Cotangent Bundle Reduction with Singularities

Miguel Rodríguez-Olmos

The University of Manchester
&
Technical University of Catalonia

RatiuFest 2010 @ CIRM

Regular Poisson and symplectic reduction

Let $(\mathcal{P}, \omega, G, \mathbf{J})$ be a Hamiltonian G-space for a free and proper G-action on $\mathcal{P}$.

Regular Poisson and symplectic reduction

Let $(\mathcal{P}, \omega, G, \mathbf{J})$ be a Hamiltonian G-space for a free and proper G-action on $\mathcal{P}$.

- The quotient $\mathcal{P}/G$ is a smooth Poisson manifold with bracket defined by

$$\{f, g\}(\pi(z)) = \omega(X_{f\circ\pi}, X_{g\circ\pi})(z)$$

where $\pi : \mathcal{P} \to \mathcal{P}/G$.

Regular Poisson and symplectic reduction

Let $(\mathcal{P}, \omega, G, \mathbf{J})$ be a Hamiltonian G-space for a free and proper G-action on $\mathcal{P}$.

- The quotient $\mathcal{P}/G$ is a smooth Poisson manifold with bracket defined by

$$\{f, g\}(\pi(z)) = \omega(X_{f\circ\pi}, X_{g\circ\pi})(z)$$

where $\pi : \mathcal{P} \to \mathcal{P}/G$.

- (Marsden, Weinstein '74) Let μ be a value of $\mathbf{J}$. The quotient $\mathbf{J}^{-1}(\mu)/G_\mu$ is a smooth symplectic manifold with 2-form ω_μ defined by

$$\pi^*\omega_\mu = \iota^*\omega, \qquad \text{where}$$

Regular Poisson and symplectic reduction

Let $(\mathcal{P}, \omega, G, \mathbf{J})$ be a Hamiltonian G-space for a free and proper G-action on $\mathcal{P}$.

- The quotient $\mathcal{P}/G$ is a smooth Poisson manifold with bracket defined by

$$\{f, g\}(\pi(z)) = \omega(X_{f\circ\pi}, X_{g\circ\pi})(z)$$

where $\pi : \mathcal{P} \to \mathcal{P}/G$.

- (Marsden, Weinstein '74) Let μ be a value of $\mathbf{J}$. The quotient $\mathbf{J}^{-1}(\mu)/G_\mu$ is a smooth symplectic manifold with 2-form ω_μ defined by

$$\pi^*\omega_\mu = \iota^*\omega, \qquad \text{where}$$

$$\begin{array}{ccc} \mathbf{J}^{-1}(\mu) & \overset{\iota}{\hookrightarrow} & \mathcal{P} \\ \pi \downarrow & & \\ \mathbf{J}^{-1}(\mu)/G_\mu & & \end{array}$$

Relationship: symplectic leaves and orbit reduction (Marle '76).

Singular Reduction

In case the G-action is not free, $\{H_i\}$ set of (compact) stabilizers and orbit type

$$\mathcal{P}_{(H_i)} = \{z \in \mathcal{P} \,:\, G_z \text{ is conjugate to } H_i\}.$$

Singular Reduction

In case the G-action is not free, $\{H_i\}$ set of (compact) stabilizers and orbit type

$$\mathcal{P}_{(H_i)} = \{z \in \mathcal{P} \ : \ G_z \text{ is conjugate to } H_i\}.$$

- $\mathcal{P}/G$ is a singular stratified space with strata

$$\mathcal{P}_{(H_i)}/G$$

which carry a Poisson bracket obtained by regular Poisson reduction.

Singular Reduction

In case the G-action is not free, $\{H_i\}$ set of (compact) stabilizers and orbit type

$$\mathcal{P}_{(H_i)} = \{z \in \mathcal{P} \ : \ G_z \text{ is conjugate to } H_i\}.$$

- $\mathcal{P}/G$ is a singular stratified space with strata

$$\mathcal{P}_{(H_i)}/G$$

which carry a Poisson bracket obtained by regular Poisson reduction.

- (Sjamaar, Lerman '91) $\mathbf{J}^{-1}(0)/G$ is a singular stratified space with strata

$$(\mathbf{J}^{-1}(0) \cap \mathcal{P}_{(H_i)})/G$$

which carry a 2-form obtained by regular symplectic reduction.

Singular Reduction

In case the G-action is not free, $\{H_i\}$ set of (compact) stabilizers and orbit type

$$\mathcal{P}_{(H_i)} = \{z \in \mathcal{P} \ : \ G_z \text{ is conjugate to } H_i\}.$$

- $\mathcal{P}/G$ is a singular stratified space with strata

$$\mathcal{P}_{(H_i)}/G$$

which carry a Poisson bracket obtained by regular Poisson reduction.

- (Sjamaar, Lerman '91) $\mathbf{J}^{-1}(0)/G$ is a singular stratified space with strata

$$(\mathbf{J}^{-1}(0) \cap \mathcal{P}_{(H_i)})/G$$

which carry a 2-form obtained by regular symplectic reduction.

$$\begin{array}{ccc} \mathcal{P}_{(H_i)} & \xhookrightarrow{\iota} & \mathcal{P} \\ {\scriptstyle \pi}\downarrow & & \\ \mathcal{P}_{(H_i)}/G & & \end{array} \qquad \begin{array}{ccc} \mathbf{J}^{-1}(0) \cap \mathcal{P}_{(H_i)} & \xhookrightarrow{\iota} & \mathcal{P} \\ {\scriptstyle \pi}\downarrow & & \\ (\mathbf{J}^{-1}(0) \cap \mathcal{P}_{(H_i)})/G & & \end{array}$$

Regular and Almost Regular Cotangent Bundle Reduction

Cotangent Bundles

If M is a smooth manifold where G acts one has a canonical construction of a Hamiltonian G-space $(\mathcal{P}, \omega, G, \mathbf{J})$, where

- $\mathcal{P} = T^*M$
- ω is the canonical 2-form $\omega_c = \mathbf{d}x^i \wedge \mathbf{d}y_i$
- G acts on T^*M by canonical lifts
- $\langle \mathbf{J}(p_x), \xi \rangle = \langle p_x, \xi_M(x) \rangle$ for all $p_x \in T^*M$, $\xi \in \mathfrak{g}$.

Cotangent Bundles

If M is a smooth manifold where G acts one has a canonical construction of a Hamiltonian G-space $(\mathcal{P}, \omega, G, \mathbf{J})$, where

- $\mathcal{P} = T^*M$
- ω is the canonical 2-form $\omega_c = \mathbf{d}x^i \wedge \mathbf{d}y_i$
- G acts on T^*M by canonical lifts
- $\langle \mathbf{J}(p_x), \xi \rangle = \langle p_x, \xi_M(x) \rangle$ for all $p_x \in T^*M$, $\xi \in \mathfrak{g}$.

These are Hamiltonian G-spaces with an extra fibered structure

$$\tau : T^*M \to M$$

Cotangent Bundles

If M is a smooth manifold where G acts one has a canonical construction of a Hamiltonian G-space $(\mathcal{P}, \omega, G, \mathbf{J})$, where

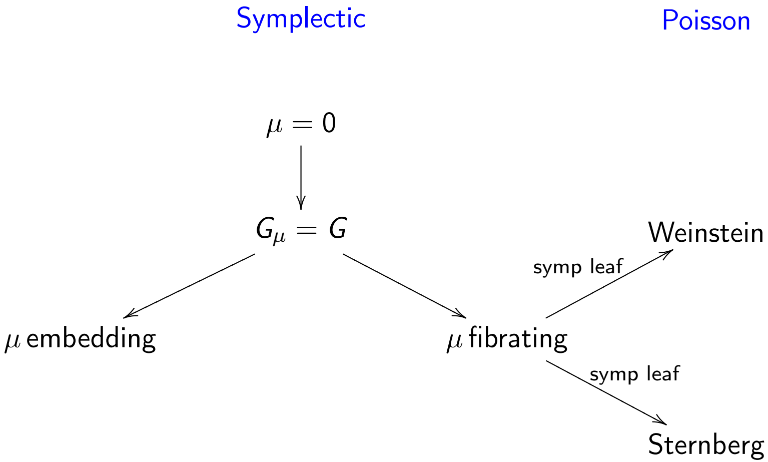
- $\mathcal{P} = T^*M$
- ω is the canonical 2-form $\omega_c = \mathbf{d}x^i \wedge \mathbf{d}y_i$
- G acts on T^*M by canonical lifts
- $\langle \mathbf{J}(p_x), \xi \rangle = \langle p_x, \xi_M(x) \rangle$ for all $p_x \in T^*M$, $\xi \in \mathfrak{g}$.

These are Hamiltonian G-spaces with an extra fibered structure

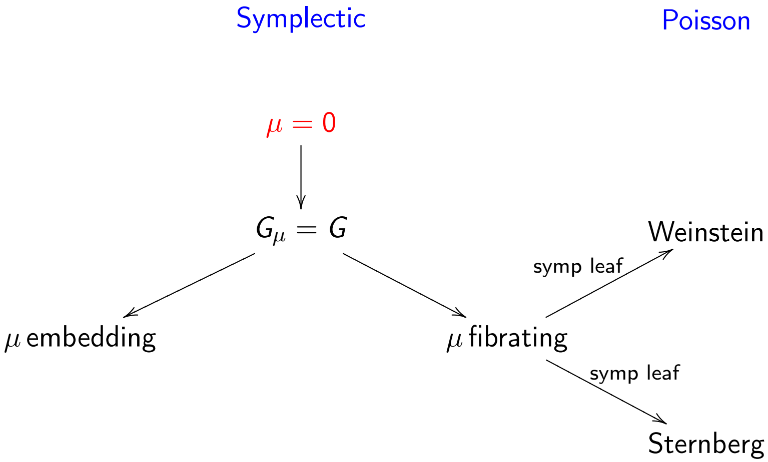
$$\tau : T^*M \to M$$

Is this fibered structure also preserved by the reduction schemes together with the symplectic or Poisson structures?

Regular Cotangent Bundle Reduction



Regular Cotangent Bundle Reduction



$\mu = 0$

- Satzer '77

$$\mathbf{J}^{-1}(0)/G \simeq T^*(M/G)$$

$\mu = 0$

- Satzer '77

$$\mathbf{J}^{-1}(0)/G \simeq T^*(M/G)$$

- Symplectic diffeomorphism with respect to the canonical symplectic form on $T^*(M/G)$.

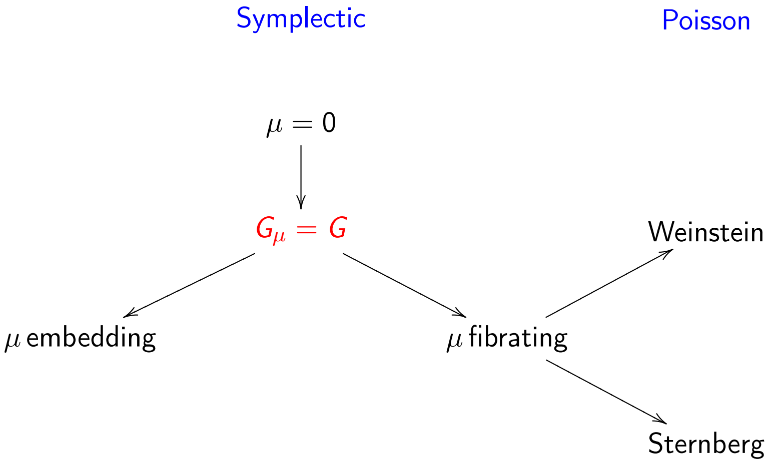
$\mu = 0$

- Satzer '77

$$\mathbf{J}^{-1}(0)/G \simeq T^*(M/G)$$

  - Symplectic diffeomorphism with respect to the canonical symplectic form on $T^*(M/G)$.
  - We can define the cotangent bundle of M/G as the symplectic quotient at $\mu = 0$ for T^*M.

Regular Cotangent Bundle Reduction



$G_\mu = G$

- Satzer '77, Abraham and Marsden '78, Kummer '81

$$\mathbf{J}^{-1}(\mu)/G_\mu \simeq T^*(M/G)$$

$G_\mu = G$

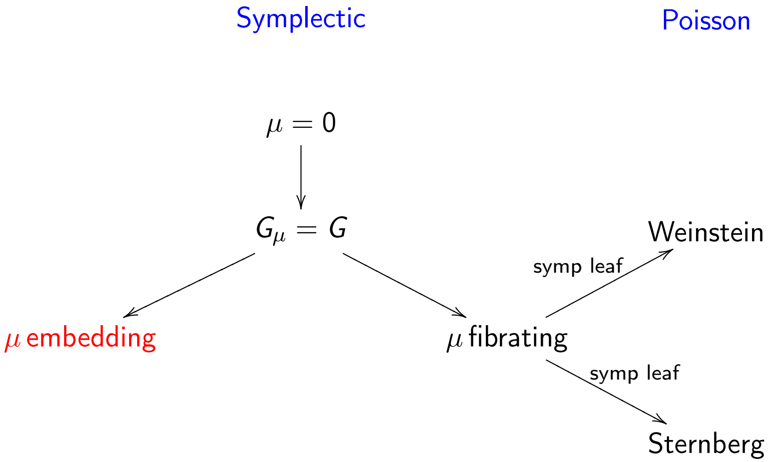
- Satzer '77, Abraham and Marsden '78, Kummer '81

$$\mathbf{J}^{-1}(\mu)/G_\mu \simeq T^*(M/G)$$

- Symplectic diffeomorphism with respect to a twisted symplectic form $\omega_c - \tau^* B_\mu$ on $T^*(M/G)$.
- $B_\mu \in \Omega^2(M/G)$ comes from the curvature of a principal connection on

$$M \to M/G$$

Regular Cotangent Bundle Reduction



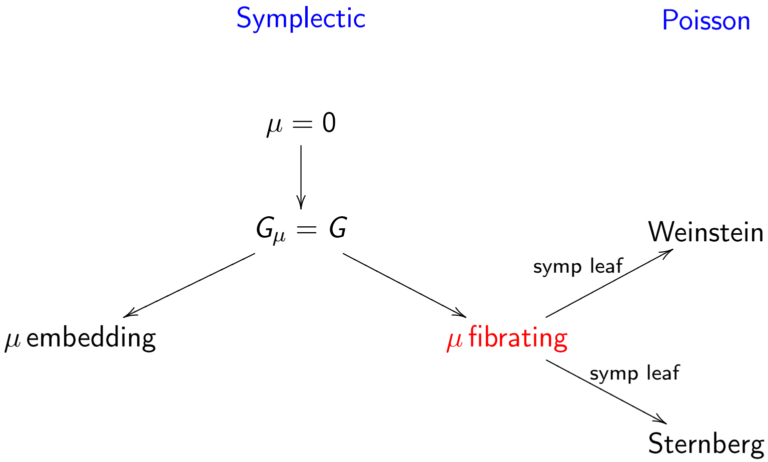
μ embedding

- Abraham and Marsden '78, Kummer '81

$$\mathbf{J}^{-1}(\mu)/G_\mu \hookrightarrow (T^*(M/G_\mu), \omega_c - \tau^* B_\mu)$$

symplectic embedding

Regular Cotangent Bundle Reduction



μ fibrating

- Zaalani '99, Marsden and Perlmutter '00

symplectic leaves of the Weinstein and Sternberg Poisson reduced spaces

$$\mathbf{J}^{-1}(\mu)/G_\mu \simeq V$$

μ fibrating

- Zaalani '99, Marsden and Perlmutter '00

symplectic leaves of the Weinstein and Sternberg Poisson reduced spaces

$$\mathbf{J}^{-1}(\mu)/G_\mu \simeq V$$

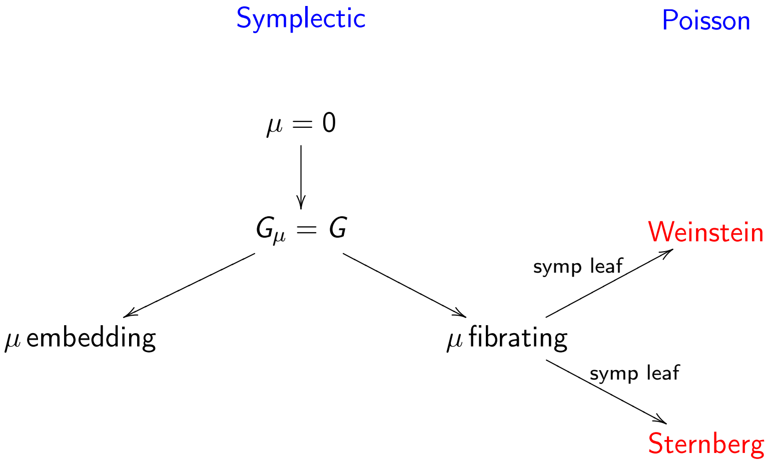
where

$$\mathcal{O}_\mu \to V \to T^*(M/G)$$

symplectic form on V:

- ω_c on $T^*(M/G)$
- KKS form on $\mathcal{O}_\mu$
- curvature term B_μ

Regular Cotangent Bundle Reduction



Weinstein and Sternberg spaces

(The Gauge Picture of Mechanics)

- Sternberg '77, Weinstein '78, Montgomery, Marsden & Ratiu '84

$$(T^*M)/G \simeq S \quad \text{or} \quad (T^*M)/G \simeq W$$

where

$$\mathfrak{g}^* \to S \to T^*(M/G)$$

Poisson structure on S (or W):

- ω_c on $T^*(M/G)$
- Lie-Poisson bracket on $\mathfrak{g}^*$
- curvature term B_μ

The construction of the Sternberg space

- Pull-back

$$\begin{array}{c} M \\ \Big\downarrow{\scriptstyle \pi} \\ M/G \end{array}$$

The construction of the Sternberg space

- Pull-back

$$\begin{array}{ccc} & & M \\ & & \downarrow \pi \\ T^*(M/G) & \xrightarrow{\tau} & M/G \end{array}$$

The construction of the Sternberg space

- Pull-back

$$\begin{array}{ccc} \widetilde{M} & \longrightarrow & M \\ \big\downarrow & & \big\downarrow \pi \\ T^*(M/G) & \xrightarrow{\tau} & M/G \end{array}$$

The construction of the Sternberg space

- Pull-back

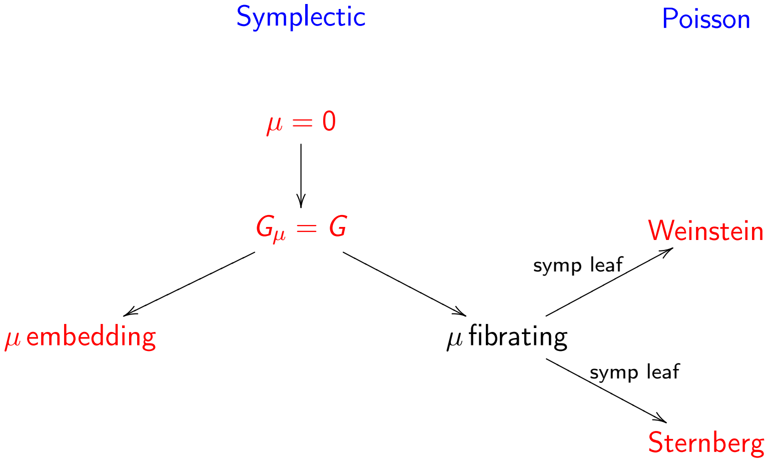
$$\begin{array}{ccc} \widetilde{M} & \longrightarrow & M \\ \downarrow & & \downarrow \pi \\ T^*(M/G) & \xrightarrow{\tau} & M/G \end{array}$$

- Association

$$\begin{array}{c} S := \widetilde{M} \times_G \mathfrak{g}^* \\ \downarrow \\ T^*(M/G) \end{array}$$

Single Orbit Type Cotangent Bundle Reduction

$$M = M_{(H)}$$

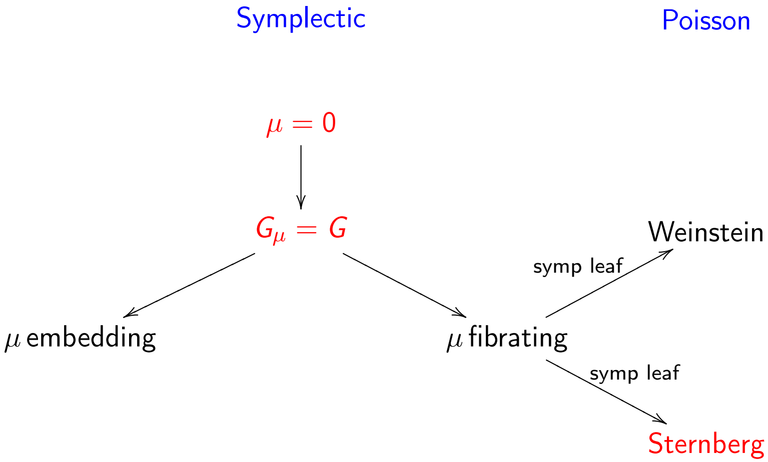


Historical Remark

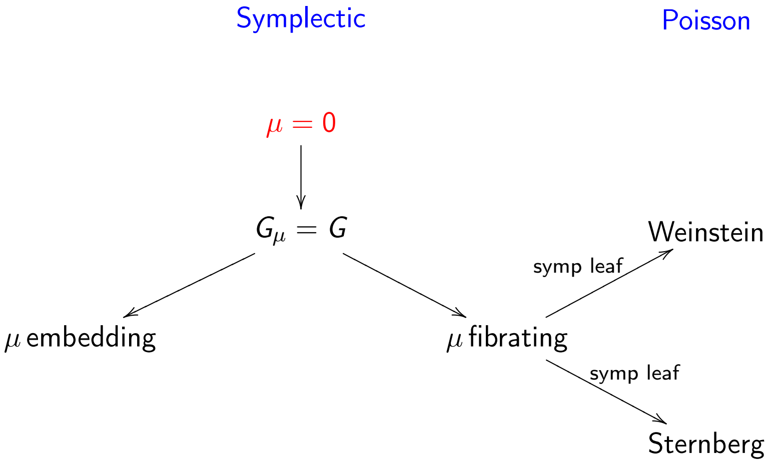
Year	Single Orbit Type	Singular Symplectic
'83	μ embedding, Montgomery	
'90	$\mu = 0$, Emmrich & Römer	
'91		$\mu = 0$, Sjamaar & Lerman
'97		$\mu \neq 0$, Bates & Lerman
'02	$G_\mu = G$, Schmah	
'06	Weinstein, Hochgerner	
'09	Sternberg, Perlmutter & MR-O	

The General Situation of Singular Cotangent Bundle Reduction

Singular Cotangent Bundle Reduction



Singular Cotangent Bundle Reduction



$\mu = 0$

- Perlmutter, Sousa-Dias & MR-O '07
  - the cotangent bundle projection $\tau : T^*M \to M$ induces a continuous projection
$$\tau^0 : \mathbf{J}^{-1}(0)/G \to M/G$$

$\mu = 0$

- Perlmutter, Sousa-Dias & MR-O '07
  - the cotangent bundle projection $\tau : T^*M \to M$ induces a continuous projection
    $$\tau^0 : \mathbf{J}^{-1}(0)/G \to M/G$$
  - M/G has a natural orbit type stratification
    $$M/G = \bigsqcup M_{(K)}/G$$

$\mu = 0$

- Perlmutter, Sousa-Dias & MR-O '07
  - the cotangent bundle projection $\tau : T^*M \to M$ induces a continuous projection
  $$\tau^0 : \mathbf{J}^{-1}(0)/G \to M/G$$
  - M/G has a natural orbit type stratification
  $$M/G = \bigsqcup M_{(K)}/G$$
  - $\mathbf{J}^{-1}(0)/G$ has the symplectic stratification
  $$\mathbf{J}^{-1}(0)/G = \bigsqcup (\mathbf{J}^{-1}(0) \cap (T^*M)_{(H)})/G$$

$\mu = 0$

- Perlmutter, Sousa-Dias & MR-O '07
  - the cotangent bundle projection $\tau : T^*M \to M$ induces a continuous projection
  $$\tau^0 : \mathbf{J}^{-1}(0)/G \to M/G$$
  - M/G has a natural orbit type stratification
  $$M/G = \bigsqcup M_{(K)}/G$$
  - $\mathbf{J}^{-1}(0)/G$ has the symplectic stratification
  $$\mathbf{J}^{-1}(0)/G = \bigsqcup (\mathbf{J}^{-1}(0) \cap (T^*M)_{(H)})/G$$

Question: Is τ^0 a stratified fibration?

$\mu = 0$

- Perlmutter, Sousa-Dias & MR-O '07
  - the cotangent bundle projection $\tau : T^*M \to M$ induces a continuous projection
$$\tau^0 : \mathbf{J}^{-1}(0)/G \to M/G$$
  - M/G has a natural orbit type stratification
$$M/G = \bigsqcup M_{(K)}/G$$
  - $\mathbf{J}^{-1}(0)/G$ has the symplectic stratification
$$\mathbf{J}^{-1}(0)/G = \bigsqcup (\mathbf{J}^{-1}(0) \cap (T^*M)_{(H)})/G$$

Question: Is τ^0 a stratified fibration?
Answer: NO!

$$\tau^0\left((\mathbf{J}^{-1}(0) \cap (T^*M)_{(H)})/G\right) = \overline{M_{(H)}/G}$$

Solution: The Coisotropic Stratification

We define the quotient manifolds

$$\mathcal{P}_{K\to H} = \mathbf{J}^{-1}(0)\cap(T^*M)_{(H)}$$

Solution: The Coisotropic Stratification

We define the quotient manifolds

$$\mathcal{P}_{K\to H} = \frac{\mathbf{J}^{-1}(0) \cap (T^*M)_{(H)} \cap T^*M\big|_{M_{(K)}}}{G}$$

Solution: The Coisotropic Stratification

We define the quotient manifolds

$$\mathcal{P}_{K\to H} = \frac{\mathbf{J}^{-1}(0)\cap (T^*M)_{(H)} \cap T^*M\big|_{M_{(K)}}}{G}$$

They satisfy the topological properties

- $\mathbf{J}^{-1}(0)/G = \bigsqcup \mathcal{P}_{K\to H}$ is a stratification
- it is a refinement of the symplectic stratification

$$\mathcal{P}_{K\to H} \subset (\mathbf{J}^{-1}(0)\cap (T^*M)_{(H)})/G$$

- $\tau^0 : \mathbf{J}^{-1}(0)/G \to M/G$ restricts to a fibration

$$\tau^0 : \mathcal{P}_{K\to H} \longrightarrow M_{(K)}/G$$

and the geometric properties

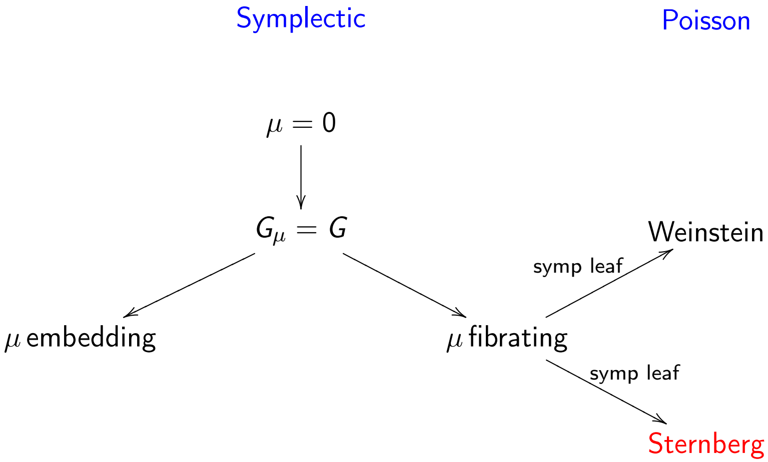
- $\mathcal{P}_{K\to H}$ is a coisotropic submanifold of $(\mathbf{J}^{-1}(0)\cap(T^*M)_{(H)})/G$
- The leaf space of $\mathcal{P}_{K\to H}$ is symplectomorphic to $(T^*(M_{(K)}/G),\omega_c)$
- In particular $\mathcal{P}_{K\to K}$ is symplectomorphic to $(T^*(M_{(K)}/G),\omega_c)$ and open and dense in $(\mathbf{J}^{-1}(0)\cap(T^*M)_{(K)})/G$

and the geometric properties

- $\mathcal{P}_{K\to H}$ is a coisotropic submanifold of $(\mathbf{J}^{-1}(0)\cap(T^*M)_{(H)})/G$
- The leaf space of $\mathcal{P}_{K\to H}$ is symplectomorphic to $(T^*(M_{(K)}/G),\omega_c)$
- In particular $\mathcal{P}_{K\to K}$ is symplectomorphic to $(T^*(M_{(K)}/G),\omega_c)$ and open and dense in $(\mathbf{J}^{-1}(0)\cap(T^*M)_{(K)})/G$

Via the coisotropic stratification the symplectic quotient $\mathbf{J}^{-1}(0)/G$ can be realized as a collection of glued cotangent bundles

Singular Cotangent Bundle Reduction



Singular Sternberg Space

- Perlmutter, Ratiu & MR-O '10

Construction of the singular Sternberg space. Step 1.

$$\begin{array}{ccc} \widetilde{M} & \longrightarrow & M \\ \downarrow & & \downarrow \pi \\ T^*(M/G) & \xrightarrow{\tau} & M/G \end{array}$$

Singular Sternberg Space

- Perlmutter, Ratiu & MR-O '10

Construction of the singular Sternberg space. Step 1.

$$\begin{array}{ccc} \widetilde{M} & \longrightarrow & N^*M \\ \big\downarrow & & \big\downarrow \pi \\ \mathbf{J}^{-1}(0)/G & \xrightarrow{\tau} & M/G \end{array}$$

Singular Sternberg Space

- Perlmutter, Ratiu & MR-O '10

Construction of the singular Sternberg space. Step 1.

$$\begin{array}{ccc} \widetilde{M} & \longrightarrow & N^*M \\ {\scriptstyle \pi^\sharp}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbf{J}^{-1}(0)/G & \xrightarrow{\tau} & M/G \end{array}$$

where we substitute

- $T^*(M/G) \longrightarrow \mathbf{J}^{-1}(0)/G$ with the coisotropic stratification (Satzer)
- $M \longrightarrow N^*M = \bigsqcup N^*M_{(K)} \subset T^*M$ with its orbit type stratification

With these stratifications each arrow is a stratified morphism.

Construction of the singular Sternberg space. Step 2.

The momentum space is $\mathfrak{g}^*$ in the case of free actions

Construction of the singular Sternberg space. Step 2.

The momentum space is $\mathfrak{g}^*$ in the case of free actions

In the case of singular actions $\mathbf{J}(T_x^*M) = \mathfrak{g}_x^\circ$. The momentum space is

$$\nu^* = \bigcup_{x \in M} \mathfrak{g}_x^\circ$$

as a singular bundle over M.

- The restriction $\nu^*_{(K)} := \nu^*\big|_{M_{(K)}}$ is a smooth bundle (slice theorem)
- The total momentum space $\nu^* = \bigsqcup \nu_{(K)}$ is endowed with the orbit type stratification

Construction of the Sternberg space. Step 3. Association.

$$\begin{array}{c} S := \widetilde{M} \times_G \nu^* \\ \Big\downarrow \pi^\sharp \\ \mathbf{J}^{-1}(0)/G \end{array} \qquad \phi_A : S \longrightarrow (T^*M)/G$$

Construction of the Sternberg space. Step 3. Association.

$$S := \widetilde{M} \times_G \nu^* \qquad \phi_A : S \longrightarrow (T^*M)/G$$

$$\Big\downarrow \pi^\sharp$$

$$\mathbf{J}^{-1}(0)/G$$

There is a natural stratification $S = \bigsqcup S^K_{H_2 \to H_1}$ with

$$S = \bigsqcup S^K_{H_2 \to H_1} = \pi^\sharp(\mathcal{P}_{H_2 \to H_1}) \times_G (\nu^*_{(H_2)})_{(K)}$$

Construction of the Sternberg space. Step 3. Association.

$$
\begin{array}{c}
S := \widetilde{M} \times_G \nu^* \\
\Big\downarrow \pi^\sharp \\
\mathbf{J}^{-1}(0)/G
\end{array}
\qquad \phi_A : S \longrightarrow (T^*M)/G
$$

There is a natural stratification $S = \bigsqcup S^K_{H_2 \to H_1}$ with

$$S = \bigsqcup S^K_{H_2 \to H_1} = \pi^\sharp(\mathcal{P}_{H_2 \to H_1}) \times_G (\nu^*_{(H_2)})_{(K)}$$

satisfying the topological properties

- It is a refinement of the Poisson stratification of $(T^*M)/G$

$$\phi_A(S^K_{H_2 \to H_1}) \subset (T^*M)_{(H_1 \cap K)}/G$$

- $\pi^\sharp$ over each stratum is a fibration over a coisotropic stratum of $\mathbf{J}^{-1}(0)/G$

$$\pi^\sharp : S^K_{H_2 \to H_1} \longrightarrow \mathcal{P}_{H_2 \to H_1}$$

and the geometric property

$$S^K_{H_2 \to H_1} \hookrightarrow (T^*M)_{(H_1 \cap K)}/G$$

is a backwards Dirac map

and the geometric property

$$S^{K}_{H_2 \to H_1} \hookrightarrow (T^*M)_{(H_1 \cap K)}/G$$

is a backwards Dirac map

So we realize the singular Poisson quotient space $(T^*M)/G$ as a stratified fiber bundle over the symplectic quotient $\mathbf{J}^{-1}(0)/G$ where the total spaces are Dirac submanifolds of the ambient Poisson strata

HAPPY BIRTHDAY TUDOR