distribution. Consider the magnitude of the screws as  $m = \sin \frac{\Delta \theta}{2} = \pm \sin \frac{\Delta \theta}{2}$ . Knowing the magnitude and pitch of screws  $S_A$  and  $S_B$ , we can solve for the magnitude  $m_C$  of a screw  $S_C$  using Eq.(18)

$$m_C = \pm \sin \frac{\Delta \theta_C}{2} = \frac{X}{\sqrt{1 + X^2}}, \quad (21)$$

where the factor  $X$  is simplified using Eq.(15) to (20) to

$$X = \frac{\|\mathbf{c}_B - \mathbf{c}_A\| p_A \tan \frac{\Delta \theta_A}{2} - \|\mathbf{c}_C - \mathbf{c}_A\| (p_A \tan \frac{\Delta \theta_A}{2} \pm p_B \tan \frac{\Delta \theta_B}{2})}{\|\mathbf{c}_B - \mathbf{c}_A\| p_C} \quad (22)$$

The positive and negative signs in the expression of  $X$  correspond to the sign of the  $\tan \frac{\theta_B}{2} - \tan \frac{\theta_A}{2}$  factor.

### 3.3 Defining the Kinematic Task as a Finite Screw System

It is now possible to define the screw system as the input task for the dimensional synthesis of the RPRP closed chain.

As a linear subspace, it is easy to shape the task at our convenience. Once we have a satisfactory screw system, any two screws from it can be selected to perform the dimensional synthesis and to obtain a finite number of solutions (in this case, just one).

We have several strategies for shaping the screw system. For instance, we can define a first relative displacement,  $\hat{S}_{12} = \cos \frac{\Delta\psi}{2} + \sin \frac{\Delta\psi}{2} (\mathbf{s}_{12} + \varepsilon \mathbf{s}_{12}^0)$ . The rotation axis of the displacement,  $\mathbf{s}_{12}$  is common to both  $\hat{S}_{12}$  and the second relative displacement. We set  $\mathbf{s}_{12} = \mathbf{s}_{13}$  and select a rotation angle to define the relative rotation  $\hat{s}_{13}$ .

We can then set the slope of the pitch distribution in order to shape the screw system. The pitch for the finite displacement screws is [20]

$$p_{1i} = \frac{\frac{\Delta t_{1i}}{2}}{\tan \frac{\Delta\psi_{1i}}{2}}, \quad (23)$$

directly calculated from the dual quaternion using  $p_{1i} = \frac{\mathbf{s}_{1i} \cdot \mathbf{s}_{1i}^0}{\mathbf{s}_{1i} \cdot \mathbf{s}_{1i}}$ . Similarly, a point on the screw axis is calculated as

$$\mathbf{c}_{1i} = \frac{\mathbf{s}_{1i} \times \mathbf{s}_{1i}^0}{\mathbf{s}_{1i} \cdot \mathbf{s}_{1i}}. \quad (24)$$

Define the slope of the distribution as  $K = \frac{p_{13} - p_{12}}{\|\mathbf{c}_{13} - \mathbf{c}_{12}\|}$ , according to Eq.(20). If we set the value of  $K$ , we can solve for  $\Delta t_{13}$  in order to define the pitch of the second relative displacement, the location of its screw axis being defined. This is one possible way of defining the screw system. Converting from this to absolute displacements we can easily check whether the trajectory for the synthesis is acceptable.

Any other strategy to basically define a triangle in space can be used. The purpose of this is to have a better control on the shape of the trajectory of the linkage than the one given by just two separate finite positions.

#### 4 Dimensional Synthesis of the RPRP Linkage for a Prescribed Screw System

The synthesis of the RP, and similarly, PR chains, is simple and yields one solution. In general, the task positions are expressed as relative displacements  $\hat{Q}_{1i} = \hat{Q}_i \hat{Q}_1^{-1}$  with respect to the first task position  $\hat{Q}_1$ . For the RP or PR chains, the maximum number of task positions that we can define is  $i = 3$ , as explained in section 2.2. Given an arbitrary relative displacement  $\hat{Q}_{12} = (q_{12}^w + \mathbf{q}_{12}) + \varepsilon(q_{12}^{w0} + \mathbf{q}_{12}^0)$  and a second displacement  $\hat{Q}_{13} = (q_{13}^w + \mathbf{q}_{13}) + \varepsilon(q_{13}^{w0} + \mathbf{q}_{13}^0)$  such that both have same direction and a given pitch distribution, we equate them to the forward kinematics in Eq.(3). We can solve for the direction of the revolute joint  $\mathbf{g}$  and the rotation angles,

$$\mathbf{g} = \frac{\mathbf{q}_{12}}{\|\mathbf{q}_{12}\|}, \quad \tan \frac{\Delta\theta_{1i}}{2} = \frac{\|\mathbf{q}_{1i}\|}{q_{1i}^w}, \quad i = 2, 3. \quad (25)$$

The equations corresponding to the dual part are linear in the moment of the revolute joint,  $\mathbf{g}^0$ ,

$$\mathbf{g}^0 = \frac{1}{\sin \frac{\Delta\theta_{1i}}{2}} \left( \mathbf{q}_{1i}^0 - \frac{\Delta d_{1i}}{2} \left( \cos \frac{\Delta\theta_{1i}}{2} \mathbf{h} + \sin \frac{\Delta\theta_{1i}}{2} \mathbf{g} \times \mathbf{h} \right) \right), \quad i = 1, 2. \quad (26)$$

Equating the solution of  $\mathbf{g}^0$  for both relative displacements, we can solve linearly for  $\mathbf{h}$  as a function of the slides  $\Delta d_{12}$ ,  $\Delta d_{13}$ . The relation between the slides is given by the pitch condition,

$$\frac{q_{12}^{w0}}{\frac{\Delta d_{12}}{2} \sin \frac{\Delta\theta_{12}}{2}} = \frac{q_{13}^{w0}}{\frac{\Delta d_{13}}{2} \sin \frac{\Delta\theta_{13}}{2}} \quad (27)$$

Imposing  $\|\mathbf{h}\| = 1$ , we can solve for the slides to obtain one solution. Using the same process, we can solve for the PR serial chain.

## 5 Examples

Two examples are presented below. The first one is performed to check the method, while for the second one, a fully general task is used.

### 5.1 First Example

In this first case, the two relative displacements are generated using the RP chain presented in Huang [27]. In this example, the revolute joint is located at the origin and the prismatic joint is located along the  $x$  direction with a twist angle of  $\alpha_{12} = \pi/5$  and link length  $a_{12} = 5$ . We use the loop condition in Eq.(14) and random values for the slide to generate the set of relative displacements of the corresponding RPRP closed chain. From those, we randomly select two displacements, the ones in Table 1.

Table 1. Goal relative displacements for the RP and PR chains, Huang's example

$$(0, 0, -0.05, 0.99) + \varepsilon(0.02, 0.37, -0.51, -0.03)$$

$$(0, 0, -0.34, 0.94) + \varepsilon(0.80, 2.23, -3.08, -1.10)$$

The resulting screw system is shown in Figure 4, where the length of each screw is proportional to its pitch. Also the corresponding trajectory of absolute displacements, considering the reference position as the identity, is shown in same Figure.

Using the synthesis procedure explained in Section 4, we obtain the RP and PR chains of Table 2. Notice that they coincide with the values given in [27], which means this is a folded RPRP overconstrained movable linkage. Figure 5 shows the linkage reaching some of the positions. Notice that the shape of the coupler link attached to the end effector depends on the position chosen as reference configuration.

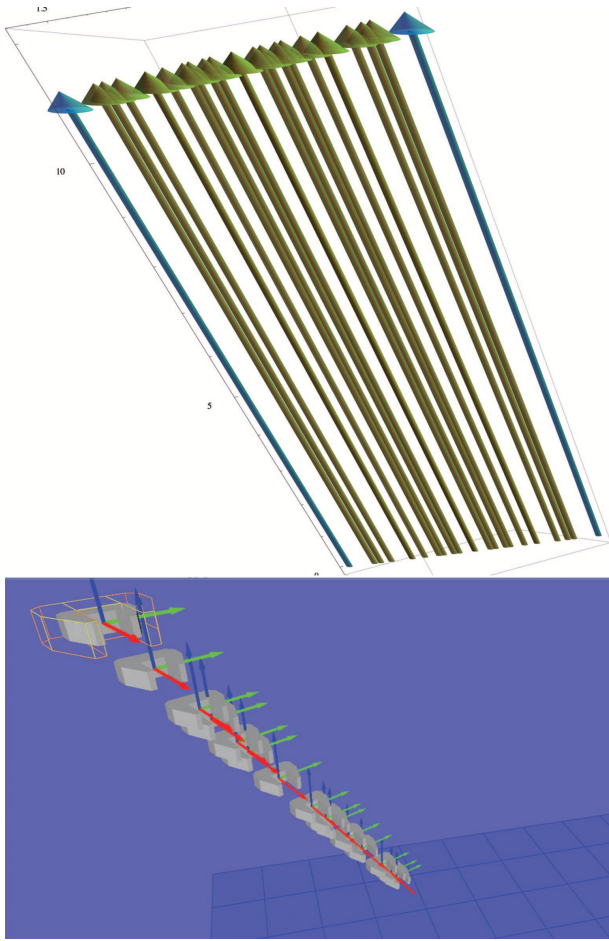


Fig. 4. Above, screw system generated by  $S_{12}$  and  $S_{13}$  (shown as first and last screws); below, corresponding absolute displacements with reference displacement being the identity

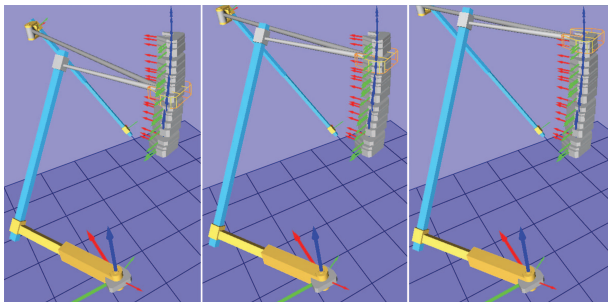


Fig. 5. RPRP linkage reaching three positions of the trajectory

## 5.2 Second example

For the second example, the dual quaternions in Table 3 have been generated as explained.  $\hat{S}_{12}$  has been randomly generated, while the rotation in  $\hat{S}_{13}$  is such that it belongs to the workspace of the chain.

We set the location of the second screw axis of the relative displacement with a point  $\mathbf{p}_{13} = (0.083, 2.159, -3.226)$ , randomly generated. Then we choose a value for the slope of the pitch distribution,  $K = 0.480$ , which allows us to create the second relative displacement.

The resulting screw system is shown in Figure 6, where

Table 2. Joint axes for the RPRP linkage at the reference configuration, Example 1

Chain	Revolute joint G	Prismatic joint $\mathbf{h}$
RP axes	$\begin{Bmatrix} 0 + 0\epsilon \\ 0 + 0\epsilon \\ 1 + 0\epsilon \end{Bmatrix}$	$\begin{Bmatrix} 0 \\ 0.59, \\ -0.81 \end{Bmatrix}$
RP Joint vars.	$\theta_{12} = -5.7$ $\theta_{13} = -39.4$	$d_{12} = 1.27$ $d_{13} = 8.08$
PR axes	$\begin{Bmatrix} 0 + 5.38\epsilon \\ 0 + 13.0\epsilon \\ -1 + 0\epsilon \end{Bmatrix}$	$\begin{Bmatrix} -0.42 \\ -0.42 \\ -0.81 \end{Bmatrix}$
PR Joint vars.	$\theta_{12} = 5.7$ $\theta_{13} = 39.4$	$d_{12} = 1.27$ $d_{13} = 8.08$

Table 3. Goal relative displacements for the RP and PR chains

$$(0.46, -0.13, -0.56, -0.67) + \epsilon(1.66, 0.34, -0.02, 1.08)$$

$$(0.13, -0.04, -0.17, 0.98) + \epsilon(0.02, -0.57, -0.92, -0.18)$$

the length of each screw is proportional to its pitch. The corresponding trajectory of absolute displacements, considering the reference position as the identity, is shown in Figure 7.

We obtain one solution for the RPRP linkage, specified in Table 4 as the Plücker coordinates of the axes and the joint variables to reach the positions.

Again, the dimensions form an overconstrained movable RPRP linkage. Figure 8 shows the chain reaching three displacements along the trajectory, considering the reference displacement as the identity.

## 6 Conclusions

This paper presents the exact workspace synthesis of an overconstrained closed linkage, the RPRP. The knowledge of the screw system that corresponds to the finite displacements of the linkage is used to generate the workspace of the linkage, which in turn ensures that the solutions of the synthesis of the RP and PR serial chains can be assembled to create a movable system. The counting of the maximum number of positions for the finite-position synthesis of the serial chain turns out to fully define the finite screw system of the linkage. This provides an easy method to shape the whole trajectory of the RPRP linkage as an input for the synthesis process. Even though the method targets the whole motion of the linkage, the synthesis equations need to be stated only at

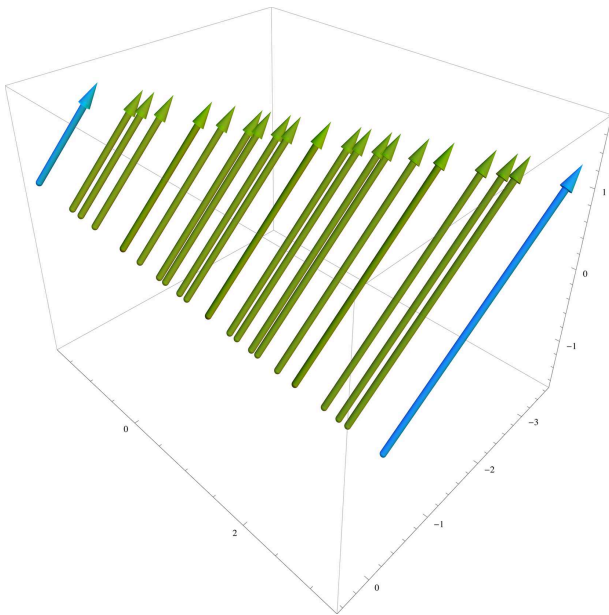


Fig. 6. Screw system generated by  $S_{12}$  and  $S_{13}$  (first and last screws)

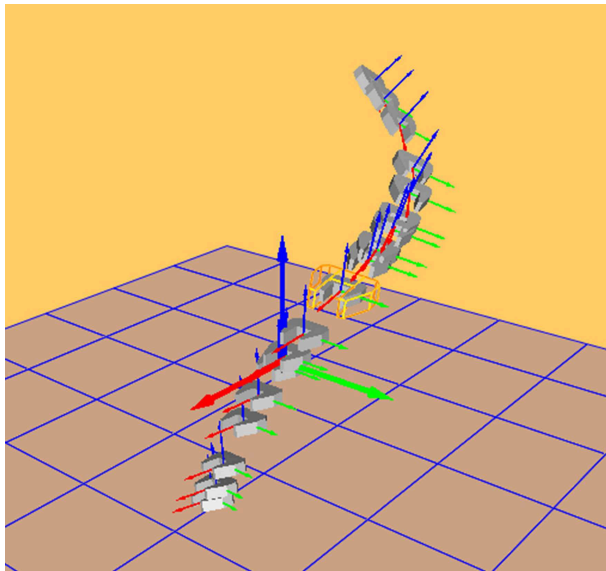


Fig. 7. The RPRP end-effector task trajectory

the task positions used to generate the linear combination of screws, hence the finite-position synthesis equations can still be used. The method yields a single RPRP linkage.

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Table 4. Joint axes for the RPRP linkage at the reference configuration, Example 2

Chain	Revolute joint $G$	Prismatic joint $h$
RP axes	$\begin{Bmatrix} -0.62 - 0.44\epsilon \\ 0.18 - 3.17\epsilon \\ 0.76 + 0.39\epsilon \end{Bmatrix}$	$\begin{Bmatrix} -0.09 \\ 0.89 \\ 0.44 \end{Bmatrix}$
RP Joint vars.	$\theta_{12} = -264.5$ $\theta_{13} = -25.2$	$d_{12} = 5.30$ $d_{13} = -3.05$
PR axes	$\begin{Bmatrix} 0.62 + 2.32\epsilon \\ -0.18 - 1.83\epsilon \\ -0.76 + 2.31\epsilon \end{Bmatrix}$	$\begin{Bmatrix} -0.27 \\ 0.06 \\ -0.96 \end{Bmatrix}$
PR Joint vars.	$\theta_{12} = 264.5$ $\theta_{13} = 25.2$	$d_{12} = -5.30$ $d_{13} = 3.05$

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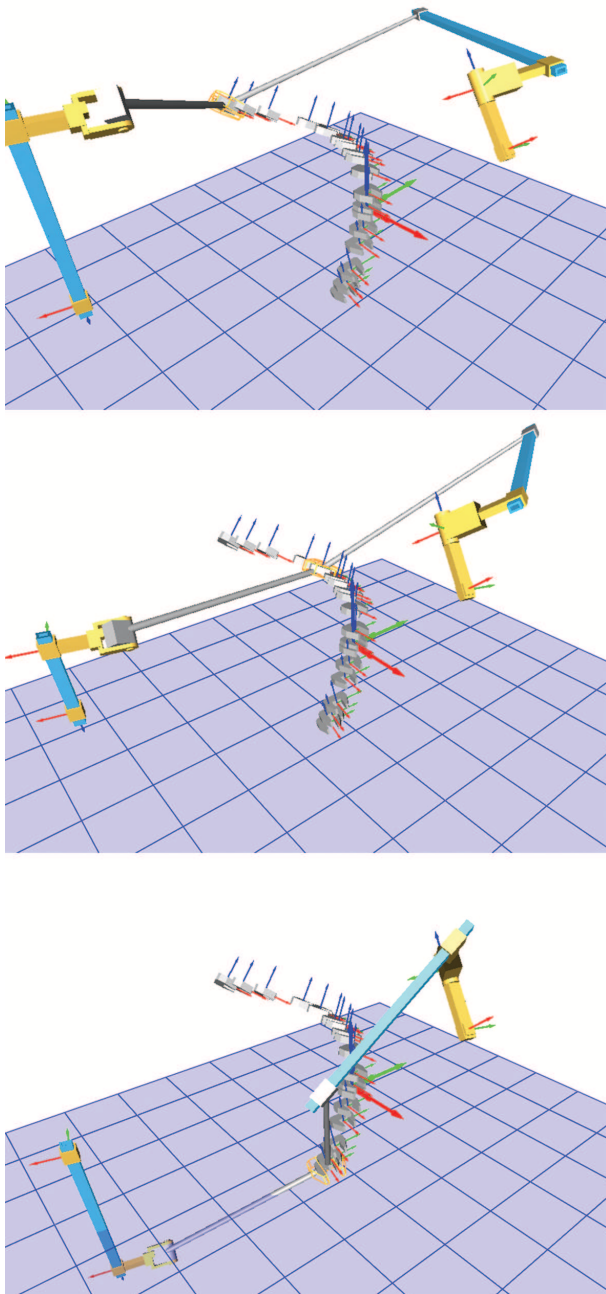


Fig. 8. The RPRP linkage reaching three positions along the trajectory

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