

Edge-Distance-Regular Graphs *

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Abstract

Edge-distance-regularity is a concept recently introduced by the authors which is similar to that of distance-regularity, but now the graph is seen from each of its edges instead of from its vertices. More precisely, a graph Γ with adjacency matrix \mathbf{A} is edge-distance-regular when it is distance-regular around each of its edges and with the same intersection numbers for any edge taken as a root. In this paper we study this concept, give some of its properties, such as the regularity of Γ , and derive some characterizations. In particular, it is shown that a graph is edge-distance-regular if and only if its k -incidence matrix is a polynomial of degree k in \mathbf{A} multiplied by the (standard) incidence matrix. Also, the analogue of the spectral excess theorem for distance-regular graphs is proved, so giving a quasi-spectral characterization of edge-distance-regularity. Finally, it is shown that every nonbipartite graph which is both distance-regular and edge-distance-regular is a generalized odd graph.

1 Introduction

Since its introduction by Biggs in the 70's (see [1]), the theory of distance-regular graphs has been widely developed in the last decades. Its importance is highlighted in the preface's comment of the comprehensive textbook of Brouwer, Cohen and Neumaier [2]: "Most finite objects bearing 'enough regularity' are closely related to certain distance-regular graphs." Thus, many characterizations of combinatorial and algebraic nature of distance-regular graphs are known (see for instance Van Dam and Haemers [19] and Fiol [9]). Moreover,

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Dalfó, Van Dam and Fiol [6] recently obtained some new characterizations of distance-regular graphs in terms of the cospectrality of their perturbed graphs. Also, distance-regular graphs have given rise to several generalizations, such as association schemes (see Brouwer and Haemers [3]) and almost distance-regular graphs [7]. When we look at the distance partition of the graph from each of its edges instead of its vertices, we arrive, in a natural way, to the concept of edge-distance-regularity. The definition of edge-distance-regular graph was introduced by Fiol and Garriga in [12]. Here we study this concept, give some of its properties and derive some characterizations. In particular, it is shown that a graph $\Gamma = (V, E)$, with adjacency matrix \mathbf{A} , is edge-distance-regular if and only if every k -incidence matrix \mathbf{B}_k (with entries $b_{ue} = 1$ if the distance between vertex $u \in V$ and edge $e \in E$ is k and $b_{ue} = 0$ otherwise) is a polynomial of degree k in \mathbf{A} multiplied by the (standard) incidence matrix $\mathbf{B} = \mathbf{B}_0$. Also, the analogue of the spectral excess theorem for distance-graphs is proved, so giving a quasi-spectral characterization of edge-distance-regularity: A graph Γ is edge-distance-regular if and only if its edge-spectral-excess (a number which can be computed from the spectrum of Γ) equals the excess of every edge (that is, the number of vertices at maximum distance from every edge). Finally, it is shown that every nonbipartite distance-regular and edge-distance-regular graph is a generalized odd graph (also known as an almost-bipartite distance-regular graph, or a regular thin near $(2D + 1)$ -gon). With this aim, the next section is devoted to present the basic notation and theoretical background on which our study is based. Apart from the basic notions on graphs and their spectra, we recall some theory on orthogonal polynomials of a discrete variable and pseudo-distance-regularity around a set. Section 3 deals with the new concept of edge-spectrum-regularity, which is closely related to walk-regularity (a concept due to Godsil and McKay [15]) and its generalizations such as m -walk-regularity (see Dalfó, Van Dam, Fiol, Garriga and Gorissen [7]). Finally, the main body of the paper is in Section 4, where we study some properties and obtain some characterizations of edge-distance-regular graphs.

2 Preliminaries

2.1 Graphs and their spectra

Throughout this paper, $\Gamma = (V, E)$ denotes a (finite, simple and connected) graph with order $|V|$ and adjacency matrix \mathbf{A} . The *distance* between two vertices u and v is denoted by $\partial(u, v)$, so that the *eccentricity* of vertex u is $\varepsilon_u = \max_{v \in V} \partial(u, v)$ and the *diameter* of the graph is $D = \max_{u \in V} \varepsilon_u$. The set of vertices at distance k , from a given vertex $u \in V$ is denoted by $\Gamma_k(u)$, for $k = 0, 1, \dots, D$. The degree of vertex u is denoted by $\delta_u = |\Gamma_1(u)|$. More generally, the distance between two subsets $U_1, U_2 \subset V$ is $\partial(U_1, U_2) = \min_{u \in U_1, v \in U_2} \{\partial(u, v)\}$ and, for a given vertex subset $C \subset V$ and some integer $k \geq 0$, we denote by $\Gamma_k(C)$ the set of vertices at distance k from C and $N_k(C) = \Gamma_0(C) \cup \Gamma_1(C) \cup \dots \cup \Gamma_k(C)$ is its k -neighborhood. The *eccentricity* of C , denoted by ε_C , is then defined as the maximum distance of any vertex of Γ from C . In Coding Theory this corresponds to the *covering radius* of C . The *antipodal* \overline{C} of C is the subset in V at maximum distance

from C , that is, $\overline{C} = \Gamma_{\varepsilon_C}(C)$.

The spectrum of Γ is denoted by $\text{sp } \Gamma = \text{sp } \mathbf{A} = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\}$, where the different eigenvalues of Γ are in decreasing order, $\lambda_0 > \lambda_1 > \dots > \lambda_d$, and the superscripts stand for their multiplicities $m_i = m(\lambda_i)$. In particular, note that $m_0 = 1$ (since Γ is connected) and $m_0 + m_1 + \dots + m_d = n$. Moreover, λ_0 has a positive eigenvector, denoted by $\boldsymbol{\nu}$, which is normalized in such a way that its minimum component is 1. For instance, if Γ is regular, we have $\boldsymbol{\nu} = \mathbf{j}$, the all-1 vector. For a given ordering of the vertices of Γ , the vector space of linear combinations (with real coefficients) of the vertices is identified with \mathbb{R}^n , with canonical basis $\{\mathbf{e}_u : u \in V\}$ and, hence, vectors and matrices are indexed by the vertices of V . The matrices $\mathbf{E}_i = \lambda_i^*(\mathbf{A})$, where λ_i^* , $i = 0, 1, \dots, d$, is the Lagrange interpolating polynomial satisfying $\lambda_i^*(\lambda_j) = \delta_{ij}$, are the (*principal*) *idempotents* of \mathbf{A} representing the orthogonal projections of \mathbb{R}^n onto the eigenspaces $\mathcal{E}_i = \text{Ker}(\mathbf{A} - \lambda_i \mathbf{I})$. These matrices satisfy the known properties: $\mathbf{E}_i \mathbf{E}_j = \delta_{ij} \mathbf{E}_i$; $\mathbf{A} \mathbf{E}_i = \lambda_i \mathbf{E}_i$; and $p(\mathbf{A}) = \sum_{i=0}^d p(\lambda_i) \mathbf{E}_i$, for any polynomial $p \in \mathbb{R}[x]$ (see for example Godsil [14, p. 28]). The (*u*-)local multiplicities of the eigenvalue λ_i are defined as

$$m_u(\lambda_i) = \|\mathbf{E}_i \mathbf{e}_u\|^2 = \langle \mathbf{E}_i \mathbf{e}_u, \mathbf{e}_u \rangle = (\mathbf{E}_i)_{uu} \quad (u \in V; i = 0, 1, \dots, d),$$

and satisfy $\sum_{i=0}^d m_u(\lambda_i) = 1$ and $\sum_{u \in V} m_u(\lambda_i) = m_i$, $i = 0, 1, \dots, d$ (see Fiol and Garriga [10]). Related to this concept, we say that Γ is *spectrum-regular* if, for any $i = 0, 1, \dots, d$, the *u*-local multiplicity of λ_i does not depend on vertex *u*. Then, the above equations imply that the (standard) multiplicities ‘split’ equitably among the *n* vertices, giving $m_u(\lambda_i) = m_i/n$, $i = 0, 1, \dots, d$.

By analogy with the local multiplicities, which correspond to the diagonal entries of the idempotents, Fiol, Garriga, and Yebra [13] defined the *crossed (uv-)local multiplicities* of eigenvalue λ_i , denoted by $m_{uv}(\lambda_i)$, as

$$m_{uv}(\lambda_i) = \langle \mathbf{E}_i \mathbf{e}_u, \mathbf{E}_i \mathbf{e}_v \rangle = \langle \mathbf{E}_i \mathbf{e}_u, \mathbf{e}_v \rangle = (\mathbf{E}_i)_{uv} \quad (u, v \in V; i = 0, 1, \dots, d).$$

Thus, in particular, $m_{uu}(\lambda_i) = m_u(\lambda_i)$. These parameters allow us to compute the number of walks of length ℓ between two vertices *u, v* in the following way:

$$a_{uv}^{(\ell)} = (\mathbf{A}^\ell)_{uv} = \sum_{i=0}^d m_{uv}(\lambda_i) \lambda_i^\ell \quad (\ell = 0, 1, \dots). \quad (1)$$

Let $a_u^{(\ell)}$ denote the number of closed walks of length ℓ rooted at vertex *u*, that is, $a_u^{(\ell)} = a_{uu}^{(\ell)}$. If these numbers only depend on ℓ , for each $\ell \geq 0$, then Γ is called *walk-regular* (a concept introduced by Godsil and McKay [15]). Notice that, as $a_u^{(2)} = \delta_u$, the degree of vertex *u*, a walk-regular graph is necessarily regular. From (1), it follows that spectrum-regularity and walk-regularity are equivalent concepts. As it is well known, any distance-regular graph, as well as any vertex-transitive graph, is walk-regular, but the converse is not true (see Godsil [14]). The above concepts were generalized by Dalfó, Fiol and Garriga [5] as follows: A graph Γ with diameter *D* is *m-walk-regular*, for some $0 \leq m \leq D$, if the number

of walks $a_{uv}^{(\ell)}$ depends only on ℓ and the distance $\partial(u, v)$, provided that $\partial(u, v) \leq m$. Thus, a 0-walk-regular graph is walk-regular, while a D -walk-regular graph is distance-regular. Analogously, we say that Γ is m -spectrum-regular if, for any $i = 0, 1, \dots, d$, the crossed local multiplicity of $m_{uv}(\lambda_i)$ does not depend on the vertices u, v but only on their distance $\partial(u, v) \leq m$. In [5], the authors showed that m -walk-regularity and m -spectrum-regularity are also equivalent concepts .

2.2 Polynomials of a discrete variable

Let $\mathcal{M} = \{\lambda_0, \lambda_1, \dots, \lambda_d\}$ be a finite set of real numbers with $\lambda_0 > \lambda_1 > \dots > \lambda_d$ and consider a weight function $g : \mathcal{M} \rightarrow (0, 1]$ such that $\sum_{i=0}^d g(\lambda_i) = 1$. Let $\mathcal{M}^* = \mathcal{M} \setminus \{\lambda_0\}$. Here, all the equalities involving polynomials are considered in $\mathbb{R}[x]/(Z)$, where (Z) is the ideal generated by the polynomial $Z = \prod_{i=0}^d (x - \lambda_i)$. As a representant of each equivalence class we consider its unique polynomial with degree at most $d = |\mathcal{M}^*|$.

In our study we make ample use of the moment-like parameters

$$\pi_i = \prod_{j=0, j \neq i}^d |\lambda_i - \lambda_j| \quad (0 \leq i \leq d) \quad (2)$$

and the polynomial $H = \frac{1}{g(\lambda_0)\pi_0} \prod_{i=1}^d (x - \lambda_i)$. Also, given the pair (\mathcal{M}, g) , we define the scalar product

$$\langle p, q \rangle = \sum_{i=0}^d g(\lambda_i) p(\lambda_i) q(\lambda_i). \quad (3)$$

An *orthogonal system* with respect to (\mathcal{M}, g) is a family of polynomials $\{r_k\}_{0 \leq k \leq d}$, with $\deg(r_k) = k$, satisfying $\langle r_s, r_t \rangle = 0$ if $s \neq t$. For our purposes we always assume that $r_0 = 1$, so that $\|r_0\| = 1$. As in every sequence of orthogonal polynomials, the following three-term recurrence applies:

$$xr_k = b_{k-1}r_{k-1} + a_k r_k + c_{k+1}r_{k+1} \quad (0 \leq k \leq d), \quad (4)$$

where the constants b_{k-1} , a_k and c_{k+1} are the Fourier coefficients of xr_k in terms of r_{k-1} , r_k and r_{k+1} , respectively, and $b_{-1} = c_{d+1} = 0$. The following proposition gives a matricial version of this recurrence, as well as other properties of special interest in our context.

Proposition 2.1 [4] *Every orthogonal system r_0, r_1, \dots, r_d satisfies the following properties:*

- (a) *There exists a tridiagonal matrix \mathbf{R} (called the recurrence matrix of the system) such*

that, in $\mathbb{R}[x]/(Z)$:

$$x\mathbf{r} = x \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \\ r_{d-2} \\ r_{d-1} \\ r_d \end{pmatrix} = \begin{pmatrix} a_0 & c_1 & 0 & & & & & & \\ b_0 & a_1 & c_2 & 0 & & & & & \\ 0 & b_1 & a_2 & \ddots & \ddots & & & & \\ & 0 & \ddots & \ddots & \ddots & 0 & & & \\ & & \ddots & \ddots & a_{d-2} & c_{d-1} & 0 & & \\ & & & 0 & b_{d-2} & a_{d-1} & c_d & & \\ & & & & 0 & b_{d-1} & a_d & & \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \\ r_{d-2} \\ r_{d-1} \\ r_d \end{pmatrix} = \mathbf{R}\mathbf{r}.$$

- (b) All the entries b_k and c_k of \mathbf{R} are nonzero and satisfy $b_k c_{k+1} > 0$.
- (c) The matrix \mathbf{R} diagonalizes with eigenvalues the elements of $\mathcal{M} = \{\lambda_0, \lambda_1, \dots, \lambda_d\}$. An eigenvector associated to λ_i is $\mathbf{r}(\lambda_i) = (r_0(\lambda_i), r_1(\lambda_i), \dots, r_{d-1}(\lambda_i), r_d(\lambda_i))^\top$.
- (d) For every $k = 1, 2, \dots, d$, the polynomial r_k has simple real roots. If \mathcal{M}_k denotes the set of the ordered roots of r_k , then (the points of) \mathcal{M}_d interlaces \mathcal{M} and, for each $k = 1, 2, \dots, d-1$, \mathcal{M}_k interlaces \mathcal{M}_{k+1} .

Moreover, it is worth noting that we can obtain the system $\{r_k\}_{0 \leq k \leq d}$ from \mathbf{R} without the iterative application of the above recurrence.

Lemma 2.2 Let \mathbf{R}_k be the principal submatrix of \mathbf{R} with diagonal entries a_0, a_1, \dots, a_k . Then $r_k = \frac{1}{c_1 c_2 \dots c_k} \phi_{k-1}$ for $k = 1, 2, \dots, d$, where ϕ_{k-1} stands for the characteristic polynomial of \mathbf{R}_{k-1} .

Proof. The result can be proved by induction on k . The first two cases $k = 1$ and $k = 2$ are proved by simple computations. For $k \geq 3$ and expanding along the last column we get:

$$\begin{aligned} \phi_{k-1} &= (x - a_{k-1})\phi_{k-2} + c_{k-1}b_{k-2}\phi_{k-3} \\ &= (x - a_{k-1})c_1 c_2 \dots c_{k-1} r_{k-1} - c_{k-1}b_{k-2}c_1 c_2 \dots c_{k-2} r_{k-2} \\ &= c_1 c_2 \dots c_{k-1} ((x - a_{k-1})r_{k-1} - b_{k-2}r_{k-2}) = c_1 c_2 \dots c_k r_k, \end{aligned}$$

and the result is obtained. \square

In our context, we use the so-called *canonical orthogonal system* $\{p_k\}_{0 \leq k \leq d}$ of polynomials associated to (\mathcal{M}, g) , characterized by the normalization condition $\|p_k\|^2 = p_k(\lambda_0)$, $k = 0, \dots, d$. (This makes sense as, from the theory of orthogonal polynomials, $p_k(\lambda_0) > 0$, see for instance [17].) In fact, the following result holds.

Proposition 2.3 [4] Let p_0, p_1, \dots, p_d be an orthogonal system with respect to the scalar product associated to (\mathcal{M}, g) and recurrence matrix \mathbf{R} . Then, the following statements are equivalent:

- (a) $\|p_k\|^2 = p_k(\lambda_0)$ for any $k = 0, 1, \dots, d$;
- (b) $p_0 + p_1 + \dots + p_d = H = \frac{1}{g(\lambda_0)\pi_0} \prod_{i=1}^d (x - \lambda_i)$;
- (c) The entries of the matrix \mathbf{R} satisfy $a_k + b_k + c_k = \lambda_0$ for any $k = 0, 1, \dots, d$ (where $c_0 = b_d = 0$).

From the above results, we can also give explicit expressions for p_d , as it is shown in the next lemma.

Lemma 2.4 [4] *The highest degree polynomial p_d of the canonical orthogonal system associated with (\mathcal{M}, g) , with $g_i = g(\lambda_i)$, satisfies:*

$$(a) \quad p_d = \left(\sum_{i=0}^d \frac{g_0 \pi_0}{g_i \pi_i^2} \right)^{-1} \sum_{i=0}^d \frac{1}{g_i \pi_i^2} \prod_{j=0, j \neq i}^d (x - \lambda_j);$$

$$(b) \quad p_d(\lambda_0) = \frac{1}{g_0} \left(\sum_{i=0}^d \frac{g_0 \pi_0^2}{g_i \pi_i^2} \right)^{-1} = \frac{\frac{1}{g_0^2 \pi_0^2}}{\sum_{i=0}^d \frac{1}{g_i \pi_i^2}};$$

$$(b) \quad p_d(\lambda_i) = (-1)^i \frac{g_0 \pi_0}{g_i \pi_i} p_d(\lambda_0) \quad (1 \leq i \leq d).$$

In our work we also use the *sum polynomials* $\{q_k\}_{0 \leq k \leq d}$ associated to (\mathcal{M}, g) , which are defined by $q_k = p_0 + p_1 + \dots + p_k$. Note that, from Proposition 2.3(b), $q_d = H$. These polynomials are optimal in the following sense:

Let $\mathcal{S} = \mathcal{S}(\mathcal{M}, g) \subset \mathbb{R}_d[x]$ denotes the sphere with antipodal points 0 and H . That is, \mathcal{S} is the sphere with center $\frac{1}{2}H$ and radius $\frac{1}{2}\|H\|$. Notice that its equation $\|p - \frac{1}{2}H\|^2 = \frac{1}{4}\|H\|^2$ can also be written as $\|p\|^2 = \langle H, p \rangle = p(\lambda_0)$. Consequently,

$$\mathcal{S} = \{p \in \mathbb{R}_d[x] : \|p\|^2 = p(\lambda_0)\} = \{p \in \mathbb{R}_d[x] : \langle H - p, p \rangle = 0\}.$$

Let $\mathcal{S}_k = \mathcal{S}_k(\mathcal{M}, g) = \mathcal{S} \cap \mathbb{R}_k[x]$. Then, we have following optimization result.

Lemma 2.5 [4] *The function $q \mapsto q(\lambda_0)$ attains its maximum in $\mathcal{S}_k(\mathcal{M}, g)$ at q_k .*

Let us now prove some more particular results about orthogonal polynomials suited to our study. Let $\mathcal{M} = \{\lambda_0, \lambda_1, \dots, \lambda_d\}$ be a set as before, with the additional conditions $\lambda_0 + \lambda_d > 0$ and a weight function $g : \mathcal{M} \rightarrow \mathbb{R}$ satisfying

$$\sum_{i=0}^d g_i = 1, \quad \sum_{i=0}^d g_i \lambda_i = 0, \quad \sum_{i=0}^d g_i \lambda_i^2 = \lambda_0, \quad (5)$$

where $g_i = g(\lambda_i)$. Note that this is the case when, with Γ being a nonbipartite regular graph with eigenvalues $\text{ev } \Gamma = \{\lambda_0, \lambda_1, \dots, \lambda_d\}$ and multiplicity function $m : \text{ev } \Gamma \rightarrow \mathbb{R}$, we consider $\mathcal{M} = \text{ev } \Gamma$ and $g = \frac{1}{|\mathbb{V}|}m$.

Let $\{p_k\}_{0 \leq k \leq d}$ be the canonical orthogonal system associated to (\mathcal{M}, g) . The conditions (5) can be expressed, respectively, by $\langle 1, 1 \rangle = 1$, $\langle x, 1 \rangle = 0$ and $\langle x, x \rangle = \lambda_0$, implying, in particular, that $p_0 = 1$ and $p_1 = x$. In this case, the recurrence matrix \mathbf{R} has entries $a_0 = 0$, $b_0 = \lambda_0$ and $c_1 = 1$. If we define $s_k = c_k p_k - b_{k-1} p_{k-1}$, using equation (4) and Proposition 2.3(c) we obtain:

$$\begin{aligned} s_k &= c_k p_k - b_{k-1} p_{k-1} = x p_{k-1} - b_{k-2} p_{k-2} - a_{k-1} p_{k-1} - (\lambda_0 - a_{k-1} - c_{k-1}) p_{k-1} \\ &= (x - \lambda_0) p_{k-1} + c_{k-1} p_{k-1} - b_{k-2} p_{k-2} = (x - \lambda_0) p_{k-1} + s_{k-1}. \end{aligned}$$

Then, since $s_1 = c_1 p_1 - b_0 p_0 = x - \lambda_0$, the above recurrence leads to:

$$s_{k+1} = c_{k+1} p_{k+1} - b_k p_k = (x - \lambda_0) q_k \quad (0 \leq k \leq d). \quad (6)$$

From (\mathcal{M}, g) , consider now the new pair (\mathcal{M}, \tilde{g}) with $\tilde{g}_i = g_i \left(1 + \frac{\lambda_i}{\lambda_0}\right)$.

Lemma 2.6 *Let $p, q \in \mathbb{R}_d[x]$ with $p = (x + \lambda_0)r$. The discrete scalar products associated to (\mathcal{M}, \tilde{g}) and (\mathcal{M}, g) satisfy*

$$\langle r, q \rangle_{\sim} = \frac{1}{\lambda_0} \langle p, q \rangle.$$

Proof. Just note that $\langle r, q \rangle_{\sim} = \sum_{i=0}^d g_i \left(1 + \frac{\lambda_i}{\lambda_0}\right) r(\lambda_i) q(\lambda_i) = \frac{1}{\lambda_0} \langle (\lambda_0 + x)r, q \rangle = \frac{1}{\lambda_0} \langle p, q \rangle$.
□

Proposition 2.7 *The canonical orthogonal system $\{\tilde{p}_k\}_{0 \leq k \leq d}$ associated to (\mathcal{M}, \tilde{g}) is given by*

$$(x + \lambda_0) \tilde{p}_k = -\lambda_0 q_k(-\lambda_0) \left(\frac{1}{p_{k+1}(-\lambda_0)} p_{k+1} - \frac{1}{p_k(-\lambda_0)} p_k \right) \quad (0 \leq k \leq d).$$

Proof. Let $\{p_k\}_{0 \leq k \leq d}$ be the canonical orthogonal system associated to (\mathcal{M}, g) and $p_{d+1} = \prod_{i=0}^d (x - \lambda_i)$. Consider the polynomials $\tilde{r}_0, \tilde{r}_1, \dots, \tilde{r}_d$ defined by

$$(x + \lambda_0) \tilde{r}_k = p_{k+1} - \frac{p_{k+1}(-\lambda_0)}{p_k(-\lambda_0)} p_k.$$

(As in the case of $p_k(\lambda_0)$, the theory of orthogonal polynomials assures that $p_k(-\lambda_0) \neq 0$ for every $k = 0, 1, \dots, d$.) Note that $\tilde{r}_0 = 1$ and \tilde{r}_k has degree k for every k . For $0 \leq h < k \leq d$, Lemma 2.6 gives

$$\langle \tilde{r}_k, \tilde{r}_h \rangle_{\sim} = \frac{1}{\lambda_0} \left\langle p_{k+1} - \frac{p_{k+1}(-\lambda_0)}{p_k(-\lambda_0)} p_k, \tilde{r}_h \right\rangle = 0.$$

Thus, $\tilde{r}_0, \tilde{r}_1, \dots, \tilde{r}_d$ is an orthogonal system associated to (\mathcal{M}, \tilde{g}) . Moreover, its corresponding canonical orthogonal system $\{\tilde{p}_k\}_{0 \leq k \leq d}$ satisfies $\tilde{p}_k = \frac{\tilde{r}_k(\lambda_0)}{\|\tilde{r}_k\|_{\tilde{g}}^2} \tilde{r}_k$. Then, our aim is to compute $\tilde{r}_k(\lambda_0)$ and $\|\tilde{r}_k\|_{\tilde{g}}^2$. Let $b_{-1} = 0$, $p_{-1} = 0$ and define c_{d+1} as the leader coefficient of p_d . Then, we can write $x p_k = b_{k-1} p_{k-1} + a_k p_k + c_{k+1} p_{k+1}$ for every $k = 0, 1, \dots, d$, and

$$\begin{aligned} \tilde{r}_k(\lambda_i) &= \frac{1}{\lambda_0 + \lambda_i} \left(p_{k+1}(\lambda_i) - \frac{p_{k+1}(-\lambda_0)}{p_k(-\lambda_0)} p_k(\lambda_i) \right) \\ &= \frac{\lambda_i p_k(\lambda_i) - b_{k-1} p_{k-1}(\lambda_i) - a_k p_k(\lambda_i) + (\lambda_0 p_k(-\lambda_0) + b_{k-1} p_{k-1}(-\lambda_0) + a_k p_k(-\lambda_0)) \frac{p_k(\lambda_i)}{p_k(-\lambda_0)}}{c_{k+1}(\lambda_0 + \lambda_i)} \\ &= \frac{1}{c_{k+1}} \left(p_k(\lambda_i) + \frac{b_{k-1}}{\lambda_0 + \lambda_i} \frac{p_{k-1}(-\lambda_0)}{p_k(-\lambda_0)} \left(p_k(\lambda_i) - \frac{p_k(-\lambda_0)}{p_{k-1}(-\lambda_0)} p_{k-1}(\lambda_i) \right) \right) \\ &= \frac{1}{c_{k+1}} \left(p_k(\lambda_i) + b_{k-1} \frac{p_{k-1}(-\lambda_0)}{p_k(-\lambda_0)} \tilde{r}_{k-1}(\lambda_i) \right). \end{aligned}$$

Thus, $\tilde{r}_k = \frac{1}{c_{k+1}} \left(p_k + b_{k-1} \frac{p_{k-1}(-\lambda_0)}{p_k(-\lambda_0)} \tilde{r}_{k-1} \right)$ for $k = 0, 1, \dots, d$. Now we can compute:

$$\begin{aligned} \|\tilde{r}_k\|_{\tilde{g}}^2 &= \langle \tilde{r}_k, \tilde{r}_k \rangle_{\tilde{g}} = \frac{1}{\lambda_0} \left\langle p_{k+1} - \frac{p_{k+1}(-\lambda_0)}{p_k(-\lambda_0)} p_k, \tilde{r}_k \right\rangle \\ &= \frac{1}{\lambda_0 c_{k+1}} \left\langle p_{k+1} - \frac{p_{k+1}(-\lambda_0)}{p_k(-\lambda_0)} p_k, p_k + b_{k-1} \frac{p_{k-1}(-\lambda_0)}{p_k(-\lambda_0)} \tilde{r}_{k-1} \right\rangle \\ &= -\frac{1}{\lambda_0 c_{k+1}} \frac{p_{k+1}(-\lambda_0)}{p_k(-\lambda_0)} p_k(\lambda_0). \end{aligned}$$

Using the definition of \tilde{r}_k , we get $\tilde{r}_k(\lambda_0) = \frac{1}{2\lambda_0} \left(p_{k+1}(\lambda_0) - \frac{p_{k+1}(-\lambda_0)}{p_k(-\lambda_0)} p_k(\lambda_0) \right)$, implying

$$\begin{aligned} (x + \lambda_0) \tilde{p}_k &= \frac{\tilde{r}_k(\lambda_0)}{\|\tilde{r}_k\|_{\tilde{g}}^2} (x + \lambda_0) \tilde{r}_k \\ &= \frac{c_{k+1}}{2} \left(1 - \frac{p_{k+1}(\lambda_0) p_k(-\lambda_0)}{p_{k+1}(-\lambda_0) p_k(\lambda_0)} \right) \left(p_{k+1} - \frac{p_{k+1}(-\lambda_0)}{p_k(-\lambda_0)} p_k \right) \\ &= \frac{1}{2} (c_{k+1} p_{k+1}(-\lambda_0) - b_k p_k(-\lambda_0)) \left(\frac{1}{p_{k+1}(-\lambda_0)} p_{k+1} - \frac{1}{p_k(-\lambda_0)} p_k \right), \end{aligned}$$

since $p_k(\lambda_0) b_k = p_{k+1}(\lambda_0) b_{k+1}$ for every $k = 0, 1, \dots, d$. Using (6), we have the equality

$$\frac{c_{k+1} p_{k+1}(-\lambda_0) - b_k p_k(-\lambda_0)}{2} = -\lambda_0 q_k(-\lambda_0),$$

which concludes the proof. \square

From Proposition 2.7 and recalling that $q_d = \frac{1}{g_0 \pi_0} \prod_{i=1}^d (x - \lambda_i)$, we get:

Corollary 2.8 For $k = 0, 1, \dots, d$,

$$\tilde{p}_k(\lambda_0) = \frac{1}{2} \left(\frac{p_k(\lambda_0)}{p_k(-\lambda_0)} - \frac{p_{k+1}(\lambda_0)}{p_{k+1}(-\lambda_0)} \right) q_k(-\lambda_0)$$

and, in particular,

$$\tilde{p}_d(\lambda_0) = \frac{1}{2g_0} \frac{\hat{\pi}_0}{\pi_0} \frac{p_d(\lambda_0)}{(-1)^d p_d(-\lambda_0)},$$

where $\hat{\pi}_0 = \prod_{i=1}^d (\lambda_0 + \lambda_i)$.

2.3 Distance-regularity around a set

Let $\Gamma = (V, E)$ be a graph with adjacency matrix \mathbf{A} , maximum eigenvalue λ_0 and corresponding positive eigenvector $\boldsymbol{\nu}$. Consider the map $\boldsymbol{\rho} : 2^V \rightarrow \mathbb{R}^n$ defined by $\boldsymbol{\rho}(C) = \boldsymbol{\rho}_C = \sum_{u \in C} \nu_u \mathbf{e}_u$ for any nonempty vertex subset C and $\boldsymbol{\rho}_\emptyset = \mathbf{0}$. Consider also the normalized vector $\mathbf{e}_C = \frac{\boldsymbol{\rho}_C}{\|\boldsymbol{\rho}_C\|}$. The C -local multiplicity of λ_i is $m_C(\lambda_i) = \|\mathbf{E}_i \mathbf{e}_C\|^2$ and $\text{ev}_C \Gamma$ denotes the set of C -eigenvalues, that is, those eigenvalues of Γ with nonzero C -multiplicity. Since \mathbf{e}_C is unitary, we have $\sum_{i=0}^d m_C(\lambda_i) = 1$. Moreover, as

$$\mathbf{E}_0 \mathbf{e}_C = \frac{\langle \mathbf{e}_C, \boldsymbol{\nu} \rangle}{\|\boldsymbol{\nu}\|^2} \boldsymbol{\nu} = \frac{1}{\|\boldsymbol{\rho}_C\|} \frac{\langle \boldsymbol{\rho}_C, \boldsymbol{\nu} \rangle}{\|\boldsymbol{\nu}\|^2} \boldsymbol{\nu} = \frac{\|\boldsymbol{\rho}_C\|}{\|\boldsymbol{\nu}\|^2} \boldsymbol{\nu} \quad \Rightarrow \quad m_C(\lambda_0) = \frac{\|\boldsymbol{\rho}_C\|^2}{\|\boldsymbol{\nu}\|^2}, \quad (7)$$

we always have $\lambda_0 \in \text{ev}_C \Gamma$. Let $\text{ev}_C^* \Gamma = \text{ev}_C \Gamma \setminus \{\lambda_0\}$ and $d_C = |\text{ev}_C^* \Gamma|$. In [11] it was shown that then the eccentricity of C satisfies $\varepsilon_C \leq d_C$. When equality is attained, we say that C is an *extremal set*.

Note that, with $\boldsymbol{\rho}_u = \boldsymbol{\rho}_{\{u\}}$, we have $\|\boldsymbol{\rho}_u\| = \nu_u$, so that we can see $\boldsymbol{\rho}$ as a function which assigns weights to the vertices of Γ . In doing so we “regularize” the graph, in the sense that the *average weighted degree* of each vertex $u \in V$ becomes a constant:

$$\delta_u^* = \frac{1}{\nu_u} \sum_{v \in \Gamma(u)} \nu_v = \lambda_0. \quad (8)$$

Using these weights, we consider the following concept: A partition \mathcal{P} of the vertex set $V = V_1 \cup \dots \cup V_m$ is called *pseudo-regular* (or *pseudo-equitable*) whenever the *pseudo-intersection numbers*

$$b_{ij}^*(u) = \frac{1}{\nu_u} \sum_{v \in \Gamma(u) \cap V_j} \nu_v \quad (1 \leq i, j \leq m) \quad (9)$$

do not depend on the chosen vertex $u \in V_i$, but only on the subsets V_i and V_j . In this case, such numbers are simply written as b_{ij}^* , and the $m \times m$ matrix $\mathbf{B}^* = (b_{ij}^*)$ is referred to as the *pseudo-quotient matrix* of \mathbf{A} with respect to the (pseudo-regular) partition \mathcal{P} . Pseudo-regular partitions were introduced by Fiol and Garriga [11], as a generalization of the so-called regular partitions, where the above numbers are defined by $b_{ij}^*(u) = |\Gamma(u) \cap V_j|$ for $u \in V_i$, and they are simply called the *intersection numbers*. A detailed study of regular partitions can be found in Godsil [14] and Godsil and McKay [15]. A graph Γ is said to be *pseudo-distance-regular around* $C \subset V$ or *C -local pseudo-distance-regular* if the *distance-partition* around C , that is $V = C_0 \cup C_1 \cup \dots \cup C_{\varepsilon_C}$ where $C_k = \Gamma_k(C)$ for $k = 0, 1, \dots, \varepsilon_C$, is pseudo-regular. In this case, C is also referred to

as a *completely pseudo-regular code*. From the characteristics of the distance-partition, it is clear that its pseudo-quotient matrix is tridiagonal with nonzero entries $c_k = b_{k-1,k}^*$, $a_k = b_{k,k}^*$ and $b_k = b_{k+1,k}^*$, $0 \leq k \leq \varepsilon_C$, with the convention $c_0 = b_{\varepsilon_C} = 0$. By (8), notice that $a_k + b_k + c_k = \lambda_0$ for $k = 0, 1, \dots, \varepsilon_C$. In particular, when Γ is regular and C consists of a single vertex u , the C -pseudo-distance-regularity coincides with the distance-regularity around u (see Brouwer, Cohen and Neumaier [2]). We refer to the pair $(\text{ev}_C \Gamma, m_C)$, constituted by the set of C -eigenvalues and the normalized weight function m_C defined by the C -local multiplicities, as the C -spectrum of Γ .

We now come back to the results on orthogonal polynomials of the previous subsection, by taking $(\mathcal{M}, g) = (\text{ev}_C \Gamma, m_C)$. Then, since $\mathbf{e}_C = \sum_{i=0}^d \mathbf{E}_i \mathbf{e}_C$ and $p(\mathbf{A})\mathbf{E}_i = p(\lambda_i)\mathbf{E}_i$, the scalar product can be written in the form

$$\langle p, q \rangle_C = \sum_{i=0}^d m_C(\lambda_i) p(\lambda_i) q(\lambda_i) = \langle p(\mathbf{A})\mathbf{e}_C, q(\mathbf{A})\mathbf{e}_C \rangle, \quad p, q \in \mathbb{R}_{d_C}[x]. \quad (10)$$

(Notice that the sum has at most d_C nonzero terms.) In what follows, the corresponding canonical orthogonal system $\{p_k\}_{0 \leq k \leq d_C}$ plays a key role. Its elements are referred to as the *predistance polynomials* of C or C -*predistance polynomials*. From their definition, they satisfy $p_k(\lambda_0) = \|p_k\|_C^2 = \|p_k(\mathbf{A})\mathbf{e}_C\|^2$. Besides, if $\text{ev}_C \Gamma = \{\mu_0 (= \lambda_0), \mu_1, \dots, \mu_{d_C}\}$, Proposition 2.3(b) and Eq. (7) yield

$$p_0 + p_1 + \dots + p_{d_C} = H_C = \frac{\|\boldsymbol{\nu}\|^2}{\|\boldsymbol{\rho}_C\|^2 \pi_0} \prod_{i=0}^{d_C} (x - \mu_i),$$

where $\pi_0 = \prod_{i=1}^{d_C} (\lambda_0 - \mu_i)$. In [11] it was shown that the predistance polynomial p_{d_C} satisfies $p_{d_C}(\lambda_0) \geq \frac{\|\boldsymbol{\rho}_{C_{d_C}}\|^2}{\|\boldsymbol{\rho}_C\|^2}$ and equality holds if and only if $p_{d_C}(\mathbf{A})\boldsymbol{\rho}_C = \boldsymbol{\rho}_{C_{d_C}}$. An equivalent result, stated in terms of the sum polynomial $q_{d_C-1} = p_0 + p_1 + \dots + p_{d_C-1}$, is the following:

Proposition 2.9 [11] *Let C be a vertex subset of a graph, with C -predistance polynomials $\{p_k\}_{0 \leq k \leq d_C}$ and let $q_{d_C-1} = \sum_{i=0}^{d_C-1} p_i$. Then, for any polynomial $r \in \mathbb{R}_{d_C-1}[x]$, we have*

$$\frac{r(\lambda_0)}{\|r\|_C} \leq \frac{\|\boldsymbol{\rho}_{N_{d_C-1}(C)}\|}{\|\boldsymbol{\rho}_C\|}, \quad (11)$$

and equality holds if and only if C is extremal and

$$\frac{1}{\|r\|_C} r(\mathbf{A})\mathbf{e}_C = \mathbf{e}_{N_{d_C-1}(C)}. \quad (12)$$

In this case, $r = \eta q_{d_C-1}$ for any $\eta \in \mathbb{R}$, whence (11) and (12) become, respectively,

$$q_{d_C-1}(\lambda_0) = \frac{\|\boldsymbol{\rho}_{N_{d_C-1}(C)}\|^2}{\|\boldsymbol{\rho}_C\|^2} \quad \text{and} \quad q_{d_C-1}(\mathbf{A})\boldsymbol{\rho}_C = \boldsymbol{\rho}_{N_{d_C-1}(C)}. \quad (13)$$

As a consequence, the following characterizations of pseudo-distance-regularity around a set were also proved by the authors.

Theorem 2.10 [11] *Let $\Gamma = (V, E)$ be a connected graph and let $C \subset V$ be a nonempty vertex subset with eccentricity ε_C and antipodal set $\overline{C} = \Gamma_{\varepsilon_C}(C)$. Then, the following assertions are equivalent:*

- (a) Γ is C -local pseudo-distance-regular.
- (b) There exist polynomials $\{r_k\}_{0 \leq k \leq \varepsilon_C}$ with $\text{dgr } r_k = k$ such that $\boldsymbol{\rho}_{C_k} = r_k(\mathbf{A})\boldsymbol{\rho}_C$.
- (c) C is extremal and there exists a polynomial $r \in \mathbb{R}_{\varepsilon_C}[x]$ such that $\boldsymbol{\rho}_{\overline{C}} = r(\mathbf{A})\boldsymbol{\rho}_C$.
- (d) The predistance polynomial of C with maximum degree satisfies $p_{d_C}(\lambda_0) = \frac{\|\boldsymbol{\rho}_{C_{d_C}}\|^2}{\|\boldsymbol{\rho}_C\|^2}$.

3 Edge-spectrum-regularity

Let $\Gamma = (V, E)$ be a graph with spectrum $\text{sp}\Gamma = (\text{ev}\Gamma, m)$, where $\text{ev}\Gamma$ is the set of different eigenvalues of Γ and $m : \text{ev}\Gamma \rightarrow \mathbb{N}$ the multiplicity function. Let λ_0 be the maximum eigenvalue and $\boldsymbol{\nu}$ the unique vector of its eigenspace with minimum component equal to one. We write $\text{ev}^*\Gamma = \text{ev}\Gamma \setminus \{\lambda_0\}$ and $d = |\text{ev}^*\Gamma|$.

Formally, we do not distinguish between an edge $e \in E$ with vertices u, v and the set $\{u, v\}$. Thus, we denote the (local) e -multiplicities of Γ as $m_e(\lambda_i) = \|\mathbf{E}_i \mathbf{e}_e\|^2$, $i = 0, 1, \dots, d$, where

$$\mathbf{e}_e = \frac{\boldsymbol{\rho}_e}{\|\boldsymbol{\rho}_e\|} = \frac{\nu_u \mathbf{e}_u + \nu_v \mathbf{e}_v}{\sqrt{\nu_u^2 + \nu_v^2}}.$$

From this, note that the relationship between the e -multiplicity and the local and crossed multiplicities of u and v is:

$$\begin{aligned} m_e(\lambda_i) &= \|\mathbf{E}_i \mathbf{e}_e\|^2 = \frac{1}{\nu_u^2 + \nu_v^2} \langle \mathbf{E}_i (\nu_u \mathbf{e}_u + \nu_v \mathbf{e}_v), \nu_u \mathbf{e}_u + \nu_v \mathbf{e}_v \rangle \\ &= \frac{1}{\nu_u^2 + \nu_v^2} (\nu_u^2 (\mathbf{E}_i)_{uu} + 2\nu_u \nu_v (\mathbf{E}_i)_{uv} + \nu_v^2 (\mathbf{E}_i)_{vv}) \\ &= \frac{1}{\nu_u^2 + \nu_v^2} (\nu_u^2 m_u(\lambda_i) + 2\nu_u \nu_v m_{uv}(\lambda_i) + \nu_v^2 m_v(\lambda_i)), \end{aligned} \quad (14)$$

and, if Γ is regular,

$$m_e(\lambda_i) = \frac{1}{2} (m_u(\lambda_i) + m_v(\lambda_i)) + m_{uv}(\lambda_i). \quad (15)$$

For a general graph, Eq. (7) yields that the e -multiplicity of λ_0 is

$$m_e(\lambda_0) = \frac{\|\boldsymbol{\rho}_e\|^2}{\|\boldsymbol{\nu}\|^2} = \frac{\nu_u^2 + \nu_v^2}{\|\boldsymbol{\nu}\|^2}. \quad (16)$$

If $|\text{ev}_e \Gamma| = d_e + 1$, the eccentricity of e , seen as a set of two vertices, satisfies $\varepsilon_e \leq d_e$. We define the *edge-diameter* of Γ by $\tilde{D} = \max_{e \in E} \varepsilon_e$. Notice that \tilde{D} coincides with the diameter of the line graph $L\Gamma$. Consequently, we have the following well-known result.

Lemma 3.1 *Let Γ be a connected graph with diameter D and edge-diameter \tilde{D} . Then, $D - 1 \leq \tilde{D} \leq D$ and, if Γ is bipartite, $\tilde{D} = D - 1$.*

We remark that $\tilde{D} = D - 1$ does not imply that Γ is bipartite. An example of this fact is any nonbipartite (connected) graph with pendant edges attached to some of its vertices.

Lemma 3.2 *The e -multiplicities of a graph $\Gamma = (V, E)$ with spectrum $\text{sp } \Gamma$ satisfy the following properties:*

$$(a) \sum_{i=0}^d m_e(\lambda_i) = 1 \text{ for every } e \in E.$$

$$(b) \text{ If } \Gamma \text{ is regular, then } \sum_{e \in E} m_e(\lambda_i) = \frac{\lambda_0 + \lambda_i}{2} m(\lambda_i) \text{ for every } \lambda_i \in \text{ev } \Gamma.$$

Proof. The first statement is known for a general vertex subset. In order to prove the second one, we have:

$$\begin{aligned} \sum_{e \in E} m_e(\lambda_i) &= \sum_{e \in E} \langle \mathbf{E}_i \mathbf{e}_e, \mathbf{e}_e \rangle = \frac{1}{2} \sum_{u \sim v} \left\langle \mathbf{E}_i \frac{1}{\sqrt{2}} (\mathbf{e}_u + \mathbf{e}_v), \frac{1}{\sqrt{2}} (\mathbf{e}_u + \mathbf{e}_v) \right\rangle \\ &= \frac{1}{4} \sum_u \sum_{v \sim u} (\mathbf{E}_i)_{uu} + \frac{1}{4} \sum_v \sum_{u \sim v} (\mathbf{E}_i)_{vv} + \frac{1}{2} \left\langle \sum_u \mathbf{E}_i \mathbf{e}_u, \sum_{v \sim u} \mathbf{e}_v \right\rangle \\ &= \frac{1}{4} \lambda_0 \sum_u m_u(\lambda_i) + \frac{1}{4} \lambda_0 \sum_v m_v(\lambda_i) + \frac{1}{2} \sum_u \langle \mathbf{E}_i \mathbf{e}_u, \mathbf{A} \mathbf{e}_u \rangle \\ &= \frac{1}{2} \lambda_0 m(\lambda_i) + \frac{1}{2} \sum_u \langle \mathbf{A} \mathbf{E}_i \mathbf{e}_u, \mathbf{e}_u \rangle = \frac{1}{2} \lambda_0 m(\lambda_i) + \frac{1}{2} \text{tr}(\mathbf{A} \mathbf{E}_i) \\ &= \frac{1}{2} \lambda_0 m(\lambda_i) + \frac{1}{2} m(\lambda_i) \lambda_i = \frac{\lambda_0 + \lambda_i}{2} m(\lambda_i), \end{aligned}$$

where we used that $\mathbf{A} \mathbf{E}_i = \lambda_i \mathbf{E}_i$ and $\text{tr } \mathbf{E}_i = m(\lambda_i)$. \square

For every eigenvalue $\lambda_i \in \text{ev } \Gamma$, the *mean vertex-multiplicity* and *mean edge-multiplicity* are, respectively,

$$g(\lambda_i) = \frac{1}{|V|} \sum_{u \in V} m_u(\lambda_i), \quad \tilde{g}(\lambda_i) = \frac{1}{|E|} \sum_{e \in E} m_e(\lambda_i).$$

Since $\sum_{u \in V} m_u(\lambda_i) = m(\lambda_i)$, we have $g(\lambda_i) = \frac{m(\lambda_i)}{|V|}$. Moreover, if Γ is regular, Lemma 3.2 gives

$$\tilde{g}(\lambda_i) = \frac{1}{|E|} \frac{\lambda_0 + \lambda_i}{2} m(\lambda_i) = \left(1 + \frac{\lambda_i}{\lambda_0}\right) \frac{m(\lambda_i)}{|V|} = \left(1 + \frac{\lambda_i}{\lambda_0}\right) g(\lambda_i). \quad (17)$$

Inspired by the concept of (vertex) spectrum-regularity, we say that Γ is *edge-spectrum-regular* if, for every $\lambda_i \in \text{ev } \Gamma$, the edge-multiplicity $m_e(\lambda_i)$ does not depend on $e \in E$. Whereas spectrum-regularity implies regularity, in the case of edge-spectrum-regularity we have the following result.

Proposition 3.3 *Let Γ be a connected edge-spectrum-regular graph. Then, Γ is either regular or bipartite biregular.*

Proof. First, let u and v be two adjacent vertices and $e = \{u, v\} \in E$. From (16), we have that $\nu_u^2 + \nu_v^2 = \|\boldsymbol{\nu}\|^2 m_e(\lambda_0) = a$ is a constant over all the edges. Now, let u and v be two vertices that can be joined by a walk of even length, that is, $u = w_0 \sim w_1 \sim \dots \sim w_{2p-1} \sim w_{2p} = v$. Then, $(-1)^k \nu_{w_k}^2 + (-1)^k \nu_{w_{k+1}}^2 = (-1)^k a$. By adding up for $k = 0, 1, \dots, 2p - 1$, we get $\nu_u^2 - \nu_v^2 = 0$ and, since all the entries of $\boldsymbol{\nu}$ are positive, $\nu_u = \nu_v$. Thus, if we consider the equivalence relation in V defined by $u \simeq v$, if and only if there exists a path of even length joining u and v , the map $u \mapsto \nu_u$ is constant over every equivalence class. Let $[u]$ and $[v]$ be two different equivalence classes. The existence of a vertex w not belonging to any of them would imply the existence of a path of even length from u to v . Hence, the quotient set has at most two classes. If there exists only one class, $\boldsymbol{\nu} = \mathbf{j}$ and the graph is regular. On the other hand, if there are two classes V_i , $i = 1, 2$, any two vertices in the same class cannot be adjacent (odd distance) and, hence, Γ is bipartite with independent sets V_1 and V_2 . Let α_i be the value of the components of $\boldsymbol{\nu}$ corresponding to vertices of V_i , $i = 1, 2$. If $u \in V_1$, then

$$\delta_u \alpha_2 = \langle \mathbf{A} \mathbf{e}_u, \boldsymbol{\nu} \rangle = \langle \mathbf{e}_u, \mathbf{A} \boldsymbol{\nu} \rangle = \lambda_0 \langle \mathbf{e}_u, \boldsymbol{\nu} \rangle = \lambda_0 \alpha_1,$$

giving $\delta_u = \frac{\alpha_1}{\alpha_2} \lambda_0$. A similar argument leads to $\delta_v = \frac{\alpha_2}{\alpha_1} \lambda_0$ for all $v \in V_2$. \square

We say that a graph Γ is *bispectrum-regular* when it is both spectrum-regular and edge-spectrum-regular. This is the case, for instance, when Γ is distance-regular. More generally, we have the following result.

Proposition 3.4 *A graph Γ is bispectrum-regular if and only if it is 1-walk-regular.*

Proof. Suppose that Γ is bispectrum-regular. In particular, the graph is regular and, from (17), $\tilde{g}(\lambda_i) = \left(1 + \frac{\lambda_i}{\lambda_0}\right) g(\lambda_i)$. Since the number of k -walks rooted at vertex u is $a_{uu}^{(k)} = \sum_{i=0}^d g(\lambda_i) \lambda^k$, this number is independent of such a vertex. Besides, for every edge

$$e = \{u, v\},$$

$$\begin{aligned} a_{uv}^{(k)} &= \langle \mathbf{A}^k \mathbf{e}_u, \mathbf{e}_v \rangle = \frac{1}{2} \langle \mathbf{A}^k (\mathbf{e}_u + \mathbf{e}_v), \mathbf{e}_u + \mathbf{e}_v \rangle - \frac{1}{2} \langle \mathbf{A}^k \mathbf{e}_u, \mathbf{e}_u \rangle - \frac{1}{2} \langle \mathbf{A}^k \mathbf{e}_v, \mathbf{e}_v \rangle \\ &= \langle \mathbf{A}^k \mathbf{e}_e, \mathbf{e}_e \rangle - a_{uu}^{(k)} = \sum_{i=0}^d \tilde{g}(\lambda_i) \lambda_i^k - \sum_{i=0}^d g(\lambda_i) \lambda_i^k = \sum_{i=0}^d g(\lambda_i) \frac{\lambda_i}{\lambda_0} \lambda_i^k \\ &= \frac{1}{\lambda_0} a_{uu}^{(k+1)} \end{aligned} \tag{18}$$

is constant. Hence, Γ is 1-walk-regular.

Conversely, let $u \in V$ and $e = \{u, v\} \in E$. Since $\langle \mathbf{A}^k \mathbf{e}_u, \mathbf{e}_u \rangle$ and $\langle \mathbf{A}^k \mathbf{e}_v, \mathbf{e}_v \rangle$ are constant over the vertices and edges, for every polynomial $p \in \mathbb{R}[x]$ the values of $\langle p(\mathbf{A}) \mathbf{e}_u, \mathbf{e}_u \rangle$ and $\langle p(\mathbf{A}) \mathbf{e}_v, \mathbf{e}_v \rangle$ do not depend on $u \in V$ and the chosen edge $\{u, v\} \in E$. From

$$\langle p(\mathbf{A}) \mathbf{e}_e, \mathbf{e}_e \rangle = \frac{1}{2} \langle p(\mathbf{A}) \mathbf{e}_u, \mathbf{e}_u \rangle + \frac{1}{2} \langle p(\mathbf{A}) \mathbf{e}_v, \mathbf{e}_v \rangle + \langle p(\mathbf{A}) \mathbf{e}_u, \mathbf{e}_v \rangle,$$

we get that $\langle p(\mathbf{A}) \mathbf{e}_e, \mathbf{e}_e \rangle$ is also constant over the edges. In particular, using the interpolating Lagrange polynomial λ_i^* giving $\mathbf{E}_i = \lambda_i^*(\mathbf{A})$, we get that $m_u(\lambda_i) = \langle \mathbf{E}_i \mathbf{e}_u, \mathbf{e}_u \rangle$ and $m_e(\lambda_i) = \langle \mathbf{E}_i \mathbf{e}_e, \mathbf{e}_e \rangle$ are independent of u and e , respectively. Hence, Γ is bispectrum-regular. \square

4 Edge-distance-regular graphs

Given a graph $\Gamma = (V, E)$ and an edge $e \in E$, we consider the partition of V induced by the distance from e , that is $V = e_0 \cup e_1 \cup \dots \cup e_{\varepsilon_e}$, where $e_k = \Gamma_k(e)$. We say that Γ is *e-local pseudo-distance-regular* if this partition is pseudo-regular. Note that all the characterizations in subsection 2.3 for completely pseudo-regular codes apply on this context. One of the advantages of considering edges is that we can see the graph from a global point of view, that is, from every edge, in the same way as we get distance-regularity by seeing the graph from every vertex.

Definition 4.1 *A graph Γ is edge-distance-regular when it is e-local pseudo-distance-regular with pseudo-intersection numbers not depending on $e \in E$.*

Examples of edge-distance-regular graphs are the odd graphs O_k (see [1]), which are distance-regular with degree k . As an example, Figure 1 shows O_4 with the intersection numbers of the distance partition induced by an edge. In an edge-distance-regular graph every edge e constitutes an extremal set with eccentricity $\varepsilon_e = \tilde{D}$, which coincides with $\tilde{d} = d_e = |\text{ev}_e \Gamma|$. Since the e -predistance polynomials are determined by the intersection matrix, there exist polynomials $\{\tilde{p}_k\}_{0 \leq k \leq \tilde{d}}$ satisfying $\tilde{p}_k(\mathbf{A}) \boldsymbol{\rho}_e = \boldsymbol{\rho}_{e_k}$ for every $e \in E$. We refer to this family of polynomials as the *edge-distance polynomials*. Since the e -multiplicities can be obtained from the recurrence matrix (see Fiol and Garriga [11]),

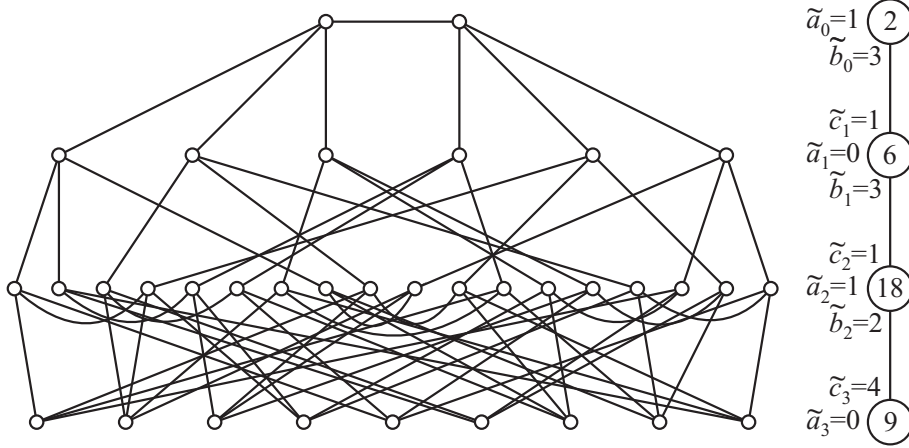


Figure 1: The odd graph O_4 and its intersection diagram induced by an edge.

the graph is also edge-spectrum-regular. Thus an edge-distance-regular graph is either regular or bipartite biregular. Assume that Γ is bipartite biregular with degrees δ_1 and δ_2 and independent sets V_1 and V_2 . If α_1 and α_2 are the values of the components of ν corresponding to vertices of V_1 and V_2 respectively, we have:

$$\frac{\delta_1 \alpha_2}{\alpha_1} = \frac{\delta_2 \alpha_1}{\alpha_2} = \lambda_0 \quad \text{and} \quad \frac{(\delta_1 - 1)\alpha_2}{\alpha_1} = \frac{(\delta_2 - 1)\alpha_1}{\alpha_2},$$

where the first equation was obtained in the proof of Proposition 3.3, using that Γ is edge-spectrum regular, and the second one follows from the definition of edge-distance-regularity (that is, the pseudo-regularity of the partition induced by any edge). By using both equalities we obtain $\delta_1 = \delta_2$, proving that the graph is regular.

Using (17), if Γ is edge-spectrum-regular, we have $m_e(\lambda_i) = \frac{1}{|E|} \frac{\lambda_0 + \lambda_i}{2} m(\lambda_i)$ for every $\lambda_i \in \text{ev } \Gamma$. In particular, if Γ is nonbipartite, then $-\lambda_0 \notin \text{ev } \Gamma$ and $\tilde{D} + 1 = |\text{ev}_e \Gamma| = |\text{ev } \Gamma| \geq D + 1$, giving that $\tilde{D} = D$, by Lemma 3.1. Moreover, if Γ is edge-distance-regular and bipartite, we get $|\text{ev } \Gamma| - 1 = |\text{ev}_e \Gamma| = \tilde{D} + 1 = D$ from the same lemma. The above reasonings are summarized in the following proposition.

Proposition 4.2 *Let Γ be an edge-distance-regular graph with diameter D and $d + 1$ distinct eigenvalues. Then,*

- (a) Γ is regular.
- (b) Γ has spectrally maximum diameter ($D = d$) and its edge-diameter satisfies:
 - (b1) If Γ is nonbipartite, then $\tilde{D} = D$.
 - (b2) If Γ is bipartite, then $\tilde{D} = D - 1$.

(c) Γ is edge-spectrum regular and, for every $e \in E$, the e -spectrum satisfies:

(c1) If Γ is nonbipartite, then $\text{ev}_e \Gamma = \text{ev} \Gamma$ and $m_e(\lambda_i) = \left(1 + \frac{\lambda_i}{\lambda_0}\right) \frac{m(\lambda_i)}{|V|}$, $\lambda_i \in \text{ev} \Gamma$.

(c2) If Γ is bipartite, then $\text{ev}_e \Gamma = \text{ev} \Gamma \setminus \{-\lambda_0\}$ and $m_e(\lambda_i) = \left(1 + \frac{\lambda_i}{\lambda_0}\right) \frac{m(\lambda_i)}{|V|}$, $\lambda_i \in \text{ev} \Gamma \setminus \{-\lambda_0\}$.

Definition 4.3 The k -incidence matrix of $\Gamma = (V, E)$ is the $(|V| \times |E|)$ -matrix $\mathbf{B}_k = (b_{ue})$ with entries $b_{ue} = 1$ if $\partial(u, e) = k$, and $b_{ue} = 0$ otherwise.

Theorem 4.4 A regular graph Γ with edge-diameter \tilde{D} is edge-distance-regular if and only if, for every $k = 0, 1, \dots, \tilde{D}$, there exists a polynomial \tilde{p}_k of degree k such that $\tilde{p}_k(\mathbf{A})\mathbf{B}_0 = \mathbf{B}_k$.

Proof. If Γ is edge-distance-regular, the edge-predistance polynomials satisfy $\tilde{p}_k(\mathbf{A})\boldsymbol{\rho}_e = \boldsymbol{\rho}_{e_k}$ for every edge $e \in E$. By arranging these equalities in columns, we get the claimed condition and the necessity is proved.

To prove sufficiency, note that the polynomial $T = \sum_{k=0}^{\tilde{D}} \tilde{p}_k$ satisfies $T(\mathbf{A})\mathbf{B}_0 = \mathbf{J}$, where \mathbf{J} stands for the all-1 $(|V| \times |E|)$ -matrix. Let e be an edge with eccentricity \tilde{D} . The scalar product

$$\langle p, q \rangle_e = \langle p\mathbf{e}_e, q\mathbf{e}_e \rangle = \sum_{i=0}^d m_e(\lambda_i) p(\lambda_i) q(\lambda_i) \quad (p, q \in \mathbb{R}_{d_e}[x])$$

satisfies

$$\langle \tilde{p}_s, \tilde{p}_k \rangle_e = \langle \tilde{p}_s(\mathbf{A})\mathbf{e}_e, \tilde{p}_k(\mathbf{A})\mathbf{e}_e \rangle = \frac{1}{2} \langle \tilde{p}_s(\mathbf{A})\boldsymbol{\rho}_e, \tilde{p}_k(\mathbf{A})\boldsymbol{\rho}_e \rangle = \frac{1}{2} \langle \boldsymbol{\rho}_{e_s}, \boldsymbol{\rho}_{e_k} \rangle = 0 \quad \text{for } s \neq k$$

and

$$\langle \tilde{p}_k, \tilde{p}_k \rangle_e = \frac{1}{2} \|\boldsymbol{\rho}_{e_k}\|^2 = \frac{|e_k|}{2} = \tilde{p}_k(\lambda_0),$$

where we used that $\boldsymbol{\rho}_e = \frac{2}{|V|}\mathbf{j} + \mathbf{z}_e$, $\boldsymbol{\rho}_{e_k} = \frac{|e_k|}{|V|}\mathbf{j} + \mathbf{z}_{e_k}$, with $\mathbf{z}_e, \mathbf{z}_{e_k} \in \mathbf{j}^\perp$, and $\tilde{p}_k(\mathbf{A})\boldsymbol{\rho}_e = \boldsymbol{\rho}_{e_k}$, implying that $\tilde{p}_k(\lambda_0) = \frac{|e_k|}{2}$. Then, $\tilde{p}_0, \tilde{p}_1, \dots, \tilde{p}_{\tilde{D}}$ are the first $\tilde{D} + 1$ polynomials of the canonical orthogonal system with respect to the scalar product $\langle \cdot, \cdot \rangle_e$. In particular, $\tilde{p}_{\tilde{D}}(\lambda_0) \neq 0$. Now, let $e \in E$ be an arbitrary edge. From $T(\mathbf{A})\boldsymbol{\rho}_e = \mathbf{j}$, we know that $\varepsilon_e \leq d_e \leq \tilde{D}$. Moreover,

$$\boldsymbol{\rho}_{e_{\tilde{D}}} = \tilde{p}_{\tilde{D}}(\mathbf{A})\boldsymbol{\rho}_e = \tilde{p}_{\tilde{D}}(\mathbf{A}) \left(\frac{2}{|V|}\mathbf{j} + \mathbf{z}_e \right) = \frac{2}{|V|}\tilde{p}_{\tilde{D}}(\lambda_0)\mathbf{j} + \tilde{p}_{\tilde{D}}(\mathbf{A})\mathbf{z}_e \neq 0.$$

Thus, $\tilde{D} \leq \varepsilon_e \leq d_e \leq \tilde{D}$. That is, all the edges have the same eccentricity $\tilde{D} = d_e$ and the polynomials $\tilde{p}_0, \tilde{p}_1, \dots, \tilde{p}_{\tilde{D}}$ constitute the canonical orthogonal system with respect to the scalar product $\langle \cdot, \cdot \rangle_e$ for every $e \in E$. Since $\boldsymbol{\rho}_{e_k} = \tilde{p}_k(\mathbf{A})\boldsymbol{\rho}_e$ for $k = 0, 1, \dots, \tilde{D}$, the distance

partition with respect to e is pseudo-regular for every edge, and since the edge-distance polynomials are the same for all the edges, all the recurrence matrices also coincide. \square

Let $\text{ev}_E \Gamma = \bigcup_{e \in E} \text{ev}_e \Gamma$ and denote by $\text{ev}_E^* \Gamma = \text{ev}_E \Gamma \setminus \{\lambda_0\}$ and $\tilde{d} = |\text{ev}_E^* \Gamma|$. Within this notation, if Γ is edge-distance-regular Proposition 4.2 establishes that, for every $e \in E$, $\text{ev}_E \Gamma = \text{ev}_e \Gamma = \text{ev} \Gamma$ if Γ is nonbipartite, and $\text{ev}_E \Gamma = \text{ev}_e \Gamma = \text{ev} \Gamma \setminus \{\lambda_0\}$ if Γ is bipartite. For a general graph, notice that the scalar product $\langle \cdot, \cdot \rangle_{\sim}$ associated to $(\text{ev}_E \Gamma, \tilde{g})$ is the average, over all edges, of the local scalar products in $(\text{ev}_e \Gamma, m_e)$ for every $e \in E$:

$$\langle p, q \rangle_{\sim} = \frac{1}{|E|} \sum_{e \in E} \langle p, q \rangle_e. \quad (19)$$

Consider the canonical orthogonal system $\{\tilde{p}_k\}_{0 \leq k \leq \tilde{d}}$ associated to $(\text{ev}_E \Gamma, \tilde{g})$, where now $\tilde{g}(\lambda_i) = m_e(\lambda_i)$ for every $e \in E$, and their sum polynomials $\{\tilde{q}_k\}_{0 \leq k \leq \tilde{d}}$. Then,

Lemma 4.5 *Let Γ be an edge-distance-regular graph. Then, for every $e \in E$,*

$$\tilde{p}_{\tilde{d}}(\lambda_0) = \frac{1}{2} |\Gamma_{\tilde{d}}(e)| \quad \text{and} \quad \tilde{q}_{\tilde{d}-1}(\lambda_0) = \frac{1}{2} |N_{\tilde{d}-1}(e)|.$$

Proof. The first equality follows from the regularity of Γ and the characterization in Theorem 2.10(d). Moreover, from Eq. (17), $\tilde{g}(\lambda_0) = \frac{2}{|V|}$. Thus, Proposition 2.3(b) establishes that $\tilde{q}_{\tilde{d}}(\lambda_0) = \frac{1}{\tilde{g}(\lambda_0)} = \frac{|V|}{2}$, giving

$$\tilde{q}_{\tilde{d}-1}(\lambda_0) = (\tilde{q}_{\tilde{d}} - \tilde{p}_{\tilde{d}})(\lambda_0) = \frac{1}{2} |N_{\tilde{d}-1}(e)|, \quad (20)$$

as claimed. \square

Godsil and Shawe-Taylor [16] defined a distance-regularised graph as that being distance-regular around each of its vertices. (These graphs are a common generalisation of distance-regular graphs and generalised polygons.) Such authors showed that distance-regularised graphs are either distance-regular or distance-biregular. Inspired by this, we introduced the following concept.

Definition 4.6 *A regular graph Γ is said to be edge-distance-regularised when it is edge-distance-regular around each of its edges.*

The next result gives an almost spectral characterization of edge-distance-regularised graphs in terms of the harmonic mean of the numbers $|N_{\tilde{d}-1}(e)|$ and the sum polynomial $\tilde{q}_{\tilde{d}-1}$.

Theorem 4.7 *Let $\Gamma = (V, E)$ be a regular graph with $\tilde{d} = |\text{ev}_E \Gamma|$. Let $H_{\tilde{d}-1}$ be the harmonic mean of the numbers $|N_{\tilde{d}-1}(e)|$ for $e \in E$. Then, Γ is edge-distance-regularised if and only if*

$$H_{\tilde{d}-1} = 2\tilde{q}_{\tilde{d}-1}(\lambda_0).$$

Proof. Lemma 4.5 establishes the necessity of the condition. Conversely, from the regularity of Γ , $C = e$ and $r = \tilde{q}_{\tilde{d}-1}$, we have that the inequality (11) of Proposition 2.9 reads $\frac{\tilde{q}_{\tilde{d}-1}(\lambda_0)}{\|\tilde{q}_{\tilde{d}-1}\|_e} \leq \frac{\sqrt{|N_{\tilde{d}-1}(e)|}}{\sqrt{2}}$, or

$$|N_{\tilde{d}-1}(e)|^{-1} \leq \frac{1}{2} \frac{\|\tilde{q}_{\tilde{d}-1}\|_e^2}{\tilde{q}_{\tilde{d}-1}^2(\lambda_0)} \quad \text{for every } e \in E. \quad (21)$$

Thus, by taking the arithmetic mean on E , we have

$$\frac{1}{|E|} \sum_{e \in E} |N_{\tilde{d}-1}(e)|^{-1} \leq \frac{1}{2\tilde{q}_{\tilde{d}-1}^2(\lambda_0)} \frac{1}{|E|} \sum_{e \in E} \|\tilde{q}_{\tilde{d}-1}\|_e^2 = \frac{\|\tilde{q}_{\tilde{d}-1}\|_E^2}{2\tilde{q}_{\tilde{d}-1}^2(\lambda_0)} = \frac{1}{2\tilde{q}_{\tilde{d}-1}(\lambda_0)},$$

where we used (19). Consequently,

$$2\tilde{q}_{\tilde{d}-1}(\lambda_0) \leq \frac{|E|}{\sum_{e \in E} |N_{\tilde{d}-1}(e)|^{-1}} = H_{\tilde{d}-1} \quad (22)$$

and the equality can only hold if and only if all the inequalities in (21) are also equalities. Hence, by Proposition 2.9, every edge is extremal and $\text{ev}_e \Gamma = \text{ev}_E \Gamma$. Consequently, by Theorem 2.10, Γ is distance-regular around each of its edges. \square

As a consequence, we can give a similar result in terms of the average of numbers of vertices at maximum distance from e , $|e_{\tilde{d}}| = |\Gamma_{\tilde{d}}(e)|$, and the highest degree polynomial $\tilde{p}_{\tilde{d}}$.

Corollary 4.8 *Let $\Gamma = (V, E)$ be a regular graph with $\tilde{d} = |\text{ev}_E \Gamma|$. Let $M_{\tilde{d}}$ be the (arithmetic) mean of the numbers $|e_{\tilde{d}}|$ for $e \in E$. Then, Γ is edge-distance-regularised if and only if*

$$M_{\tilde{d}} = 2\tilde{p}_{\tilde{d}}(\lambda_0).$$

Proof. The necessity follows again from Lemma 4.5. To prove sufficiency, note that, as $\tilde{q}_{\tilde{d}-1} = \tilde{q}_{\tilde{d}} - \tilde{p}_{\tilde{d}}$, inequation (22) gives

$$\begin{aligned} 2\tilde{p}_{\tilde{d}}(\lambda_0) &\geq 2\tilde{q}_{\tilde{d}}(\lambda_0) - H_{\tilde{d}-1} = |V| - \frac{|E|}{\sum_{e \in E} (|V| - |e_{\tilde{d}}|)^{-1}} \\ &\geq |V| - \frac{\sum_{e \in E} (|V| - |e_{\tilde{d}}|)}{|E|} = \frac{1}{|E|} \sum_{e \in E} |e_{\tilde{d}}| = M_{\tilde{d}}, \end{aligned}$$

where we used that the harmonic mean is always smaller than or equal to the arithmetic mean. Thus, in case of equality, Theorem 4.7 applies and we complete the proof. \square

As another consequence of Theorem 4.7, we also have an almost spectral characterization of edge-distance-regularity.

Theorem 4.9 *A regular graph $\Gamma = (V, E)$ with $\tilde{d} = |\text{ev}_E \Gamma|$ is edge-distance-regular if and only if, for every edge $e \in E$,*

$$|e_{\tilde{d}}| = 2\tilde{p}_{\tilde{d}}(\lambda_0).$$

Proof. Again, we only need to prove sufficiency. Since the hypothesis of Theorem 4.7 holds, we have already seen that all the inequalities in (21) are equalities and that every edge is extremal. Moreover, by Proposition 2.9, there exist constants $\eta_e \in \mathbb{R}$ such that $q_e = \eta_e \tilde{q}_{\tilde{d}-1} \in \mathcal{S}_{\tilde{d}-1}(\text{ev}_E \Gamma, m_e) = \mathcal{S}_{\tilde{d}-1}(\text{ev}_E \Gamma, m_e)$ for every edge $e \in E$. Also, since $|e_{\tilde{d}}| = 2\tilde{p}_{\tilde{d}}(\lambda_0)$, Eq. (20) holds and, together with Proposition 2.9, we have

$$\eta_e \frac{1}{2} |N_{\tilde{d}-1}(e)| = \eta_e \tilde{q}_{\tilde{d}-1}(\lambda_0) = q_e(\lambda_0) \leq \frac{1}{2} |N_{\tilde{d}-1}(e)|,$$

thus $\eta_e \leq 1$, or, equivalently, for every edge $e \in E$ there exists $\xi_e \geq 1$ such that $\tilde{q}_{\tilde{d}-1} = \xi_e q_e$ with $q_e \in \mathcal{S}_{\tilde{d}-1}(\text{ev}_E \Gamma, m_e)$. Consider the norms $\|\cdot\|_{\sim}$ associated to $(\text{ev}_E \Gamma, \tilde{g})$ and $\|\cdot\|_e$ associated to $(\text{ev}_E \Gamma, m_e)$. From

$$\begin{aligned} \tilde{q}_{\tilde{d}-1}(\lambda_0) &= \|\tilde{q}_{\tilde{d}-1}\|_{\sim}^2 = \sum_{i=0}^d \tilde{g}(\lambda_i) (\tilde{q}_{\tilde{d}-1}(\lambda_i))^2 = \frac{1}{|E|} \sum_{i=0}^d \left[\sum_{e \in E} m_e(\lambda_i) \right] (\tilde{q}_{\tilde{d}-1}(\lambda_i))^2 \\ &= \frac{1}{|E|} \sum_{e \in E} \xi_e^2 \sum_{i=0}^d m_e(\lambda_i) q_e^2(\lambda_i) = \frac{1}{|E|} \sum_{e \in E} \xi_e^2 \|q_e\|_e^2 = \frac{1}{|E|} \sum_{e \in E} \xi_e^2 q_e(\lambda_0) \\ &= \frac{1}{|E|} \sum_{e \in E} \xi_e \tilde{q}_{\tilde{d}-1}(\lambda_0) = \left[\frac{1}{|E|} \sum_{e \in E} \xi_e \right] \tilde{q}_{\tilde{d}-1}(\lambda_0) \geq \tilde{q}_{\tilde{d}-1}(\lambda_0), \end{aligned}$$

we get that $\xi_e = 1$ for every $e \in E$. Consequently, $q_e = \tilde{q}_{\tilde{d}-1}$ and $q_e(\lambda_0) = \frac{1}{2} |N_{\tilde{d}-1}(e)|$ for every edge $e \in E$. Then, Proposition 2.9 gives $\tilde{q}_{\tilde{d}-1}(\mathbf{A})\boldsymbol{\rho}_e = \boldsymbol{\rho}_{e_{\tilde{d}-1}}$ or, equivalently, $\tilde{p}_{\tilde{d}}(\mathbf{A})\boldsymbol{\rho}_e = \boldsymbol{\rho}_{e_{\tilde{d}}}$. Theorem 2.10 and the independence of the result from the chosen edge complete the proof. \square \square

Note that, using Lemma 2.4 applied to $(\text{ev}_E \Gamma, \tilde{g})$, we can specify the value of $\tilde{p}_{\tilde{d}}(\lambda_0)$. First recall that $\tilde{d} = |\text{ev}_E \Gamma|$ is either $d = |\text{ev} \Gamma|$ if Γ is nonbipartite or $d - 1 = |\text{ev} \Gamma \setminus \{-\lambda_0\}|$ otherwise. Thus, in order to give a statement valid for both situations, let us consider the set $\overline{\text{ev}} \Gamma = \text{ev} \Gamma \cup \{-\lambda_0\}$ and

$$\bar{\pi}_i = (\lambda_i + \lambda_0) \prod_{j=0, j \neq i}^{\tilde{d}} |\lambda_i - \lambda_j| = (\lambda_i + \lambda_0) \bar{\pi}_i \quad (0 \leq i \leq d),$$

where the $\tilde{\pi}_i$'s correspond to the moment-like parameters of (2) defined on $\text{ev}_E \Gamma$. Using Lemma 2.4 we get:

$$\begin{aligned}
|e_{\tilde{d}}| &= 2\tilde{p}_{\tilde{d}}(\lambda_0) = \frac{2}{\tilde{g}^2(\lambda_0)\tilde{\pi}_0^2} \left(\sum_{i=0}^{\tilde{d}} \frac{1}{\tilde{g}(\lambda_i)\tilde{\pi}_i^2} \right)^{-1} \\
&= 2 \frac{|V|^2}{4} \frac{4\lambda_0^2}{\pi_0^2} \left(\sum_{i=0}^{\tilde{d}} \frac{\lambda_0}{\lambda_0 + \lambda_i} \frac{|V|}{m(\lambda_i)} \frac{(\lambda_0 + \lambda_i)^2}{\pi_i^2} \right)^{-1} \\
&= \frac{2\lambda_0}{\pi_0^2} \left(\sum_{i=0}^{\tilde{d}} \frac{\lambda_0 + \lambda_i}{m(\lambda_i)\pi_i^2} \right)^{-1} |V| = \frac{2\lambda_0}{\pi_0^2} \left(\sum_{i=0}^d \frac{\lambda_0 + \lambda_i}{m(\lambda_i)\pi_i^2} \right)^{-1} |V| \\
&= \frac{4|E|}{\pi_0^2} \left(\sum_{i=0}^d \frac{\lambda_0 + \lambda_i}{m(\lambda_i)\pi_i^2} \right)^{-1},
\end{aligned}$$

since the (possible) term corresponding to $-\lambda_0$ is null. Thus, we have the following characterization which can be seen as the analogue of the spectral excess theorem [10, 18, 8] for edge-distance-regularity.

Theorem 4.10 *Let $\Gamma = (V, E)$ be a regular graph with $d + 1$ distinct eigenvalues, and spectrally maximum edge-diameter $\tilde{D} = \tilde{d}$. Then, Γ is edge-distance-regular if and only if, for every edge $e \in E$,*

$$|e_{\tilde{D}}| = \frac{4|E|}{\pi_0^2} \left(\sum_{i=0}^d \frac{\lambda_0 + \lambda_i}{m(\lambda_i)\pi_i^2} \right)^{-1}.$$

Let Γ be a nonbipartite regular graph. If we apply Corollary 2.8 to the pairs $(\text{ev} \Gamma, \frac{1}{|V|}m)$ and $(\text{ev} \Gamma, \frac{1}{\lambda_0|V|}(x + \lambda_0)m)$, we get the following equation relating the highest degree polynomials of their canonical orthogonal systems (recall that $\hat{\pi}_0 = \prod_{i=1}^d (\lambda_0 + \lambda_i)$):

$$(-1)^d p_d(-\lambda_0) = \frac{\hat{\pi}_0 p_d(\lambda_0) |V|}{\pi_0 \tilde{p}_d(\lambda_0) 2}. \quad (23)$$

Proposition 4.11 *Let Γ be a λ_0 -regular graph with edge-diameter $\tilde{D} = |\text{ev}^* \Gamma| = d$. Assume that, for every vertex $u \in V$ and every edge $e \in E$,*

$$\frac{|e_d|}{|u_d|} = \frac{\hat{\pi}_0}{\pi_0} \frac{|V|}{(-1)^d p_d(-\lambda_0)},$$

where p_d is the d -th predistance polynomial of Γ . Then, Γ is edge-distance-regular if and only if it is distance-regular.

Proof. Note that the condition $\tilde{D} = d$ implies that $\tilde{D} = D$. Thus, by Lemma 3.1, Γ is nonbipartite. Distance-regularity is equivalent to $p_d(\lambda_0) = |u_d|$ for every vertex $u \in V$ and edge-distance-regularity is equivalent to $2\tilde{p}_d(\lambda_0) = |e_d|$. The equivalence of both conditions follows from equation (23). \square

As an immediate consequence, we have the following result.

Corollary 4.12 *Let Γ be a distance-regular and edge-distance-regular graph with $\tilde{D} = D$. Then, for every vertex $u \in V$ and every edge $e \in E$,*

$$\frac{|e_d|}{|u_d|} = \frac{\hat{\pi}_0}{\pi_0} \frac{|V|}{(-1)^d p_d(-\lambda_0)}.$$

Now, in order to give a characterization of those distance-regular graphs which are also edge-distance-regular, we have the two following technical results.

Lemma 4.13 *Given $d \geq 1$, let $b_0, b_1, \dots, b_{d-1}, c_1, c_2, \dots, c_d$ be positive real numbers and $c_0 = b_d = 0$. Let $I_i = [b_i + c_i, \infty)$, $0 \leq i \leq d$. The map $F : I_0 \times I_1 \times \dots \times I_d \subset \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ defined by the determinant*

$$F(\mathbf{x}) = \begin{vmatrix} x_0 & c_1 & & & & \\ b_0 & x_1 & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & & \ddots & x_{d-1} & c_d \\ & & & & b_{d-1} & x_d \end{vmatrix}$$

is nonnegative and it only vanishes at $(b_0 + c_0, b_1 + c_1, \dots, b_d + c_d)$.

Proof. Let $\mathbf{x}_0 = (b_0 + c_0, b_1 + c_1, \dots, b_d + c_d)$ and let \mathbf{B} be the matrix with determinant $F(\mathbf{x}_0)$. The product $\mathbf{u}\mathbf{B}$ with $\mathbf{u} = (1, -1, 1, -1, \dots, (-1)^d)$ is zero, so $F(\mathbf{x}_0) = 0$. It remains to prove that, for every $d \geq 1$, if $\mathbf{x} = (x_0, x_1, \dots, x_d) \neq \mathbf{x}_0$ satisfies $x_i \geq b_i + c_i$ for every i , then $F(\mathbf{x}) > 0$. The case $d = 1$ is straightforward. Assume that the claim holds for all values smaller than d . The mean value theorem ensures that there exists

$\boldsymbol{\xi} \in \{\mathbf{x}_0 + t\mathbf{x} \mid 0 < t < 1\}$ such that

$$\begin{aligned}
F(\mathbf{x}) &= \sum_{k=0}^d \frac{\partial F}{\partial x_k}(\boldsymbol{\xi})(x_k - b_k - c_k) \\
&= \sum_{k=0}^d \left| \begin{array}{cccccccc}
\xi_0 & c_1 & & & & & & \\
b_0 & \ddots & \ddots & & & & & \\
& \ddots & \ddots & \ddots & & & & \\
& & b_{k-2} & \xi_{k-1} & & & & \\
& & & & \xi_{k+1} & c_{k+2} & & \\
& & & & b_{k+1} & \ddots & \ddots & \\
& & & & & \ddots & \ddots & c_d \\
& & & & & & b_{d-1} & \xi_d
\end{array} \right| (x_k - b_k - c_k).
\end{aligned}$$

Since $\xi_{k-1} \geq b_{k-1} + c_{k-1} > c_{k-1}$ and $\xi_{k+1} \geq b_{k+1} + c_{k+1} > b_{k+1}$, we obtain $\frac{\partial F}{\partial x_k}(\boldsymbol{\xi}) > 0$ for every k . Thus, $F(\mathbf{x}) > 0$. \square

Lemma 4.14 *Given $d \geq 1$, let $b_0, b_1, \dots, b_{d-1}, c_1, c_2, \dots, c_d$ be positive real numbers and $c_0 = b_d = 0$. Let $\lambda_0 \geq b_k + c_k$ for $k = 0, 1, \dots, d$ and $a_k = \lambda_0 - b_k - c_k \geq 0$. The tridiagonal determinant*

$$\left| \begin{array}{cccccccc}
\lambda_0 + a_0 & c_1 & & & & & & \\
b_0 & \lambda_0 + a_1 & \ddots & & & & & \\
& \ddots & \ddots & \ddots & & & & \\
& & \ddots & \ddots & \ddots & & & \\
& & & \ddots & \lambda_0 + a_{d-1} & c_d & & \\
& & & & b_{d-1} & \lambda_0 - a_d & &
\end{array} \right|$$

vanishes if and only if $a_0 = a_1 = \dots = a_{d-1} = 0$.

Proof. Applying Lemma 4.13, we obtain that the determinant vanishes if and only if $\lambda_0 + a_k = b_k + c_k$ for $k = 0, 1, \dots, d-1$, which is equivalent to $a_0 = a_1 = \dots = a_{d-1} = 0$, and $\lambda_0 - a_d = b_d + c_d$. \square

A distance-regular graph Γ with diameter D and odd-girth (that is, the shortest cycle of odd length) $2D + 1$ is called a *generalized odd graph*, also known as an *almost-bipartite distance-regular graph* or a *regular thin near $(2D + 1)$ -gon*. (In particular, these conditions are fulfilled by the mentioned odd graphs O_k .) Notice that, in this case, the intersection parameters of Γ satisfy $a_0 = a_1 = \dots = a_{d-1} = 0$ and $a_d \neq 0$. Recently, Van Dam and Haemers [20] showed that any connected regular graph with $d + 1$ distinct eigenvalues and odd-girth $2d + 1$ is a generalized odd graph. Here we show that the same result holds when Γ is both distance-regular and edge-distance-regular.

Proposition 4.15 *Let Γ be a distance-regular graph with intersection numbers c_k, a_k, b_k , $0 \leq k \leq d$. Suppose that $a_d \neq 0$. Then, Γ is edge-distance-regular if and only if it is a generalized odd graph.*

Proof. In a distance-regular graph, $a_d \neq 0$ is equivalent to $\tilde{D} = d$. Indeed, on one hand we have $\tilde{D} \leq D = d$ and, on the other hand, if we take $u \in V$, $a_d \neq 0$ implies that any edge between vertices of u_d is at distance d from u , thus $\tilde{D} \geq d$.

Using Proposition 4.11 and Corollary 4.12, we obtain that edge-distance-regularity is equivalent to the equality:

$$\begin{aligned} (-1)^d p_d (-\lambda_0) a_d &= \frac{\widehat{\pi}_{\lambda_0}}{\pi_{\lambda_0}} \lambda_0 |V| = \frac{|V|}{2\pi_{\lambda_0}} \det(\lambda_0 \mathbf{I} + \mathbf{D}) \\ &= \frac{|V|}{2\pi_{\lambda_0}} \begin{vmatrix} \lambda_0 + a_0 & c_1 & \cdots & 0 & 0 \\ b_0 & \lambda_0 + a_1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \lambda_0 + a_{d-1} & c_d \\ 0 & 0 & \cdots & b_{d-1} & \lambda_0 + a_d \end{vmatrix}, \end{aligned}$$

where D is the intersection matrix of Γ . Using Lemma 2.2, we get

$$\begin{aligned} p_d &= \frac{1}{c_1 c_2 \cdots c_d} \begin{vmatrix} x - a_0 & -c_1 & \cdots & 0 \\ -b_0 & x - a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x - a_{d-1} \end{vmatrix} \\ &= \frac{|V|}{\pi_{\lambda_0}} \begin{vmatrix} x - a_0 & -c_1 & \cdots & 0 \\ -b_0 & x - a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x - a_{d-1} \end{vmatrix}, \end{aligned}$$

where the second equality is due to the fact that p_d and $p_0 + p_1 + \cdots + p_d = \frac{|V|}{\pi_0} \prod_{i=1}^d (x - \lambda_i)$ have the same leading coefficient. Then, equation (24) can be rewritten as

$$\begin{aligned} &\begin{vmatrix} \lambda_0 + a_0 & c_1 & \cdots & 0 & 0 & 0 \\ b_0 & \lambda_0 + a_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_0 + a_{d-2} & c_{d-1} & 0 \\ 0 & 0 & \cdots & b_{d-2} & \lambda_0 + a_{d-1} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 2a_d \end{vmatrix} \\ &= \begin{vmatrix} \lambda_0 + a_0 & c_1 & & & & \\ b_0 & \lambda_0 + a_1 & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \lambda_0 + a_{d-1} & c_d & \\ & & & b_{d-1} & \lambda_0 + a_d & \end{vmatrix}. \end{aligned}$$

Thus,

$$\begin{vmatrix} \lambda_0 + a_0 & c_1 & & & & & \\ & b_0 & \lambda_0 + a_1 & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \lambda_0 + a_{d-1} & c_d & \\ & & & & b_{d-1} & \lambda_0 - a_d & \end{vmatrix} = 0$$

and the result follows from Lemma 4.14. \square

For a distance-regular graph it is immediate to check that $a_k = 0$, $k = 0, 1, \dots, d-1$, is equivalent to the parity of the distance polynomials; more precisely, $p_k(-x) = (-1)^k p_k(x)$.

Proposition 4.16 *Let Γ be a nonbipartite distance-regular graph with intersection array*

$$\begin{pmatrix} 0 & c_1 & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & \cdots & b_{d-1} & 0 \end{pmatrix}$$

and distance polynomials $\{p_k\}_{0 \leq k \leq d}$. Then, the following statements are equivalent:

- (a) Γ is edge-distance-regular;
- (b) $a_0 = a_1 = \cdots = a_{d-1} = 0$ and $a_d \neq 0$;
- (c) For every $k = 0, 1, \dots, d$, p_k has even parity for even k and odd parity for odd k .

In this case, the edge-distance polynomials $\{\tilde{p}_k\}_{0 \leq k \leq d}$ and the edge-intersection array are:

$$\tilde{p}_k = p_k - p_{k-1} + p_{k-2} - \cdots + (-1)^k p_0 \quad (0 \leq k \leq d-1),$$

$$\tilde{p}_d = \frac{1}{2}(p_d - p_{d-1} + p_{d-2} - \cdots + (-1)^d p_0),$$

$$\begin{pmatrix} 0 & \tilde{c}_1 & \cdots & \tilde{c}_{d-2} & \tilde{c}_{d-1} & \tilde{c}_d \\ \tilde{a}_0 & \tilde{a}_1 & \cdots & \tilde{a}_{d-2} & \tilde{a}_{d-1} & \tilde{a}_d \\ \tilde{b}_0 & \tilde{b}_1 & \cdots & \tilde{b}_{d-2} & \tilde{b}_{d-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & c_1 & \cdots & c_{d-2} & c_{d-1} & 2c_d \\ c_1 & c_2 - c_1 & \cdots & c_{d-1} - c_{d-2} & c_d - c_{d-1} & a_d - c_d \\ b_1 & b_2 & \cdots & b_{d-1} & a_d & 0 \end{pmatrix}.$$

Proof. The equivalences follow from Proposition 4.15 and the above remark.

Let $n_r = p_r(\lambda_0)$ and recall that $b_r n_r = c_{r+1} n_{r+1}$. Since $a_r = 0$ for $r = 0, 1, \dots, d-1$, we have

$$\begin{aligned} \lambda_0 q_k(-\lambda_0) &= \lambda_0 \sum_{r=0}^k p_r(-\lambda_0) = \lambda_0 \sum_{r=0}^k (-1)^r n_r = \sum_{r=0}^k (-1)^r (b_r + c_r) n_r \\ &= \sum_{r=0}^k (-1)^r b_r n_r + \sum_{r=0}^k (-1)^r c_r n_r = \sum_{r=0}^k (-1)^r b_r n_r + \sum_{r=1}^k (-1)^r b_{r-1} n_{r-1} \\ &= \sum_{r=0}^k (-1)^r b_r n_r - \sum_{r=0}^{k-1} (-1)^r b_r n_r = (-1)^k b_k n_k \end{aligned}$$

for every $k = 0, 1, \dots, d-1$. Using Proposition 2.7 and equation (6), we have

$$\begin{aligned} (x + \lambda_0) \tilde{p}_k &= -\lambda_0 q_k(-\lambda_0) \left(\frac{1}{p_{k+1}(-\lambda_0)} p_{k+1} - \frac{1}{p_k(-\lambda_0)} p_k \right) \\ &= (-1)^{k+1} b_k n_k \left(\frac{1}{n_{k+1}} p_{k+1}(-x) - \frac{1}{n_k} p_k(-x) \right) \\ &= (-1)^{k+1} (c_{k+1} p_{k+1}(-x) - b_k p_k(-x)) \\ &= (-1)^{k+1} (-x - \lambda_0) q_k(-\lambda_0) = (-1)^k (x + \lambda_0) q_k(-\lambda_0), \end{aligned}$$

implying that $\tilde{p}_k = (-1)^k q_k(-x) = p_k - p_{k-1} + p_{k-2} - \dots + (-1)^k p_0$ for $k = 0, 1, \dots, d-1$.

Since the edge distance-polynomials $\{\tilde{p}_k\}_{0 \leq k \leq d}$ are the canonical orthogonal system with respect to $(\text{ev } \Gamma, \tilde{g})$, using Proposition 2.3 we can compute \tilde{p}_d :

$$\begin{aligned} \tilde{p}_d &= \frac{1}{\tilde{g}_0 \pi_0} \prod_{i=1}^d (x - \lambda_i) - \sum_{k=0}^{d-1} \tilde{p}_k = \frac{1}{2\tilde{g}_0 \pi_0} \prod_{i=1}^d (x - \lambda_i) - \sum_{k=0}^{d-1} \sum_{s=0}^k (-1)^{k+s} p_s \\ &= \frac{1}{2} \sum_{s=0}^d p_s - \sum_{s=0}^{d-1} (-1)^s p_s \sum_{k=s}^{d-1} (-1)^k = \frac{1}{2} \sum_{s=0}^d p_s - \frac{1}{2} \sum_{s=0}^{d-1} (1 - (-1)^{d+s}) p_s \\ &= \frac{1}{2} \left(p_d + (-1)^d \sum_{s=0}^{d-1} (-1)^s p_s \right) = \frac{1}{2} (p_d - p_{d-1} + p_{d-2} - \dots + (-1)^d p_0). \end{aligned}$$

Consider the column matrices $\mathbf{P} = (p_k)_{0 \leq k \leq d}$ and $\tilde{\mathbf{P}} = (\tilde{p}_k)_{0 \leq k \leq d}$. The recurrence matrices of the canonical orthogonal systems of $(\text{ev } \Gamma, g)$ and $(\text{ev } \Gamma, \tilde{g})$, \mathbf{R} and $\tilde{\mathbf{R}}$, satisfy $x\mathbf{P} = \mathbf{R}\mathbf{P}$ and $x\tilde{\mathbf{P}} = \tilde{\mathbf{R}}\tilde{\mathbf{P}}$ in the quotient ring $\mathbb{R}[x]/\mathcal{I}$, where \mathcal{I} is the ideal generated by $\prod_{i=1}^d (x - \lambda_i)$. The previous expression can be expressed in terms of these matrices by $\tilde{\mathbf{P}} = \mathbf{M}\mathbf{P}$, where

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{d-1} & (-1)^{d-2} & (-1)^{d-3} & \cdots & 1 & 0 \\ \frac{1}{2}(-1)^d & \frac{1}{2}(-1)^{d-1} & \frac{1}{2}(-1)^{d-2} & \cdots & -\frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

with inverse matrix

$$M^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 2 \end{pmatrix}.$$

From $\tilde{R}\tilde{P} = x\tilde{P} = xMP = MxP = MRP = MRM^{-1}\tilde{P}$, we obtain $\tilde{R} = MRM^{-1}$. That is,

$$\tilde{R} = \begin{pmatrix} c_1 & c_1 & 0 & \cdots & 0 & 0 & 0 \\ b_1 & c_2 - c_1 & c_2 & \cdots & 0 & 0 & 0 \\ 0 & b_2 & c_3 - c_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_{d-1} - c_{d-2} & c_{d-1} & 0 \\ 0 & 0 & 0 & \cdots & b_{d-1} & c_d - c_{d-1} & 2c_d \\ 0 & 0 & 0 & \cdots & 0 & a_d & a_d - c_d \end{pmatrix},$$

as claimed. \square

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