OUTPUT OBSERVABILITY OF TIME-INVARIANT SINGULAR LINEAR SYSTEMS

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Abstract
In this paper finite-dimensional singular linear discrete-time-invariant systems in the form $Ex(k+1) = Ax(k) + Bu(k)$, $y(k) = Cx(k)$ where $E, A \in M = M_n(\mathbb{C})$, $B \in M_{n\times m}(\mathbb{C})$, $C \in M_{p\times n}(\mathbb{C})$, describing convolutional codes are considered and the notion of output observability is analyzed.

Key words
Singular systems, feedback proportional and derivative, output injection proportional and derivative, output observability.

1 Introduction

Let us consider a finite-dimensional singular linear discrete-time-invariant system $Ex(k+1) = Ax(k) + Bu(k)$, $y(k) = Cx(k)$ where $E, A \in M_n(\mathbb{C})$, $B \in M_{n\times m}(\mathbb{C})$, $C \in M_{p\times n}(\mathbb{C})$, describing convolutional codes. For simplicity, we denote the systems as a quadruples of matrices (E, A, B, C) and we denote by $\mathcal{M}$ the set of this kind of systems. In the case where $E = I_n$ the system is standard and we denote as a triple (A, B, C).

For simplicity but without loss of generality, we consider that matrix B has column full rank and $\operatorname{rank} B = m$ and C has row full rank and $\operatorname{rank} C = p$, so $0 < p, m \leq n$.

It is well known that there is a close connection between linear systems and convolutional codes and there is a large literature about that as for example [F. R. Gantmacher, (1959), M. Kuijper, R. Pinto, (2009), J.L Massey, M.K. Sain, (1967), J. Rosenthal, J.M. Schumacher, E.V. York, (1996)].

Ch. Fragouli and R. D. Wesel [Ch. Fragouli, R.D. Wesel, (1999)] give the following definition of output observable for standard systems.

Definition 1.1. *The standard system (A, B, C) is said to be output observable if the state sequence $\{x_0, x_1, \dots, x_{n-1}\}$ is uniquely determined by the knowledge of the output sequence $\{y_0, y_1, \dots, y_{n-1}\}$ for a finite number of steps $n-1$.*

Taking into account that

$$y_k = CA^k x_0 + CA^{k-1}Bu_0 + \dots + CABu_{k-2} + CBu_{k-1}$$

the output observability is characterized by the following proposition.

Proposition 1.1 ([Ch. Fragouli, R.D. Wesel, (1999)]). *The system $(A, B, C) \in \mathcal{M}$ is output observable if and only if the following matrix*

$$M = \begin{pmatrix} C & 0 & 0 & \dots & 0 \\ CA & CB & 0 & \ddots & \vdots \\ CA^2 & CAB & CB & \ddots & \\ \vdots & \vdots & & \ddots & \\ CA^{n-1} & CA^{n-2}B & CA^{n-3}B & \dots & CB \end{pmatrix} \in M_{pn\times(n+(n-1)m)}(\mathbb{C})$$

has full rank.

In this paper, we generalize the notion of output observability given for standard linear systems to the singular linear systems, and we characterize the set of output observable systems.

We remark that the observability of singular systems has been widely discussed by T. Kaczorek in [T. Kaczorek, (1992)].

2 Preliminaries

We consider quadruples of matrices $(E, A, B, C) \in \mathcal{M}$, representing singular discrete time invariant linear systems, a manner to understand the properties of the system is treating it by purely algebraic techniques. The main aspect of this approach is defining an equivalence relation preserving these properties, many interesting and useful equivalence relations between singular systems have been defined. We deal with the equivalence relation accepting one or more, of the following transformations: basis change in the state space, input space, output space, operations of state and derivative feedback, state and derivative output injection and to premultiply matrices E, A, B (i.e. the first equation of the system), by an invertible matrix. That is to say.

Definition 2.1. *Two quadruples* $(E_i, A_i, B_i, C_i) \in \mathcal{M}, i = 1, 2$, *are equivalent if and only if there exist matrices* $P \in Gl(n; \mathbb{C})$, $Q \in Gl(p; \mathbb{C})$, $R \in Gl(m; \mathbb{C})$, $S \in Gl(q; \mathbb{C})$, $F_E^B, F_A^B \in M_{m\times n}(\mathbb{C})$, $F_E^C, F_A^C \in M_{p\times q}(\mathbb{C})$ *such that*

$$\begin{array}{l} E_2 = QE_1P + QB_1F_E^B + F_E^C C_1 P, \\ A_2 = QA_1P + QB_1F_A^B + F_A^C C_1 P, \\ B_2 = QB_1R, \\ C_2 = SC_1P. \end{array} \tag{1}$$

Given a quadruple of matrices $(E, A, B, C) \in \mathcal{M}$, we can associate the following matrix pencil

$$H(\lambda) = \begin{pmatrix} \lambda E + A & \lambda B & B \\ \lambda C & 0 & 0 \\ C & 0 & 0 \end{pmatrix},$$

and we have the following result.

Proposition 2.1. *Two quadruples are equivalent under equivalent relation considered if and only if the associated matrix pencils are strictly equivalent.*

So, we can apply Kronecker's theory of singular pencils (see [F. R. Gantmacher, (1959)], for more details).

3 Output-Observability

In this section we generalize the output observability condition to singular systems.

Definition 3.1. *The singular system* (E, A, B, C) *is said to be output observable if the state sequence* $\{x_0, x_1, \ldots, x_{n-1}\}$ *is uniquely determined by the knowledge of the output sequence* $\{y_0, y_1, \ldots, y_{n-1}\}$ *for a finite number of steps* $n-1$.

Output observability is characterized by the following proposition.

Proposition 3.1. *The system* $(E, A, B, C) \in \mathcal{M}$ *is output observable if and only if the following matrix*

$$M = \begin{pmatrix} A & B & E & 0 & 0 & & & \\ C & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & A & B & E & & & \\ 0 & 0 & C & 0 & 0 & & & \\ & & & & & \ddots & & \\ & & & & & & A & B & E \\ & & & & & & C & 0 & 0 \\ & & & & & & 0 & 0 & C \end{pmatrix} \in M_{x\times y}(\mathbb{C}),$$

$x = (n-1)n + np, y = n^2 + (n-1)m$, *has full rank.*

Proof. It suffice to observe that

$$\begin{pmatrix} A & B & -E & 0 & 0 & & & \\ C & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & A & B & -E & & & \\ 0 & 0 & C & 0 & 0 & & & \\ & & & & & \ddots & & \\ & & & & & & A & B & -E \\ & & & & & & C & 0 & 0 \\ & & & & & & 0 & 0 & C \end{pmatrix} \begin{pmatrix} x(0) \\ u(0) \\ x(1) \\ u(1) \\ \vdots \\ x(n-2) \\ u(n-2) \\ x(n-1) \end{pmatrix} = \begin{pmatrix} 0 \\ y(0) \\ 0 \\ y(1) \\ \vdots \\ 0 \\ y(n-2) \\ y(n-1) \end{pmatrix}.$$

□

In the case where the system (E, A, B, C) is standard, (i.e. $E = I_n$), making elementary transformations to the matrix M we obtain

$$\operatorname{rank} M = n(n-1) + \operatorname{rank} \begin{pmatrix} C & 0 & 0 & \dots & 0 \\ CA & CB & 0 & \ddots & \vdots \\ CA^2 & CAB & CB & \ddots & \\ \vdots & \vdots & & \ddots & \\ CA^{n-1} & CA^{n-2}B & CA^{n-3}B & \dots & CB \end{pmatrix}.$$

So, the definition is a generalization of the definition given for standard systems.

Proposition 3.2. *Let* (E, A, B, C) *be a system. Then, the rank of the matrix* M *is invariant under equivalence relation considered.*

Proof. Let (E_1, A_1, B_1, C_1) be an equivalent system, then there exist matrices $P \in Gl(n; C)$, $Q \in Gl(p; C)$, $R \in Gl(m; C)$, $S \in Gl(q; C)$, $F_E^B, F_A^B \in M_{m\times n}(C)$, $F_E^C, F_A^C \in M_{p\times q}(C)$ such that (1) is verified.

Denoting by

$$\mathbf{Q} = \begin{pmatrix} Q & F_A^C & 0 & F_E^C & & & \\ 0 & S & 0 & 0 & & & \\ 0 & 0 & Q & F_A^C & & & \\ 0 & 0 & 0 & S & & & \\ & & & & \ddots & & \\ & & & & & Q & F_A^C & F_E^C \\ & & & & & 0 & S & 0 \\ & & & & & 0 & 0 & S \end{pmatrix}$$

and

$$\mathbf{P} = \begin{pmatrix} P & 0 & 0 & 0 & 0 & & & \\ F_A^B & R & F_E^C & 0 & 0 & & & \\ 0 & 0 & P & 0 & 0 & & & \\ 0 & 0 & F_A^B & R & F_E^B & & & \\ 0 & 0 & 0 & 0 & P & & & \\ & & & & & \ddots & & \\ & & & & & & P & 0 & 0 \\ & & & & & & F_A^B & R & F_E^B \\ & & & & & & 0 & 0 & P \end{pmatrix}.$$

So,

$$\operatorname{rank} \mathbf{Q} \cdot \begin{pmatrix} A & B & E & 0 & 0 \\ C & 0 & 0 & 0 & 0 \\ 0 & 0 & A & B & E \\ 0 & 0 & C & 0 & 0 \\ & & & & & \ddots \\ & & & & & & A & B & E \\ & & & & & & C & 0 & 0 \\ & & & & & & 0 & 0 & C \end{pmatrix} \cdot \mathbf{P} =$$

$$= \operatorname{rank} \begin{pmatrix} A_1 & B_1 & E_1 & 0 & 0 \\ C_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_1 & B_1 & E_1 \\ 0 & 0 & C_1 & 0 & 0 \\ & & & & & \ddots \\ & & & & & & A_1 & B_1 & E_1 \\ & & & & & & C_1 & 0 & 0 \\ & & & & & & 0 & 0 & C_1 \end{pmatrix}.$$

□

4 Qualitative properties of the systems

In this section we will go to analyze the qualitative properties characterizing output observable systems in the case of standard systems.

Having defined an equivalence relation, the standard procedure then is to look for a canonical form, that is to say to look for a regularisable system which is equivalent to a given system and which has a simple form from which we can directly read off the properties and invariants of the corresponding singular system. In this case it is well known the following proposition.

For simplicity we consider $m, p \leq n$ and $\operatorname{rank} B = m$, $\operatorname{rank} C = p$.

Proposition 4.1. *Let (E, A, B, C) be a regularizable system, then it is equivalent to (E_c, A_c, B_c, C_c) with*

$$E_c = \begin{pmatrix} I_1 & & & & \\ & I_2 & & & \\ & & I_3 & & \\ & & & I_4 & \\ & & & & N_4 \end{pmatrix}, A_c = \begin{pmatrix} N_1 & & & & \\ & N_2 & & & \\ & & N_3 & & \\ & & & J & \\ & & & & I_5 \end{pmatrix},$$

$$B_c = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \\ 0 & B_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_c = \begin{pmatrix} 0 & C_1 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 & 0 \end{pmatrix},$$

with
$N_1 = \operatorname{diag}(N_1^1, \ldots, N_r^1) \in M_{n_1}(\mathbb{C})$
$N_2 = \operatorname{diag}(N_1^2, \ldots, N_s^2) \in M_{n_2}(\mathbb{C})$
$N_3 = \operatorname{diag}(N_1^3, \ldots, N_t^3) \in M_{n_3}(\mathbb{C})$
$N_4 = \operatorname{diag}(N_1^4, \ldots, N_v^4) \in M_{n_4}(\mathbb{C})$
$J = \operatorname{diag}(J_1, \ldots, J_u) \in M_{n_5}(\mathbb{C})$
$B_1 = \operatorname{diag}(B_1^1, \ldots, B_r^1) \in M_{n_1 \times m_1}(\mathbb{C})$
$B_2 = \operatorname{diag}(B_1^2, \ldots, B_t^2) \in M_{n_2 \times m_2}(\mathbb{C})$
$C_1 = \operatorname{diag}(C_1^1, \ldots, C_s^1) \in M_{p_1 \times n_1}(\mathbb{C})$
$C_2 = \operatorname{diag}(C_1^2, \ldots, C_t^2) \in M_{p_2 \times n_3}(\mathbb{C})$
$N_i^1 = \begin{pmatrix} 0 & 0 \\ I_{k_i - 1} & 0 \end{pmatrix} \in M_{k_i}(\mathbb{C}), \quad 1 \leq i \leq r$
$B_i^1 = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix}^t \in M_{k_i \times 1}(\mathbb{C}), \quad 1 \leq i \leq r$
$N_i^2 = \begin{pmatrix} 0 & I_{l_i - 1} \\ 0 & 0 \end{pmatrix} \in M_{l_i}(\mathbb{C}), \quad 1 \leq i \leq s$
$C_i^1 = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix} \in M_{1 \times l_i}(\mathbb{C}), \quad 1 \leq i \leq s$
$N_i^3 = \begin{pmatrix} 0 & I_{m_i - 1} \\ 0 & 0 \end{pmatrix} \in M_{m_i}(\mathbb{C}), \quad 1 \leq i \leq t$
$N_i^4 = \begin{pmatrix} 0 & I_{v_i - 1} \\ 0 & 0 \end{pmatrix} \in M_{v_i}(\mathbb{C}), \quad 1 \leq i \leq q$
$B_i^2 = \begin{pmatrix} 0 & 0 & \ldots & 1 \end{pmatrix}^t \in M_{m_i \times 1}(\mathbb{C}), \quad 1 \leq i \leq t$
$C_i^2 = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix} \in M_{1 \times m_i}(\mathbb{C}), \quad 1 \leq i \leq t$
J_{4_i} is the endomorphism in its Jordan form.

The canonical form can be obtained directly from the initial system, without knowing transformations that permit us to reduce the system to its reduced form, (see [Mª I. García-Planas, (2009), Mª I. García-Planas, M.D. Magret, (1999)]).

Theorem 4.1. *The system (E, A, B, C) is output observable if and only if*

1. *if $p \leq m \leq n$, the system has not observable non controllable part, that is to say the reduced form is in the form* $\left(\begin{pmatrix} I_1 & & & \\ & I_3 & & \\ & & I_5 & \\ & & & N_4 \end{pmatrix}, \begin{pmatrix} N_1 & & & \\ & N_3 & & \\ & & J & \\ & & & I_4 \end{pmatrix}, \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & C_2 & 0 & 0 \end{pmatrix} \right).$
2. *if $m < p \leq n$, the system has only controllable and observable part and observable non controllable part with $C = \begin{pmatrix} C_1 & \\ & I_{p_2} \end{pmatrix}$, $A = \begin{pmatrix} N_2 & \\ & 0_{p_2} \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ I_{p_2} \end{pmatrix}$, and $E = I$.*

Proof. First of all we observe that

i) if $p \leq m \leq n$ then $(n-1)n + np \leq n^2 + (n-1)m$
ii) if $m < p \leq n$ then $(n-1)n + np > n^2 + (n-1)m$

Proposition 3.2 permit us to consider the system in its reduced form and for study output observability. So, making elementary transformations to the matrix M, we have

$$\operatorname{rank} M = \sum \operatorname{rank} M_i$$

where

$$\operatorname{rank} M_1 = \operatorname{rank} \begin{pmatrix} J & I & & \\ & J & I & \\ & & \ddots & \ddots \\ & & & J & I \end{pmatrix} = (n-1)n_5$$

$$\operatorname{rank} M_2 = \operatorname{rank} \begin{pmatrix} I & N_4 & & \\ & I & N_4 & \\ & & \ddots & \ddots \\ & & & I & N_5 \end{pmatrix} = (n-1)n_4$$

$$\operatorname{rank} M_3 = \operatorname{rank} \begin{pmatrix} N_1 & B_1 & I & & & \\ & & N_1 & B_1 & I & \\ & & & \ddots & \ddots & \\ & & & & N_1 & B_1 & I \end{pmatrix} =$$
$(n-1)n_1$

$$\operatorname{rank} M_4 = \operatorname{rank}\begin{pmatrix} N_2 & I & & & & & \\ C_1 & 0 & & & & & \\ & N_2 & I & & & & \\ & C_1 & 0 & & & & \\ & & & \ddots & \ddots & & \\ & & & & N_2 & I & \\ & & & & C_1 & 0 & \\ & & & & & C_1 & \end{pmatrix} =$$

$$\operatorname{rank}\begin{pmatrix} I_{n_2} & & & \\ & \ddots & & \\ & & I_{n_2} & \\ & & & C_1 \\ & & & C_1N_2 \\ & & & \vdots \\ & & & C_1N_2^{n-1} \end{pmatrix} =$$

$$(n-1)n_2 + \operatorname{rank}\begin{pmatrix} C_1 \\ C_1N_2 \\ \vdots \\ C_1N_2^{n-1} \end{pmatrix} =$$

$$(n-1)n_2$$

$$\operatorname{rank} M_5 = \operatorname{rank}\begin{pmatrix} N_3 & B_2 & I & & & & \\ C_2 & 0 & 0 & & & & \\ & & N_3 & B_2 & I & & \\ & & C_2 & 0 & & & \\ & & & \ddots & \ddots & & \\ & & & & N_3 & B_2 & I \\ & & & & C_2 & 0 & 0 \\ & & & & 0 & 0 & C_2 \end{pmatrix} =$$

$$\operatorname{rank}\begin{pmatrix} I_{n_3} & & & & & \\ & \ddots & & & & \\ & & I_{n_3} & & & \\ & & & C_2 & & \\ & & & C_2N_3 & C_2B_2 & \\ & & & \vdots & \vdots & \ddots \\ & & & C_2N_3^{n-1} & C_2N_3^{n-2}B_2 & \dots & C_2B_2 \end{pmatrix} =$$

$$(n-1)n_3 + \operatorname{rank}\begin{pmatrix} C_2 & & & \\ C_2N_3 & C_2B_2 & & \\ \vdots & \vdots & \ddots & \\ C_2N_3^{n-1} & C_2N_3^{n-2}B_2 & \dots & C_2B_2 \end{pmatrix} =$$

$$(n-1)n_3 + n\operatorname{rank} C_2$$

Suppose now $p \leq m \leq n$ so, the system is output observable if and only if $\operatorname{rank} M = (n-1)n + np$, i. e. if and only if $n_2 + n\operatorname{rank} C_2 = np$ and that is only possible if and only if $n_2 = 0$.

If $m < p \leq n$, the system is output observable if and only if $\operatorname{rank} M = n^2 + (n-1)m$, i.e. if and only if $(n-1)n + n_2 + n\operatorname{rank} C_2 = n^2 + (n-1)m$, so $n_1 = n_4 = n_5 = 0$ consequently $m = \operatorname{rank} C_2$, $n = n_2 + n_3$ and $(n_2+n_3)m - (n_2+n_3) + n_2 = (n_2+n_3)m - m$ and that is only possible if $m = n_3$. $\square$

Example 4.1. *1. The system (I, A, B, C) with $A = \begin{pmatrix} 0&1&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&a \end{pmatrix}$, $B = \begin{pmatrix} 0&0 \\ 1&0 \\ 0&0 \\ 0&1 \\ 0&0 \end{pmatrix}$ and $C = \left(\begin{smallmatrix} 1&0&0&0&0 \\ 0&0&1&0&0 \end{smallmatrix}\right)$, is output observable.*

2. The system (A, B, C) with $A = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}$, $B = \begin{pmatrix} 0&0 \\ 1&0 \\ 0&0 \\ 0&1 \end{pmatrix}$ and $C = \left(\begin{smallmatrix} 1&0&0&0 \\ 0&1&0&0 \end{smallmatrix}\right)$, is not output observable.

3. The system (A, B, C) with $A = \begin{pmatrix} 0&1&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $C = \left(\begin{smallmatrix} 1&0&0 \\ 0&0&1 \end{smallmatrix}\right)$ is output observable.

4. The system (A, B, C) with $A = 0$, $B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $C = \left(\begin{smallmatrix} 0&1&0 \\ 0&0&1 \end{smallmatrix}\right)$ is not output observable.

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