Perturbation analysis of eigenvalues of polynomial matrices smoothly depending on parameters

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Abstract

Let $P(\lambda)=\sum_{i=0}^{k}\lambda^i A_i(p)$ be a family of monic polynomial matrices smoothly dependent on a vector of real parameters $p=(p_1,\dots,p_n)$. In this work we study behavior of a multiple eigenvalue of the monic polynomial family $P(\lambda)$.

Key Words: Polynomial matrix, Eigenvalues, Perturbation.

1. Introduction

Given a polynomial matrix $P(\lambda)=\sum_{i=0}^{k}\lambda^i A_i$ where A_i are square matrices over real or complex field, it is an important question from both the theoretical and the practical points of view to know how the eigenvalues and eigenvectors change when the elements of $P(\lambda)$ are subjected to small perturbations.

Eigenvalue problem for polynomial matrices $P(\lambda)v=0$, appears (among many other applications) modeling physical and engineering problems by means systems of k-order linear ordinary differential equations. The values of eigenvalues can correspond among others, to frequencies of vibration, critical values of stability parameters, or energy levels of atoms.

The eigenvalues of some matrices are sensitive to perturbations, it is well know that the eigenvalues of monic polynomial matrices are continuous functions of the entries of the matrix coefficients of the polynomial, but Small changes in the matrix elements can lead to large changes in the multiplicity of eigenvalues. For example a little perturbation of the matrix $\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$ as $\begin{pmatrix}\lambda & 1\\ \varepsilon & \lambda\end{pmatrix}$ the double eigenvalue $\lambda=0$ is perturbed to two different eigenvalues $\lambda=\pm\sqrt{\varepsilon}$ changing completely the structure of the polynomial matrix. Obviously if we consider the perturbation $\begin{pmatrix}\lambda & 1+\varepsilon\\ 0 & \lambda\end{pmatrix}$ there are not changes in the structure.

Given a square complex matrix A, it is an important question from both the theoretical and the practical points of view to know how the eigenvalues and eigenvectors change when the elements of A are subjected to small perturbations. The usual formulation of the problem introduces a perturbation parameter ε belonging to some neighborhood of zero, and writes the perturbed matrix as $A+\varepsilon B$ for an arbitrary matrix B. In this situation, it is well known [8] section II.1.2, that each eigenvalue or eigenvector of $A+\varepsilon B$ admits an expansion in fractional powers of ε, whose zero-th order term is an eigenvalue or eigenvector of the unperturbed matrix A.

In this paper, in section 1 we present an overview over polynomial matrices $P(\lambda)$ and the analysis of perturbation of simple eigenvalue λ_0 of $P(\lambda)$ such that 0 is a simple eigenvalue of the linear map $P(\lambda_0)$. Finally, in section 3, we study the perturbation of a multiple eigenvalue with a simple eigenvector of a monic polynomial matrix smoothly depending on parameters.

The study of behavior of simple and multiple eigenvalues of a matrix depending smoothly of parameters has a great interest for its many applications. Perturbation theory for eigenvalues and eigenvectors of regular pencils is well established see [1],[10] for example and for vibrational systems in [9]. In this paper we extend some of these results to polynomial matrices.

2. Preliminaries

A square polynomial matrix of size n and degree k is a polynomial of the form

$$P(\lambda)=\sum_{i=0}^{k}\lambda^i A_i,\quad A_0,\dots,A_k\in M_n(\mathbb{F}),\qquad (1)$$

where $\mathbb{F}$ is the field of real or complex numbers. Our focus is on monic polynomial matrices. A square polynomial matrix $P(\lambda)$ is said to be monic if $A_k=I_n$ is

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identically. The polynomial matrix (1) naturally arises associated with linear systems of differential equations

$$A_k x^{(k)}(t)+A_{k-1}x^{(k-1)}(t)+\ldots+A_1x^1(t)+A_0x(t) = f(t) \quad (2)$$

where $x(t)$ is a vector-valued function (unknown) with n coordinates, $x^{(j)}(t)$ denotes the j-th derivative of $x(t)$ and $f(t)$ is another vector-valued function with n coordinates. Of particular relevance is the case of linear systems of second order, appearing in many engineering applications.

The eigenvalues of a polynomial matrix $P(\lambda)$ are the zeros of the nk-degree scalar polynomial $\det P(\lambda)$.

Let λ_0 be an eigenvalue of polynomial matrix $P(\lambda)$, then there exists a vector $v_0 \neq 0$ such that $P(\lambda_0)(v_0) = 0$, this vector is called an eigenvector.

We will call a Jordan chain of length $k+1$ for $P(\lambda)$ corresponding to complex number λ_0 to the sequence of n-dimensional vectors $v_0, \ldots, v_k$ such that

$$\sum_{\ell=0}^{i} \frac{1}{\ell!} P^{(\ell)}(\lambda_0) v_{i-\ell} = 0, \quad i = 0, \ldots, k \quad (3)$$

where $P^{(\ell)}$ denotes the ℓ-derivative of $P(\lambda)$ with respect the variable λ. If λ_0 is an eigenvalue there exists a Jordan chain of length at least 1 formed by the eigenvector.

Let λ_0 be an eigenvalue of $P(\lambda)$, then $\det P^t(\lambda_0) = \det P(\lambda_0) = 0$, so λ_0 is an eigenvalue of $P^t(\lambda)$. For this eigenvalue there exists an eigenvector u_0, that is $P^t(\lambda_0)(u_0) = 0$, equivalently $u_0^t P(\lambda_0) = 0$. The vector u_0 is called left eigenvector corresponding to the eignevalue λ_0 of $P(\lambda)$.

For more information see [4], or [7] for example.

Let $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ be now, a polynomial matrix and we assume that the matrices A_i smoothly depend on the vector of real parameters $p = (p_1, \ldots, p_r)$. The function $P(\lambda; p) = \sum_{i=0}^{k} \lambda^i A_i(p)$ is called a multiparameter family of polynomial matrices. Eigenvalues of the polynomial matrix function are continuous functions of the vector of parameters. We are going to review the behavior of a simple eigenvalue of the family of polynomial matrices $P(\lambda; p)$.

Let $\lambda(p)$ be a simple eigenvalue of the polynomial matrix $P(\lambda; p)$. Since $\lambda(p)$ is a simple root of the scalar polynomial $\det P(\lambda)$, we have

$$\frac{\partial}{\partial \lambda} \det P(\lambda; p) \neq 0. \quad (4)$$

The expression (4) permit us to make use the implicit function theorem to the equation $\det P(\lambda; p) = 0$, and we observe that the eigenvalue $\lambda(p)$ of the family of polynomial matrices smoothly depends on the vector of parameters, and its derivatives with respect to parameters are

$$\frac{\partial \lambda(p)}{\partial p_i} = -\frac{\dfrac{\partial}{\partial p_i} \det P(\lambda; p)}{\dfrac{\partial}{\partial \lambda} \det P(\lambda; p)}, \quad i = 1, \ldots, r. \quad (5)$$

Taking into account that $\lambda(p)$ is a simple eigenvalue and that the sum of the lengths of Jordan chains in a canonical set is the multiplicity of the eigenvalue as zero of $\det P(\lambda; p)$, we have that the Jordan chains consist only of the eigenvectors.

The eigenvector $v_0(p)$ corresponding to the simple eigenvalue $\lambda(p)$ is determined up to a nonzero scaling factor α. This eigenvector determines a one-dimensional null-subspace of the matrix operator $P(\lambda(p); p)$ smoothly dependent on p. Hence, the eigenvector $v_0(p)$ can be chosen as a smooth function of the parameters.

An approximation of the eigenvalues as well of the corresponding eigenvectors by means their derivatives is given by the following result.

Theorem 1.

$$\frac{\partial \lambda}{\partial p_i}_{|(\lambda_0; p_0)} = -\frac{u_0^t \dfrac{\partial P(\lambda; p)}{\partial p_i}_{|(\lambda_0, p_0)} v_0(p_0)}{u_0^t P'(\lambda_0; p_0) v_0(p_0)} \quad (6)$$

and

$$\frac{\partial v_0(p)}{\partial p_i}_{|(\lambda_0, p_0)} = -T_0^{-1} \left(\frac{\partial \lambda}{\partial p_i} (P'(\lambda; p)) + \frac{\partial P(\lambda; p)}{\partial p_i} \right)_{|(\lambda_0, p_0)} v_0(p_0). \quad (7)$$

where $T_0 = P(\lambda_0); p_0) + u_0 u_0^t P'(\lambda_0; p_0)$, *and*

$$\frac{\partial^2 \lambda}{\partial p_i \partial p_j}_{|(\lambda_0, p_0)} = -\frac{a}{b},$$

with

$$a = \Bigg(u_0^t \left(\frac{\partial \lambda}{\partial p_i} \frac{\partial \lambda}{\partial p_j} P'(\lambda; p) + \frac{\partial \lambda}{\partial p_i} \frac{\partial P'(\lambda; p)}{\partial p_j} + \frac{\partial P'(\lambda; p)}{\partial p_i} \frac{\partial \lambda}{\partial p_j} + \frac{\partial^2 P(\lambda; p)}{\partial p_i \partial p_j} \right) v_0(p) + u_0^t \left(P'(\lambda; p) \frac{\partial \lambda}{\partial p_j} + \frac{\partial P(\lambda; p)}{\partial p_j} \right) \frac{\partial v_0}{\partial p_i} + u_0^t \left(P'(\lambda; p) \frac{\partial \lambda}{\partial p_i} + \frac{\partial P(\lambda; p)}{\partial p_i} \right) \frac{\partial v_0}{\partial p_j} \Bigg)_{|(\lambda_0, p_0)},$$

and

$$b = u_0^t P'(\lambda_0; p_0) v_0(p_0).$$

$$\frac{\partial^2 v_0(p)}{\partial p_i \partial p_j}_{|(\lambda_0,p_0)} =$$
$$T_0^{-1}\left(\frac{\partial^2\lambda}{\partial p_i\partial p_j}P'(\lambda;p)v_0(p)+\right.$$
$$\left(\frac{\partial\lambda}{\partial p_i}\frac{\partial\lambda}{\partial p_j}P'(\lambda;p)+\frac{\partial\lambda}{\partial p_i}\frac{\partial P'(\lambda;p)}{\partial p_j}\right.$$
$$\left.+\frac{\partial P'(\lambda;p)}{\partial p_i}\frac{\partial\lambda}{\partial p_j}+\frac{\partial^2 P(\lambda;p)}{\partial p_i\partial p_j}\right)v_0(p)$$
$$+\left(P'(\lambda;p)\frac{\partial\lambda}{\partial p_j}+\frac{\partial P(\lambda;p)}{\partial p_j}\right)\frac{\partial v_0}{\partial p_i}$$
$$\left.+\left(P'(\lambda;p)\frac{\partial\lambda}{\partial p_i}+\frac{\partial P(\lambda;p)}{\partial p_i}\right)\frac{\partial v_0}{\partial p_j}\right)_{|(\lambda_0,p_0)}.$$

The proof is analogous to that given in [9] for matrix pencils and for vibrational systems.

3. Perturbation of eigenvalue of arbitrary multiplicity with single eigenvector

Let $P(\lambda;p)=\lambda^2 I_2+A(p)$ with $A(p)=\begin{pmatrix}-1 & p\\ p & 0\end{pmatrix}$ be a one parameter family of polynomial matrices. The eigenvalues are

$$\lambda_i=\pm\sqrt{\frac{1\pm\sqrt{1+4p^2}}{2}}, \qquad (8)$$

that they are branches of one quadruple-valued analytic function $\lambda(p)=\sqrt{\frac{1+\sqrt{1+4p^2}}{2}}$

the exceptional points are:

- $p=\frac{1}{2}i$ and the eigenvalues are $\pm\frac{\sqrt{2}}{2}$ both being double.

- $p=-\frac{1}{2}i$ and the eigenvalues are $\pm\frac{\sqrt{2}}{2}$ both being double.

-$p=0$ and the eigenvalues are $+1$, -1 both being simple and 0 being double.

We observe that for $p=0$, the polynomial matrix $P(\lambda;p)$ has a single eigenvector up to a non-zero scaling factor for the double eigenvalue $\lambda=0$.

We next consider the behavior of the eigenvalues in the neighborhood of one of the exceptional points. Concretely we take $p=0$. In this case the eigenvalues are not differentiable functions of the parameter at $p=0$, just where the double eigenvalue appears. Therefore the analysis of perturbations of multiple eigenvalues with single eigenvector, must be treated in a different manner.

Let $P(\lambda;p)$ be a monic polynomial matrix family and λ_0 an eigenvalue of arbitrary multiplicity ℓ with single eigenvector up to a non-zero scaling factor at the point $p=p_0$, then, there exists a Jordan chain v_0, …, $v_{\ell-1}$ such that

$$\begin{gathered}P(\lambda_0,p_0)v_0=0,\\ P'(\lambda_0,p_0)v_0+P(\lambda_0,p_0)v_1=0,\\ \frac{1}{(\ell-1)!}P^{\ell-1}(\lambda_0,p_0)v_0+\ldots+P(\lambda_0,p_0)v_{\ell-1}=0,\end{gathered} \qquad (9)$$

and, there exists a left Jordan chain u_0, …, $u_{\ell-1}$ such that

$$\begin{gathered}u_0^tP(\lambda_0,p_0)=0,\\ u_0^tP'(\lambda_0,p_0)+u_1^tP(\lambda_0,p_0)=0,\\ \frac{1}{(\ell-1)!}u_0^tP(\lambda_0,p_0)+\ldots+u_{\ell-1}^tP(\lambda_0,p_0)=0.\end{gathered} \qquad (10)$$

Remark 1. *a)* $u_0^tP'(\lambda_0,p_0)v_0=0$,

b) $u_1^tP'(\lambda_0,p_0)v_0=0 \Leftrightarrow u_1^tP(\lambda_0,p_0)v_1=0 \Leftrightarrow u_0^tP'(\lambda_0,p_0)v_1=0$,

c) $u_0^tP'(\lambda_0;p_0)v_1=u_1^tP'(\lambda_0;p_0)v_0$.

In order to analyze the behavior of two eigenvalues $\lambda(p)$ that merge to λ_0 at p_0, we consider a perturbation of the parameter along a smooth curve $p=p(\varepsilon)$, where $\varepsilon\geq 0$ is a small real perturbation parameter and $p(0)=p_0$.

Along the curve $p(\varepsilon)=(p_1(\varepsilon),\ldots,p_r(\varepsilon))$ we have a one parameter matrix family $P(\lambda,p(\varepsilon))$, which can be represented in the form of Taylor expansion

$$P(\lambda,p(\varepsilon))=P_0+\varepsilon P_1+\varepsilon^2P_2+\ldots,$$

with $P_0=P(\lambda,p_0)$, $P_1=\sum_{i=1}^r\frac{\partial P(\lambda,p(\varepsilon))}{\partial p_i}\frac{dp_i}{d\varepsilon}$,

$$P_2=\frac{1}{2}\left(\sum_{i=1}^r\frac{\partial P(\lambda,p(\varepsilon))}{\partial p_i}\frac{d^2p_i}{d\varepsilon^2}+\sum_{i,j=1}^r\frac{\partial^2P(\lambda,p(\varepsilon))}{\partial p_i\partial p_j}\frac{dp_i}{d\varepsilon}\frac{dp_j}{d\varepsilon}\right),$$

where the derivatives are evaluated at p_0.

Taking into account that $P(\lambda,p(\varepsilon))=\sum_{i=0}^k\lambda iA_i(p(\varepsilon)\ (A_k(p(\varepsilon)=I_n)$, we have that

$$P(\lambda,p(\varepsilon))=\sum_{i=0}^k\lambda^i(A_{i_0}+\varepsilon A_{i_1}+\varepsilon^2A_{i_2}+\ldots) \qquad (11)$$

where $A_{k_0}+\varepsilon A_{k_1}+\varepsilon^2A_{k_2}+\ldots=I_n$, $A_{\ell_0}=A_\ell(p_0)$, $A_{\ell_1}=\sum_{i=1}^r\frac{\partial A_\ell(p(\varepsilon))}{\partial p_i}\frac{dp_i}{d\varepsilon}$, $A_{\ell_2}=\frac{1}{2}\left(\sum_{i=1}^r\frac{\partial A_\ell(p(\varepsilon))}{\partial p_i}\frac{d^2p_i}{d\varepsilon^2}+\sum_{i,j=1}^r\frac{\partial^2A_\ell(p(\varepsilon))}{\partial p_i\partial p_j}\frac{dp_i}{d\varepsilon}\frac{dp_j}{d\varepsilon}\right)$. and the derivatives are evaluated at p_0.

If λ_0 is a ℓ-multiplicity eigenvalue of $P(\lambda;p_0)$ having a unique eigenvector v_0 up to a non-zero scaling factor

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the perturbation theory (see [8], for example) tell us that the ℓ-fold eigenvalue λ_0 generally splits into ℓ of simple eigenvalues λ under perturbation of the polynomial matrix $P(\lambda;p_0)$. These eigenvalues λ and the corresponding eigenvectors v can be represented in the form of the Puiseux series:

$$\begin{array}{l} \lambda = \lambda_0 + \varepsilon^{1/\ell}\lambda_1 + \varepsilon^{2/\ell}\lambda_2 + \varepsilon^{3/\ell}\lambda_3 + \varepsilon^{4/\ell}\lambda_4 + \dots \\ v = v_0 + \varepsilon^{1/\ell}w_1 + \varepsilon^{2/\ell}w_2 + \varepsilon^{3/\ell}w_3 + \varepsilon^{4/\ell}w_4 + \dots \end{array} \quad (12)$$

Lemma 1. *Let p_0 be a point such that $\lambda(p_0) = \lambda_0$ is a ℓ-multiplicity eigenvalue with single eigenvector $v_0(p_0)$ and u_0 a corresponding left eigenvector. Then, $[u_0]^\perp = Im\, P(\lambda_0, p_0)$.*

Proof. Let $z \in \operatorname{Im} P(\lambda_0, p_0)$, then there exists a vector x such that $P(\lambda_0, p_0)x = z$. So

$$u_0^t z = u_0^t P(\lambda_0, p_0)x = 0^t x = 0,$$

consequently $\operatorname{Im} P(\lambda_0; p_0) \subset [u_0]^\perp$. And taking into account that

$$\operatorname{rank} P(\lambda_0, p_0) = \dim \operatorname{Im} P(\lambda_0, p_0) = n - 1 = \dim[u_0]^\perp,$$

we conclude the result. □

Corollary 1. *With the same conditions as the previous lemma, we have.* $\frac{1}{\ell!}u_0^t P^\ell(\lambda_0;p_0)v_0 + \frac{1}{(\ell-1)!}u_0^t P^{\ell-1}(\lambda_0;p_0)v_1 + \dots + u_0^t P'(\lambda_0;p_0)v_{\ell-1} \neq 0.$

Proof. Suppose $\frac{1}{\ell!}u_0^t P^\ell(\lambda_0;p_0)v_0 + \dots + u_0^t P'(\lambda_0;p_0)v_{\ell-1} \neq 0$. Then $\frac{1}{\ell!}P^\ell(\lambda_0;p_0)v_0 + \frac{1}{(\ell-1)!}P^{\ell-1}(\lambda_0;p_0)v_1 + \dots + P'(\lambda_0;p_0)v_{\ell-1} \in \operatorname{Im} P(\lambda_0, p_0)$, and $\frac{1}{\ell!}P^\ell(\lambda_0;p_0)v_0 + \frac{1}{(\ell-1)!}P^{\ell-1}(\lambda_0;p_0)v_1 + \dots + P'(\lambda_0;p_0)v_{\ell-1} = P(\lambda_0;p_0)x$. Equivalently:

$$\begin{array}{l} \frac{1}{\ell!}P^\ell(\lambda_0;p_0)v_0 + \frac{1}{(\ell-1)!}P^{\ell-1}(\lambda_0;p_0)v_1 + \dots \\ +P'(\lambda_0;p_0)v_{\ell-1} + P(\lambda_0;p_0)(-x) = 0, \end{array} \quad (13)$$

but the Jordan chains of the $P(\lambda;p_0)$ for $\lambda = \lambda_0$ are length ℓ, so there is no vector x verifying (13). □

4-1. Perturbation of double eigenvalue with single eigenvector

Firstly and for a more understanding, we analyze the case where $\ell = 2$

Substituting (12) into (11) we obtain

$$\begin{array}{l} P(\lambda;p(\varepsilon)) = \\ (\lambda_0^k I_n + \lambda_0^{k-1}A_{k-1_0} + \dots + \lambda_0 A_{1_0} + A_{0_0}) + \\ \varepsilon^{1/2}(k\lambda_0^{k-1}\lambda_1 I_n + (k-1)\lambda_0^{k-2}\lambda_1 A_{k-1_0} + \dots + \lambda_1 A_{1_0}) + \\ \varepsilon((k\lambda_0^{k-1}\lambda_2 + \frac{1}{2}k(k-1)\lambda_0\lambda_1^2)I_n + ((k-1)\lambda_0^{k-2}\lambda_2 + \\ \quad \frac{1}{2}(k-1)(k-2)\lambda_0\lambda_1^2)A_{k-1_0} + \lambda_0^{k-1}A_{k-1_1} + \\ \quad \lambda_2 A_{1_0} + \lambda_0 A_{1_1} + \dots + A_{0_1}) + \dots \end{array}$$

If v is an eigenvector for the eigenvalue λ, we have that

$$P(\lambda;p(\varepsilon))v = P(\lambda;p(\varepsilon))(v_0 + \varepsilon^{1/2}w_1 + \varepsilon w_2 + \dots) = 0.$$

Then, we find the chain of equations for the unknowns $\lambda_1, \lambda_2, \dots$ and w_1, w_2, ...

$$P(\lambda_0, p_0)v_0 = 0, \quad (14)$$

$$\lambda_1 P'(\lambda_0;p_0)v_0 + P(\lambda_0;p_0)w_1 = 0, \quad (15)$$

$$\begin{array}{l} P(\lambda_0;p_0)w_2 + \lambda_1 P'(\lambda_0;p_0)w_1 + \frac{1}{2}\lambda_1^2 P''(\lambda_0;p_0)v_0 + \\ \quad \lambda_2 P'(\lambda_0;p_0)v_0 + P_1(\lambda_0;p_0)v_0 = 0, \end{array} \quad (16)$$

$$\begin{array}{l} P(\lambda_0;p_0)w_3 + \lambda_1 P'(\lambda_0;p_0)w_2 + \frac{1}{2}\lambda_1^2 P''(\lambda_0;p_0)w_1 + \\ \lambda_2 P'(\lambda_0;p_0)w_1 + P_1(\lambda_0;p_0)w_1 + \lambda_1\lambda_2 P''(\lambda_0;p_0)v_0 + \\ \lambda_1^3 \frac{1}{3!}P'''(\lambda_0;p_0)v_0 + \lambda_3 P'(\lambda_0;p_0)v_0 + \lambda_1 P_1'(\lambda_0;p_0)v_0 = 0, \end{array} \quad (17)$$

where $P_1(\lambda_0;p_0) = \lambda_0^{k-1}A_{k-1_1} + \lambda_0 A_{k-2_1} + \dots + \lambda_0 A_{11} + A_{01}$.

Equation (14) is satisfied because v_0 is an eigenvector corresponding to the eigenvalue λ_0. Comparing equation (15) with (3) for $i = 1$ we observe that $w_1 = \lambda_1 v_1 + \beta v_0$ for all β is a solution, we take $w_1 = \lambda_1 v_1$.

To find the value of λ_1 we premultiply equation (16) by u_0^t, using the given value for w_1 and taking into account $u_0^t P(\lambda_0;p_0) = 0$ and $u_0^t P'(\lambda_0;p_0)v_0 = 0$ we obtain

$$\lambda_1^2(u_0^t P'(\lambda_0;p_0)v_1 + \frac{1}{2}u_0^t P''(\lambda_0;p_0)v_0) + u_0^t P_1(\lambda_0;p_0)v_0 = 0.$$

Taking into account corollary 1 we can find

$$\lambda_1 = \pm\sqrt{\frac{-u_0^t P_1(\lambda_0;p_0)v_0}{u_0^t P'(\lambda_0;p_0)v_1 + \frac{1}{2}u_0^t P''(\lambda_0;p_0)v_0}}. \quad (18)$$

If $u_0^t P_1(\lambda_0;p_0)v_0 \neq 0$ we have two values of λ_1 that determine leading terms in expansions for two different eigenvalues λ that bifurcate from the double eigenvalue λ_0.

Suppose then, that $u_0^t P_1(\lambda_0;p_0)v_0 \neq 0$. Premultiplying (17) by u_0^t,

$$\lambda_1 u_0^t P'(\lambda_0,p_0)w_2 + \frac{1}{2}\lambda_1^3 u_0^t P''(\lambda_0;p_0)v_1 + \lambda_1\lambda_2 u_0^t P'(\lambda_0;p_0)v_1 + \lambda_1 u_0^t P_1(\lambda_0;p_0)v_1 + \lambda_1\lambda_2 u_0^t P''(\lambda_0;p_0)v_0 + \lambda_1^3 \frac{1}{3!} u_0^t P'''(\lambda_0;p_0)v_0 + \lambda_1 u_0^t P_1'(\lambda_0;p_0)v_0 = 0.$$

Premultiplying (16) by u_1^t and according to 1, we have:

$$u_0^t P'(\lambda_0;p_0)w_2 = \lambda_1 u_1^t P'(\lambda_0;p_0)w_1 + \frac{1}{2}\lambda_1^2 u_1^t P''(\lambda_0;p_0)v_0 + \lambda_2 u_1^t P'(\lambda_0;p_0)v_0 + u_1^t P_1(\lambda_0;p_0)v_0.$$

So, taking into account (18)

$$\lambda_1\lambda_2(2u_0^t P'(\lambda_0;p_0)v_1 + u_0^t P''(\lambda;p_0)v_0) = -(\lambda_1^3(u_1^t P'(\lambda_0;p_0)v_1 + \frac{1}{2}u_1^t P''(\lambda_0;p_0)v_0 + \frac{1}{2}u_0^t P''(\lambda_0;p_0)v_1 + \frac{1}{3!}u_0^t P'''(\lambda_0;p_0)v_0) + \lambda_1(u_1^t P_1(\lambda_0;p_0)v_0 + u_0^t P_1(\lambda_0;p_0)v_1 + u_0^t P_1'(\lambda_0;p_0)v_0))$$

Since $\lambda_1(u_0^t P'(\lambda_0;p_0)v_1 + \frac{1}{2}u_0^t P''(\lambda_0;p_0)v_0) \neq 0$ we obtain

$$\lambda_2 = -\frac{\lambda_1^2(\frac{1}{2}u_0^t P''(\lambda_0;p_0)v_1 + \frac{1}{3!}u_0^t P'''(\lambda_0;p_0)v_0}{2(u_0^t P'(\lambda_0;p_0)v_1 + \frac{1}{2}u_0^t P''(\lambda_0;p_0)v_0)} - \frac{\lambda_1^2(u_1^t P'(\lambda_0;p_0)v_1 + \frac{1}{2}u_1^t P''(\lambda_0;p_0)v_0)}{2(u_0^t P'(\lambda_0;p_0)v_1 + \frac{1}{2}u_0^t P''(\lambda_0;p_0)v_0)} + \frac{u_0^t P_1(\lambda_0;p_0)v_1 + u_0^t P_1'(\lambda_0;p_0)v_0 + u_1^t P_1(\lambda_0;p_0)v_0}{2(u_0^t P'(\lambda_0;p_0)v_1 + \frac{1}{2}u_0^t P''(\lambda_0;p_0)v_0)}. \tag{19}$$

Now, we can compute w_2. We have

$$P(\lambda_0;p_0)w_2 = -\lambda_1 P'(\lambda_0;p_0)w_1 - \frac{1}{2}\lambda_1^2 P''(\lambda_0;p_0)v_0 - \lambda_2 P'(\lambda_0;p_0)v_0 + P_1(\lambda_0;p_0)v_0 \tag{20}$$

Lemma 2. *Following condition $u_0^t P_1(\lambda_0;p_0)v_0 \neq 0$ we have that $P(\lambda_0;p_0) + u_0 u_0^t P_1(\lambda_0;p_0)v_0 v_0^t$ is an invertible matrix.*

Proof. Let $x = \alpha v_0 + w$ with $w \in [v_0]^\perp$, be a vector in the null space, then $(P(\lambda_0;p_0) + u_0 u_0^t P_1(\lambda_0;p_0)v_0 v_0^t)x = 0$.

Premultiplying by u_0^t we have

$$u_0^t(P(\lambda_0;p_0) + u_0 u_0^t P_1(\lambda_0;p_0)v_0 v_0^t)x = 0,$$

$$0 = u_0 u_0^t P_1(\lambda_0;p_0)v_0 v_0^t(\alpha v_0 + w) = |\alpha| \|u_0\|^2 \|v_0\|^2 u_0^t P_1(\lambda_0;p_0)v_0.$$

Then $\alpha = 0$.

Consequently, $x = w \in [v_0]^\perp$ and $x \in \operatorname{Ker} u_0 u_0^t P_1(\lambda_0;p_0)v_0 v_0^t$, so $x \in \operatorname{Ker} P(\lambda_0;p_0)$ and $x = \beta v_0$, but $x \in [v_0]^\perp$, then $\beta = 0$. ☐

Now we consider the normalization condition $v_0^t w_2 = 0$, and adding $u_0 u_0^t P_1(\lambda_0;p_0)v_0 v_0^t$ from the left to equation (20) and using lemma 2, we find vector w_2.

Using these calculations we have the following theorem.

Theorem 2. *Let λ_0 be a double eigenvalue of the polynomial matrix $P(\lambda;p_0)$, with a single eigenvector up to a non-zero scaling factor, and let v_0, v_1 be a Jordan chain and u_0, u_1 a left Jordan chain. We consider a perturbation of the parameter vector along the curve $p(\varepsilon)$ starting at p_0 satisfying the condition $\lambda_1 \neq 0$.*

Then, the double eigenvalue λ_0 bifurcates into two simple eigenvalues given by the relation

$$\lambda = \lambda_0 + \varepsilon^{1/2}\lambda_1 + \varepsilon\lambda_2 + o(\varepsilon),$$

with λ_1 and λ_2 as (18) *and* (19) *respectively.*

4-2. Perturbation of a ℓ-multiplicity eigenvalue with single eigenvector

Now, we analyze the general case.

Analogously, substituting (12) into (11) we obtain

$$P(\lambda;p(\varepsilon)) = (\lambda_0 + \varepsilon^{1/\ell}\lambda_1 + \varepsilon^{2/\ell}\lambda_2 + \ldots + \varepsilon\lambda_\ell + \ldots)^k I_n + (\lambda_0 + \varepsilon^{1/\ell}\lambda_1 + \ldots)^{k-1}(A_{k-1_0} + \ldots + \varepsilon^\ell A_{k-1_\ell} + \ldots) + \ldots + (\lambda_0 + \varepsilon^{1/\ell}\lambda_1 + \varepsilon^{2/\ell}\lambda_2 + \ldots)(A_{1_0} + \varepsilon A_{1_1} + \varepsilon^2 A_{1_2} + \ldots) + A_{0_0} + \varepsilon A_{0_1} + \varepsilon^2 A_{0_2} + \ldots = (\lambda_0^k I_n + \lambda_0^{k-1} A_{k-1_0} + \ldots + \lambda_0 A_{1_0} + A_{0_0}) + \varepsilon^{1/\ell}(k\lambda_0^{k-1}\lambda_1 I_n + (k-1)\lambda_0^{k-2}\lambda_1 A_{k-1_0} + \ldots + \lambda_1 A_{1_0}) + \varepsilon^{2/\ell}((k\lambda_0^{k-1}\lambda_2 + \frac{1}{2}k(k-1)\lambda_0\lambda_1^2)I_n + ((k-1)\lambda_0^{k-2}\lambda_2 + \frac{1}{2}(k-1)(k-2)\lambda_0\lambda_1^2)A_{k-1_0} + \ldots + \lambda_2 A_{1_0}) + \ldots$$

If v is an eigenvector for the eigenvalue λ we have that

$$P(\lambda;p(\varepsilon))v = P(\lambda;p(\varepsilon))(v_0 + \varepsilon^{1/\ell}w_1 + \varepsilon^{2/\ell}w_2 + \ldots) = 0$$

Then, we find the chain of equations for the unknowns $\lambda_1, \lambda_2, \ldots$ and $w_1, w_2, \ldots$.

$$P(\lambda_0,p_0)v_0 = 0, \tag{21}$$

$$\lambda_1 P'(\lambda_0;p_0)v_0 + P(\lambda_0;p_0)w_1 = 0, \tag{22}$$

$$P(\lambda_0;p_0)w_2 + \lambda_1 P'(\lambda_0;p_0)w_1 + \frac{1}{2}\lambda_1^2 P''(\lambda_0;p_0)v_0 + \lambda_2 P'(\lambda_0;p_0)v_0 = 0, \tag{23}$$

$$\lambda_3 P'(\lambda_0, p_0)v_0 + \frac{1}{3!}\lambda_1^3 P'''(\lambda_0; p_0)v_0 + \frac{1}{2}\lambda_1\lambda_2 P''(\lambda_0; p_0)v_0 + \lambda_2 P'(\lambda_0; p_0)w_1 + \frac{1}{2}\lambda_1^2 P''(\lambda_0; p_0)w_1 + \lambda_1 P'(\lambda_0; p_0)w_2 + P(\lambda_0; p_0)w_3 = 0, \quad (24)$$

...

$$P(\lambda_0; p_0)w_\ell + \lambda_1 P'(\lambda_0; p_0)w_{\ell-1} + \frac{1}{2}\lambda_1^2 P''(\lambda_0; p_0)w_{\ell-2} + \lambda_2 P'(\lambda_0; p_0)w_1 + \ldots + \lambda_{\ell-1} P'(\lambda_0; p_0)w_1 + P_1(\lambda_0; p_0)v_0 = 0, \quad (25)$$

where $P_1(\lambda_0; p_0) = \lambda_0^{k-1} A_{k-1_1} + \lambda_0 A_{k-2_1} + \ldots + \lambda_0 A_{11} + A_{01}$.

Equation (21) is satisfied because v_0 is an eigenvector corresponding to the eigenvalue λ_0. Comparing equation (22) with (3) for $i = 1$ we observe that $w_1 = \lambda_1 v_1 + \beta v_0$ is a solution, comparing equation (23) with (3) for $i = 2$ $w_2 = \lambda_1^2 v_2 + \lambda_2 v_1$ is a solution, following in this sense $w_3 = \lambda_1^3 v_3 + \lambda_1\lambda_2 v_2 + \lambda_3 v_1$ etc.

Theorem 3. *Let λ_0 be a ℓ-multiplicity eigenvalue of the polynomial matrix $P(\lambda; p_0)$, with a single eigenvector up to a non-zero scaling factor, and let $v_0, \ldots, v_{\ell-1}$ be a Jordan chain and $u_0, \ldots, u_{\ell-1}$ a left Jordan chain. We consider a perturbation of the parameter vector along the curve $p(\varepsilon)$ starting at p_0. Suppose $u_0^t P_1(\lambda_0; p_0)v_0 \neq 0$, then, the eigenvalue λ_0 bifurcates into ℓ simple eigenvalues given by the relation*

$$\lambda = \lambda_0 + \varepsilon^{1/\ell}\lambda_1 + o(\varepsilon),$$

with

$$\lambda_1 = \sqrt[\ell]{\frac{-u_0^t P_1(\lambda_0; p_0)v_0}{\frac{1}{\ell!}u_0^t P^\ell(\lambda_0; p_0)v_0 + \ldots + u_0^t P'(\lambda_0; p_0)v_{\ell-1}}}.$$

Remark 2. *Condition $u_0^t P_1(\lambda_0; p_0)v_0 \neq 0$ holds for almost all perturbations.*

Proof. To find the value of λ_1 using $w_1 = \lambda_1 v_1 + \beta v_0$ in equation (16) and premultiply it by u_0^t and taking into account remark 1 and normalization condition $u_0^t P'(\lambda_0; p_0)v_i = 0$, we obtain

$$\lambda_1^\ell\left(\frac{1}{\ell!}u_0^t P^\ell(\lambda_0; p_0)v_0 + \frac{1}{(\ell-1)!}u_0^t P^{\ell-1}(\lambda_0; p_0)v_1 + \ldots + u_0^t P'(\lambda_0; p_0)v_{\ell-1}\right) + u_0^t P_1(\lambda_0; p_0)v_0 = 0.$$

Now, corollary 1 ensures the result. □

5. Conclusion

In this paper the perturbation of a multiple eigenvalue with a simple eigenvector of a monic polynomial matrix smoothly depending on parameters is analyzed.

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