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By A. W. Moser

A friend recently brought to my attention the case of a mortgage company which was to be absorbed by another. It was agreed that all the mortgages should be taken over at their book value. The securities in question aggregated about $500,000 and were composed partly of mortgages with serial redemptions or repayment in instalments. Accountants were called in to verify the book values, with the astonishing result that one found their total to be $5,000 higher than the amount established by the other.

Although an accountant, in general, is not often called upon to do this kind of work, which is more of an actuarial nature, yet it sometimes happens that his advice or services are sought in such cases, and it may prove useful for him to be acquainted with certain fundamentals involved. With this idea in mind I propose to discuss some features of book values of securities such as bonds and mortgages whose redemption at the end of a specified term of years may be assumed to be certain.

Whenever a security is bought at a price differing from the nominal amount, it is said to be purchased at a premium or a discount. The expression "bonus," often used with respect to mortgages, is a synonym of discount. In all these cases the rate of interest which will be earned on the investment differs from the nominal or dividend rate named in the security. It is this that causes complications when the question arises as to how the security should be dealt with on an investment basis.

Good accounting practice not only demands that the capital originally invested be restored unaltered at the end of the term, but also that the income derived be properly assigned to the individual interest periods. To do this requires us to remember that the dividend from securities acquired at a premium or a discount consists, in the one case, partly of interest and partly of capital applicable to the gradual reduction of the premium, and, in the second case, the dividend does not represent the whole of the interest earned. Failure to reflect this on the books of account would lead, in the first case, to an impairment of principal to the extent of the premium paid, since the repayment at maturity will be by that much less than was the purchase price; while in the case of a discount there would result a gain of principal to the extent of this discount, which should have been treated as income, properly spread over the term of the security, in order that earnings at the corresponding income rate, which in this instance is higher than the dividend rate, may be realized in full. It appears therefore, next, that the amount invested in a redeemable security should periodically be written up or written down, as the case may be, by a sum which, if accumulated at the income rate, will amount at the date of redemption to the premium or discount in question. These accumulations deducted from or added at any time during the life of the security to the purchase price give what is commonly called the book value and will bring the purchase price down, in the case of a premium, or up, in the case of a discount, to the redemption value by the end of the term, i.e., the book value will then have reached the same amount.

After these general considerations let the following symbols be noted:

j to represent the nominal or dividend rate of interest;
i " " the income or investment rate of interest;
t " " the number of years at the expiration of which the security becomes redeemable;
V " " the purchase price of the security;
Vr" " the book value r years after acquisition;
$\pm h$ " " the premium or discount involved, respectively;
$S_{\overline{t,i|}} = \frac{(1+i)^t-1}{i}$ = amount of an immediate annuity of one unit a year, payable at the end of each year, at interest rate i.

The sum per interval which, if accumulated at the income rate, will amount at the end of the term to the premium or discount may conveniently be called amortization factor and is given by the expression

$$\alpha = \frac{\pm h}{S_{\overline{t,i|}}},$$

as far as securities with one redemption are concerned, since this annuity per interval will amount t years hence to the value h. The total of these accumulations at the end of the r^{th} year is then given by $h\frac{S_{\overline{r,i|}}}{S_{\overline{t,i|}}}$, so that as book value of the redeemable security at that same moment results

$$Vr = V \mp h\frac{S_{\overline{r,i|}}}{S_{\overline{t,i|}}} \tag{1}$$

This is the book value obtained by what is known as the exact or scientific method. Its determination always requires that the investment rate, or income rate, be known or determinable.

It should be well understood at this point that the method followed is slightly defective from a practical point of view, as far as purchases at a premium are concerned, because of the assumption implicitly made that the amortization factors can and will be invested at the rate of interest earned by the principal of the security. The amortization factors are relatively small sums, compared with the principal, and small sums do not as a rule command as high a rate as larger sums. The specified income rate is consequently seldom fully realized. This criticism does not extend, however, to purchases at a discount, because in such instances the periodical interest payments do not include any part of principal to be reinvested, the discount accumulations merely representing bookkeeping charges against income receivable, which will be realized at the moment the redemption price is received.

In view of these conditions, it is common practice to accumulate the amortization factors at another rate, usually smaller than the income rate, which signifies that a somewhat higher amount α than would be necessary if interest at the income rate were fully earned must periodically be provided in order to make up for the slightly reduced interest accretions, and the buyer at a premium will fail to that extent to realize the rate of interest upon which the price is based. The rule is even carried so far, probably for the sake of convenience and simplicity, that in many cases the interest element is entirely disregarded and α taken as the quotient $\frac{\pm h}{t}$. This procedure, simple and often referred to as the pro-rata method of amortization, will ordinarily prove satisfactory in cases where the security is held as an investment until maturity. It will then not matter much if the accumulations applicable to the individual intervals differ a little from what they would have been according to the scientific method, if they reach the required total by the end of the term. When, however, a sale or purchase is contemplated at any intermediate date, it is evidently desirable, as the example given at the beginning will indicate, to have a more accurate value upon which to base calculations.

From this discussion it appears that the expression "book value" as commonly used, i.e. without qualification as to the method employed for its calculation, does not convey a definite idea as to the value it represents. It would, therefore, be an advantage to have that book value which is obtained on the basis of the specified investment rate distinguished by designating it, for instance, as mathematical book value. As pointed out before, for its computation it is necessary to know or to determine first the income rate of interest.

The calculation of the investment rate, or income rate of interest, produced by a given security purchased at a premium or a discount, is in any case a relatively simple task. Even in the case of a security redeemable in instalments and acquired at a premium or discount (or bonus) the problem of determining the income rate is not a very difficult matter.

In view of the fact that the pro-rata method of establishing book values is so simple and easily applicable to any case, it deserves, of course, to be retained for many practical purposes. Its application will naturally furnish an amount different from the mathematical book value, and the relative importance of this difference and its nature (whether positive or negative) may now be further investigated, because even the mere knowledge that one of the two book values will be higher or lower than the other in certain circumstances is sometimes useful.

As a first step it is well to prepare an amortization schedule showing the amounts to be written off or added to the invested capital at the end of each period. The periodical dividends, decreased or increased by these amounts, will give the correct interest for each period. In the schedule on the following page the process of amortization of a premium is demonstrated in detail, with both the mathematical and the pro-rata book values indicated.

While from this schedule it can be learned at a glance that the mathematical book value is throughout the life of the security higher than the book value determined by the pro-rata method, it may not be amiss to establish a more general exposition of the relations between the two.

As has been shown before, the mathematical book value at the end of the r^{th} year (for a security with a single redemption) amounts to

$$V_r = V \mp h \frac{S_{\overline{r},i|}}{S_{\overline{t},i|}}.$$

Similarly, the pro-rata book value would at that time reach the sum of

$$V_r = V \mp h\frac{r}{t}.$$

The difference between the two values is given by

$$\mp h\left[\frac{(1+i)^r-1}{(1+i)^t-1} - \frac{r}{t}\right] = \mp h\left[\frac{t(1+i)^r - t - r(1+i)^t + r}{t(1+i)^t - t}\right] \quad (2)$$

SCHEDULE I

$100,000 6% BONDS, 15 YEARS, INTEREST PAYABLE HALF-YEARLY, TO NET INCOME OF 5%
Purchase price, $110,465.15

End of period	Dividend rec'd	Interest on book value at 2½%	For amortization $\alpha = \frac{h}{s_{t,2\frac{1}{2}\%}}$	Mathematical book value (a)	For amortization $\alpha = \frac{h}{t}$	Book value as per pro-rata method (b)	Difference (a–b)
0				$110,465.15		$110,465.15	
1	$3,000.00	$2,761.63	$238.37	110,226.78	$348.838	110,116.31	$110.47
2	"	2,755.67	244.33	109,982.45	"	109,767.47	214.98
3	"	2,749.56	250.44	109,732.01	"	109,418.63	313.38
4	"	2,743.30	256.70	109,475.31	"	109,069.80	405.51
5	"	2,736.88	263.12	109,212.19	"	108,720.96	491.23
6	"	2,730.30	269.70	108,942.49	"	108,372.12	570.37
7	"	2,723.56	276.44	108,666.05	"	108,023.28	642.77
8	"	2,716.65	283.35	108,382.70	"	107,674.45	708.25
9	"	2,709.57	290.43	108,092.27	"	107,325.61	766.66
10	"	2,702.31	297.69	107,794.58	"	106,976.77	817.81
11	"	2,694.86	305.14	107,489.44	"	106,627.93	861.51
12	"	2,687.24	312.76	107,176.68	"	106,279.09	897.59
13	"	2,679.41	320.59	106,856.09	"	105,930.26	925.83
14	"	2,671.40	328.60	106,527.49	"	105,581.42	946.07
15	"	2,663.19	336.81	106,190.68	"	104,232.59	958.09
16	"	2,654.76	345.24	105,845.44	"	104,883.75	961.69
17	"	2,646.14	353.86	105,491.58	"	104,534.91	956.67
18	"	2,637.29	362.71	105,128.87	"	104,186.07	942.80
19	"	2,628.22	371.78	104,757.09	"	103,837.23	919.86
20	"	2,618.92	381.08	104,376.01	"	103,488.39	887.62
21	"	2,609.40	390.60	103,985.41	"	103,139.55	845.86
22	"	2,599.64	400.36	103,585.05	"	102,790.71	794.34
23	"	2,589.62	410.38	103,174.67	"	102,441.88	732.79
24	"	2,579.36	420.64	102,754.03	"	102,093.04	660.99
25	"	2,568.85	431.15	102,322.88	"	102,744.20	578.68
26	"	2,558.07	441.93	101,880.95	"	101,395.36	485.59
27	"	2,547.02	452.98	101,427.97	"	101,046.52	381.45
28	"	2,535.69	464.31	100,963.66	"	100,697.68	295.98
29	"	2,524.09	475.91	100,487.75	"	100,348.84	138.91
30	"	2,512.19	487.81	100,000.00*	"	100,000.00	......

* Including an adjustment of 6 cents.

From this equation the time of maximum difference may be established. When

$$y=t\ (1+i)^r-t-r\ (1+i)^t+r,$$

or simplified,

$$y=t+tir+\frac{tr\ (r-1)\ i^2}{2}-t-r\ (1+i)^t+r$$

it follows by differentiation that

$$\frac{dy}{dr}=ti+tri^2-\frac{ti^2}{2}-(1+i)^t+1$$

and for $\frac{dy}{dr}=0$,

$$r=\frac{1}{ti}\left[\frac{(1+i)^t\ -1}{i}+\frac{ti}{2}-t\right],$$

or approximately,

$$r=\frac{1}{ti^2}\left[1+ti+\frac{t(t-1)}{2}i^2-1+\frac{ti^2}{2}-ti\right]=\frac{t}{2}.$$

Inserting now any value of r expressed in terms of t, for instance $\frac{t}{2}$, in the equation for the difference between the two book values (formula 2), this will read

$$\mp h\left[\frac{(1+i)^{\frac{t}{2}}-1}{(1+i)^t-1}-\frac{1}{2}\right]=\mp h\left[\frac{S_{\overline{\frac{t}{2}}|}}{S_{\overline{t}|}}-\frac{1}{2}\right]=\pm h\left[\frac{1}{2}-\frac{S_{\overline{\frac{t}{2}}|}}{S_{\overline{t}|}}\right] \qquad (3)$$

An analysis of this formula reveals that because of the fact that the fraction $\frac{S_{\overline{\frac{t}{2}}|}}{S_{\overline{t}|}}$ slowly decreases with increasing t, starting from a somewhat higher value for smaller i's, while always remaining smaller than $\frac{1}{2}$, the expression in parentheses represents a gradually increasing positive quantity. Recalling that $+h$ stands for a premium and $-h$ for a discount, (formula 3) and the preceding deductions as to the maximum difference permit drawing the following conclusions:

(a) With a premium involved, the mathematical book value is greater than the one derived by the pro-rata method, at any time during the life of the security.

(b) With a discount involved, the mathematical book value is smaller than the pro-rata book value, at any time during the life of the security.

(c) The difference between the two book values, besides being proportional to the amount *h*, gradually increases until it reaches a maximum near the middle of the term, and from that point it decreases until the end of the term, when both values become equal at namely the redemption price.

(d) Other things being equal, the difference between the two book values becomes more pronounced with higher interest rates and with longer terms.

(e) A rough idea of the relative importance of the difference may be gained from the example given, a 6% 15-year bond to net 5% and entailing a premium of 10.46%. In this case there was a maximum difference, near the middle of the term, of 0.96% of the face of the security.

To show the accumulation of discounts, let us construct a schedule for a bond issue of $200,000, with interest at 5%, payable half yearly, a first redemption of $100,000 to take place after five years, and further redemptions to be made at the end of each following year at the rate of $20,000 each, the purchase price being $180,000. Under these conditions the income rate is 3.4825% for a half year. In this schedule too, both book values will be shown, the one obtained on the basis of the income rate of 3.4825% for a half year and the other by the pro-rata method, which disregards entirely interest for the discount accumulations. How the pro-rata method works in such cases is generally well known. It is sufficient to state here that as the principal outstanding is being reduced from time to time, there should be assigned to each period an accumulation factor proportional to the principal outstanding during the period. This may be established by adding the capitals corresponding to each period and multiplying the capital used during the individual periods with the reciprocal value of that total sum and with the amount of *h* in question. In the case under consideration, the sum of capitals for the period in use is

10×200,000 =	2,000,000
2×100,000 =	200,000
2× 80,000 =	160,000
2× 60,000 =	120,000
2× 40,000 =	80,000
2× 20,000 =	40,000
Total,	$2,600,000

SCHEDULE II

Accumulation of Discount on $200,000 Bonds, Interest at 5% Payable Semi-Annually

Purchase price, $180,000; income rate, 6.9650% nominal

End of period	Principal outstanding	Dividend received	Interest on book value at 3.4825%	For accumulation	Mathematical book value (a)	For accumulation by pro-rata method	Book value by pro-rata method (b)	Difference (a–b)
0	$200,000.00				$180,000.00		$180,000.00	
1	"	$5,000.00	$6,268.50	$1,268.50	181,268.50			
2	"	5,000.00	6,312.68	1,312.68	182,581.18	$3,076.92	183,076.92	*$495.74*
3	"	5,000.00	6,358.38	1,358.38	183,939.56			
4	"	5,000.00	6,405.71	1,405.71	185,345.27	3,076.92	186,153.84	*808.57*
5	"	5,000.00	6,454.64	1,454.64	186,799.91			
6	"	5,000.00	6,505.31	1,505.31	188,305.22	3,076.92	189,230.76	*925.54*
7	"	5,000.00	6,557.72	1,557.72	189,862.94			
8	"	5,000.00	6,611.98	1,611.98	191,474.92	3,076.92	192,307.68	*832.76*
9	"	5,000.00	6,668.12	1,668.12	193,143.04			
10	"	5,000.00	6,726.20	1,726.20	194,869.24	3,076.92	195,384.60	*515.36*
	$100,000.00				$94,869.24		$95,384.60	
11		2,500.00	3,303.81	803.81	95,673.05			
12		2,500.00	3,331.81	831.81	96,504.86	1,538.46	96,923.06	*418.20*
	$80,000.00				$76,504.86		$76,923.06	
13		2,000.00	2,664.28	664.28	77,169.14			
14		2,000.00	2,687.41	687.41	77,856.55	1,230.77	78,153.83	*297.28*
	$60,000.00				$57,856.55		$58,153.83	
15		1,500.00	2,014.84	514.84	58,371.39			
16		1,500.00	2,032.77	532.77	58,904.16	923.08	59,076.91	*172.75*
	$40,000.00				$38,904.16		$39,076.91	
17		1,000.00	1,354.81	354.83	39,258.99			
18		1,000.00	1,367.19	367.19	39,626.18	615.39	39,692.30	*66.12*
	$20,000.00				$19,626.18		$19,692.30	
19		500.00	683.47	183.47	19,809.65			
20		500.00	689.88	189.88	19,999.53*	307.70	20,000.00	.....

* Difference of 47 cents to be adjusted.

Consequently, the amount to be accumulated for a period with, say, \$200,000 in use would be $\frac{\$200,000}{2,600,000}$ 20,000 = \$1538.46; for a period with \$100,000 in use $\frac{\$100,000}{2,600,000}$ 20,000 = \$769.23, etc.

It appears therefore that the book value of bonds and mortgages, whether with single or serial redemptions, if established by the pro-rata or a related method, will differ to some extent from the exact book value obtained by the so-called scientific method, and that the difference in question, although relatively small, may nevertheless assume substantial proportions in cases of larger principals. Neglect of this will not do much harm, if the security is held as an investment. When, however, a transfer is intended it may be desirable and well worth while to determine exactly the book value or the income rate of interest.