

**Mixture parameters of a trivariate normal superposition  
from sample cumulants: Application to the local  
stellar velocity distribution <sup>1</sup>**

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# Mixture parameters of a trivariate normal superposition from sample cumulants: Application to the local stellar velocity distribution

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## Abstract

The velocity distribution of nearby stars can be studied as a mixture of two main population components. In order to determine the mixing proportions and the population parameters a combined geometric-statistical method has been developed. The overall distribution is approximated from a superposition of two trivariate normal velocity density functions. The peculiar velocity is projected on a plane containing the global centroid (mean of the distribution), which is ortogonal to the direction  $D$  through both population subcentroids, obtaining two linear independent projected velocities. The statistical moments of these new variables are computed from second, third and fourth-order sample cumulants. The symmetric behaviour of the distribution around the direction  $D$  allows to determine it working only from third cumulants. Finally the overall set of projected peculiar velocity moments is used to determine the population covariance matrices, population means, and mixture proportions. The method does not require any extra hypotheses such as those concerning to prior population parameters, or specific symmetries of the distribution.

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## 1. Introduction and notation

The galactic system is usually represented as a superposition of several stellar populations. Its behavior is described, in nearly all the regions, as a conservative dynamic system where the superposition principle is satisfied. Thus a composite velocity distribution function can be assumed, where each component is associated with an ideal stellar population. According to the theory of gas dynamics, the stellar velocity distribution of each component is taken of normal type in the peculiar velocities. Moreover, in the solar neighborhood a two-component model, for thin disk and thick disk stars, is enough to explain most relevant astrophysical data (Cubarsi 1990).

Let  $f(\mathbf{V})$  be the velocity distribution function, that is the phase space distribution function for fixed time and position. The following quantities are defined:

- Stellar density,

$$N = \int_{\mathbf{V}} f(\mathbf{V}) d\mathbf{V} \quad (1)$$

- Mean velocity (velocity of the centroid),

$$\mathbf{v} \equiv \langle \mathbf{V} \rangle = \frac{1}{N} \int_{\mathbf{V}} \mathbf{V} f(\mathbf{V}) d\mathbf{V} \quad (2)$$

- Tensor of the  $n$ -order central moments,

$$\mathbf{M}_n \equiv \langle (\mathbf{V} - \mathbf{v})^n \rangle = \frac{1}{N} \int_{\mathbf{V}} (\mathbf{V} - \mathbf{v})^n f(\mathbf{V}) d\mathbf{V} \quad (3)$$

where  $(\cdot)^n$  denotes the  $n$ -tensor power and the difference  $\mathbf{u} = \mathbf{V} - \mathbf{v}$  is the peculiar velocity (velocity referred to the centroid). The symmetric tensor  $\mathbf{M}_n$  of the  $n$ -order central moments has  $\binom{n+2}{2}$  components according to the following expression,

$$\mu_{\alpha_1 \alpha_2 \dots \alpha_n} = \langle u_{\alpha_1} u_{\alpha_2} \dots u_{\alpha_n} \rangle \quad (4)$$

where the indices, depending on the velocity components, belong to the set  $\{1, 2, 3\}$ .

The purpose of this work is, by using the global central moments and cumulants for a local stellar velocity distribution which is composed of two trivariate normal populations, to describe a method to obtain the covariance tensor  $\mathbf{M}_2$  of the population components, the respective means, and mixing proportions.

## 2. Single component

The velocity distribution function of a single stellar population is assumed of normal type:

$$f(\mathbf{V}) = e^{-\frac{1}{2}Q}, \quad Q = \mathbf{u}^T \cdot \mathbf{M}_2^{-1} \cdot \mathbf{u}, \quad (5)$$

In this case all the odd-order central moments are zero and the even central moments can be computed from the second ones (Orús, 1977). Thus, the fourth central moments satisfy by components the following relationship:

$$\mu_{ijkl} = \mu_{ij}\mu_{kl} + \mu_{ik}\mu_{jl} + \mu_{il}\mu_{jk}; \quad i, j, k \in \{1, 2, 3\} \quad (6)$$

In order to simplify the algebraic notation of following sections, these relationships can be written in a more compact tensor notation (Cubarsi, 1992). If  $\mathbf{A}_m$  and  $\mathbf{B}_n$  are two  $m$ - and  $n$ -rank symmetric tensors, we define the tensor  $\mathbf{A}_m \star \mathbf{B}_n$  as the obtained by symmetrizing the tensor product  $\mathbf{A}_m \otimes \mathbf{B}_n$ , and by normalizing with respect to the number of summation terms. The result is a  $(m+n)$ -rank symmetric tensor, which components are

$$\mathbf{A}_m \star \mathbf{B}_n |_{i_1 i_2 \dots i_{m+n}} = \frac{n!m!}{(m+n)!} \sum_{\alpha} A_{\alpha i_1 \dots \alpha i_m} B_{\alpha i_{m+1} \dots \alpha i_{m+n}} \quad (7)$$

where  $\alpha$  belongs to the symmetric group  $\mathcal{S}(m+n)$ , with the indices satisfying  $\alpha i_1 < \dots < \alpha i_m$ , and  $\alpha i_{m+1} < \dots < \alpha i_{m+n}$ . Then Eq. 6 simply becomes

$$\mathbf{M}_4 = 3 \mathbf{M}_2 \star \mathbf{M}_2 \quad (8)$$

On the other hand, the cumulants can also be used in order to describe the velocity distribution instead of the central moments. They present some attractive properties coming from their symmetry and, particularly, the population cumulants have as unbiased estimators the corresponding  $k$ -statistics. Then, taking into account the relationships between central moments and the corresponding cumulants (see e.g. Kendall, Stuart & Ord, 1987), for second, third and fourth cumulants we have

$$\begin{aligned} \mathbf{K}_2 &= \mathbf{M}_2 \\ \mathbf{K}_3 &= \mathbf{M}_3 \\ \mathbf{K}_4 &= \mathbf{M}_4 - 3 \mathbf{M}_2 \star \mathbf{M}_2 \end{aligned} \quad (9)$$

In particular for a multivariate normal distribution the following important property is satisfied:

$$\mathbf{K}_n = \mathbf{0}; \quad n \geq 3 \quad (10)$$

hence the only non-vanishing cumulants are the second ones, which values completely characterize the distribution.

### 3. Moments and cumulants of the mixture

The total distribution function is obtained by superposition of two normal distribution functions associated with the corresponding populations –(') or (") for the first or second population–, according to

$$f = f' + f'' \quad (11)$$

We shall deduce the quantities  $N, \mathbf{v}$  and  $\mathbf{M}_n$  –defined in the first section– corresponding to the total velocity distribution, starting from those of the partial distributions. For the stellar densities, from Eq. 1, Eq. 11, and by defining  $n' = \frac{N'}{N}$ ,  $n'' = \frac{N''}{N}$ , we get

$$1 = n' + n'' \quad (12)$$

For the respective mean velocities, according to Eq. 2, Eq. 11 and Eq. 12, we obtain the following equation

$$\mathbf{v} = n'\mathbf{v}' + n''\mathbf{v}'' \quad (13)$$

Furthermore, the total central moments can be computed from the partial ones, taking into account the Eq. 3 and Eq. 11, by using the centroid differential velocity,

$$\mathbf{w} = \mathbf{v}' - \mathbf{v}'' \quad (14)$$

Thus, with the notation introduced from Eq. 7, the tensor of the  $n$ -order central moments of the total sample, expressed from the partial ones, has the following form:

$$\mathbf{M}_n = \sum_{k=0}^n \binom{n}{k} \{n'n''^k \mathbf{M}'_{n-k} + (-n')^k n'' \mathbf{M}''_{n-k}\} \star (\mathbf{w})^k \quad (15)$$

We explicitly write the total central moments up to fourth-order in the particular case of a two normal component mixture. Obviously for  $n = 0$ , we have  $\mathbf{M}_0 = \mathbf{M}'_0 = \mathbf{M}''_0 = 1$  and, for  $n = 1$ ,  $\mathbf{M}_1 = \mathbf{M}'_1 = \mathbf{M}''_1 = 0$ .

- Second moments, with six different components,

$$\mathbf{M}_2 = n'\mathbf{M}'_2 + n''\mathbf{M}''_2 + n'n''(\mathbf{w})^2 \quad (16)$$

- Third moments, with ten different components,

$$\mathbf{M}_3 = 3n'n''(\mathbf{M}'_2 - \mathbf{M}''_2) \star \mathbf{w} + n'n''(n'' - n')(\mathbf{w})^3 \quad (17)$$

- Fourth moments, with fifteen different components,

$$\mathbf{M}_4 = 3n'\mathbf{M}'_2 \star \mathbf{M}'_2 + 3n''\mathbf{M}''_2 \star \mathbf{M}''_2 + 6n'n''(n''\mathbf{M}'_2 + n'\mathbf{M}''_2) \star (\mathbf{w})^2 + n'n''(1 - 3n'n'')(\mathbf{w})^4 \quad (18)$$

It is possible to express above relationships in a shorter form by introducing the cumulants, according to Eq. 9, and the following new auxiliar variables,

$$\mathbf{D} = \sqrt{n'n''} \mathbf{w}; \quad q = \sqrt{n'/n''} - \sqrt{n''/n'} \quad (19)$$

and new symmetric tensors,

$$\begin{aligned} \mathbf{a}_2 &= n'\mathbf{K}'_2 + n''\mathbf{K}''_2 \\ \mathbf{C}_2 &= \frac{1}{\sqrt{q^2+4}}(\mathbf{K}'_2 - \mathbf{K}''_2) - q(\mathbf{D})^2 \end{aligned} \quad (20)$$

(we can appoint the populations so that  $n' \geq n''$ ; then  $q$  is non-negative).

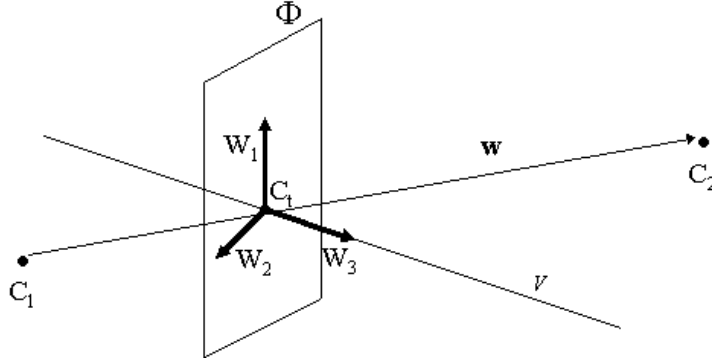
Now, with latter definitions, Eq. 16, Eq. 17 and Eq. 18 can be rewritten in this way:

$$\begin{aligned} \mathbf{K}_2 &= \mathbf{a}_2 + (\mathbf{D})^2 \\ \mathbf{K}_3 &= 3\mathbf{C}_2 \star \mathbf{D} + 2q(\mathbf{D})^3 \\ \mathbf{K}_4 &= 3\mathbf{C}_2 \star \mathbf{C}_2 - 2(q^2 + 1)(\mathbf{D})^4 \end{aligned} \quad (21)$$

Thus, our problem can be definitively described as follows. We know the total cumulants and the mean of a two-component multivariate normal distribution, and we want to determine the unknown parameters of the partial distributions. These unknowns are the partial cumulants –reduced to six components of  $\mathbf{K}'_2$  and six of  $\mathbf{K}''_2$ , or the equivalent ones  $\mathbf{a}_2$  and  $\mathbf{C}_2$ –, the percentage of populations –for example  $n'$  or  $q$ – and the three components of the centroid differential velocity  $\mathbf{w}$ , or the components of  $\mathbf{D}$ . Sixteen unknowns in total. Moreover, we have a set of thirty-one non-linear scalar equations involved in Eq. 21. Thus we must also find a set of fifteen constraint equations, which can provide us with a useful test of the method.

#### 4. Projections of the peculiar velocity

We are interested in the general case where the centroid differential velocity (and vector  $\mathbf{D}$ ) is not null. Hence let us assume  $D_3 \neq 0$  (we can suppose that this component corresponds to  $\max_i |D_i|$ ), and let us define a normalized vector  $\mathbf{d} = \mathbf{D}/D_3$  with the direction containing both subcentroids  $C_1$  and  $C_2$ . Since every normal population distribution is symmetric with respect to its centroid, the total velocity distribution will be symmetric in whatever direction within a plane  $\Phi$ , containing the global centroid  $C_t$ , orthogonal to the vectors  $\mathbf{d}$  and  $\mathbf{w}$ . Thus, in order to simplify the problem, it is convenient to work with some transformed vector  $\mathbf{W}$ , which is the sum of three scaled vector projections of the peculiar velocity  $\mathbf{u}$  on the following directions:  $\mathbf{W}_1 = (0, d_3, -d_2)^t$ ,  $\mathbf{W}_2 = (-d_3, 0, d_1)^t$ , on the plane  $\Phi$ , and another independent direction  $\mathbf{W}_3 = (0, 0, d_3)^t$ .



The new vector  $\mathbf{W}$  is the obtained from the following isomorphic transformation of vector  $\mathbf{u}$ ,

$$\mathbf{W} = \mathbf{H}_2 \cdot \mathbf{u}; \quad \mathbf{H}_2 = \begin{pmatrix} 0 & d_3 & -d_2 \\ -d_3 & 0 & d_1 \\ 0 & 0 & d_3 \end{pmatrix} \quad (22)$$

Note that  $\det(\mathbf{H}_2) = d_3^3$ , and  $d_3 = 1$ , but in case of permuting indices it is helpful to maintain this notation. Also note that  $\mathbf{W}$  and  $\mathbf{u}$  have zero mean.

Now we can calculate the moments—in particular the third and fourth moments—of  $\mathbf{W}$ , in function of the components  $\mu_{ijk}$  and  $\mu_{ijkl}$ . From Eq. 3 we obtain expressions in the form,

$$\begin{aligned} \langle W_\alpha W_\beta W_\gamma \rangle &= H_{\alpha i} H_{\beta j} H_{\gamma k} \mu_{ijk}; & \alpha, \beta, \gamma, \delta, i, j, k, l \in \{1, 2, 3\} \\ \langle W_\alpha W_\beta W_\gamma W_\delta \rangle &= H_{\alpha i} H_{\beta j} H_{\gamma k} H_{\delta l} \mu_{ijkl} \end{aligned} \quad (23)$$

(We use Einstein's summation criterion). In this way it is possible to establish several relationships between the total cumulants, which enable us to solve the problem. Thus, we use the twenty five components of the following *two-dimensional* tensors and vectors  $\mathbf{o}_3$ ,  $\mathbf{p}_2$ ,  $\mathbf{s}$ ,  $\mathbf{X}_4$ ,  $\mathbf{Y}_3$ ,  $\mathbf{Z}_2$  and  $\mathbf{T}$ , with the indices  $\alpha, \beta, \gamma, \delta \in \{1, 2\}$  and  $i, j, k, l \in \{1, 2, 3\}$ :

$$\begin{aligned}
\mathbf{o}_3|_{\alpha\beta\gamma} = o_{\alpha\beta\gamma} &\equiv H_{\alpha i} H_{\beta j} H_{\gamma k} \kappa_{ijk} & \mathbf{X}_4|_{\alpha\beta\gamma\delta} = X_{\alpha\beta\gamma\delta} &\equiv H_{\alpha i} H_{\beta j} H_{\gamma k} H_{\delta l} \kappa_{ijkl} \\
\mathbf{p}_2|_{\alpha\beta} = p_{\alpha\beta} &\equiv H_{\alpha i} H_{\beta j} \kappa_{ij3} & \mathbf{Y}_3|_{\alpha\beta\gamma} = Y_{\alpha\beta\gamma} &\equiv H_{\alpha i} H_{\beta j} H_{\gamma k} \kappa_{ijk3} \\
\mathbf{s}|_{\alpha} = s_{\alpha} &\equiv \frac{1}{2} H_{\alpha i} \kappa_{i33} & \mathbf{Z}_2|_{\alpha\beta} = Z_{\alpha\beta} &\equiv H_{\alpha i} H_{\beta j} \kappa_{ij33} \\
\kappa_{333} & & \mathbf{T}|_{\alpha} = T_{\alpha} &\equiv H_{\alpha i} \kappa_{i333} \\
& & \kappa_{3333} &
\end{aligned} \tag{24}$$

These quantities are here explicitly written:

$$\begin{aligned}
o_{111} &= -\kappa_{333} d_1^3 + 3\kappa_{133} d_1^2 d_3 - 3\kappa_{113} d_1 d_3^2 + \kappa_{111} d_3^3 = 0 \\
o_{222} &= -\kappa_{333} d_2^3 + 3\kappa_{332} d_3 d_2^2 - 3\kappa_{322} d_3^2 d_2 + \kappa_{222} d_3^3 = 0 \\
o_{112} &= -\kappa_{333} d_1^2 d_2 + \kappa_{332} d_1^2 d_3 + 2\kappa_{133} d_1 d_3 d_2 - 2\kappa_{132} d_1 d_3^2 - \kappa_{113} d_3^2 d_2 + \kappa_{112} d_3^3 = 0 \\
o_{122} &= -\kappa_{333} d_1 d_2^2 + \kappa_{133} d_3 d_2^2 + 2\kappa_{332} d_1 d_3 d_2 - 2\kappa_{132} d_3^2 d_2 - \kappa_{322} d_1 d_3^2 + \kappa_{122} d_3^3 = 0 \\
p_{22} &= \kappa_{333} d_2^2 - 2\kappa_{332} d_3 d_2 + \kappa_{322} d_3^2 \\
p_{12} &= \kappa_{333} d_1 d_3 d_2 - \kappa_{332} d_1 d_3 - \kappa_{133} d_3 d_2 + \kappa_{132} d_3^2 \\
p_{11} &= \kappa_{333} d_1^2 - 2\kappa_{133} d_1 d_3 + \kappa_{113} d_3^2 \\
s_2 &= \frac{1}{2} (-\kappa_{333} d_2 + \kappa_{332} d_3) \\
s_1 &= \frac{1}{2} (-\kappa_{333} d_1 + \kappa_{133} d_3) \\
X_{2222} &= \kappa_{3333} d_2^4 - 4\kappa_{3332} d_3 d_2^3 + 6\kappa_{3322} d_3^2 d_2^2 - 4\kappa_{3222} d_3^3 d_2 + \kappa_{2222} d_3^4 \\
X_{1222} &= \kappa_{3333} d_1 d_2^3 - \kappa_{1333} d_3 d_2^3 - 3\kappa_{3332} d_1 d_3 d_2^2 + 3\kappa_{1332} d_3^2 d_2^2 + 3\kappa_{3322} d_1 d_3^2 d_2 - 3\kappa_{1322} d_3^3 d_2 - \\
&\quad \kappa_{3222} d_1 d_3^3 + \kappa_{1222} d_3^4 \\
X_{1122} &= \kappa_{3333} d_1^2 d_2^2 - 2\kappa_{3332} d_1^2 d_3 d_2 - 2\kappa_{1333} d_1 d_3 d_2^2 + \kappa_{3322} d_1^2 d_3^2 + 4\kappa_{1332} d_1 d_3^2 d_2 + \\
&\quad \kappa_{1133} d_3^2 d_2^2 - 2\kappa_{1322} d_1 d_3^3 - 2\kappa_{1132} d_3^3 d_2 + \kappa_{1122} d_3^4 \\
X_{1112} &= \kappa_{3333} d_1^3 d_2 - \kappa_{3332} d_1^3 d_3 - 3\kappa_{1333} d_1^2 d_3 d_2 + 3\kappa_{1332} d_1^2 d_3^2 + 3\kappa_{1133} d_1 d_3^2 d_2 - 3\kappa_{1132} d_1 d_3^3 - \\
&\quad \kappa_{1113} d_3^3 d_2 + \kappa_{1112} d_3^4 \\
X_{1111} &= \kappa_{3333} d_1^4 - 4\kappa_{1333} d_1^3 d_3 + 6\kappa_{1133} d_1^2 d_3^2 - 4\kappa_{1113} d_1 d_3^3 + \kappa_{1111} d_3^4 \\
Y_{222} &= -\kappa_{3333} d_2^3 + 3\kappa_{3332} d_3 d_2^2 - 3\kappa_{3322} d_3^2 d_2 + \kappa_{3222} d_3^3 \\
Y_{122} &= -\kappa_{3333} d_1 d_2^2 + 2\kappa_{3332} d_1 d_3 d_2 + \kappa_{1333} d_3 d_2^2 - \kappa_{3322} d_1 d_3^2 - 2\kappa_{1332} d_3^2 d_2 + \kappa_{1322} d_3^3 \\
Y_{112} &= -\kappa_{3333} d_1^2 d_2 + 2\kappa_{1333} d_1 d_3 d_2 + \kappa_{3332} d_1^2 d_3 - 2\kappa_{1332} d_1 d_3^2 - \kappa_{1133} d_2 d_3^2 + \kappa_{1132} d_3^3 \\
Y_{111} &= -\kappa_{3333} d_1^3 + 3\kappa_{1333} d_1^2 d_3 - 3\kappa_{1133} d_1 d_3^2 + \kappa_{1113} d_3^3 \\
Z_{22} &= \kappa_{3333} d_2^2 - 2\kappa_{3332} d_3 d_2 + \kappa_{3322} d_3^2 \\
Z_{12} &= \kappa_{3333} d_1 d_2 - \kappa_{3332} d_1 d_3 - \kappa_{1333} d_3 d_2 + \kappa_{1332} d_3^2 \\
Z_{11} &= \kappa_{3333} d_1^2 - 2\kappa_{1333} d_1 d_3 + \kappa_{1133} d_3^2 \\
T_2 &= -\kappa_{3333} d_2 + \kappa_{3332} d_3 \\
T_1 &= -\kappa_{3333} d_1 + \kappa_{1333} d_3
\end{aligned}$$



## 5. Mixture parameters

We summarize the main properties and the method to obtain the parameters of the mixture and population distributions. By substitution of expressions  $\mathbf{K}_3$  and  $\mathbf{K}_4$  (Eq. 21 ) in Eq. 24, we get the following results:

- There exist four independent vanishing linear combinations of the third  $\mathbf{W}$ -cumulants

$$o_{\alpha\beta\gamma} = 0; \alpha, \beta, \gamma \in \{1, 2\} \quad (25)$$

These are the third cumulants of  $W_1$  and  $W_2$  vector components. From above equations we can compute  $d_1$  and  $d_2$ . The algorithm we use is to transform these equations in a linear overdeterminate system around the solution, which is solved iteratively by weighted least squares. With these values it is already possible to calculate all the tensor components of defined in Eq. 24.

- The tensor components of  $\mathbf{p}_2$  and  $\mathbf{s}$  satisfy the following equalities,

$$\begin{aligned} p_{\alpha\beta} &= D_3 H_{\alpha i} H_{\beta j} C_{ij} \\ s_{\alpha} &= D_3 H_{\alpha i} C_{i3} \end{aligned} \quad (26)$$

These expressions enables us to write the tensor  $\mathbf{K}_4$  as a function of the third cumulants.

- The second, third and fourth total cumulants are constrained by a set of fourteen relationships with  $\alpha, \beta, \gamma, \delta \in \{1, 2\}$  in the following form,

$$\begin{aligned} X_{\alpha\beta\gamma\delta} &= \mathcal{A}(p_{\alpha\beta}p_{\gamma\delta} + p_{\alpha\gamma}p_{\beta\delta} + p_{\alpha\delta}p_{\beta\gamma}) \\ Y_{\alpha\beta\gamma} &= \mathcal{A}(p_{\alpha\beta}s_{\gamma} + p_{\alpha\gamma}s_{\beta} + p_{\beta\gamma}s_{\alpha}) \\ Z_{\alpha\beta} &= 2\mathcal{A}s_{\alpha}s_{\beta} + \mathcal{B}p_{\alpha\beta} \\ T_{\alpha} &= 3\mathcal{B}s_{\alpha} \end{aligned} \quad (27)$$

where  $\mathcal{A} = D_3^{-2}$  and  $\mathcal{B} = C_{33}D_3^{-1}$ .

This is an overdeterminate linear system which is solved also by weighted least squares in order to find optimal values for  $\mathcal{A}$  and  $\mathcal{B}$ . A diagonal covariance matrix of errors has been use. These step provides us with the absolute values of  $C_{33}$  and  $D_3$ .

- The mixing proportions we can be obtained from the variable  $q$ , so that the following relationships are fulfilled. Note that this provides a new constraint equation.

$$\begin{aligned} \kappa_{333} &= 3C_{33}D_3 + 2qD_3^3 \\ \kappa_{3333} &= 3C_{33}^2 - 2(q^2 + 1)D_3^4 \end{aligned} \quad (28)$$

From the first of these equations it is possible to determine the sign of  $C_{33}$  and  $D_3$ . Usually this equation provides the better estimation of the parameter, since third moments have smaller errors than fourth.

- The remaining five unknowns of tensor  $\mathbf{C}_2$  can be computed from Eq. 26, which involves the total third cumulants.
- From Eq. 12, Eq. 13, Eq. 14, Eq. 19, and Eq. 20 we can determine the original unknowns  $n'$ ,  $\mathbf{w}$ ,  $\mathbf{v}'$ ,  $\mathbf{v}''$ ,  $\mathbf{K}'_2$ , and  $\mathbf{K}''_2$ .

## 6. Numerical application

The method has been converted into a numerical procedure, where the result is very sensible to the errors of entering data. Therefore attention must be paid to the error propagation and correct assignation of weights to the corresponding equations. In this work we have adopted some standard techniques, like statistical propagation of errors (instead of interval analysis or linear approximation of errors), and weighted least squares from the respective error covariance matrices. Nevertheless the main source of errors comes from an unapropriate selection of the stellar sample, in particular when the sample contains more than two populations, or a few non-representative stars. Thus an improvement of the method, based on a maximum entropy sampling filter, is also presented at this Conference (Alcobé *et al.*, 2000).

The method has been applied to samples composed of data coming from synthetic and astronomical catalogues. The population moments and cumulants are obtained from the corresponding statistics, and their standard errors are estimated from higher-order sampling cumulants. Here we present an example with synthetic populations.

Both populations are respectively composed of 1727 and 213 virtual stars (mixture proportion of 89%). They are generated of normal type from standard algorithms (Press *et al.* 1992) according to the following characteristic values (in this case simulating an astronomical local stellar sample),

- Population I:  
 -standard deviations  $\sigma_1 : \sigma_2 : \sigma_3 = 34 : 20 : 18$ ,  $\text{diag}(\mathbf{M}_2) = (1156, 400, 324)$   
 -means  $V_1 : V_2 : V_3 = -1.5 : -1.6 : -0.4$

-rotation around 3rd-axis=  $9^\circ$ ;  $\mu_{12} = 108$

• Population I:

-standard deviations  $\sigma_1 : \sigma_2 : \sigma_3 = 58 : 40 : 38$ ,  $\text{diag}(\mathbf{M}_2) = (3364, 1600, 1444)$

-means  $V_1:V_2:V_3 = -23:-40:-6$

-rotation around 3rd-axis=  $-1^\circ$ ;  $\mu_{12} = -28$

After application of the method the following characteristic parameters for the normal components are obtained:

	$\mu_{11}$	$\mu_{22}$	$\mu_{33}$	$\mu_{12}$	$\mu_{13}$	$\mu_{23}$	$V_1$	$V_2$	$V_3$	%
<b>Pop. I</b>	1154	444	313	110	4	-2	-1.1	-1.5	1.4	0.89
std. error	142	154	50	103	69	61	0.9	0.8	0.5	0.02
<b>Pop. II</b>	3954	1531	1428	-43	-215	-110	-23.5	-38.9	-11.9	0.11
std. error	1007	1184	360	776	507	455	2.7	4.3	1.6	0.02

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Total central moments and cumulants are shown in the following table:

CENTRAL MOMENTS, CUMULANTS AND STANDARD ERRORS  
OF THE SYNTHETIC SAMPLE

1ST ORDER MOMENTS			3RD ORDER MOMENTS		
100	-3.61	.88	300	-19461.71	6909.45
010	-5.66	.60	210	-11214.74	2315.66
001	-.06	.48	120	-3718.86	1967.42
2ND ORDER MOMENTS			030	-16216.48	2664.90
200	1518.62	61.93	201	-6422.67	2624.37
110	176.30	29.95	111	393.37	1108.16
020	705.63	33.35	021	-2093.26	908.81
101	9.23	25.33	102	-2115.27	1398.11
011	36.06	17.79	012	-4388.47	891.47
002	455.55	19.58	003	-1739.82	1131.43
4TH ORDER MOMENTS			4TH ORDER CUMULANTS		
400	9747406.67	1398124.15	1111	2828776.01	1435584.00
310	1136010.04	315333.94	1112	332812.25	330122.92
220	1771782.24	197718.78	1113	46578.52	476070.02
130	539569.71	212039.35	1122	638026.98	219584.78
040	2655683.83	339920.61	1123	141105.51	131684.21
301	88649.01	471245.21	1133	552434.83	219767.08
211	199119.00	128218.74	1222	166359.87	217141.36
121	31183.56	80853.98	1223	11953.66	83086.71
031	198377.45	125026.98	1233	76725.17	75878.08
202	1244416.72	211990.51	1333	-102733.98	123222.71
112	157704.88	73911.14	2222	1161922.14	349559.38
022	615429.51	72668.70	2223	122047.48	127277.40
103	-90113.78	121539.68	2233	291375.66	78264.73
013	93706.36	70373.71	2333	44428.33	72009.49
004	950914.19	113519.57	3333	328330.53	117647.62