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ON THE INTEGRABILITY OF POLYNOMIAL FIELDS IN THE PLANE BY MEANS OF PICARD-VESSIOT THEORY

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ABSTRACT. We study the integrability of polynomial vector fields using Galois theory of linear differential equations when the associated foliations is reduced to a Riccati type foliation. In particular we obtain integrability results for some families of quadratic vector fields, Liénard equations and equations related with special functions such as Hypergeometric and Heun ones. We also study the Poincaré problem for some of the families.

INTRODUCTION

We consider the *polynomial differential system* in $\mathbb{C}^2$ defined by

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{y} = Q(x, y), \tag{1}$$

where P, Q are complex polynomial of degree at most m, namely $P, Q \in \mathbb{C}_m[x, y]$. We associate to system (1) the differential vector field

$$X = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}. \tag{2}$$

Integral curves of vector field (2) correspond to solutions of system (1). We are mainly interesting not in the behavior of these integral curves with respect to time t but in the orbits which are solutions of the first order differential equation (foliation)

$$y' = \frac{Q(x, y)}{P(x, y)}. \tag{3}$$

Here $'$ denotes derivative with respect to x. In this geometrical language of foliations, the latter expression is usually written as a Pfaff equation

$$\Omega = 0, \tag{4}$$

Ω being the differential 1-form $\Omega = Q(x, y)dx - P(x, y)dy$. Remind that the connection between the vector field X and the 1-form Ω is given by $L_X\Omega = 0$, which means that the field is tangent to the leaves of the foliation (orbits) defined by (4). From a dynamical point of view, the general solution of equation (4), $H(x, y) = C$, where C is any constant, is given by a *first integral* H of the original vector field X, i.e., a non-constant function that is constant

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along any solution $(x(t), y(t))$ of system (1). This is equivalent to say that $X(H) = 0$ and, since $L_X\Omega = 0$, to $\Omega \wedge dH = 0$ as well and, therefore, to $\Omega = f dH$ for some suitable function f. We remind that the *Liouvillian first integrals* $H(x, y)$ are first integrals obtained from $\mathbb{C}(x, y)$ by means of a combination of algebraic functions, quadratures and exponential of quadratures.

In this work we will be concerned with the study of the *integrability* of some families of equations (4) inside the complex analytical category, we mean, when the original vector field X is complex polynomial (i.e. $P, Q \in \mathbb{C}[x, y]$) or can be reduced to a complex polynomial field. Note that for polynomial fields there are many open problems; for instance, two classical problems are the following:

(i) Study the existence of invariant algebraic curves of system (1) (or of algebraic solutions of equation (3)).
(ii) Study the existence of Liouvillian first integrals of system (1) (or study if the general solution of equation (3) is Liouvillian).

For general polynomial fields, problems (i) and (ii) are difficult ones and today we are very far away to obtain any effective method to decide if an arbitrary polynomial field has or has not an invariant curve or an elementary or a Liouvillian first integral. In fact, problem (i) is connected with the classical unsolved *Poincaré problem*: bound the degree of possible invariant algebraic curves as a function of the degree of the field (or of the associated foliation defined by (3)). As we will see, these two problems are not independent: they are related by Darboux integrability approach and adjacent results due to Prelle–Singer and Singer ones [43, 46].

The aim of this paper is to decide about the integrability in closed form of equation (3) (or the equivalent equation (4)) when it can be reduced to a Riccati type equation,

$$y' = a_1(x) + a_2(x)y + a_3(x)y^2, \tag{5}$$

being a_1, a_2 and a_3 rational functions with complex coefficients. For Riccati's equations there is a very nice theory of integrability in the context of the Differential Galois theory of the associated second order linear differential equation. We will like to stress that this is the natural framework where it must be considered several results about the integrability in closed form of equation (5). Moreover in 1986, Kovacic in [27] obtained an effective algorithm to decide if the equation (5) has an algebraic solution or not. Additionally by a theorem of J. Liouville, the existence of an algebraic solution is exactly the definition of the integrability for the equation (5) in the context of the Galois theory of linear differential equations. Thus, *for foliations of type* (5) *problems (i) and (ii) are equivalent* and Kovacic's algorithm is an extremely powerful tool to solve them.

In some sense, this work can be considered as a very particular case of the Malgrange approach to the Galois theory of codimension–one foliations [35, 36, 6], ie, for Riccati's codimension–one foliations on the complex plane. Our aim is not to obtain general theoretical classification results, but effective criteria of integrability of this kind of foliations. As we shall see we obtain integrability criteria of several families of vector fields some quadratic fields and some Liénard equations involving special functions, which allows us to recover previous results of several authors. For instance, we will solve completely the integrability problem for the family of Liénard's type equations (see [44]),

$$yy' = (a(2m+k)x^{2k} + b(2m-k)x^{m-k-1})y - (a^2mx^{4k} + cx^{2k} + b^2m)x^{2m-2k-1},$$

with a, b, c, m, k complex parameters; we note that if the above equation comes from a polynomial vector field then m and k must be rational numbers.

We also study the Poincaré problem for some families, see for example Theorem 3.4.

The paper is structured as follows. In Section 1 we present the most necessary results of the two theories of integrability that are used in this work: Galois theory of linear differential equations and Darboux theory of integrability of polynomial vector fields. In Section 2 we consider several useful remarks about Riccati equation. Section 3 is devoted to applications. For completeness we include in the Appendixes Kovacic's algorithm and the necessary information about some special functions.

1. Two notions of integrability for planar polynomial vector fields

1.1. **Darboux theory of Integrability.** In this section we give a very brief overview of Darboux's integrability ideas [17], his terminology and some essential results.

Consider vector field (2) and an irreducible polynomial $f \in \mathbb{C}[x, y]$. The curve $f = 0$ is called an *invariant algebraic curve* of vector field (2) if it satisfies

$$\dot{f}|_{f=0} = 0.$$

This condition is equivalent to the existence of a polynomial $K \in \mathbb{C}_{m-1}[x, y]$, called *cofactor*, such that

$$X(f(x,y)) = P(x,y)\frac{\partial f}{\partial x} + Q(x,y)\frac{\partial f}{\partial y} = K(x,y)f(x,y),$$

or, equivalently,

$$\frac{X(f)}{f} = X(\log(f)) = K. \tag{6}$$

From this expression it follows that the curve $f = 0$ is formed by leaves and critical points of the vector field $X = (P, Q)$ defined by (2). We stress that if the polynomial system (1) has degree m, that is $m = \max\{\deg P, \deg Q\}$, then we have that $\deg K \leq m-1$, independently of the degree of the curve $f(x, y) = 0$. From definition (6) it follows that if the cofactor of an invariant curve $f = 0$ is vanishes identically then the polynomial f is a first integral of the vector field X. In terms of the associated foliation, this invariant curve $f = 0$ is a particular solution of $y' = Q/P$ and $Qdx - Pdy = 0$.

Besides, an analytic $\mathbb{C}$-valued function R is called an *integrating factor* of system (1) if it is not constant and satisfies that

$$\frac{X(R)}{R} = X(\log(R)) = -\operatorname{div} X,$$

where $\operatorname{div} X = (\partial P/\partial x) + (\partial Q/\partial y)$ is the *divergence* of the vector field $X = (P, Q)$. In case that the domain of definition of X is simply connected, from the integrating factor R it follows that

$$H(x,y) = \int R(x,y)P(x,y)\,dy + \varphi(x)$$

is a first integral of X, provided that $\partial H/\partial y = -RQ$.

To ensure the existence of a first integral for a system (1) is, in general, a very difficult problem. In [17], Darboux introduced a method to detect and construct first integrals using invariant algebraic curves. Darboux proved that any planar polynomial differential system of degree m having, at least, $m(m+1)/2$ invariant algebraic curves, admits a first integral or an integrating factor which can be obtained using them (see also Jouanolou [24] for a study in a general context for codimension-1 foliations).

Darboux's original ideas have been improved considering the multiplicity of the invariant algebraic curves, see [16] for more details. Related to the multiplicity of the algebraic curves some other invariant objects have appeared (see [11]) they are the so called *exponential factors*: given $h, g \in \mathbb{C}[x, y]$ relatively prime, the function $F = \exp(g/h)$ is called an *exponential factor* of the polynomial system (1) if there exists a polynomial $\widetilde{K} \in \mathbb{C}_{m-1}[x, y]$ (also called *cofactor*) that satisfies the equation

$$\frac{X(F)}{F} = X\left(\frac{g}{h}\right) = \widetilde{K}. \tag{7}$$

It is known that if h is not a constant polynomial then $h = 0$ is an invariant algebraic curve of (1) of cofactor K_h satisfying that $X(g) = gK_h + h\widetilde{K}$.

The following Theorem (starting from Darboux) shows how the construction of first integrals and integrating factors of (2) can be carrying out of the invariant algebraic curves of it.

Theorem 1.1. *We consider a planar polynomial system* (1) *of degree* m*, having*

- p *invariant algebraic curves* $f_i = 0$ *with cofactors* K_i*, for* $i = 1, \ldots, p$ *and*
- q *exponential factors* $F_j = \exp(g_j/h_j)$ *with cofactors* $\widetilde{K}_j$*,* $j = 1, \ldots, q$*.*

Then the following assertions hold:

(*a*) *There exist constants* $\lambda_i, \widetilde{\lambda}_j \in \mathbb{C}$ *not all vanishing such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \widetilde{\lambda}_j \widetilde{K}_j = 0$$

if and only if the multivalued function

$$f_1^{\lambda_1} \ldots f_p^{\lambda_p} F_1^{\widetilde{\lambda}_1} \ldots F_q^{\widetilde{\lambda}_q} \tag{8}$$

is a (Darboux) first integral of system (1)*.*

(*b*) *There exist constants* $\lambda_i, \widetilde{\lambda}_j \in \mathbb{C}$ *not all vanishing such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \widetilde{\lambda}_j \widetilde{K}_j + div\, X = 0$$

if and only if the function defined by (8) *is a (Darboux) integrating factor of* X*.*

For more recent versions of Theorem 1.1 see [33, 34]. For a generalization of Theorem 1.1 to a class of non-autonomous vector fields see [30].

Functions of the form (8) are called *Darboux functions*. We say that the polynomial system (1) is *Darboux integrable* if it admits a first integral or an integrating factor which is given by a Darboux function.

Remark 1.2. *Prelle and Singer* [43] *showed that if system* (1) *admits an* elementary first integral *then it admits an integrating factor which is the* n*-th root of a rational function (a slightly improved version of this result can be found in* [29, Corollary 4]*). Later, Singer in* [46] *proved that if system* (1) *admits a Liouvillian first integral then it has an integrating factor which is given by a Darboux function. This is an important argument to motivate sentences like "Darboux functions capture Liouvillian integrability", or "Liouvillian first integrals are either Darboux first integrals or integrals coming from a Darboux integrating factor".*

Given a polynomial system (1) of degree m, the computation of all its invariant algebraic curves is a very difficult problem since nothing is known a priori about the maximum degree of these curves. This makes necessary to impose additional conditions either on the structure of the system (1) or on the nature of such curves, see for instance, [5, 7, 14] or references therein. This difficulty has motivated the study of different types of inverse problems of the Darboux theory of integrability [39, 14, 12, 15, 13, 31].

1.2. **Picard-Vessiot theory.** Picard-Vessiot theory is the Galois theory of linear differential equations. We will just remind here some of its main definitions and results but we refer the reader to [45] for a wide theoretical background.

We start recalling some basic notions on algebraic groups and, afterwards, Picard-Vessiot theory will be introduced.

An algebraic group of matrices 2×2 is a subgroup $G \subset \mathrm{GL}(2,\mathbb{C})$ defined by means of algebraic equations in its matrix elements and in the inverse of its determinant. That is, there exists a set of polynomials $P_i \in \mathbb{C}[x_1, \ldots, x_5]$, for $i \in I$, such that $A \in \mathrm{GL}(2,\mathbb{C})$ given by

$$A = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix},$$

belongs to G if and only if $P_i\left(x_{11}, x_{12}, x_{21}, x_{22}, (\det A)^{-1}\right) = 0$ for all $i \in I$ and where $\det A = x_{11}x_{22} - x_{21}x_{12}$. It is said that G is an algebraic manifold endowed with a group structure.

Recall that a group G is called *solvable* if and only if there exists a chain of normal subgroups

$$e = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$$

satisfying that the quotient G_i/G_j is abelian for all $n \geq i \geq j \geq 0$.

It is well known that any algebraic group G has a unique connected normal algebraic subgroup G^0 of finite index. In particular, the *identity connected component* G^0 of G is defined as the largest connected algebraic subgroup of G containing the identity. In case that $G = G^0$ we say that G is a *connected group.* Moreover, if G^0 is solvable we say that G is *virtually solvable.*

The following result provides the relation between virtual solvability of an algebraic group and its structure.

Theorem 1.3 (Lie-Kolchin)**.** *Let* $G \subseteq \mathrm{GL}(2,\mathbb{C})$ *be a virtually solvable group. Then,* G^0 *is triangularizable, that is, it is conjugate to a subgroup of upper triangular matrices.*

Now, we briefly introduce Picard-Vessiot Theory.

First, we say that $(\mathcal{K},\ ')$ - or, simply, $\mathcal{K}$ - is a *differential field* if $\mathcal{K}$ is a commutative field of characteristic zero, depending on x and $'$ is a derivation on $\mathcal{K}$ (that is, satisfying that

$(a+b)' = a' + b'$ and $(a \cdot b)' = a' \cdot b + a \cdot b'$ for all $a, b \in \mathcal{K}$). We denote by $\mathcal{C}$ the *field of constants of* $\mathcal{K}$, defined as $\mathcal{C} = \{c \in \mathcal{K} \mid c' = 0\}$.

We will deal with second order linear homogeneous differential equations, that is, equations of the form

$$y'' + ay' + by = 0, \quad a, b \in \mathcal{K}, \tag{9}$$

and will be concerned with the algebraic structure of their solutions. Moreover, along this work, we will refer the current differential field as the smallest one containing the field of coefficients of this differential equation.

Let us suppose that y_1, y_2 is a basis of solutions of equation (9), i.e., y_1, y_2 are linearly independent over $\mathcal{K}$ and every solution is a linear combination over $\mathcal{C}$ of these two. Let $\mathcal{L} = \mathcal{K}\langle y_1, y_2\rangle = \mathcal{K}(y_1, y_2, y_1', y_2')$ be the differential extension of $\mathcal{K}$ such that $\mathcal{C}$ is the field of constants for $\mathcal{K}$ and $\mathcal{L}$. In this terms, we say that $\mathcal{L}$, the smallest differential field containing $\mathcal{K}$ and $\{y_1, y_2\}$, is the *Picard-Vessiot extension* of $\mathcal{K}$ for the equation (9).

The group of all the differential automorphisms of $\mathcal{L}$ over $\mathcal{K}$ that commute with the derivation $\,'$ is called the *differential Galois group* of $\mathcal{L}$ over $\mathcal{K}$ and is denoted by $\mathrm{Gal}(\mathcal{L}/\mathcal{K})$. This means, in particular, that for any $\sigma \in \mathrm{Gal}(\mathcal{L}/\mathcal{K})$, $\sigma(a') = (\sigma(a))'$ for all $a \in \mathcal{L}$ and that $\sigma(a) = a$ for all $a \in \mathcal{K}$. Thus, if $\{y_1, y_2\}$ is a fundamental system of solutions of (9) and $\sigma \in \mathrm{Gal}(\mathcal{L}/\mathcal{K})$ then $\{\sigma y_1, \sigma y_2\}$ is also a fundamental system. This implies the existence of a non-singular constant matrix

$$A_\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}),$$

such that

$$\sigma \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \sigma(y_1) \\ \sigma(y_2) \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix} A_\sigma.$$

This fact can be extended in a natural way to a system

$$\sigma \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \begin{pmatrix} \sigma(y_1) & \sigma(y_2) \\ \sigma(y_1') & \sigma(y_2') \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} A_\sigma,$$

which leads to a faithful representation $\mathrm{Gal}(\mathcal{L}/\mathcal{K}) \to \mathrm{GL}(2, \mathbb{C})$ and makes possible to consider $\mathrm{Gal}(\mathcal{L}/\mathcal{K})$ as a subgroup of $\mathrm{GL}(2, \mathbb{C})$ depending (up to conjugacy) on the choice of the fundamental system $\{y_1, y_2\}$.

One of the fundamental results of the Picard-Vessiot Theory is the following theorem (see [25, 28]).

Theorem 1.4. *The differential Galois group* $\mathrm{Gal}(\mathcal{L}/\mathcal{K})$ *is an algebraic subgroup of* $\mathrm{GL}(2, \mathbb{C})$.

We say that equation (9) is *integrable* if the Picard-Vessiot extension $\mathcal{L} \supset \mathcal{K}$ is obtained as a tower of differential fields $\mathcal{K} = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_m = \mathcal{L}$ such that $\mathcal{L}_i = \mathcal{L}_{i-1}(\eta)$ for $i = 1, \ldots, m$, where either

(*i*) η is *algebraic* over $\mathcal{L}_{i-1}$, that is η satisfies a polynomial equation with coefficients in $\mathcal{L}_{i-1}$.
(*ii*) η is *primitive* over $\mathcal{L}_{i-1}$, that is $\eta' \in \mathcal{L}_{i-1}$.
(*iii*) η is *exponential* over $\mathcal{L}_{i-1}$, that is $\eta'/\eta \in \mathcal{L}_{i-1}$.

We remark that in the usual Differential Algebra terminology to say that an equation (9) is integrable is equivalent to say that the corresponding Picard-Vessiot extension is *Liouvillian*. Moreover,

Theorem 1.5 (Kolchin)**.** *Equation* (9) *is integrable if and only if* $\mathrm{Gal}(\mathcal{L}/\mathcal{K})$ *is virtually solvable, that is, its identity component* $(\mathrm{Gal}(\mathcal{L}/\mathcal{K}))^0$ *is solvable.*

For instance, for the case $a = 0$ in equation (9), i.e. $y'' + by = 0$, it is very well known [25, 28, 45] that $\mathrm{Gal}(\mathcal{L}/\mathcal{K})$ is an algebraic subgroup of $\mathrm{SL}(2, \mathbb{C})$ (remind that $A \in \mathrm{SL}(2, \mathbb{C}) \Leftrightarrow A \in \mathrm{GL}(2, \mathbb{C})$ and $\det A = 1$). For a more detailed study about this case we refer the reader to Appendix A.

2. Some remarks about Riccati equation

Ricatti equation is probably one of the most studied equations. However, its rôle in our study of the Darboux and Picard-Vessiot integrability has induced us to devote this section to some of their properties. Even though the results of Section (2) are known, their proofs have been included for completeness. We divide these properties in two types: the first one (see Subsection 2.1) concerning transformations leading a general second order differential equation into a Riccati equation (written in the so-called reduced form) which becomes the starting point of the celebrated Kovacic algorithm (see Appendix A); a second one, Darboux-like, that studies first integrals and integrating factors for a Riccati equation, see Subsection 2.2.

2.1. **Transformations related to Riccati equations.**

It is known that any second order differential equation can be led into a general Riccati equation through a classical logarithmic change of variable (see, for instance [44, 23]). Next proposition recall it and summarises some other related transformations.

Proposition 2.1. *Let $\mathcal{K}$ be a differential field and consider functions $a_0(x)$, $a_1(x)$, $a_2(x)$, $r(x)$, $\rho(x)$, $b_0(x)$, $b_1(x)$ belonging to $\mathcal{K}$ that, for simplicity, will be denoted without their explicit dependence on x. Consider now the following forms associated to any second order ordinary differential equation (ode) and Riccati equation:*

(*i*) *Second order ode in general form:*

$$y'' + b_1 y' + b_0 y = 0. \tag{10}$$

(*ii*) *Second order ode in reduced form:*

$$\xi'' = \rho \xi. \tag{11}$$

(*iii*) *Riccati equation in general form:*

$$v' = a_0 + a_1 v + a_2 v^2, \quad a_2 \neq 0. \tag{12}$$

(*iv*) *Riccati equation in reduced form:*

$$w' = r - w^2, \tag{13}$$

Then, there exist transformations $\mathcal{T}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{R}$ leading some of these equations into the other ones, as showed in the following diagram:

$$\begin{array}{ccc} v' = a_0 + a_1 v + a_2 v^2 & \xrightarrow{\mathcal{T}} & w' = r - w^2\ . \\ {\scriptstyle B}\downarrow & & \downarrow{\scriptstyle \mathcal{R}} \\ y'' + b_1 y' + b_0 y = 0 & \xrightarrow{\mathcal{S}} & \xi'' = \rho\xi \end{array}$$

The new independent variables are defined by means of

$$\mathcal{T}:\ v = -\left(\frac{a_2'}{2a_2^2} + \frac{a_1}{2a_2}\right) - \frac{1}{a_2}w, \qquad \mathcal{B}:\ v = -\frac{1}{a_2}\frac{y'}{y},$$

$$\mathcal{S}:\ y = \xi \mathrm{e}^{-\frac{1}{2}\int b_1 dx}, \qquad \mathcal{R}:\ w = \frac{\xi'}{\xi},$$

and the functions r, ρ, b_0 and b_1 are given by

$$r = \frac{1}{\beta}\left(a_0 + a_1\alpha + a_2\alpha^2 - \alpha'\right), \tag{14}$$

$$\alpha = -\left(\frac{a_2'}{2a_2^2} + \frac{a_1}{2a_2}\right), \quad \beta = -\frac{1}{a_2}, \tag{15}$$

$$b_1 = -\left(a_1 + \frac{a_2'}{a_2}\right), \quad b_0 = a_0 a_2, \tag{16}$$

$$\rho = r = \frac{b_1^2}{4} + \frac{b_1'}{2} - b_0. \tag{17}$$

Proof. The proof is quite standard.

[$\mathcal{T}$]: Applying the change $v = \alpha + \beta w$ we get the equation

$$\alpha' + \beta' w + \beta w' = a_0 + a_1\alpha + a_1\beta w + a_2\alpha^2 + 2a_2\alpha\beta w + a_2\beta^2 w^2$$

that, regrouping terms, leads to

$$w' = \frac{1}{\beta}\left(a_0 + a_1\alpha + a_2\alpha^2 - \alpha'\right) + \left(a_1 + 2a_2\alpha - \frac{\beta'}{\beta}\right)w + a_2\beta w^2.$$

Since $a_2 \neq 0$ we can take $\beta = -1/a_2$ and, therefore, $a_2\beta = -1$. Having this into account, the value of α satisfying that the coefficient in w vanishes is given by

$$\alpha = \frac{1}{2a_2}\left(\frac{\beta'}{\beta} - a_1\right).$$

The expressions for α, β and r follow straightforwardly,

$$r = \frac{1}{\beta}\left(a_0 + a_1\alpha + a_2\alpha^2 - \alpha'\right), \quad \alpha = -\left(\frac{a_2'}{2a_2^2} + \frac{a_1}{2a_2}\right), \quad \beta = -\frac{1}{a_2}.$$

Moreover, it is clear that α, β and r belong to $\mathcal{K}$.

$[\mathcal{B}]$: Imposing $\alpha = 0$ and taking $\beta = -1/a_2$ in transformation $\mathcal{T}$, this is $v = -w/a_2$, we obtain the Riccati equation

$$w' = -a_0 a_2 + \left(a_1 + \frac{a_2'}{a_2}\right) w - w^2.$$

Performing now the change of variables $w = (\log y)'$ (or, equivalently, $v = -a_2 y'/y$) we get the differential equation $y'' + b_1 y' + b_0 y = 0$ with

$$b_1 = -\left(a_1 + \frac{a_2'}{a_2}\right), \quad b_0 = a_0 a_2.$$

Obviously, b_0 and b_1 belong to $\mathcal{K}$.

$[\mathcal{S}]$: The change of variable $y = \mu\xi$, with $\mu = \mu(x)$ and $\xi = \xi(x)$, lead us to

$$\xi'' + \left(2\frac{\mu'}{\mu} + b_0\right)\xi' + \left(\frac{\mu''}{\mu} + b_0\frac{\mu'}{\mu} + b_1\right)\xi = 0.$$

In order to obtain the equation $\xi'' = \rho\xi$ we need to impose

$$2\frac{\mu'}{\mu} + b_0 = 0, \qquad \frac{\mu''}{\mu} + b_0\frac{\mu'}{\mu} + b_1 = -\rho,$$

which gives rise to

$$\mu = \mathrm{e}^{-\frac{1}{2}\int \mathrm{b}_0}, \qquad \rho = \frac{\mathrm{b}_0^2}{4} + \frac{\mathrm{b}_0'}{2} - \mathrm{b}_1.$$

Moreover, it is straightforward to check that $\rho \in \mathcal{K}$.

$[\mathcal{R}]$: This is a particular case of transformation $[\mathcal{B}]$ with the particular choice $a_0 = r$, $a_1 = 0$ and $a_2 = -1$.

Finally, composing the transformations provided by $[\mathcal{B}]$, $[\mathcal{R}]$ and $[\mathcal{S}]$:

$$-a_2 v = \frac{y'}{y}, \qquad y = \xi e^{-\frac{1}{2}\int b_0}, \qquad b_0 = -\left(a_1 + \frac{a_2'}{a_2}\right) \qquad \frac{\xi'}{\xi} = w,$$

we recover the result given by $[\mathcal{T}]$,

$$v = -\left(\frac{a_1}{2a_2} + \frac{a_2'}{2a_2^2}\right) - \frac{1}{a_2}w = \alpha + \beta w,$$

which implies that, in some sense and taking $\rho = r$, the diagram commutes: $\mathcal{S} \circ \mathcal{B} = \mathcal{R} \circ \mathcal{T}$.

Note, from this Lemma, that the function v is algebraic over $\mathcal{K}$ if and only if the function w is also algebraic over $\mathcal{K}$. Furthermore, in such case, the degree over $\mathcal{K}$ of both functions v and w is the same.

It is known that a Riccati equation (12) has an algebraic solution on $\mathcal{K}$ if and only if the differential equation (10) is integrable in a Picard-Vessiot sense. If this situation holds we will say, from now on, that the Riccati equation is integrable over $\mathcal{K}$. This is the starting point of Kovacic algorithm (see Appendix A).

2.2. **Integrating factor and first integrals for Riccati vector fields.** In this Subection we briefly show some relations between the existence of invariant curves of a certain type of vector fields and the integrability, via Kovacic algorithm (see Appendix A), of its associated Riccati foliation.

We recall (see Remark 1.2) that from Singer's work [46] we have that if a planar polynomial vector field (2) admits a Liouvillian first integral then it has also an integrating factor given by a Darboux function. However, very few results are known about the relation between the existence of an algebraic invariant curve of a general planar vector field and the Liouvillian integrability of its foliation.

Let us consider the family of planar vector fields of the form

$$X = (p(x) - q(x)w^2)\frac{\partial}{\partial w} + q(x)\frac{\partial}{\partial x}, \tag{18}$$

with $p(x), q(x) \in \mathbb{C}[x]$ complex polynomials. Introducing an independent variable t, usually called *time*, we can associate to the vector field (18) the following system of differential equations

$$\begin{array}{rcl} \dot{w} & = & p(x) - q(x)w^2, \\ \dot{x} & = & q(x), \end{array}$$

where we denote by $\dot{} = d/dt$. Its foliation is governed by the equation

$$w' = \frac{dw}{dx} = \frac{p(x) - q(x)w^2}{q(x)} = \frac{p(x)}{q(x)} - w^2,$$

which is a Riccati equation given in reduced form $w' = r(x) - w^2$ provided that $r = p/q \in \mathbb{C}(x)$. As it will be showed in the next lemma, the integrability of this "Riccati foliation" is closely related to the existence of an algebraic invariant curve of its vector field (18). A similar approach for this problema can be found in [21, 22].

Lemma 2.2. *Let $w_1 = w_1(x)$ be a solution of a Riccati equation in reduced form*

$$w' = r(x) - w^2,$$

with $r(x) = p(x)/q(x) \in \mathbb{C}[x]$. Then the associated vector field (18) *has an integrating factor given by*

$$R_1 = \frac{e^{2\int(-w_1(x)dx)}}{(-w + w_1(x))^2}. \tag{19}$$

Proof. It is straightforward to check that if $w_1(x)$ is a solution of $w' = p/q - w^2$ then it holds $X(f_1) = K_1 f_1$ with $f_1(w, x) = -w + w_1(x) = 0$ and $K_1 = -w - w_1(x)$. In addition, $F(x) = \mathrm{e}^{\int(-\omega_1(\mathrm{x})\mathrm{dx})}$ satisfies $X(F) = L_1 F$ with $L_1 = -\omega_1$. Note that X has divergence $\mathrm{div}X = -2w$, and additionally it holds $-2K_1 + 2L_1 + \mathrm{div}X = 0$. Similarly to Theorem 1.1, vector field (18) admits the integrating factor

$$R_1 = \frac{F_1^2}{f_1^2} = \frac{\left(e^{\int(-w_1(x)dx)}\right)^2}{(-w + w_1(x))^2},$$

as it was claimed.

Remark 2.3. *We would like to stress the fact that the result in Lemma 2.2 is independent of the nature of the solution $w = w_1(x)$. Note that the integral $\int \omega_1(x)dx$ is an abelian integral.*

The important fact is that, conversely, Picard-Vessiot theory and in particular, Kovacic algorith, supply information about first integrals and integrating factors of the equation $w' = r(x) - w^2$ from the knowledge of some of its solutions, w_1, w_2, w_3. More precisely, from the first three cases in Kovacic algorithm [27] (the integrable ones) we obtain the following types of first integrals, see also Weil [50] and Żołądek [52].

Proposition 2.4. *The following statements hold.*

Case 1: *One has two possibilities:*

- *If only $w_1 \in \mathbb{C}(x)$ then $H(x, y)$ is of Darboux–Schwarz–Christoffel type.*
- *If both $w_1, w_2 \in \mathbb{C}(x)$ then $H(x, y)$ is of Darboux type. In particular, from Lemma 2.2 we can construct two integrating factors R_1 and R_2 and so R_1/R_2 is a first integral of X. Thus, we have*

$$H(x, y) = \frac{(-w + w_2(x))}{(-w + w_1(x))} e^{\int [(w_2(x) - w_1(x))dx]}.$$

Case 2: *If w_1 is a solution of a quadratic polynomial then a first integral is of hyperelliptic type.*

Case 3: *If all w_1, w_2, w_3 are algebraic over $\mathbb{C}(x)$ then X admits a rational first integral.*

3. Applications

In this section we analyze some cases concerning to integrability and nonintegrability of some Riccati planar vector fields.

3.1. **Quadratic polynomials fields.** The study of the integrability of the quadratic polynomial vector field

$$\begin{aligned} x' &= a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}, \\ y' &= b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{10}x + b_{01}y + b_{00}, \end{aligned}$$

being $a_{ij}, b_{i,j} \in \mathbb{C}$, is in general a hard problem. In this section we only consider some special cases for these parameters.

It is known (see [32, Prop.3], for instance) that the study of linear-quadratic planar systems having a finite equilibrium point can be reduced to consider the following two families of systems: a first type, using the notation introduced in [32], denoted by (S1),

$$\begin{aligned} x' &= x, \\ y' &= \varepsilon x + \lambda y + b_{20}x^2 + b_{11}xy + b_{02}y^2, \end{aligned} \qquad (S1)$$

and a second type denoted by (S2),

$$\begin{aligned} x' &= y, \\ y' &= \varepsilon x + \lambda y + b_{20}x^2 + b_{11}xy + b_{02}y^2. \end{aligned} \qquad (S2)$$

In [32], the authors proved that the linear-quadratic systems admiting a global analytic first integral are those satisfying:

(a_1) $b_{02} = \lambda = 0$.
(b_1) $b_{02} = 0$ and $\lambda = -p/q \in \mathbb{Q}^-$,

in the case of (S1)-type systems and

(a_2) $b_{20} = b_{02} = \lambda = 0$ and $\varepsilon b_{11} \neq 0$.

(b_2) $b_{20} = b_{11} = \lambda = 0$ and $\varepsilon b_{02} \neq 0$.
(c_2) $b_{11} = \lambda = 0$ and $b_{20} \neq 0$,

for (S2)-type systems. Furthermore, they also provide the explicit form of the corresponding first integrals. It is important to notice that all of them are of Darboux type (and, of course, are Liouvillian first integral).

Our aim is to show that these results can be recovered easily using arguments of the Differential Galois theory. We start first with the (S1)-case. Thus, we consider the associated foliation of system (S1):

$$\frac{dy}{dx} = (\varepsilon + b_{20}x) + \left(\frac{\lambda + b_{11}x}{x}\right) y + \frac{b_{02}}{x} y^2, \tag{20}$$

which is a Riccati equation. By Lemma 2.1 equation (20) can be transformed to a Riccati equation $w' = r(x) - w^2$, with

$$r(x) = \frac{1}{4} - \frac{\kappa}{x} + \frac{4\mu^2 - 1}{4x^2}, \quad \kappa = \frac{1}{\sqrt{b_{11}^2 - 4b_{20}b_{02}}} \left(b_{02}\varepsilon + \frac{b_{11}}{2}(1-\lambda)\right), \quad \mu = \frac{\lambda}{2}, \tag{21}$$

provided $b_{11}^2 - 4b_{20}b_{02} \neq 0$.

We note that equation $\xi'' = r(x)\xi$ with r as in (21) is a Whittaker equation, (see Appendix B).

Now, we can apply Martinet-Ramis Theorem B.2 for Whittaker equations which asserts that our Whittaker equation is integrable if and only if at least one the following conditions is verified:

$$\pm\kappa \pm \mu \in \frac{1}{2} + \mathbb{N},$$

or, equivalently (and more suitable for the expressions derived of κ and μ)

$$2\,(\kappa \pm \mu) \in 2\mathbb{Z} + 1,$$

i.e., an integer odd number. In our case this condition reads

$$2\,(\kappa \pm \mu) = \frac{2b_{20}\varepsilon + b_{11}(1-\lambda)}{\sqrt{b_{11}^2 - 4b_{20}b_{02}}} \pm \lambda.$$

Notice that case (a_1) in (S1)-type equations corresponds to $2(\kappa \pm \mu) = 1 \in 2\mathbb{Z}+1$ and case (b_1) to $2(\kappa + \mu) = (1 + (p/q)) + (-p/q) = 1 \in 2\mathbb{Z}+1$. Thus *Galois* recovers the integrability result asserted in [32, Thm.1].

Let us consider now the case of a (S2)-type system, namely

$$\begin{aligned} x' &= y, \\ y' &= \varepsilon x + \lambda y + b_{20}x^2 + b_{11}xy + b_{02}y^2, \end{aligned}$$

Its associated differential equation in the foliation reads

$$\frac{dy}{dx} = (\lambda + b_{11}x) + \left(\varepsilon x + b_{20}x^2\right)\frac{1}{y} + b_{02}y. \tag{22}$$

Thus, equation (22) falls into the following situations:

(i) If $\lambda = b_{11} = 0$ it leads to

$$\frac{dy}{dx} = \left(\varepsilon x + b_{20}x^2\right)\frac{1}{y} + b_{02}y,$$

which corresponds to a Bernoulli equation. This case corresponds essentially to cases (b_2) and (c_2) for (S2)-systems.

(ii) If $\varepsilon = b_{20} = 0$ we obtain the linear equation (and integrable in a Liouville sense)

$$\frac{dy}{dx} = (\lambda + b_{11}x) + b_{02}y.$$

This possibility is not taken into account by Llibre and Valls [32] since this resulting equation is not, strictly speaking, a Riccati.

(iii) case (a_2)) gives rise to $dy/dx = b_{11}x + \varepsilon xy^{-1}$, which is a separable equation (Bernoulli as well). Is solutions, easily computable, are all Liouvillian.

(iv) If $b_{02} = 0$ we obtain a particular case of Liénard equation

$$y\frac{dy}{dx} = (\lambda + b_{11}x)\, y + \left(\varepsilon x + b_{20}x^2\right),$$

that will be considered in the following section.

3.2. **An application of orthogonal polynomials.** We recall that Hypergeometric equation, including confluences, is a particular case of the differential equation

$$y'' + \frac{L}{Q}y' + \frac{\lambda}{Q}y, \quad \lambda \in \mathbb{C}, \quad L = a_0 + a_1x, \quad Q = b_0 + b_1x + b_2x^2. \tag{23}$$

It is well known that *classical orthogonal polynomials* and *Bessel polynomials* are solutions of equation (23), see for example [10]:

- Hermite, denoted by H_n,
- Chebyshev of first kind, denoted by T_n,
- Chebyshev of second kind, denoted by U_n,
- Legendre, denoted by P_n,
- Laguerre, denoted by L_n,
- associated Laguerre, denoted by $L_n^{(m)}$,
- Gegenbauer, denoted by $C_n^{(m)}$,
- Jacobi polynomials, denoted by $\mathcal{P}_n^{(m,\nu)}$ and
- Bessel polynomials, denoted by B_n.

In the following table we present Q, L and λ corresponding to equation (23) for classical orthogonal polynomials and Bessel polynomials.

Polynomial	$\boldsymbol{Q}$	$\boldsymbol{L}$	$\boldsymbol{\lambda}$
H_n	1	$-2x$	$2n$
T_n	$1-x^2$	$-x$	n^2
U_n	$1-x^2$	$-3x$	$n(n+2)$
P_n	$1-x^2$	$-2x$	$n(n+1)$
L_n	x	$1-x$	n
$L_n^{(m)}$	x	$m+1-x$	n
$C_n^{(m)}$	$1-x^2$	$-(2m+1)x$	$n(n+2m)$
$\mathcal{P}_n^{(m,\nu)}$	$1-x^2$	$\nu-m-(m+\nu+2)x$	$n(n+1+m+\nu)$
B_n	x^2	$2(x+1)$	$-n(n+1)$

We remark that integrability conditions and solutions of differential equations with solutions orthogonal polynomials, including Bessel polynomials, after reduction, can be obtained applying Kovacic's algorithm, specifically case 1 of the algorithm. The same results can be obtained via Kimura's Theorem and Martinet-Ramis Theorem and the *parabolic cylinder equation* (see [19, 26, 37]).

Theorem 3.1. *Consider Q, L and λ as in the previous table. The vector field*

$$\left\{\begin{array}{lcl} x' & = & -Q, \\ y' & = & \lambda + Ly + Qy^2, \end{array}\right.$$

has invariant curves with unbounded degree.

Proof. Only in case 1 of Kovacic's algorithm one solution of the equation (11) can be obtained as $y = P_n e^{\int \omega}$, where P_n is one of the polynomials listed in the table. □

3.3. **Liénard equation.** We consider the first order differential equation of the foliation associated to Liénard equation in the following form

$$yy' = f(x)y + g(x), \tag{24}$$

where $y = y(x)$. We will call it the *Liénard equation*. We are mainly interested in the case where $f(x)$ and $g(x)$ are rational functions.

Problem: Obtain criteria for the polynomials $f(x)$ and $g(x)$ in order to reduce the equation (24) to a Riccati equation.

This problem is difficult and, as far as we know, only some partial answers are known.

We will give some examples of these reductions from the handbook [44]. A first example is given by the family (1.3.3.11 in [44])

$$yy' = (a(2m+k)x^{2k} + b(2m-k)x^{m-k-1})y - (a^2 m x^{4k} + cx^{2k} + b^2 m)x^{2m-2k-1}, \tag{25}$$

being a, b, c, m, k complex parameters; if the equation (25) comes from a polynomial vector field then m and k must be rational numbers.

The change $w = x^k$, $y = x^m(z + ax^k + bx^{-k})$ convert (25) to a Riccati equation

$$(-mz^2 + 2abm - c)w'(z) = bk + kzw + akw^2, \tag{26}$$

being the associated second order linear equation a Riemann equation; more concretely, using Lemma 2.1 it can be write as a Legendre equation:

$$(1-t^2)u''(t) - 2tu'(t) + \left(\nu(\nu+1) - \frac{\mu^2}{1-t^2}\right)u(t) = 0, \tag{27}$$

with

$$\mu = -\frac{m+k}{2m},$$

and ν a solution of

$$\nu^2+\nu+\frac{m^2-k^2}{4m^2}-\frac{abk^2}{mc-2abm^2}=0.$$

The difference of exponents in (27) is μ, μ and $2\nu-1$. Thus we are in conditions to apply Kimura's Theorem (Appendix B).

Proposition 3.2. *The Legendre equation* (27) *is integrable if and only if, either*

(1) $\mu\pm\nu\in\mathbb{Z}$ *or* $\nu\in\mathbb{Z}$, *or*

(2) $\pm\mu$, $\pm\mu$, $\pm 2\nu+1$ *belong to one of the following seven families*

- *(a)* $\mu\in\frac{1}{2}+\mathbb{Z}$, $\nu\in\mathbb{C}$
- *(b)* $\mu\in\mathbb{Z}\pm\frac{1}{3}$, $\nu\in\frac{1}{2}\mathbb{Z}\pm\frac{1}{3}$ *and* $\mu\pm\nu\in\mathbb{Z}+\frac{1}{6}$
- *(c)* $\mu\in\mathbb{Z}\pm\frac{2}{5}$, $\nu\in\frac{1}{2}\mathbb{Z}\pm\frac{3}{10}$ *and* $\mu\pm\nu\in\mathbb{Z}+\frac{1}{10}$
- *(d)* $\mu\in\mathbb{Z}\pm\frac{1}{3}$, $\nu\in\frac{1}{2}\mathbb{Z}\pm\frac{2}{5}$ *and* $\mu\pm\nu\in\mathbb{Z}+\frac{1}{10}$
- *(e)* $\mu\in\mathbb{Z}\pm\frac{1}{5}$, $\nu\in\frac{1}{2}\mathbb{Z}\pm\frac{2}{5}$ *and* $\mu\pm\nu\in\mathbb{Z}+\frac{1}{10}$
- *(f)* $\mu\in\mathbb{Z}\pm\frac{2}{5}$, $\nu\in\frac{1}{2}\mathbb{Z}\pm\frac{1}{3}$ *and* $\mu\pm\nu\in\mathbb{Z}+\frac{1}{6}$

Proof. The difference of exponents is given by μ, μ and $2\nu+1$, thus by Kimura's Theorem we have the above possibilities, which corresponds to the cases 1,3,11,12,13,15 of Kimura's table. □

The equation

$$\frac{dx}{dw}=A(x)+B(x)w,$$

by means of the change of variable

$$w=y-\frac{A}{B},$$

is transformed into the Liénard's equation

$$y\frac{dy}{dx}=\frac{1}{B}+\frac{d}{dx}\left(\frac{A}{B}\right)y,$$

for any functions A and B.

In particular, for

$$A=A(x)=a+bx+cx^2,\quad B=B(x)=\alpha+\beta x+\gamma x^2,$$

the Liénard's equations falls into the Riccatti equation (12) where $a_0=c+\gamma x$, $a_1=b+\beta x$ and $a_2=a+\alpha x$.

Applying the transformation T we obtain the reduced Riccatti equation (13), and by transformation R we arrive to the normalized second order differential equation (11)

$$\rho(x)=\frac{\beta^2-4\alpha\gamma}{4}x^2-\frac{2a\gamma+2\alpha c-b\beta}{2}x-\frac{4ac-b^2}{4}+\frac{b\gamma-\beta c}{2(\gamma x+c)}+\frac{3\gamma^2}{4(\gamma x+c)^2}. \tag{28}$$

We focus on the second order differential equation $\xi''=r(x)\xi$. By the change of variable $\tau=\gamma x+c$ we obtain the equation

$$\xi''=\rho(\tau)\xi, \tag{29}$$

where

$$r(\tau) = \frac{\beta^2 - 4\alpha\gamma}{4\gamma^2}\tau^2 - \frac{2a\gamma^2 - 2\alpha c\gamma - \beta b\gamma + \beta^2 c}{2\gamma^2}\tau + \frac{b^2\gamma^2 - 2b\beta c\gamma + \beta^2 c^2}{4\gamma^2} + \frac{b\gamma - \beta c}{2\tau} + \frac{3\gamma^2}{4\tau^2}. \quad (30)$$

By the change of variable $z = \sqrt[4]{\frac{\beta^2-4\alpha\gamma}{4\gamma^2}}\tau$, we obtain the equation

$$\psi'' = \phi(z)\psi, \quad \phi(z) = z^2 + \delta_1 z + \frac{\delta_1^2}{4} - \delta_2 + \frac{\delta_3}{2z} + \frac{\delta_0^2 - 1}{4z^2}, \quad (31)$$

where δ_i are algebraic functions in a, b, c, α, β and γ. We can see that equation (31) is exactly the *biconfluent Heun equation* and its integrability is analyzed in Appendix B.2.

Assuming $\beta = \gamma = 0$ we obtain a Liénard equation which is transformable to a reduced second order differential equation with $r \in \mathbb{C}[x]$ and $\deg r = 1$. This means that the equation is not integrable, see [27] and the following subsection. As a particular case, we have a Liénard equation that can be reduced to a Riccati equation given in [41, equation 1.3.2.1]:

$$2yy' = (ax + b)y + 1. \quad (32)$$

Now, doing some restrictions over the parameters of biconfluent Heun equation we can obtain the Whittaker equation. For instance, the Liénard equation of the previous section

$$y\frac{dy}{dx} = (\lambda + b_{11}x)\,y + \left(\varepsilon x + b_{20}x^2\right),$$

falls into a Whittaker equation for some special values of the parameters.

Remark 3.3. *The following hold.*

(a) *It is well–known that via the change $z(x) = \int f(x)dx$ (with inverse $x = x(z)$), the Liénard equation (24) can be transformed to the equation*

$$yy\prime(z) = y + h(z), \quad (33)$$

with

$$h(z) := \frac{g(x(z))}{f(x(z))}.$$

In a similar way, with the change $z(x) = \int g(x)dx$, we reduce the Liénard equation to

$$yy\prime(z) = h(z)y + 1, \quad (34)$$

with

$$h(z) := \frac{f(x(z))}{g(x(z))}.$$

But if we do these changes, then, in general, for functions $f(x)$ and $g(x)$ in a differential field, the transformed function $h(z)$ could not be in the same differential field; ie, in general, the differential field of coefficients is not preserved by these changes. For this reason we do not use these reductions of equation (24).

(b) *Sometimes equation (24) is called Abel equation of second kind. The reason is that by the change $y = 1/w$ it is reduced to an Abel equation*

$$w' = -f(x)w^2 - g(x)w^3.$$

In Cheb–Terrab [8, 9] (see also references therein) are considered some families of Abel equations reducible to Riccati equation.

3.4. **Other families.** a)*Polynomial Riccati equations*

The Riccati equation
$$v' = r(x) - v^2, \quad r(x) \in \mathbb{C}[x]$$
has been studied for several authors, see for example [2, 52, 49]. The Galois group for its associated second order linear differential is connected and can be either, $\mathrm{SL}(2,\mathbb{C})$ or the Borel group, see [1, 3, 2]. In the first case, the tangent field associated to the Riccati equation has not invariant curve and in the second case there is not exist rational first integral for this field.

In particular, the reduced form for the *triconfluent Heun equation* is of this type and is given by

$$y'' = r(x)y, \quad r(x) = \frac{9x^4}{4} + \frac{3}{2}\delta_2 x^2 - \delta_1 x + \frac{\delta_2^2}{4} - \delta_0. \tag{35}$$

b) *Lamé families.*
In the Lamé equation

$$\frac{d^2\xi}{dx^2} + \frac{f'(x)}{2f(x)}\frac{d\xi}{dx} - \frac{n(n+1)x + B}{f(x)}\xi = 0, \tag{36}$$

where $f(x) = 4x^3 - g_2 x - g_3$, with n, B, g_2 and g_3 parameters such that the discriminant of f, $27g_3^2 - g_2^3$ is non-zero (see Appendix B) we consider the change to the Riccati equation

$$y = -\frac{\xi'}{c\xi}, \tag{37}$$

being $c = c(x)$ any non-zero rational function, we obtain the family of Riccati equations associated to the Lamé equation

$$y' = -\frac{n(n+1)x + B}{cf} - \left(\frac{f'(x)}{2f(x)} + \frac{c'}{c}\right) y + cy^2. \tag{38}$$

For the Lamé case, (i.1) of B.3, with $B = B_i$, from Remark B.5 we have one polynomial solution of equation (38), but the general solution of this equation is not algebraic. Furthermore, it is clear that for a fixed n we have associated Riccati equations (38) with arbitrary degree, because the non-zero rational function $c = c(x)$ is arbitrary. Now the Lamé functions correspond here to algebraic solutions of (38), and moving the natural number n, we obtain algebraic invariants curves

$$y + \frac{E'(x)}{c(x)E(x)} = 0,$$

of unbounded degree.

We have proved the following:

Theorem 3.4. *Given a fixed degree in the Riccati family (38) associated to the Lamé equation (36) there are invariant algebraic curves of any tangent field X to the corresponding foliation with unbounded degree. Furthermore the first integral of X is not rational.*

Appendix

Appendix A. Kovacic's Algorithm

This algorithm is devoted to solve the reduced linear differential equation (RLDE) $\xi'' = r\xi$ and is based on the algebraic subgroups of $\mathrm{SL}(2,\mathbb{C})$. For more details see [27]. Although improvements for this algorithm are given in [19, 48], we follow the original version given by Kovacic in [27].

Theorem A.1. *Let G be an algebraic subgroup of $\mathrm{SL}(2,\mathbb{C})$. Then one of the following four cases can occur.*

(1) G is triangularizable.
(2) G is conjugate to a subgroup of infinite dihedral group (also called meta-abelian group) and case 1 does not hold.
(3) Up to conjugation G is one of the following finite groups: Tetrahedral group, Octahedral group or Icosahedral group, and cases 1 and 2 do not hold.
(4) $G = \mathrm{SL}(2,\mathbb{C})$.

Each case in Kovacic's algorithm is related with each one of the algebraic subgroups of $\mathrm{SL}(2,\mathbb{C})$ and the associated Riccatti equation

$$\theta' = r - \theta^2 = \left(\sqrt{r} - \theta\right)\left(\sqrt{r} + \theta\right), \quad \theta = \frac{\xi'}{\xi}.$$

According to Theorem A.1, there are four cases in Kovacic's algorithm. Only for cases 1, 2 and 3 we can solve the differential equation the RLDE, but for the case 4 we have not Liouvillian solutions for the RLDE. It is possible that Kovacic's algorithm can provide us only one solution (ξ_1), so that we can obtain the second solution (ξ_2) through

$$\xi_2 = \xi_1 \int \frac{dx}{\xi_1^2}. \tag{39}$$

Notations. For the RLDE given by

$$\frac{d^2\xi}{dx^2} = r\xi, \qquad r = \frac{s}{t}, \quad s, t \in \mathbb{C}[x],$$

we use the following notations.

(1) Denote by Γ' be the set of (finite) poles of r, $\Gamma' = \{c \in \mathbb{C} : t(c) = 0\}$.
(2) Denote by $\Gamma = \Gamma' \cup \{\infty\}$.
(3) By the order of r at $c \in \Gamma'$, $\circ(r_c)$, we mean the multiplicity of c as a pole of r.
(4) By the order of r at ∞, $\circ(r_\infty)$, we mean the order of ∞ as a zero of r. That is $\circ(r_\infty) = deg(t) - deg(s)$.

A.1. **The four cases. Case 1.**

In this case $[\sqrt{r}]_c$ and $[\sqrt{r}]_\infty$ means the Laurent series of $\sqrt{r}$ at c and the Laurent series of $\sqrt{r}$ at ∞ respectively. Furthermore, we define $\varepsilon(p)$ as follows: if $p \in \Gamma$, then $\varepsilon(p) \in \{+, -\}$. Finally, the complex numbers $\alpha_c^+, \alpha_c^-, \alpha_\infty^+, \alpha_\infty^-$ will be defined in the first step. If the differential equation has not poles it only can fall in this case.

Step 1. Search for each $c \in \Gamma'$ and for ∞ the corresponding situation as follows:

(c_0)**:** If $\circ(r_c)=0$, then
$$\left[\sqrt{r}\right]_c=0,\quad \alpha_c^{\pm}=0.$$
(c_1)**:** If $\circ(r_c)=1$, then
$$\left[\sqrt{r}\right]_c=0,\quad \alpha_c^{\pm}=1.$$
(c_2)**:** If $\circ(r_c)=2$, and
$$r=\cdots+b(x-c)^{-2}+\cdots,\quad \text{then}$$
$$\left[\sqrt{r}\right]_c=0,\quad \alpha_c^{\pm}=\frac{1\pm\sqrt{1+4b}}{2}.$$
(c_3)**:** If $\circ(r_c)=2v\geq 4$, and
$$r=\left(a\,(x-c)^{-v}+...+d\,(x-c)^{-2}\right)^2+b(x-c)^{-(v+1)}+\cdots,\quad \text{then}$$
$$\left[\sqrt{r}\right]_c=a\,(x-c)^{-v}+...+d\,(x-c)^{-2},\quad \alpha_c^{\pm}=\frac{1}{2}\left(\pm\frac{b}{a}+v\right).$$
(∞_1)**:** If $\circ(r_\infty)>2$, then
$$\left[\sqrt{r}\right]_\infty=0,\quad \alpha_\infty^{+}=0,\quad \alpha_\infty^{-}=1.$$
(∞_2)**:** If $\circ(r_\infty)=2$, and $r=\cdots+bx^2+\cdots$, then
$$\left[\sqrt{r}\right]_\infty=0,\quad \alpha_\infty^{\pm}=\frac{1\pm\sqrt{1+4b}}{2}.$$
(∞_3)**:** If $\circ(r_\infty)=-2v\leq 0$, and
$$r=\left(ax^v+...+d\right)^2+bx^{v-1}+\cdots,\quad \text{then}$$
$$\left[\sqrt{r}\right]_\infty=ax^v+...+d,\quad \text{and}\quad \alpha_\infty^{\pm}=\frac{1}{2}\left(\pm\frac{b}{a}-v\right).$$

Step 2. Find $D\neq\emptyset$ defined by
$$D=\left\{m\in\mathbb{Z}_+ : m=\alpha_\infty^{\varepsilon(\infty)}-\sum_{c\in\Gamma'}\alpha_c^{\varepsilon(c)},\forall\left(\varepsilon\left(p\right)\right)_{p\in\Gamma}\right\}.$$
If $D=\emptyset$, then we should start with the case 2. Now, if $\#D>0$, then for each $m\in D$ we search $\omega\in\mathbb{C}(x)$ such that
$$\omega=\varepsilon\left(\infty\right)\left[\sqrt{r}\right]_\infty+\sum_{c\in\Gamma'}\left(\varepsilon\left(c\right)\left[\sqrt{r}\right]_c+\alpha_c^{\varepsilon(c)}(x-c)^{-1}\right).$$

Step 3. For each $m\in D$, search for a monic polynomial P_m of degree m with
$$P_m''+2\omega P_m'+(\omega'+\omega^2-r)P_m=0.$$
If success is achieved then $\xi_1=P_m e^{\int\omega}$ is a solution of the differential equation RLDE. Else, Case 1 cannot hold.

Case 2. Search for each $c\in\Gamma'$ and for ∞ the corresponding situation as follows:

Step 1. Search for each $c\in\Gamma'$ and ∞ the sets $E_c\neq\emptyset$ and $E_\infty\neq\emptyset$. For each $c\in\Gamma'$ and for ∞ we define $E_c\subset\mathbb{Z}$ and $E_\infty\subset\mathbb{Z}$ as follows:

(c_1): If $\circ(r_c)=1$, then $E_c=\{4\}$.
(c_2): If $\circ(r_c)=2$, and $r=\cdots+b(x-c)^{-2}+\cdots$, then

$$E_c=\left\{2+k\sqrt{1+4b}:k=0,\pm 2\right\}.$$

(c_3): If $\circ(r_c)=v>2$, then $E_c=\{v\}$.
(∞_1): If $\circ(r_\infty)>2$, then $E_\infty=\{0,2,4\}$.
(∞_2): If $\circ(r_\infty)=2$, and $r=\cdots+bx^2+\cdots$, then

$$E_\infty=\left\{2+k\sqrt{1+4b}:k=0,\pm 2\right\}.$$

(∞_3): If $\circ(r_\infty)=v<2$, then $E_\infty=\{v\}$.

Step 2. Find $D\neq\emptyset$ defined by

$$D=\left\{m\in\mathbb{Z}_+:\quad m=\frac{1}{2}\left(e_\infty-\sum_{c\in\Gamma'}e_c\right),\forall e_p\in E_p,\ p\in\Gamma\right\}.$$

If $D=\emptyset$, then we should start the case 3. Now, if $\#D>0$, then for each $m\in D$ we search a rational function θ defined by

$$\theta=\frac{1}{2}\sum_{c\in\Gamma'}\frac{e_c}{x-c}.$$

Step 3. For each $m\in D$, search a monic polynomial P_m of degree m, such that

$$P_m'''+3\theta P_m''+(3\theta'+3\theta^2-4r)P_m'+\left(\theta''+3\theta\theta'+\theta^3-4r\theta-2r'\right)P_m=0.$$

If P_m does not exist, then Case 2 cannot hold. If such a polynomial is found, set $\phi=\theta+P'/P$ and let ω be a solution of

$$\omega^2+\phi\omega+\frac{1}{2}\left(\phi'+\phi^2-2r\right)=0.$$

Then $\xi_1=e^{\int\omega}$ is a solution of the differential equation RLDE.

Case 3. Search for each $c\in\Gamma'$ and for ∞ the corresponding situation as follows:

Step 1. Search for each $c\in\Gamma'$ and ∞ the sets $E_c\neq\emptyset$ and $E_\infty\neq\emptyset$. For each $c\in\Gamma'$ and for ∞ we define $E_c\subset\mathbb{Z}$ and $E_\infty\subset\mathbb{Z}$ as follows:

(c_1): If $\circ(r_c)=1$, then $E_c=\{12\}$.
(c_2): If $\circ(r_c)=2$, and $r=\cdots+b(x-c)^{-2}+\cdots$, then

$$E_c=\left\{6+k\sqrt{1+4b}:\quad k=0,\pm1,\pm2,\pm3,\pm4,\pm5,\pm6\right\}.$$

(∞): If $\circ(r_\infty)=v\geq 2$, and $r=\cdots+bx^2+\cdots$, then

$$E_\infty=\left\{6+\frac{12k}{n}\sqrt{1+4b}:\quad k=0,\pm1,\pm2,\pm3,\pm4,\pm5,\pm6\right\},\quad n\in\{4,6,12\}.$$

Step 2. Find $D \neq \emptyset$ defined by

$$D = \left\{ m \in \mathbb{Z}_+ : \quad m = \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma'} e_c\right), \forall e_p \in E_p,\ p \in \Gamma \right\}.$$

In this case we start with $n = 4$ to obtain the solution, afterwards $n = 6$ and finally $n = 12$. If $D = \emptyset$, then the differential equation has not Liouvillian solution because it falls in the case 4. Now, if $\#D > 0$, then for each $m \in D$ with its respective n, search a rational function

$$\theta = \frac{n}{12}\sum_{c\in\Gamma'} \frac{e_c}{x-c},$$

and a polynomial S defined as

$$S = \prod_{c\in\Gamma'} (x-c).$$

Step 3. Search for each $m \in D$, with its respective n, a monic polynomial $P_m = P$ of degree m, such that its coefficients can be determined recursively by

$$P_{-1} = 0, \quad P_n = -P,$$

$$P_{i-1} = -SP_i' - ((n-i)\,S' - S\theta)\,P_i - (n-i)\,(i+1)\,S^2 r P_{i+1},$$

where $i \in \{0, 1 \ldots, n-1, n\}$. If P does not exist, then the differential equation has not Liouvillian solution because it falls in Case 4. Now, if P exists search ω such that

$$\sum_{i=0}^{n} \frac{S^i P}{(n-i)!}\omega^i = 0,$$

then a solution of the differential equation the RLDE is given by

$$\xi = e^{\int \omega},$$

where ω is solution of the previous polynomial of degree n.

Appendix B. Some Special Functions

B.1. **Hypergeometric families.**

B.1.1. *Kimura's Theorem.* The hypergeometric (or Riemann) equation is the more general second order linear differential equation over the Riemann sphere with three regular singular singularities. If we place the singularities at $x = 0, 1, \infty$ it is given by

$$\begin{aligned} &\frac{d^2\xi}{dx^2} + \left(\frac{1-\alpha-\alpha'}{x} + \frac{1-\gamma-\gamma'}{x-1}\right)\frac{d\xi}{dx} \\ &\qquad + \left(\frac{\alpha\alpha'}{x^2} + \frac{\gamma\gamma'}{(x-1)^2} + \frac{\beta\beta' - \alpha\alpha'\gamma\gamma'}{x(x-1)}\right)\xi = 0, \end{aligned} \tag{40}$$

where (α, α'), (γ, γ'), (β, β') are the exponents at the singular points and must satisfy the Fuchs relation $\alpha + \alpha' + \gamma + \gamma' + \beta + \beta' = 1$.

Now, we will briefly describe Kimura's Theorem that provides necessary and sufficient conditions for the integrability of the hypergeometric equation. Let be $\lambda = \alpha - \alpha'$, $\mu = \beta - \beta'$ and $\nu = \gamma - \gamma'$.

Theorem B.1 (Kimura, [26]). *The hypergeometric equation (40) is integrable if and only if either*

(i) At least one of the four numbers $\lambda + \mu + \nu$, $-\lambda + \mu + \nu$, $\lambda - \mu + \nu$, $\lambda + \mu - \nu$ *is an odd integer, or*

(ii) The numbers λ *or* $-\lambda$, μ *or* $-\mu$ *and* ν *or* $-\nu$ *belong (in an arbitrary order) to some of the following fifteen families*

1	$1/2 + l$	$1/2 + m$	*arbitrary complex number*	
2	$1/2 + l$	$1/3 + m$	$1/3 + q$	
3	$2/3 + l$	$1/3 + m$	$1/3 + q$	$l + m + q$ *even*
4	$1/2 + l$	$1/3 + m$	$1/4 + q$	
5	$2/3 + l$	$1/4 + m$	$1/4 + q$	$l + m + q$ *even*
6	$1/2 + l$	$1/3 + m$	$1/5 + q$	
7	$2/5 + l$	$1/3 + m$	$1/3 + q$	$l + m + q$ *even*
8	$2/3 + l$	$1/5 + m$	$1/5 + q$	$l + m + q$ *even*
9	$1/2 + l$	$2/5 + m$	$1/5 + q$	$l + m + q$ *even*
10	$3/5 + l$	$1/3 + m$	$1/5 + q$	$l + m + q$ *even*
11	$2/5 + l$	$2/5 + m$	$2/5 + q$	$l + m + q$ *even*
12	$2/3 + l$	$1/3 + m$	$1/5 + q$	$l + m + q$ *even*
13	$4/5 + l$	$1/5 + m$	$1/5 + q$	$l + m + q$ *even*
14	$1/2 + l$	$2/5 + m$	$1/3 + q$	$l + m + q$ *even*
15	$3/5 + l$	$2/5 + m$	$1/3 + q$	$l + m + q$ *even*

Here l, m, q *are integers.*

B.1.2. *Confluent hypergeometric.* The *confluent Hypergeometric equation* is a degenerate form of the Hypergeometric differential equation where two of the three regular singularities merge into an irregular singularity. The following are two classical forms:

- *Kummer's* form

$$y'' + \frac{c - x}{x} y' - \frac{a}{x} y = 0, \qquad a, c \in \mathbb{C} \tag{41}$$

- *Whittaker's* form

$$y'' = \left(\frac{1}{4} - \frac{\kappa}{x} + \frac{4\mu^2 - 1}{4x^2} \right) y, \tag{42}$$

where the parameters of the two equations are linked by $\kappa = \frac{c}{2} - a$ and $\mu = \frac{c}{2} - \frac{1}{2}$. Furthermore, using the expression (10), we can see that the Whittaker's equation is the reduced form of the Kummer's equation (41). The Galoisian structure of these equations has been deeply studied in [37, 19].

Theorem B.2 (Martinet & Ramis, [37]). *The Whittaker's differential equation* (42) *is integrable if and only if either,* $\kappa + \mu \in \frac{1}{2} + \mathbb{N}$, *or* $\kappa - \mu \in \frac{1}{2} + \mathbb{N}$, *or* $-\kappa + \mu \in \frac{1}{2} + \mathbb{N}$, *or* $-\kappa - \mu \in \frac{1}{2} + \mathbb{N}$.

The *Bessel's equation* is a particular case of the confluent Hypergeometric equation and is given by

$$y'' + \frac{1}{x}y' + \frac{x^2 - n^2}{x^2}y = 0. \tag{43}$$

Under a suitable transformation, the reduced form of the Bessel's equation is a particular case of the Whittaker's equation (42).

Corollary B.3. *The Bessel's differential equation* (43) *is integrable if and only if* $n \in \frac{1}{2} + \mathbb{Z}$.

B.2. **Heun's families.** The Heun's equation is the generic differential equation with four regular singular points at 0, 1, c and ∞. In its reduced form, the Heun's equation is $y'' = r(x)y$, where

$$r(x) = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-c} + \frac{D}{x^2} + \frac{E}{(x-1)^2} + \frac{F}{(x-c)^2}, \tag{44}$$

$$A = -\frac{\alpha\beta}{2} - \frac{\alpha\gamma}{2c} + \frac{\delta\eta h}{c}, \quad B = \frac{\alpha\beta}{2} - \frac{\beta\gamma}{2(c-1)} - \frac{\delta\eta(h-1)}{c-1},$$

$$C = \frac{\alpha\gamma}{2c} + \frac{\beta\gamma}{2(c-1)} - \frac{\delta\eta(c-h)}{c(c-1)}, \quad D = \frac{\alpha}{2}\left(\frac{\alpha}{2} - 1\right), \quad E = \frac{\beta}{2}\left(\frac{\beta}{2} - 1\right),$$

$$F = \frac{\gamma}{2}\left(\frac{\gamma}{2} - 1\right), \quad \text{with} \quad \alpha + \beta + \gamma - \delta - \eta = 1.$$

To our purposes we write the determinant $\Pi_{d+1}(a, b, u, v, \xi, w)$ as in [19]:

$$\begin{vmatrix} w & u & 0 & 0 & 0 & \dots & 0 \\ d\xi w + 1 & v & 2(u+b) & 0 & 0 & \dots & 0 \\ 0 & (d-1)\xi & w + 2(v+a) & 3(u+2b) & 0 & \dots & 0 \\ 0 & 0 & (d-2)\xi & w + 3(v+2a) & 4(u+3b) & \dots & 0 \\ \vdots & & & & & \dots & \\ 0 & \dots & & \dots & 2\xi & w + (d-1)(v + (d-2)a) & d(u + (d-1)b) \\ 0 & \dots & & \dots & 0 & \xi & w + d(v + (d-1)a) \end{vmatrix}$$

B.2.1. *Biconfluent Heun.* The equation

$$y'' = (x^2 + \delta_1 x + \frac{\delta_1^2}{4} - \delta_2 + \frac{\delta_3}{2x} + \frac{\delta_0^2 - 1}{4x^2})y, \tag{45}$$

is the well known *biconfluent Heun equation* which has been deeply analyzed by Duval and Loday-Richaud in [19, p. 236].

Theorem B.4. [19]. *The biconfluent Heun equation* (45) *has Liouvillian solutions if and only if falls in case 1 of Kovacic algorithm and one of the following conditions is fullfilled:*

(1) $\delta_0^2 = 1$, $\delta_3 = 0$ *and* $\delta_2 \in 2\mathbb{Z} + 1$.

(2) $\delta_0^2 = 1$, $\delta_3 \neq 0$ *and* $\delta_2 \in 2\mathbb{Z}^* + 1$ *with* $|\delta_2| \geq 3$, *and if* $\varepsilon = \mathsf{sign}\,\delta_2$, *then*

$$\Pi_{(|\delta_2|-1)/2}\left(0, 1, 2, \varepsilon\delta_1, -2\varepsilon, \varepsilon\delta_1 - \frac{\delta_3}{2}\right) = 0.$$

(3) $\delta_0 \neq \pm 1$, $\pm\delta_0 \pm \delta_2 \in 2\mathbb{Z}^*$ *and if* $\varepsilon_0, \varepsilon_\infty \in \{\pm 1\}$ *are such that* $\varepsilon_\infty\delta_2 - \varepsilon_0\delta_0 = 2d^* \in 2\mathbb{N}^*$ *then*

$$\Pi_{d^*}\left(0, 1, 1 + \varepsilon_0\delta_0, \varepsilon_\infty\delta_1, -2\varepsilon_\infty, \frac{1}{2}(\varepsilon_\infty\delta_1(1 + \varepsilon_0\delta_0) - \delta_3)\right) = 0.$$

B.3. **Lamé equation.** The algebraic form of the Lamé Equation is [42, 51]

$$\frac{d^2\xi}{dx^2}+\frac{f'(x)}{2f(x)}\frac{d\xi}{dx}-\frac{n(n+1)x+B}{f(x)}\xi=0, \tag{46}$$

where $f(x)=4x^3-g_2x-g_3$, with n, B, g_2 and g_3 parameters such that the discriminant of f, $27g_3^2-g_2^3$ is non-zero. This equation is a Fuchsian differential equation with four singular points over the Riemann sphere: the roots, e_1,e_2,e_3, of f and the point at the infinity.

Now the known mutually exclusive cases of closed form solutions of the Lamé equation (46) are as follows:

(i) The *Lamé-Hermite* case [42, 51]. In this case $n\in\mathbf{N}$ the three other parameters are arbitrary.

(ii) The *Brioschi-Halphen-Crawford* solutions [20, 42]. Now $m:=n+\frac{1}{2}\in\mathbb{N}$ and the parameters B, g_2 and g_3 must satisfy an algebraic equation

$$0=Q_m(g_2/4,g_3/4,B)\in\mathbb{Z}[g_2/4,g_3/4,B],$$

where Q_m has degree m in B. This polynomial is known as the Brioschi determinant.

(iii) The *Baldassarri* solutions [4]. The condition on n is $n+\frac{1}{2}\in\frac{1}{3}\mathbb{Z}\cup\frac{1}{4}\mathbb{Z}\cup\frac{1}{5}\mathbb{Z}-\mathbb{Z}$, with additional (involved) algebraic restrictions on the other parameters.

It is possible to prove that the only integrable cases for the Lamé equation are cases (i)–(iii) above [38]. For cases (ii) and (iii) the general solution of (46) is algebraic and the Galois group is finite.

Case (i) split in two subcases [42, 51]:
(i.1) The Lamé case. For a fixed natural n, the Lamé equation has a solution (Lamé function)

$$E(x)=\prod_{i=1}^{3}(x-e_i)^{k_i}P_m(x), \tag{47}$$

being P_m a monic polynomial of degree $m=n/2-(k_1+k_2+k_3)$ and k_i are 0 or 1/2; as m must be a natural, we have four possible choices for the k_i, two for n is even and the other two for odd n, these are the four classes of Lame's functions. Moreover the parameter B is one of the $m+1$ different roots $B_1,...,B_{m+1}$ of certain irreducible polynomial of degree $m+1$. Furthermore, the numbers B_i are reals.

(i.2) The Hermite case. Here we are not in case (i.1) and n is an arbitrary natural number. We are also in case 1 of Kovacic's algorithm, but with a diagonal Galois group.

Remark B.5. *We remark that the polynomial P_m in (i.1) satisfies a second order linear differential equation similar to the one in the case 1 of Kovacic algorithm. In fact it is possible to obtain the above passing to normal form and using Kovacic algorithm.Then the second linear independent solution is not algebraic and the Riccati associated equation has not a rational first integral.*

We are interested in the Lamé case. For fixed n we have

$$2n+1$$

Lamé different equations that fall in Lamé case, corresponding to the different choices of the real numbers $B = B_i$.

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